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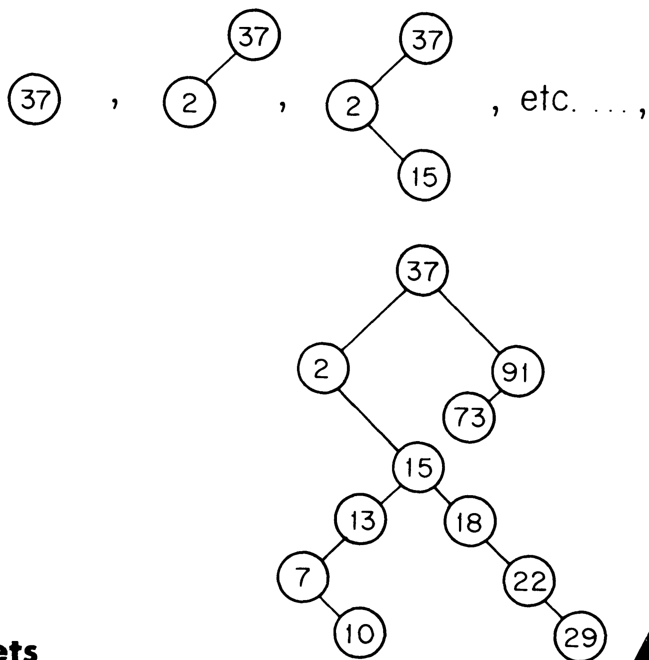
"Why don't you just . . . ?"



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BONNESEN-STYLE ISOPERIMETRIC INEQUALITIES

ROBERT OSSERMAN

The classical isoperimetric inequality states that, for a simple closed curve C of length L in the plane, the area A enclosed by C satisfies

$$L^2 \geqslant 4\pi A. \tag{1}$$

Since equality holds when C is a circle, it follows that the circle encloses maximum area among all curves of the same length. It does *not* follow that the circle is the *only* curve enclosing maximum area. That statement normally requires a separate proof.

During the 1920's, Bonnesen proved a series of inequalities of the form

$$L^2 - 4\pi A \geqslant B \tag{2}$$

where the quantity B on the right-hand side is an expression having the following three basic properties:

1. B is non-negative;
2. B can vanish only when C is a circle;
3. B has geometric significance.

Because of Property 1, any Bonnesen inequality implies the isoperimetric inequality (1).

From Property 2, it follows that equality can hold in (1) only when C is a circle.

The effect of Property 3 is to give a measure of the curve's "deviation from circularity."

Our purpose here is, first, to review what is known for plane domains. In particular, we include ten different inequalities of the form (2), all of which have Property 1 of Bonnesen's inequality and hence imply the isoperimetric inequality. These inequalities have been obtained by various authors using a variety of methods. We show that in fact nine of the ten follow in an elementary fashion from one basic inequality; only the last needs a separate proof.

Next we note that certain of the inequalities given hold more generally for domains on curved surfaces, provided the Gauss curvature is nowhere positive.

Finally, we show that certain of these inequalities generalize to arbitrary curved surfaces.

To elaborate on this last point, let us note one form that the isoperimetric problem can take for domains on surfaces. Let D be a simply connected domain of area A, bounded by a simple closed curve of length L. Suppose that the Gauss curvature K is bounded above on D by a constant M. Let D' be a geodesic disk on the complete simply connected surface S of constant curvature M, and choose the radius so that the area of D' equals the area A of D. (Thus S is a sphere if $M>0$, S is the plane if $M=0$, and S is (up to a constant factor) the hyperbolic plane if $M<0$.) Let L' be the length of the boundary of D'. Then the isoperimetric property is that $L \geqslant L'$, with equality if and only if D is isometric to D'.

Since the length of a geodesic circle on a constant curvature surface and the area enclosed are both easily calculated, the inequality $L \geqslant L'$ can be written explicitly. It takes the form

$$L^2 \geqslant 4\pi A - MA^2$$

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His initial work was on Riemann surfaces and conformal mapping; later he became interested in minimal surfaces and differential geometry; more recently he has turned to the study of isoperimetric inequalities and of geometric influences on the spectrum of the Laplacian.—*Editors*

where $K \leqslant M$ on D. This is the isoperimetric inequality on curved surfaces. A Bonnesen-style inequality would therefore be of the form

$$L^2 - 4\pi A + MA^2 \geqslant B$$

where B has the three properties listed at the outset. The second property in this case is that B can vanish only if D is isometric to a geodesic disk with constant curvature $K \equiv M$.

Since the history of this subject has been the source of much confusion and a certain amount of amusement in itself, the plan of presentation adopted here is the following.

Part I is largely expository. We present the various Bonnesen-style inequalities together with some indications about their proofs. Section A deals with plane domains and Section B with surfaces.

Part II is historical, and includes precise references where detailed proofs can be found.

Part III presents further elaborations of the basic themes of Part I. Section A contains a discussion of how the various expressions for B in inequality (2) compare quantitatively, as well as indications about related inequalities. Section B gives applications to the problem of estimating the lowest eigenvalue λ_1 of the Laplacian in a domain. Finally, in Section C we note how various of the results presented generalize to higher dimensions.

We call attention to the special status of Section B; it could be subtitled "How the geometry and topology of a drum affects its fundamental tone." This section may be omitted by a reader interested in purely geometric questions. On the other hand, it may be read separately by those interested in eigenvalue problems, as it provides a review of some basic results and shows how several of them may be derived in an elementary fashion from the isoperimetric inequalities presented here. However, this section is probably best seen in the context of the whole paper, since it illustrates how the purely geometric results may be applied to other problems and how certain of those problems in fact provided the impetus that led to the discovery of new isoperimetric inequalities.

A word about references: Whenever an author is referred to in the text there is a corresponding entry in the Bibliography. No further notation is given except when an author has several entries in the Bibliography.

It is a pleasure to thank Michael Beeson, Marcel Berger, Isaac Chavel, and David Singmaster for helpful conversations and suggestions during the preparation of this paper; also Yu. D. Burago for providing a key argument for handling the case $M > 0$ of Theorem 6 and its corollary.

I. Bonnesen-style Inequalities

A. Plane domains. Barring indications to the contrary, the following notation will be used throughout this section.

D is a simply connected domain,
C a simple closed curve, the boundary of D,
A the area of D,
L the length of C,
R the circumradius of D: the radius of the circumscribed circle,
ρ the inradius of D: the radius of an inscribed circle.

Each domain has a unique circle of smallest radius that encloses it; that is the circumscribed circle. There may not be a unique inscribed circle—that is, a unique circle of maximum radius lying in $D \cup C$—but the maximum radius of all such circles is well-defined, and is the inradius of D.

We begin with a statement of the results, and then discuss how they are related and their proofs.

LEMMA 1. *Let r, L, A be any three positive numbers. Then the following inequalities are equivalent*:

$$L^2 - 4\pi A > (L - 2\pi r)^2 \tag{3}$$

$$L^2-4\pi A>\left(\frac{A}{r}-\pi r\right)^2 \tag{4}$$

$$L^2-4\pi A>\left(L-\frac{2A}{r}\right)^2 \tag{5}$$

$$L-2\pi r>\frac{A}{r}-\pi r \tag{6}$$

$$L-\frac{2A}{r}>\pi r-\frac{A}{r} \tag{7}$$

$$\frac{(L-2\pi r)^2-\left(L-\frac{2A}{r}\right)^2}{\left(\frac{A}{r}-\pi r\right)}>0 \qquad (A\neq\pi r^2) \tag{8}$$

$$rL>A+\pi r^2 \tag{9}$$

$$\frac{L-\sqrt{L^2-4\pi A}}{2\pi}<r<\frac{L+\sqrt{L^2-4\pi A}}{2\pi} \qquad (L^2>4\pi A). \tag{10}$$

REMARK. This lemma seems at first sight quite paradoxical. The first three inequalities, (3), (4), and (5), all give lower bounds for the same quantity: $L^2-4\pi A$. The next three inequalities, (6), (7), (8), imply that the lower bounds in the first three are all different. Yet the first three are all algebraically equivalent, and, what is more, they are equivalent to the inequalities asserting that they are different.

THEOREM 1. *Let D be a plane domain bounded by a rectifiable Jordan curve of length L. A,ρ,R denote the area, inradius, and circumradius of D. Then if r satisfies $\rho<r<R$, the inequalities* (3)–(10) *are all valid.*

Theorem 1 is the main result from which the various Bonnesen inequalities may be derived. It is more convenient to formulate two separate theorems (Theorems 2 and 3 below), which together are equivalent to Theorem 1. For later reference, it is convenient to replace the set of equivalent strict inequalities of Lemma 1 by two separate sets of weak inequalities.

LEMMA 2. *For any positive numbers ρ,L,A, the following inequalities are equivalent*:

$$L^2-4\pi A\geqslant(L-2\pi\rho)^2 \tag{11}$$

$$L^2-4\pi A\geqslant(A-\pi\rho^2)^2/\rho^2 \tag{12}$$

$$L^2-4\pi A\geqslant\left(L-\frac{2}{\rho}A\right)^2 \tag{13}$$

$$\rho L\geqslant A+\pi\rho^2. \tag{14}$$

These imply the further inequality

$$\rho\geqslant\frac{L-\sqrt{L^2-4\pi A}}{2\pi}=\frac{2A}{L+\sqrt{L^2-4\pi A}}. \tag{15}$$

If $L\geqslant 2\pi\rho$, and $L^2\geqslant 4\pi A$, then (15) implies the previous inequalities, and (11)–(15) are all equivalent.

THEOREM 2. *If a rectifiable Jordan curve of length L bounds a domain of area A with inradius ρ, then* (11)–(15) *are satisfied.*

THEOREM 3. *If a rectifiable Jordan curve of length L bounds a domain of area A and circumradius R, then the following inequalities are valid*:

$$L^2-4\pi A\geqslant(2\pi R-L)^2 \tag{16}$$

$$L^2-4\pi A \geqslant (\pi R^2-A)^2/R^2 \tag{17}$$

$$L^2-4\pi A \geqslant \left(L-\frac{2}{R}A\right)^2 \tag{18}$$

$$RL \geqslant A+\pi R^2 \tag{19}$$

$$R \leqslant \frac{L+\sqrt{L^2-4\pi A}}{2\pi}=\frac{2A}{L-\sqrt{L^2-4\pi A}}. \tag{20}$$

THEOREM 4. *With L, A, ρ, and R as above, the following inequalities hold*:

$$L^2-4\pi A \geqslant \pi^2(R-\rho)^2 \tag{21}$$

$$L^2-4\pi A \geqslant A^2\left(\frac{1}{\rho}-\frac{1}{R}\right)^2=\frac{A^2}{R^2\rho^2}(R-\rho)^2 \tag{22}$$

$$L^2-4\pi A \geqslant \frac{L^2}{(R+\rho)^2}(R-\rho)^2. \tag{23}$$

Before turning to the proofs, let us make several comments.

First, note that we have exhibited nine inequalities of Bonnesen type: (11)–(13), (16)–(18), and (21)–(23). The last three obviously have all three properties of a Bonnesen inequality, since the right-hand side can vanish only if $R=\rho$, in which case the curve must be a circle of radius R. Of the other inequalities, (12) and (17) most clearly have the same property, since A equal to $\pi\rho^2$ or πR^2 can only happen if the curve coincides with its inscribed or circumscribed circle, respectively. The same property holds for (11) and (18), but *not* for (13) and (16). For example the right-hand side of (13) vanishes for any polygon circumscribed about a circle. (See Singmaster for a full description of curves satisfying $L/A=2/\rho$; Also Bonnesen and Fenchel, p. 82.)

Next, we may well ask what the point is of this welter of inequalities. As stated earlier, one point is that they have all appeared in different places under different guises, and it is of some interest to note that they all follow from the single inequality (9), valid for $\rho<r<R$. (That inequality for the case of convex curves appears in Bonnesen's original paper in 1921.) Another point is that various inequalities prove to be more appropriate for different purposes, as we shall see later.

Let us prove our way backward through these results starting with Theorem 4.

Subtracting (15) from (20) gives (21).

From (13) and (18), we have

$$\sqrt{L^2-4\pi A} \geqslant \frac{2}{\rho}A-L, \qquad \sqrt{L^2-4\pi A} \geqslant L-\frac{2}{R}A. \tag{24}$$

Adding these two gives (22).

Finally, multiplying the first inequality in (24) by ρ, the second by R, and then adding, yields (23).

Thus Theorem 4 is an immediate consequence of Theorems 2 and 3.

To prove Theorems 2 and 3, let us consider first the case where C is a convex polygon. For any positive number t, let C_t be the *exterior parallel curve* to C at distance t, i.e., the set of all points lying outside D at a distance t from C. $C(t)$ consists of a set of line segments, each parallel to a given side of C and of the same length, together with circular arcs of radius t with centers at each vertex. Since these circular arcs together make up one complete circle, one sees that the length $L(t)$ of C_t is given by

$$L(t)=L+2\pi t, \tag{25}$$

while the area of the domain D_t enclosed by C_t is equal to

$$A(t)=A+Lt+\pi t^2. \tag{26}$$

The critical point of the proof is the observation that if t lies in the interval $\rho<t<R$, then every circle of radius t whose center lies in D_t must intersect C. That is clear, since by the definition of ρ

and R such a circle cannot lie inside D nor surround it; while by the definition of D_t, it can also not lie totally in the exterior of C.

We now use this property of D_t to get a different estimate of its area $A(t)$. We let P be an *arbitrary* polygon (closed or not), and let E be defined by: $p \in E \Leftrightarrow$ the circle of radius t centered at p intersects P. Divide E into subsets E_k defined by $p \in E_k \Leftrightarrow$ the circle of radius t centered at p intersects P in exactly k points. Then we shall show that the areas A_k of E_k and the length L of P are related by

$$\Sigma k A_k = 4tL. \tag{27}$$

Observe that in this formula and in subsequent arguments we may ignore circles tangent to a side of P or passing through a vertex, since the centers of such circles will lie on a union of line segments and circular arcs and will contribute nothing to the areas A_k. In view of this remark, if we consider *closed* polygons P, then every circle will intersect P in an even number of points, so that $A_k = 0$ whenever k is odd. In that case, (27) implies that the total area A of the set E will satisfy

$$2A = 2(A_2 + A_4 + A_6 + \cdots) \leqslant 2A_2 + 4A_2 + 6A_6 + \cdots = 4tL. \tag{28}$$

Combining this with (26) yields

$$2tL \geqslant A(t) = A + Lt + \pi t^2 \tag{29}$$

for every t in the interval $\rho < t < R$. Letting t tend to ρ and R respectively yields (14) and (19).

Before proceeding further with the argument, let us prove (27). This is easily done by induction. First, if P is a single line segment S of length s, then a circle can only intersect it in one or two points. Denote by e_1, e_2 the corresponding set of centers, and by a_1, a_2 their areas. Then (27) takes the form

$$a_1 + 2a_2 = 4ts. \tag{30}$$

To verify this, note that the total set E in this case consists of a rectangle of dimensions s by $2t$, surmounted by two semicircles of radius t. If Δ_1, Δ_2 denote the full disks of radius t centered at the endpoints of S, then one has

$$e_1 = (\Delta_1 \cup \Delta_2) \setminus (\Delta_1 \cap \Delta_2),$$
$$e_2 = E \setminus (\Delta_1 \cup \Delta_2).$$

We now make the following observation. Consider two copies of the set E placed one on top of the other; remove the disk Δ_1 from one sheet and Δ_2 from the other. Then what is left will cover each point of e_1 exactly once and each point of e_2 twice. Its total area therefore corresponds to the left side of (30). On the other hand, the area of each sheet is exactly that of the $s \times 2t$ rectangle, and this proves (30).

Finally, suppose that (27) holds for all n-sided polygons. Adjoining an extra side S, we have the equation (30) for this single side. Adding (27) to (30), we see that the right-hand sides add up to the correct expression for the $(n+1)$-sided polygon P', while the sets e_1, e_2 corresponding to S are distributed among the sets E_k in such a manner that they increase the coefficient k by one or two, respectively, and thus the left-hand sides of (27) and (30) add up to the correct expression for the polygon P'. Thus (27) holds for all polygons.

We can now quickly complete the proof of Theorem 3. We have shown that (19) holds for any convex polygon. Let P be an arbitrary simple closed polygon. Its convex hull will be bounded by a convex polygon P'. The quantities L', A', R' associated with P' satisfy (19); i.e.,

$$R'L' \geqslant A' + \pi (R')^2.$$

But $L' \leqslant L$, $A' \geqslant A$, and $R' = R$. Thus (19) also holds for the original polygon P.

Finally, given an arbitrary rectifiable Jordan curve, one needs only a suitable approximation by Jordan polygons.

LEMMA 3. *Let D be a domain bounded by a rectifiable Jordan curve with L, A, and R defined as*

usual. Then there exists a sequence of Jordan polygons P_n such that the associated quantities L_n, A_n, R_n satisfy

$$L_n \leqslant L, \qquad A_n \to A, \qquad R_n \to R.$$

We omit the proof of the lemma; the main point is that the rectifiability of C has two consequences: first, one can approximate C by polygons whose length is arbitrarily close to L; and second, the area of C is zero. Removing closed loops from the approximating polygon in a suitable manner leaves a Jordan polygon with the desired properties.

Combining Lemma 3 with the fact that (19) holds for all Jordan polygons gives the validity of (19) for arbitrary rectifiable curves. By Lemma 2, the Bonnesen-type inequalities (16), (17), (18) must also hold. Since (20) is also an immediate consequence, Theorem 3 is proved.

One can proceed along similar lines to prove Theorem 2, but there is an additional complication. In order to prove (19) for arbitrary polygons it is sufficient to do it for convex polygons, since R is the same for a curve and its convex hull. However, ρ will generally increase under formation of the convex hull, so that it is necessary to prove (14) directly for arbitrary polygons. This can be done by an elaboration of the above argument, using (27) together with a generalized formula substituting for (26). (See Fejes-Tóth.) However, we shall instead give a brief indication of another method of attack.

LEMMA 4. *Let D be a domain bounded by a Jordan polygon C. For each r in the interval $0 \leqslant r \leqslant \rho$, let D_r be the set of points in D whose distance to C is greater than r, and let C_r be the boundary of D_r. If $L(r)$ is the length of C_r, then*

$$L(r) \leqslant L - 2\pi r, \tag{31}$$

and

$$A = \int_0^\rho L(r)\,dr. \tag{32}$$

We shall discuss the proof of this lemma in more detail in Part II. For the present we make only the following comments.

First, every point of C_r is at distance r from the polygon C; hence C_r consists of a finite union of circular arcs of radius r and line segments parallel to the sides of C.

Second, when r is small, (31) follows easily by simple arguments of elementary geometry. Also, when C is convex, (31) follows easily for all $r \leqslant \rho$. However, when C is non-convex, more elaborate arguments are needed.

Finally, inserting (31) into (32) yields inequality (14) for Jordan polygons, and an approximation argument analogous to Lemma 3 proves that (14) holds for rectifiable Jordan curves. Combining this with Lemma 2 concludes the proof of Theorem 2.

Theorem 1 is an immediate consequence: Since the function $\pi r^2 - Lr + A$ is a convex function of r, and it is non-positive for $r = \rho$ and $r = R$ by (14) and (19), it must be strictly negative for $\rho < r < R$. This proves (9), and by Lemma 1, the remaining inequalities (3)–(10).

Concerning the proofs of Lemmas 1 and 2, they are completely elementary, as can be seen by reducing each inequality to (9) and (14), respectively. More details about these lemmas and their significance will be given in Part III, Section A.

We conclude our discussion of the basic Bonnesen inequalities by stating one final example that is of particular interest historically.

For a convex curve C, let d be the minimum width of circular annuli containing C. Then one has the further inequality

$$L^2 - 4\pi A \geqslant 4\pi d^2. \tag{33}$$

We shall discuss this inequality and its proof in Part II.

B. Domains on surfaces. The notation in this section is the same as in Section A, except that

domains D will be considered to lie on a surface or two-dimensional Riemannian manifold S. We denote by K the Gauss curvature of S. The notions of inradius and circumradius require more careful analysis, since one does not have inscribed and circumscribed circles in general.

For each point p in D, if d_p is the distance from p to the boundary of D, then for all $r \leqslant d_p$ we define the *metric disk* of radius r and center p to be the set of points in D whose distance to p is less than r. The *inradius* ρ of D is the maximum of d_p as p varies over D, or alternatively, the maximum radius of metric disks lying in D.

In case D is a plane domain, the above definition clearly coincides with the notion of the inradius given earlier. The problem in general is that a metric disk may not look much like a disk, even in the case that D is simply connected. For example, if S is a half-cylinder capped by a hemisphere, and D is a domain on S including the hemisphere and a long part of the cylinder, then for a point p halfway along the cylinder, metric disks will look like disks for small r, but become doubly connected domains as r grows larger. If one varies the topography of S by adding a number of mushroom-shaped bumps, then even though D is simply connected, metric disks on D can have arbitrarily high connectivity. A further complication is that even when a metric disk centered at a point p is simply connected, the "radii" of the disk may intersect each other; in other words, two geodesic segments starting at p may intersect again before distance r. (Take a cylinder of radius a and height $h=5a$, capped by a hemisphere of radius a. Consider the metric disk of radius $r=4a$ centered at a point where the hemisphere meets the cylinder.)

If a metric disk with center p and radius r has the property that no two geodesics starting at p intersect before a distance r, then we shall call it a *geodesic disk*. In the current standard terminology of differential geometry, a metric disk is the surjective image of a disk of radius r under the exponential map at p, and it is a geodesic disk if this map is bijective.

Happily, there is a large class of domains for which none of these complications arise. In fact, for simply connected domains whose Gauss curvature K is nowhere positive, it follows immediately from the Gauss–Bonnet theorem that distinct geodesics at p cannot intersect again, and hence every metric disk is a geodesic disk. In particular, if D is simply connected, with inradius ρ, and if $K \leqslant 0$ on D, then there exists a geodesic disk of radius ρ lying in D.

Inside a geodesic disk one can introduce polar geodesic coordinates, and they will be basic for the following discussion.

We start with a well-known and elementary lemma.

Lemma 5. *Let D_ρ be a geodesic disk of radius ρ on a surface S, and let A_ρ be its area. If $K \leqslant 0$ on D_ρ, then $A_\rho \geqslant \pi\rho^2$, with equality if and only if D_ρ is isometric to a plane disk of radius ρ.*

Proof. Using polar geodesic coordinates in D_ρ, the metric can be written as $ds^2 = dr^2 + g(r,\theta)d\theta^2$, where for each θ, the function $f(r) = \sqrt{g}\,(r,\theta)$ satisfies $f(0)=0, f'(0)=1$. Since the Gauss curvature is given in geodesic coordinates by

$$K = -\frac{1}{\sqrt{g}}\frac{\partial^2}{\partial r^2}\sqrt{g}\,, \tag{34}$$

the condition $K \leqslant 0$ implies $f''(r) \geqslant 0$. In view of the initial conditions, $f(r) \geqslant r$ for $0 \leqslant r \leqslant \rho$, and hence

$$A_\rho = \int_0^{2\pi}\int_0^{\rho}\sqrt{g}\,(r,\theta)\,dr\,d\theta \geqslant \pi\rho^2.$$

Equality can only hold if $f(r) \equiv r$ for all θ; i.e., $g(r,\theta) \equiv r^2$. In that case the metric is the standard Euclidean metric.

Theorem 5. *If D is a simply connected domain, and $K \leqslant 0$ on D, then inequalities* (11)–(15) *are valid.*

Corollary. *The isoperimetric inequality $L^2 \geqslant 4\pi A$ holds for simply connected domains with $K \leqslant 0$, and equality holds only when $K \equiv 0$ and the domain is a disk.*

Proof of the Corollary. Inequality (12) states that

$$L^2 - 4\pi A \geqslant \frac{(A-\pi\rho^2)^2}{\rho^2}.$$

Thus $L^2 \geqslant 4\pi A$, with equality only if $A = \pi\rho^2$. By definition of ρ, there exists a geodesic disk D_ρ of radius ρ with $D_\rho \subset D$. Thus, using Lemma 5,

$$A \geqslant A_\rho \geqslant \pi\rho^2.$$

If $A = \pi\rho^2$, then $A = A_\rho$, so that $D = D_\rho$, and $A_\rho = \pi\rho^2$, so that D_ρ is isometric to a plane disk.

Concerning the proof of Theorem 5, it is based on a suitable generalization of Lemma 4. The basic idea is to start with a curve C in a suitable class, guaranteeing that the level curves C_r will be sufficiently regular so that $L(r)$ is defined and (32) holds. Then observe that the derivative $L'(r)$ is equal to the integral of the geodesic curvature of C_r along smooth parts together with an appropriate term at corners. Finally use the Gauss-Bonnet theorem, together with the condition $K \leqslant 0$ and a careful analysis of what happens at a corner, to show that $L'(r) \leqslant -2\pi$ for $0<r<\rho$. This yields (31) and proves Lemma 4. Again, inequality (14) is an immediate consequence.

In order to state corresponding theorems for surfaces satisfying $K \leqslant M$ for some constant $M \neq 0$, let us introduce some more notation; let

D_ρ^M = geodesic disk of radius ρ on the complete simply-connected surface of constant curvature $K \equiv M$,
A_ρ^M = area of D_ρ^M,
L_ρ^M = length of boundary of D_ρ^M.

The explicit expressions for these quantities are:

	$M=-\alpha^2<0$	$M=0$	$M=\alpha^2>0$
L_ρ^M	$2\pi\dfrac{\sinh\alpha\rho}{\alpha}$	$2\pi\rho$	$2\pi\dfrac{\sin\alpha\rho}{\alpha}$
A_ρ^M	$4\pi\dfrac{\sinh^2\frac{1}{2}\alpha\rho}{\alpha^2}$ $=2\pi\dfrac{\cosh\alpha\rho-1}{\alpha^2}$	$\pi\rho^2$	$4\pi\dfrac{\sin^2\frac{1}{2}\alpha\rho}{\alpha^2}$ $=2\pi\dfrac{1-\cos\alpha\rho}{\alpha^2}$

The following equation is easily verified in all three cases.

$$\left(L_\rho^M\right)^2 - 4\pi A_\rho^M + M\left(A_\rho^M\right)^2 = 0 \tag{35}$$

Thus, the isoperimetric inequality on a surface of constant curvature $K \equiv M$ takes the form

$$L^2 \geqslant 4\pi A - MA^2. \tag{36}$$

Namely, given a domain D of area A, if ρ is chosen so that A_ρ^M equals A, then (35) and (36) imply $L \geqslant L_\rho^M$, so that the disk D_ρ^M has minimum boundary length among all domains of the same area.

The dual statement requires a bit more care. The function $f(x) = 4\pi x - Mx^2$ satisfies $f'(x) = 4\pi - 2Mx$. Thus $f(x)$ is monotone increasing for all $x \geqslant 0$ when $M \leqslant 0$, and for all x in the interval $0 \leqslant x \leqslant 2\pi/M$ when $M>0$. It follows from (35) and (36) that if $L = L_\rho^M$, then $A \leqslant A_\rho^M$ provided either $M \leqslant 0$, or else when $M>0$, if $A \leqslant 2\pi/M$. That this latter condition is necessary is clear from the case of a sphere of radius r, where $M = 1/r^2$ and the condition becomes $A \leqslant 2\pi r^2$, which is the area of a hemisphere. Indeed, a Jordan curve of length L on the sphere will bound two domains, one of which will have area at most $2\pi r^2$, and for that one the inequality $A \leqslant A_\rho^M$ holds where A_ρ^M is the smaller of the two areas bounded by the geodesic circle of length L. It may be worth noting that if S is a sphere of radius r, and if we denote by A and $\bar{A}$ the area of the two domains bounded by the curve C, then (36) can be written as

$$L^2 \geqslant \frac{1}{r^2}A(4\pi r^2 - A) = \frac{A\bar{A}}{r^2} \tag{37}$$

in which the two domains enter with complete symmetry.

We want now a Bonnesen-style version of (36), implying in particular that equality can hold *only* for a geodesic disk. Also, we want to allow surfaces of variable curvature. First, a preliminary lemma.

LEMMA 6. *Let $\hat{L},\hat{A},M$ be any three numbers satisfying $\hat{A}>0$ and*

$$\hat{L} - 4\pi\hat{A} + M\hat{A}^2 = 0. \tag{38}$$

Then for any positive numbers L and A, the following are equivalent:

$$\hat{L}L \geqslant 2\pi(A+\hat{A}) - M\hat{A}A, \tag{39}$$

$$L^2 - 4\pi A + MA^2 \geqslant \left(L - \frac{\hat{L}}{\hat{A}}A\right)^2 \tag{40}$$

$$L^2 - 4\pi A + MA^2 \geqslant (L-\hat{L})^2 + M(A-\hat{A})^2. \tag{41}$$

If furthermore, $MA<4\pi$, then these are equivalent to

$$L^2 - 4\pi A + MA^2 \geqslant \left[\frac{2\pi}{\hat{L}}(A-\hat{A})\right]^2, \qquad \hat{L}>0. \tag{42}$$

Proof. Both (40) and (41) reduce to (39) by adroit use of (38). To prove (42), note that (38) implies that $M\hat{A} \leqslant 4\pi$, since $\hat{A}>0$, and $\hat{A}(4\pi - M\hat{A}) = \hat{L}^2 \geqslant 0$. If also $MA<4\pi$, then

$$\frac{1}{A} + \frac{1}{\hat{A}} > \frac{M}{2\pi},$$

which implies that the right-hand side of (39) is positive. Hence (39) implies $\hat{L}>0$. Squaring both sides of (39) and combining with (38) yields (42).

THEOREM 6. *Let D be a simply connected domain whose Gauss curvature K satisfies $K \leqslant M$. Let ρ be the inradius of D, A the area of D, L the length of its boundary. If $MA<4\pi$, then* (38)–(42) *are all valid, with $\hat{L}=L_\rho^M$, $\hat{A}=A_\rho^M$.*

Note that when $M=0$, inequalities (39)–(42) reduce to (11)–(14). Hence Theorem 6 is a direct generalization of Theorem 2.

COROLLARY. *The isoperimetric inequality* (36) *holds for a simply connected domain D with Gauss curvature $K \leqslant M$. Equality holds if and only if $K \equiv M$ and D is a geodesic disk.*

Proof of the Corollary. If $MA \geqslant 4\pi$, then (36) holds trivially, with strict inequality. If $MA<4\pi$, then we may apply Theorem 6, and (36) follows from (40) or (42). Equality in (36) implies $A=\hat{A}=A_\rho^M$, using (42), (or (41) in case $M>0$). If D_ρ is a geodesic disk of radius ρ included in D, and A_ρ its area, then

$$A \geqslant A_\rho \geqslant A_\rho^M$$

where the second inequality follows from the fact that $K \leqslant M$. (See the corollary to Lemma 7 below.) Furthermore, $A_\rho = A_\rho^M$ implies $K \equiv M$ on D_ρ, and $A=A_\rho$ implies that $D=D_\rho$. This proves the corollary under the assumption that there exists a geodesic disk of radius ρ lying in D. When $M \leqslant 0$, such a disk always exists, as we have noted before Lemma 5, and there is nothing further to prove. When $M>0$, we have to use a result of Burago [27] that if p is a point of D whose distance to the boundary is ρ, then there exists a geodesic disk D_r in D with center p and radius $r=\min\{\rho,\pi/\alpha\}$, where $M=\alpha^2$, $\alpha>0$. If $\rho<\pi/\alpha$, then $r=\rho$ and D_r is the desired geodesic disk D_ρ. The alternative is $\rho \geqslant \pi/\alpha$, in which case $D_r = D_{\pi/\alpha}$, and its area $A_{\pi/\alpha}$ satisfies

$$A \geqslant A_{\pi/\alpha} \geqslant A_{\pi/\alpha}^{\alpha^2} = \frac{4\pi}{\alpha^2} = \frac{4\pi}{M}$$

by the corollary to Lemma 7 below. This contradicts the assumption $MA < 4\pi$, and completes the proof of the corollary.

We next prove, in a slightly stronger form than is needed in the argument above, a standard comparison lemma. We shall need this stronger form for subsequent arguments.

LEMMA 7. *Let $f(r)$ be continuously differentiable on the interval $0 \leqslant r \leqslant r_0$ and suppose that, except at a finite number of points in the interval, $f''(r)$ exists and satisfies*

$$f''(r) + cf(r) \geqslant 0, \qquad f(0) = 0, \qquad f'(0) = a \tag{43}$$

for some constants, a, c. Let $h(r)$ be the unique solution of

$$h''(r) + ch(r) = 0, \qquad h(0) = 0, \qquad h'(0) = 1. \tag{44}$$

Let s be any number such that $h(r) > 0$ for $0 < r < s$, and let $r_1 = \min\{r_0, s\}$. Then

$$f(r) \geqslant ah(r), \quad \text{for } 0 \leqslant r \leqslant r_1, \tag{45}$$

and

$$f'(r) \geqslant ah'(r), \quad \text{for } 0 \leqslant r \leqslant \min\left\{r_0, \frac{s}{2}\right\}. \tag{46}$$

Proof. Let $\phi(r) = f(r)/h(r)$. Then

$$(h^2\phi')' = (hf' - fh')' = hf'' - fh'' \geqslant 0$$

except at the singular points. By the mean value theorem, $h^2\phi'$ is a (weakly) monotone increasing function, and hence (even including singular points)

$$(h^2\phi')(r) \geqslant (h^2\phi')(0) = 0, \qquad 0 \leqslant r \leqslant r_1. \tag{47}$$

Thus $\phi'(r) \geqslant 0$, and hence

$$\frac{f(r)}{h(r)} = \phi(r) \geqslant \lim_{r \to 0} \frac{f(r)}{h(r)} = \frac{f'(0)}{h'(0)} = a, \quad \text{for } 0 \leqslant r \leqslant r_1.$$

This proves (45). Returning to (47), $hf' \geqslant fh'$, or

$$f'(r) \geqslant \frac{f(r)}{h(r)} h'(r) \geqslant ah'(r), \quad \text{providing } h'(r) \geqslant 0. \tag{48}$$

But $h(r)$ can be written explicitly, and in fact, using the notation above (35),

$$h(r) = \frac{1}{2\pi} L_r^c. \tag{49}$$

Thus for $c \leqslant 0$: $h(r) > 0$ and $h'(r) > 0$ for all $r > 0$. Then (45) and (46) hold in the whole interval $0 \leqslant r \leqslant r_0$. For $c = \alpha^2 > 0$: $h(r) > 0$ for $0 < r < \pi/\alpha$, and $h'(r) > 0$ for $0 < r < \pi/2\alpha$. Thus (45) and (46) hold, with $s = \pi/\alpha$. This proves the lemma.

COROLLARY. *Let D_ρ be a geodesic disk of radius ρ, and let A_ρ be its area. If $K \leqslant M$ on D_ρ, then*

$$A_\rho \geqslant A_\rho^M, \tag{50}$$

and equality holds if and only if $K \equiv M$ on D_ρ.

Proof. As in the proof of Lemma 5, introduce geodesic polar coordinates in D_ρ, and for fixed θ, let $f(r) = \sqrt{g}\,(r, \theta)$. Then using (34), and the condition $K \leqslant M$, $f(r)$ satisfies (43), with $a = 1$, $c = M$, and $r_0 = \rho$. Thus we may apply the lemma; from (45) and (49) we have

$$A_{r_1} = \int_0^{2\pi} \int_0^{r_1} \sqrt{g}\,(r, \theta)\, dr\, d\theta \geqslant 2\pi \int_0^{r_1} h(r)\, dr = A_{r_1}^M. \tag{51}$$

Furthermore, the equality holds if and only if $\sqrt{g}\,(r, \theta) \equiv h(r)$, in which case $K \equiv M$, by (34).

This proves the corollary provided $r_1 = \rho$. That is always the case if $M \leqslant 0$. When $M = \alpha^2 > 0$, then $r_1 = \min\{\rho, \pi/\alpha\}$, so that $r_1 = \rho$ unless $\rho > \pi/\alpha$. But in that case, (51) implies that

$$A_\rho > A_{\pi/\alpha} \geqslant A^{\alpha^2}_{\pi/\alpha} = \frac{4\pi}{\alpha^2} = \max_r A_r^{\alpha^2}$$

so that (50) holds with strict inequality, and the corollary is proved in all cases.

In order to prove Theorem 6, one needs another comparison lemma, essentially a non-homogeneous version of Lemma 7.

LEMMA 8. *Let $F(r)$ be continuously differentiable for $0 \leqslant r \leqslant r_0$, and suppose that except for a finite set of points, $F''(r)$ exists and satisfies*

$$F''(r) + cF(r) \leqslant b, \qquad F(0) = 0, \qquad F'(0) = a, \tag{52}$$

for some constants a, b, c. Let $H(r)$ be the unique solution of

$$H''(r) + cH(r) = b, \qquad H(0) = 0, \qquad H'(0) = a. \tag{53}$$

Then the inequality

$$F(r) \leqslant H(r) \tag{54}$$

holds for $0 \leqslant r \leqslant r_0$ when $c \leqslant 0$, and for $0 \leqslant r \leqslant \min\{r_0, \pi/\alpha\}$, when $c = \alpha^2$, $\alpha > 0$.

Proof. For $\varepsilon > 0$, define $H_\varepsilon(r)$ by

$$H_\varepsilon''(r) + cH_\varepsilon(r) = b, \qquad H_\varepsilon(0) = 0, \qquad H_\varepsilon'(0) = a + \varepsilon.$$

Let $f(r) = H_\varepsilon(r) - F(r)$. Then f satisfies (43), with $a = \varepsilon$. Hence by Lemma 7,

$$f(r) \geqslant \varepsilon h(r) > 0 \quad \text{for } 0 < r < r_1,$$

where $h(r)$ is given by (49). Since this is true for every $\varepsilon > 0$, the lemma is proved.

Let us note that the solutions $H(r)$ of (53) can be written explicitly as

$$H(r) = \frac{1}{2\pi}\left(aL_r^c + bA_r^c\right) \tag{55}$$

in terms of the length and area functions defined above (35).

Finally, we need one further result.

LEMMA 9. *Let D be a simply connected domain on a two-dimensional real analytic Riemannian manifold with analytic metric. Suppose D is bounded by an analytic Jordan curve C of length L. Let A be the area of D, and*

$$F(r) = \text{area }\{p \in D \mid d(p, C) < r\}.$$

If $K \leqslant M$ on D, then $F(r)$ satisfies the hypotheses of Lemma 8 with

$$a = L, \qquad b = MA - 2\pi, \qquad c = M, \tag{56}$$

and where $r_0 = \rho$, the inradius of D.

Let us assume for the moment that Lemma 9 is proved. In case $M = \alpha^2 > 0$, then as we have seen in the proof of the Corollary to Theorem 6, the assumption $MA < 4\pi$ implies that $\rho < \pi/\alpha$. Hence by Lemma 8, (54) holds for $0 \leqslant r \leqslant \rho$. By (55) and (56):

$$A = F(\rho) \leqslant H(\rho) = \frac{1}{2\pi}\left(LL_\rho^M + (MA - 2\pi)A_\rho^M\right),$$

or

$$LL_\rho^M \geqslant 2\pi\left(A + A_\rho^M\right) - MAA_\rho^M. \tag{57}$$

But this is exactly (39), and together with (35) and Lemma 6, this proves Theorem 6 in the analytic case. In order to prove Theorem 6 more generally, there are two possibilities: either approximate a given domain and metric by domains with analytic metrics and analytic boundary curves, or else prove Lemmas 8 and 9 under more general hypotheses.

The proof of Lemma 9 depends on the following fundamental result:

LEMMA 10. *Under the hypotheses of Lemma* 9, *let*

$$C_r = \{ p \in D \mid d(p,C) = r \}.$$

Then C_r is a rectifiable curve for $0 \leqslant r < \rho$. The length $L(r)$ of C_r is a continuous function on $0 \leqslant r < \rho$, and

$$F(r) = \int_0^r L(t)\,dt. \tag{58}$$

Finally, $L'(r)$ exists except at a finite number of values of r, and satisfies

$$L'(r) \leqslant \int_{D_r}\!\!\int K - 2\pi, \tag{59}$$

where

$$D_r = \{ p \in D \mid d(p,C) > r \}.$$

To see that Lemma 9 follows from Lemma 10, note that area $(D_r) = A - F(r)$, so that if $K \leqslant M$ on D, then from (58) and (59):

$$F''(r) = L'(r) \leqslant M(A - F(r)) - 2\pi,$$

which is the conclusion of Lemma 9.

Concerning the proof of Lemma 10, a fair amount of delicate analysis is needed to show that C_r is piecewise-smooth and $L(r)$ is continuous. (In fact $L(r)$ can be discontinuous even for a C^∞-plane Jordan curve.) Equation (58) is then standard, and (59) follows from a careful application of the Gauss–Bonnet formula, taking into account what happens at corners of the curve C_r.

This completes our discussion of the straightforward generalization of Bonnesen inequalities to domains on curved surfaces. A slight variation of the method gives other inequalities that are also of interest. In order to state them, let us use the following notation: for any number x, let

$$x^+ = \max\{x, 0\}.$$

and let

$$\omega^+ = \int\!\!\int_D K^+.$$

THEOREM 7. *Let D be a simply connected domain with inradius ρ. Then*

$$\rho L \geqslant A + \left(\pi - \tfrac{1}{2}\omega^+\right)\rho^2. \tag{60}$$

For $\omega^+ < 2\pi$, this is equivalent to

$$L^2 - 4\pi A + 2\omega^+ A \geqslant (L - (2\pi - \omega^+)\rho)^2. \tag{61}$$

COROLLARY. *For simply-connected domains one has the isoperimetric inequality*

$$L^2 \geqslant 4\pi A - 2\omega^+ A. \tag{62}$$

The proof of Theorem 7 is an easy consequence of Lemma 10. Namely, from (59),

$$L'(r) \leqslant \int_{D_r}\!\!\int K - 2\pi \leqslant \int_{D_r}\!\!\int K^+ - 2\pi \leqslant \omega^+ - 2\pi,$$

so that

$$L(t) = L + \int_0^t L'(r)\,dr \leqslant L + (\omega^+ - 2\pi)t,$$

and hence from (58),

$$A = F(\rho) = \int_0^\rho L(t)\,dt \leqslant \rho L + \tfrac{1}{2}(\omega^+ - 2\pi)\rho^2.$$

Multiplying this by $2(\omega^+ - 2\pi)$, and adding L^2 gives (61), provided $\omega^+ - 2\pi < 0$.

Inequality (62) is an immediate consequence of (61) if $\omega^+ < 2\pi$, and it is trivially true for $\omega^+ \geqslant 2\pi$.

Note that Theorem 7 is essentially a generalization of Theorem 5, since the hypothesis $K \leqslant 0$ implies $\omega^+ = 0$, and then inequalities (60) and (61) reduce to the original Bonnesen inequalities (14) and (11), respectively.

Finally, we note a general inequality that contains both (60) and (57) as special cases.

For any real number c, set

$$\omega_c^+ = \int\int_D (K-c)^+.$$

Then from (59),

$$\begin{aligned} L'(r)+2\pi &\leqslant \int_{D_r}\int K = \int_{D_r}\int (K-c) + c\,\text{Area}\,(D_r) \\ &\leqslant \int_{D_r}\int (K-c)^+ + c(A-F(r)) \\ &\leqslant \omega_c^+ + cA - cF(r). \end{aligned}$$

Since from (58), $F''(r) = L'(r)$, we have the differential inequality

$$F''(r) + cF(r) \leqslant \omega_c^+ + cA - 2\pi.$$

Using Lemmas 8 and 9 as before, with $b = \omega_c^+ + cA - 2\pi$, we find

$$LL_\rho^c \geqslant 2\pi(A + A_\rho^c) - cAA_\rho^c - \omega_c^+ A_\rho^c. \tag{63}$$

If we choose $c=0$, then $\omega_c^+ = \omega^+$, and (63) reduces to (60). On the other hand, if $K \leqslant M$ and we choose $c = M$, then $\omega_c^+ = 0$, and (63) reduces to (57). There may well be cases where an intermediate value of c gives the optimal inequality.

Inequality (63) can again be Bonnesenized, and written in the equivalent form

$$L^2 - 4\pi A + 2\omega_c^+ A + cA^2 \geqslant (A_\rho^c L - L_\rho^c A)^2/(A_\rho^c)^2, \tag{64}$$

which in turn yields the general isoperimetric inequality

$$L^2 \geqslant 4\pi A - 2\omega_c^+ A - cA^2. \tag{65}$$

When $c=0$, (65) reduces to (62), while when $K \leqslant M$ and $c = M$, (65) reduces to (36).

II. History

A. Plane curves. It is peculiarly appropriate to the strange history of the circle of ideas discussed here, that the first Bonnesen-style inequality should have been proved not by Bonnesen, but by F. Bernstein in 1905, and that, furthermore, Bernstein started by considering curves on the sphere, and then obtained a "Bonnesen inequality" for plane curves as a limiting case. Thus, the first inequality proved by Bernstein was of the form given in Theorem 6:

$$L^2 - 4\pi A + MA^2 \geqslant B, \tag{66}$$

where

$$M = \alpha^2 = \frac{1}{t^2}$$

is the constant curvature of a sphere of radius t. Bernstein considered convex curves on the sphere, and the width d of the narrowest circular annulus on the sphere containing the curve. He then proved (66) with the value

$$B = (2tg(t))^2(2\pi + g(t)^2); \qquad g(t) = \sin\frac{d}{4(1+2\pi)t}. \tag{67}$$

Since $B \geqslant 0$, with equality only if $d=0$, this gives the isoperimetric inequality for convex curves on a

sphere, with equality only for a circle. Then, since

$$\lim_{t\to\infty} B(t) = \frac{\pi}{2(1+2\pi)^2} d^2,$$

it follows from (66) that for plane convex curves:

$$L^2 - 4\pi A \geq \frac{\pi}{2(1+2\pi)^2} d^2. \tag{68}$$

In other words, this is precisely Bonnesen's inequality (33), but with a weaker constant. In his first paper in 1921, Bonnesen gave a different proof of this inequality, improving the constant to π^2, and still later, in 1924, he obtained the constant 4π, which he showed by an example was the best possible.

Of the various inequalities in Theorems 1–4, the first one to be stated explicitly was (14), proved by Liebmann in 1919 (p. 289–290) for convex curves in the plane. This inequality was also given by Bonnesen in 1921, and an equivalent one appears in Chisini. However, none of these authors noticed the equivalent "Bonnesen forms" (11)–(13) until Bonnesen himself in 1926 gave (11). What Bonnesen did in his first paper was to prove (14) and (19), and use them to derive the Bonnesen inequality (21).

All of these results were for convex curves only, and the extension to non-convex curves required essentially new methods.

The first results are due to Erhard Schmidt in 1939. Using analytic rather than geometric methods, he derives several Bonnesen-type inequalities for plane domains bounded by an arbitrary rectifiable Jordan curve ([**68**, pp. 690–694]). He does not, however, obtain the inequalities of Theorems 1 and 2 above. The first method to succeed here was integral geometry. The book of Blaschke (p. 26) gives a proof of (11) and (16) for convex curves, due to Santaló. Also using integral geometry, Hadwiger in 1941 [**41**] obtained results equivalent to inequalities (15) and (20) for arbitrary rectifiable Jordan curves. He does not appear to notice the connections with Bonnesen inequalities, however, until a later paper [**42**], where he derives the inequalities (12), (13), (17), (18), (22) and (23), but only for convex domains.

In the meanwhile, in the same volume of the journal that contains the first of Hadwiger's papers, there appeared a fundamental paper of Fiala. In it Fiala develops another method for proving Bonnesen inequalities for non-convex curves. That is the method of interior parallels, and, except for the proof of Theorem 3 above, it is the method used here. Fiala's principal focus is on obtaining isoperimetric inequalities on curved surfaces (see Section B below), but his paper applies in particular to the plane and is the first to give explicitly (on p. 336) (11) and (14) for non-convex curves. His proof is for analytic Jordan curves. One could then obtain the result for more general curves by approximation.

Returning to the integral geometry approach, this was developed in a whole series of papers by Santaló, starting around 1935. For detailed references, and for further proofs of Bonnesen inequalities, see his two books: [**65**, pp. 38–42] and [**66**, pp. 119–124]. Let us note here that a proof of Bonnesen's inequalities for non-convex curves can be obtained by combining Lemma 2 above with Scherk's observation that the argument used in a paper by Santaló [**63**] actually gives inequality (15).

Into this picture, although apparently unaware of any part of it, stepped Besicovitch in 1949. Making the normalization $\rho = 1$, he gives a proof of (14) for arbitrary Jordan curves; and furthermore, he gives the first characterization of those curves for which equality holds in (14).

To Fejes-Tóth in 1950, we owe the observation that by working with polygons one can drop the machinery of integral geometry and give direct elementary proofs of the formulas needed to prove Bonnesen's inequalities. It is his argument that we have given above in the proof of Theorem 3.

Meanwhile, not much progress had been made since Fiala in the use of interior parallels. A paper by Bol in 1941 gives a careful discussion of the method for convex curves, and proves the inequality $L'(r) \leq 2\pi$. He combines this with (32) to prove the standard isoperimetric inequality, but he does not observe that it also has the consequence (31), which, combined with (32), yields the stronger Bonnesen inequality.

The problem in dealing with curves that are neither convex nor polygons nor analytic is that the behavior of the inner parallels becomes quite difficult to describe. A paper of Sz.-Nagy finds an ingenious way of avoiding the whole problem of the rectifiability of the parallel curves C_r. He works directly with the area function $F(r)$ defined in Lemma 9, which is a monotone function, and shows that its right and left derivates at each point satisfy the same inequality (31) as $L(r)$, and $F(r)+\pi r^2$ is a concave function for $0 \leqslant r \leqslant \rho$. Since $F(0)=0$, $F(\rho)=A$, and $F'_+(0) \leqslant L$, it follows that $F(r)+\pi r^2$ lies below the straight line through the origin with slope L, and hence $F(\rho)+\pi\rho^2 \leqslant L_\rho$, which is (14). The proof of these facts is given first for regions formed by unions of circles, and then in great generality by an approximation argument.

A complete discussion of the parallel curves themselves, and the function $L(r)$, was given by Hartman in 1964. He points out that $L(r)$ need not be continuous if one leaves the class of analytic curves. For example, consider the polygonal domain consisting of the union of a 1×2 rectangle and a 2×2 square with the smaller side of the rectangle placed against the middle of the square. For $r=\frac{1}{2}$, the function $L(r)$ has a jump discontinuity. The same is true for the C^∞ curve obtained by rounding out the corners of this domain. However, Hartman is able to show that except for a set of measure zero in $[0,\rho]$, the curves C_r are piecewise smooth, and the function $L(r)$ is well defined and continuously differentiable. Furthermore, (31) and (32) both hold. Then (14) is an immediate consequence, and also the Bonnesen inequalities (11)–(13).

More recently, using analytic methods reminiscent of those of Schmidt [**68**], Benson [**13, 14**] has derived several Bonnesen-type inequalities for rectifiable Jordan curves.

B. Domains on surfaces. The first result in the direction of the inequalities discussed in Part I, Section B came from an unexpected direction. Using complex variable techniques, Carleman in 1921 proved that the inequality $L^2 \geqslant 4\pi A$ is valid for simply connected minimal surfaces, with equality only if the surface lies in a plane, and the domain is a circular disk. His method was generalized in 1933 by Beckenbach and Radó. Using potential theory they showed that Carleman's result was true not only for minimal surfaces, but for all surfaces satisfying $K \leqslant 0$. Thus they obtained the Corollary to Theorem 5.

For the case of positively curved surfaces, the only results known were for the sphere. There, as we have already noted, Bernstein had proved the first Bonnesen-type inequality (66). Bonnesen gave an improved version of this inequality, with

$$B=\left(2\pi t \tan\frac{d}{2t}\right)^2 \tag{69}$$

on a sphere of radius t, (See [**22**, pp. 80–82].) Using this expression for B in (66), and letting $t\to\infty$, gives the inequality (33), with the constant π^2 instead of 4π.

Extending his previous work on Euclidean space [**68**] Erhard Schmidt in 1940 ([**70**, p. 745]) proved inequality (39) for curves on a sphere, where $\hat{L}$ and $\hat{A}$ are the length and area of the circle on the sphere having the same breadth as the original curve (in some direction). In a previous paper ([**69**, p. 209]), he obtained the same result for the hyperbolic plane, and he thus deduces the isoperimetric inequality (36) for all constant curvature surfaces.

Santaló in 1942 [**61**] derived (41) for a convex curve on a sphere, where $\hat{L}$ and $\hat{A}$ are the length and area of either the inscribed or circumscribed circles. The following year [**62**] he showed that (40) holds in the hyperbolic plane, where again $\hat{L}$ and $\hat{A}$ may be chosen to correspond to either the inscribed or circumscribed circle. He also obtains a number of other Bonnesen-type inequalities for the case of constant negative curvature. Still later, in 1949, he derived an inequality for convex domains on surfaces of variable curvature ([**64**], p. 373) which in the case of constant curvature reduces to (39).

The first proof of the isoperimetric inequality (36) for surfaces of variable curvature appears to be due to Bol in 1941 (p. 230). In fact, using the method of interior parallels he proves (58) and (59), a key step toward proving the stronger inequality (57), but he misses the comparison argument of

Lemma 8, which is the final step. That was carried out much later by Ionin in 1969, using the same basic ideas, but in the context of the Alexandroff theory, to be discussed below. (See also Burago **[26]***).

A number of quite different proofs have been given for the inequality (36). Schmidt **[71]** gives a purely analytic proof for a special class of metrics. Karcher (**[51**, p. 93]) gives a geometric argument via Alexandroff angle comparison theorems, in the case of convex curves. Recently Aubin **[7]** has obtained a proof that holds in great generality.

Chavel and Feldman **[31]** obtain a semi-Bonnesen version of (36) in the form $L^2 - 4\pi A + MA^2 \geqslant B$, where B can vanish only if $K \equiv M$. However, they must then quote the result in the constant curvature case that equality can hold only for a geodesic disk. (It was, in fact, this inequality of Chavel and Feldman that sparked the search for a true Bonnesen inequality on arbitrary curved surfaces, leading eventually to the formulation given in Theorem 6 above.)

Theorem 7 and its corollary were first proved by Fiala in 1941 (p. 336) under the hypothesis that the curve and the metric are analytic. He also assumes $K \geqslant 0$, but makes no essential use of the assumption. His proof is the one given here.

Bol proves the corollary to Theorem 7 under less restrictive hypotheses. It is later proved under very weak hypotheses by A. Huber **[47]**, using potential-theoretic methods, and by Aleksandroff (Alexandrow) and Strel'tsov **[3, 4]** using the Aleksandroff theory. Finally, Burago and Zalgaller give a complete proof of Theorem 7, including a characterization of domains where equality holds.

The method of these authors is the theory of "two-dimensional manifolds of bounded curvature" due to Aleksandroff. (See Aleksandroff and Zalgaller.) This method permits one to prove isoperimetric inequalities under very general conditions, allowing singularities in the metric, which are sometimes necessary to characterize cases of equality. In the first application of the method, Aleksandroff in 1945 **[1]** proves the isoperimetric inequality (36). Later, in the papers of Aleksandroff and Strel'tsov referred to above, (62) is proved, and it is shown that equality holds if and only if the domain is a right circular cone having the given boundary length and area. (The total curvature ω^+ is then concentrated at the vertex.) By rounding off the cone at the vertex it follows that (62) cannot be improved even for smooth metrics, although in that case equality cannot hold unless $\omega^+ = 0$. The more general inequality of the form (65) is considered by Aleksandroff (**[2]**, p. 514), and finally inequality (65) together with its stronger Bonnesen version (63) is proved by Ionin.

For a different treatment of these inequalities, together with applications, see recent papers by Bandle.

Lemma 7 is proved in the case $c < 0$ and applied in a different manner to the study of isoperimetric inequalities on minimal surfaces in recent work of Feinberg.

Finally we note that a paper by Huber **[48]** compares the geometric approach of Aleksandroff theory with his own potential-theoretic approach, and shows the essential overlap in the class of surfaces treated by the two methods.

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*Professor Burago has pointed out that inequalities (2) and (9) of his paper are not correct in general, and should be replaced by

$$F(t) \leqslant \sup_{0 \leqslant t' \leqslant t} \psi(t', L, \omega_K^+ + KF - 2\pi\chi, K), \tag{2'}$$

$$F(t) \leqslant \sup_{0 \leqslant t' \leqslant t} \psi(t', F'(0), A, K). \tag{9'}$$

In our notation, the result is $F(t) \leqslant \max_{0 \leqslant t' \leqslant t} H(t')$. Inequality (39) (or (57)) is equivalent to $A = F(\rho) \leqslant H(\rho)$, and we have shown that to be true if $M \leqslant 0$, or if $M = \alpha^2$, $\alpha > 0$, and $\rho < \pi/\alpha$. In the remaining case, when $\rho \geqslant \pi/\alpha$, Burago's result is $A \leqslant H(t_0) = \max_{0 \leqslant t \leqslant \rho} H(t)$. Using the explicit expression (55) for $H(t)$, one obtains the following supplement to Theorem 6: under the same hypotheses, but with the assumption $MA \geqslant 4\pi$, inequalities (38)–(41) still hold, with $\hat{L} = L_r^M$ and $\hat{A} = A_r^M$, for a suitable value of r; namely, $r = \rho$ if $\rho \leqslant \pi/\alpha$, and $r = (1/\alpha)\cot^{-1}[(2\pi - MA)/\alpha L]$ if $\rho > \pi/\alpha$, where $M = \alpha^2$, $\alpha > 0$.

III.

A. Amplifications. A natural question, given the plethora of Bonnesen inequalities, is whether certain of them are quantitatively superior to the others. We shall examine Lemma 1 in this context.

Let us note first that in proving Lemma 1 the idea is to reduce each of the inequalities to (9). In particular, (8) follows from the identity

$$(L-2\pi r)^2-\left(L-\frac{2A}{r}\right)^2=\frac{4}{r}(rL-A-\pi r^2)\left(\frac{A}{r}-\pi r\right). \tag{70}$$

To see which of the inequalities (3), (4), (5) is best, we must consider two cases.

Case 1.

$$\pi r^2 \leqslant A.$$

Then from (70) we have

$$(L-2\pi r)^2 \geqslant \left(L-\frac{2A}{r}\right)^2,$$

while from (6),

$$L-2\pi r > \frac{A}{r}-\pi r \geqslant 0$$

so that

$$(L-2\pi r)^2 > \left(\frac{A}{r}-\pi r\right)^2.$$

Thus, for those values of r, inequality (3) is stronger than (4) and (5).

Case 2.

$$\pi r^2 > A.$$

Then it follows from (70) that

$$\left(L-\frac{2A}{r}\right)^2 \geqslant (L-2\pi r)^2,$$

and from (7) that

$$L-\frac{2A}{r} \geqslant \pi r-\frac{A}{r} \geqslant 0. \tag{71}$$

Hence

$$\left(L-\frac{2A}{r}\right)^2 \geqslant \left(\pi r-\frac{A}{r}\right)^2,$$

and for these values of r, inequality (5) is best of the three. Since in particular,

$$\pi\rho^2 \leqslant A \leqslant \pi R^2,$$

it follows that of the six inequalities, (11)–(13), (16)–(18), the strongest numerically must be either (11) or (18). Which one of these is better depends on the particular example.

Next we note that although our orientation has been toward isoperimetric inequalities, the results of the theorems presented here can also be viewed from a broader viewpoint as constraints on the four quantities, L, A, ρ, R, in order for them to represent the length, area, inradius, and circumradius of a domain. In fact, we have restricted our attention to simply connected domains, and one can consider also the connectivity k (or equivalently the Euler characteristic $\chi=2-k$, in the case of genus zero) of the domain as another parameter. Then the inequalities of Theorems 2, 3, and 4 provide bounds for any of these quantities when the others are given. For example, fixing L and A, we get a

lower bound on the inradius and an upper bound on the circumradius from (15) and (20). For a discussion of these and related results from this point of view, see Bonnesen and Fenchel (pp. 75–85).

One consequence of (71) is the simple inequality

$$\frac{L}{A} \geqslant \frac{2}{R}. \tag{72}$$

In the case of *convex* domains one has a companion inequality

$$\frac{L}{A} \leqslant \frac{2}{\rho}, \tag{73}$$

where equality holds for a class of domains circumscribed around a circle (see Bonnesen and Fenchel, p. 82, or Singmaster).

An immediate consequence of (14) is

$$\frac{L}{A} > \frac{1}{\rho} \tag{74}$$

for arbitrary simply connected plane domains. The weak inequality

$$\frac{L}{A} \geqslant \frac{1}{\rho} \tag{75}$$

holds for doubly connected domains, and more generally, the inequality

$$\frac{L}{A} \geqslant \frac{2}{k}\frac{1}{\rho} \tag{76}$$

for plane domains of connectivity k was proved recently by Osserman.

Furthermore, (75) holds for simply connected domains on an arbitrary surface, provided $\omega^+ \leqslant 2\pi$, by virtue of (60).

For negatively curved surfaces, there are a number of analogous results. We collect them in the form of a theorem.

THEOREM 8. *Let D be a domain on a surface S whose Gauss curvature satisfies $K \leqslant -\alpha^2$, $\alpha > 0$. Then* (a) *if D is simply connected or doubly connected, or if S is simply connected and D is arbitrary, then*

$$\frac{L}{A} \geqslant \alpha; \tag{77}$$

(b) *if D is simply connected, then*

$$\frac{L}{A} > \alpha \coth \alpha\rho > \alpha; \tag{78}$$

(c) *if D is arbitrary and S is complete and simply connected, then*

$$\frac{L}{A} \geqslant \alpha \coth \alpha R > \alpha; \tag{79}$$

(d) *inequality* (77) *does not hold without some restriction on the topology of D or S.*

Proof. (a) The isoperimetric inequality (36) gives

$$L^2 \geqslant 4\pi A + \alpha^2 A^2 > \alpha^2 A^2$$

for simply connected domains with $K \leqslant -\alpha^2$, so that (77) holds in that case. If S is simply connected and D is an arbitrary subdomain, then $D \subset D'$, where D' is a simply connected subdomain bounded by a single boundary curve of D. Since (77) holds for D', it holds a fortiori for D. Finally, if D is doubly connected, we use an extension of (36) to domains of Euler characteristic χ:

$$L^2 \geqslant 4\pi\chi A + \alpha^2 A^2 \tag{80}$$

(see Ionin, Theorem 3). Since $\chi = 0$ for doubly connected domains, (77) follows.

(b) By Theorem 6, for simply connected domains the Bonnesen-style inequality (39) holds with

$$\hat{L}=L_\rho^M=2\pi\frac{\sinh\alpha\rho}{\alpha},\qquad \hat{A}=A_\rho^M=4\pi\frac{\sinh^2\frac{1}{2}\alpha\rho}{\alpha^2}.$$

Thus (39) becomes

$$L\geqslant A\alpha\coth\alpha\rho+\frac{2\pi}{\alpha}\tanh\frac{\alpha\rho}{2},\tag{81}$$

and (78) follows.

(c) Using an argument of Yau (p. 498), the distance function from a fixed point on a complete simply connected surface S on which $K\leqslant-\alpha^2<0$ satisfies

$$\Delta r\geqslant\alpha\coth\alpha r.\tag{82}$$

Choosing the fixed point to be the center of the circumscribed circle around D, one has $r\leqslant R$ on D and hence $\Delta r\geqslant\alpha\coth\alpha R$. Integrating over D and applying the divergence theorem yields (79).

(d) If S is an arbitrary compact surface of genus $g\geqslant 2$, then there exists a metric of constant negative curvature on S. Choosing D to be the complement of an arbitrarily small disk on S, the left-hand side of (77) can be made arbitrarily close to zero.

B. Applications. There have been many applications of isoperimetric inequalities to other problems. For older results, see for example Pólya and Szegö. Some recent examples can be found in Aubin, Bandle [**8–10**], Chavel and Feldman [**31**], Cheng [**34**], Kohler–Jobin, Payne, and Yau.

One of the most basic recent results is that of Cheeger relating the first eigenvalue λ_1 of the Laplacian on a manifold to certain isoperimetric constants. For a domain D on a two-dimensional surface, Cheeger considers the quantity

$$h=\inf_{D'\in F}\frac{L'}{A'}\tag{83}$$

where F is the family of relatively compact subdomains of D, and L',A' are the boundary length and area of the subdomain D'. Cheeger's result is the inequality

$$\lambda_1(D)\geqslant\tfrac{1}{4}h^2.\tag{84}$$

It follows immediately from Theorem 8(a) that if S is a simply connected surface with $K\leqslant-\alpha^2<0$, then for any domain D on S,

$$\lambda_1(D)\geqslant\frac{\alpha^2}{4}.\tag{85}$$

If one defines

$$\lambda_1(S)=\inf_{D\subset S}\lambda_1(D),$$

it follows that

$$\lambda_1(S)\geqslant\frac{\alpha^2}{4}\tag{86}$$

This is a result of McKean. This inequality is sharp, with equality if $K\equiv-\alpha^2$, and S is complete. (See McKean, or (107) below.)

An interesting corollary of the above derivation of McKean's theorem is that it shows that the constant $1/4$ in Cheeger's inequality (84) cannot be improved. Namely, if S is the hyperbolic plane, then $\lambda_1(S)=\frac{1}{4}$, and given $\varepsilon>0$, any sufficiently large domain D satisfies $\lambda_1(D)<\frac{1}{4}+\varepsilon$. But by the isoperimetric inequality in the hyperbolic plane (or by Theorem 8(a)), $L\geqslant A$ for every subdomain of D, and hence $h\geqslant 1$. Thus

$$\frac{\lambda_1(D)}{h^2}<\frac{1}{4}+\varepsilon.$$

A corresponding statement (in even stronger form) has been obtained recently for compact surfaces by Buser.

Using Theorem 8(b) and 8(c), one obtains sharper forms of (85).

THEOREM 9. *Let S be a complete simply connected surface with $K \leqslant -\alpha^2$, $\alpha > 0$. Then for any domain $D \subset S$, if R is its circumradius,*

$$\lambda_1(D) - \frac{\alpha^2}{4} \geqslant \frac{\alpha^2}{4}\operatorname{csch}^2 \alpha R. \tag{87}$$

If D is simply connected and ρ is its inradius, then one has the stronger inequality

$$\lambda_1(D) - \frac{\alpha^2}{4} \geqslant \frac{\alpha^2}{4}\operatorname{csch}^2 \alpha\rho. \tag{88}$$

COROLLARY. *For simply connected domains D in the hyperbolic plane, in order for $\lambda_1(D)$ to be arbitrarily close to its lower limit $1/4$, D must contain an arbitrarily large geodesic disk.*

Proof. Inequality (87) follows immediately from (79), (83), (84), since for any subdomain D' of D, its circumradius R' satisfies $R' \leqslant R$, $\coth R' \geqslant \coth \alpha R$. To prove (88), one needs (78), together with the observation that, when D is simply connected, it is sufficient to restrict the family F in (83) to simply connected subdomains (see Osserman).

The analog of the Corollary for plane domains was first proved by Hayman. Note that, as $\alpha \to 0$, (88) tends to

$$\lambda_1(D) \geqslant \frac{1}{4\rho^2}. \tag{89}$$

This inequality is in fact valid for simply connected plane domains, and is proved by the method of Theorem 9, using (74). In fact, each case where one of the inequalities (74)–(79) holds yields a corresponding bound on λ_1 (see Osserman). Using a different argument, Hayman obtained an inequality of the form (89) with a much smaller constant on the right.

As a further comment, let us note that for (85) to hold it is not sufficient to assume that $K \leqslant -\alpha^2$, without some restriction on the topology of D or the containing surface S. Namely, given an arbitrary compact surface S of genus greater than one, there exists a metric of constant negative curvature -1 on S. For any point $p \in S$ and any $\varepsilon > 0$, let D_ε be the closed geodesic disk of radius ε centered at p, and let D'_ε be the complement of D_ε. Then $\lim_{\varepsilon\to 0}\lambda_1(D'_\varepsilon) = 0$. (See Chavel–Feldman [32].)

We conclude this section with two theorems adapted from recent work of Pinsky. Although not directly related to Bonnesen inequalities, they provide interesting counterparts to Cheeger's Theorem and Theorem 9.

The point of Cheeger's Theorem is that if the inequality

$$L \geqslant hA$$

holds for a suitable class of subdomains of a given domain D, then one has the lower bound

$$\lambda_1(D) \geqslant \tfrac{1}{4}h^2.$$

The following result goes in the other direction.

THEOREM 10. *Let S be a complete simply connected surface such that there exist global geodesic coordinates centered at some point p. (For example, if $K \leqslant 0$ on S.) Let D_ρ be the geodesic disk of radius ρ with center at p; let A_ρ be the area of D_ρ, and L_ρ the length of its boundary. Suppose*

$$\frac{d^2}{d\rho^2}L_\rho \leqslant H\frac{d^2}{d\rho^2}A_\rho, \qquad \rho_0 \leqslant \rho \leqslant \rho_1, \tag{90}$$

for some constant H. Then

$$\lambda_1(D_{\rho_1}) \leqslant \frac{H^2}{4} + \left(\frac{\pi}{\rho_1 - \rho_0}\right)^2. \tag{91}$$

COROLLARY. *If under the same hypotheses,* (90) *holds for all* $\rho \geqslant \rho_0 > 0$ *then*

$$\lambda_1(S) \leqslant \frac{H^2}{4}. \tag{92}$$

Proof. Let

$$F(\rho) = e^{\frac{-H}{2}\rho} \sin \pi \frac{\rho - \rho_0}{\rho_1 - \rho_0}, \qquad \rho_0 \leqslant \rho \leqslant \rho_1.$$

F satisfies

$$F''(\rho) + HF'(\rho) + cF(\rho) = 0, \tag{93}$$

with

$$c = \left(\frac{H}{2}\right)^2 + \left(\frac{\pi}{\rho_1 - \rho_0}\right)^2. \tag{94}$$

Denoting L_ρ by L, $F(\rho)$ by F, we multiply (93) by LF and add $L'FF'$ to both sides, giving

$$LFF'' + L'FF' + cLF^2 = (L' - HL)FF'$$

or

$$\frac{d}{d\rho}(LFF') - L(F')^2 + cLF^2 = \frac{1}{2}\frac{d}{d\rho}[(L' - HL)F^2] - \frac{1}{2}(L'' - HL')F^2. \tag{95}$$

Since

$$L_\rho = \frac{d}{d\rho} A_\rho, \tag{96}$$

(90) is equivalent to $L'' \leqslant HL'$. Using this in the last term of (95), integrating from ρ_0 to ρ_1, and noting that $F(\rho_0) = F(\rho_1) = 0$, we find

$$\int_{\rho_0}^{\rho_1} L_\rho [F'(\rho)]^2 d\rho \leqslant c \int_{\rho_0}^{\rho_1} L_\rho [F(\rho)]^2 d\rho$$

or

$$\int_0^{2\pi}\int_{\rho_1}^{\rho} |\nabla F|^2 \sqrt{g}\,(\rho,\theta)\, d\rho\, d\theta \leqslant c \int_0^{2\pi}\int_{\rho_0}^{\rho_1} F^2 \sqrt{g}\,(\rho,\theta)\, d\rho\, d\theta, \tag{97}$$

since

$$L(\rho) = \int_0^{2\pi} \sqrt{g}\,(\rho,\theta)\, d\theta \tag{98}$$

in terms of the geodesic polar coordinates, $ds^2 = d\rho^2 + g(\rho,\theta)\,d\theta^2$. But if we denote by $D_{0,1}$ the domain $\rho_0 < \rho < \rho_1$, and by Φ the family of smooth functions in $D_{0,1}$ vanishing on the boundary, then

$$\lambda_1(S) \leqslant \lambda_1(D_{\rho_1}) \leqslant \lambda_1(D_{0,1}) = \inf_{\substack{\phi \in \Phi \\ \phi \not\equiv 0}} \left(\int_{D_{0,1}}\!\!\int |\nabla\phi|^2 dA\right) \Big/ \left(\int_{D_{0,1}}\!\!\int \phi^2 dA\right)$$

$$\leqslant \left(\int_{D_{0,1}}\!\!\int |\nabla F|^2 dA\right) \Big/ \left(\int_{D_{0,1}}\!\!\int F^2 dA\right) \leqslant c = \left(\frac{H}{2}\right)^2 + \left(\frac{\pi}{\rho_1 - \rho_0}\right)^2,$$

by (94) and (97). This proves (91), and letting $\rho_1 \to \infty$, also (92).

REMARK. There are several alternative forms for the hypothesis (90) of Theorem 10. First of all, by (96), one can write (90) in the form

$$\frac{d^2}{d\rho^2} L_\rho \leqslant H \frac{d}{d\rho} L_\rho. \tag{99}$$

Next, if we introduce the *total curvature* ω_ρ of D_ρ, we have by (34) and (98) that

$$\begin{aligned}\omega_\rho &= \int_{D_\rho}\!\!\int K dA = \int_0^{2\pi}\!\!\int_0^\rho K\sqrt{g}\,(r,\theta)\,dr\,d\theta \\ &= -\int_0^{2\pi}\!\!\int_0^\rho \frac{\partial^2}{\partial r^2}\sqrt{g}\,(r,\theta)\,dr\,d\theta = 2\pi - \int_0^{2\pi}\frac{\partial}{\partial r}\sqrt{g}\,(r,\theta)\,d\theta \\ &= 2\pi - \frac{d}{d\rho} L_\rho.\end{aligned}$$

It follows that if $\omega_\rho < 2\pi$ (and in particular, if $K \leqslant 0$ on S) then (99) is equivalent to a growth condition on ω_ρ: specifically, a bound on the logarithmic derivative:

$$\frac{d}{d\rho}\log(2\pi - \omega_\rho) \leqslant H. \tag{100}$$

Next, we give a counterpart of Theorem 9 involving curvature bounds.

THEOREM 11. *Let S be a simply connected complete surface with $K \leqslant 0$ everywhere. Let D_ρ be a geodesic disk of radius ρ, and D'_ρ the complement of D_ρ. If*

$$\alpha^2 = \inf_{D_\rho}(-K), \qquad \beta^2 = \sup_{D_\rho}(-K), \qquad 0 < \alpha \leqslant \beta, \tag{101}$$

then

$$\lambda_1(D_\rho) \leqslant \left(\frac{\beta^2}{2\alpha\coth\alpha\rho}\right)^2 + \left(\frac{\pi}{\rho}\right)^2. \tag{102}$$

If

$$\alpha^2 = \inf_{D'_\rho}(-K), \qquad \beta^2 = \sup_{D'_\rho}(-K), \qquad 0 < \alpha \leqslant \beta, \tag{103}$$

then for $0 < \varepsilon < \alpha$, there exists $\rho_0 \geqslant \rho$ such that for all $\rho_1 > \rho_0$,

$$\lambda_1(D'_\rho) \leqslant \lambda_1(D_{0,1}) \leqslant \left(\frac{\beta^2}{2(\alpha-\varepsilon)}\right)^2 + \left(\frac{\pi}{\rho_1 - \rho_0}\right)^2, \tag{104}$$

when $D_{0,1}$ is the geodesic annulus, $\rho_0 < \rho < \rho_1$.

COROLLARY 1. *If S is simply connected, complete, with $K \leqslant 0$ everywhere and if*

$$0 < \alpha^2 \leqslant -K \leqslant \beta^2 \tag{105}$$

outside of some compact set, then

$$\lambda_1(S) \leqslant \frac{\beta^4}{4\alpha^2}. \tag{106}$$

COROLLARY 2. *If under the same hypotheses, $K \to -\alpha^2 < 0$ at infinity, then*

$$\lambda_1(S) \leqslant \frac{\alpha^2}{4}. \tag{107}$$

Proof of Theorem 11. Since $K \leqslant -\alpha^2$ in D_ρ, we may apply the comparison argument of Lemma 7, and by (49) combined with the first inequality in (48),

$$\frac{\partial}{\partial r}\sqrt{g}\,(r,\theta) \geqslant (\alpha\coth\alpha r)\sqrt{g}\,(r,\theta), \qquad 0 < r < \rho.$$

By (98),

$$\frac{d}{dr}L_r \geqslant (\alpha\coth\alpha r)L_r \geqslant (\alpha\coth\alpha\rho)L_r, \qquad 0<r<\rho.$$

Then, since

$$\frac{\frac{\partial^2}{\partial r^2}\sqrt{g}\,(r,\theta)}{\sqrt{g}\,(r,\theta)} = -K \leqslant \beta^2, \qquad 0<r<\rho$$

we have,

$$\frac{d^2L_r}{dr^2} \leqslant \beta^2 L_r \leqslant \frac{\beta^2}{\alpha\coth\alpha\rho}\frac{d}{dr}L_r.$$

Thus (99) is satisfied, with

$$H = \frac{\beta^2}{\alpha\coth\alpha\rho}.$$

Using Theorem 10 with this value of H and letting $\rho_0 \to 0$ in (91) yields (102).

The proof of (104) is similar, but somewhat more delicate.

Fix θ, and let $f(r) = \sqrt{g}\,(r,\theta)$. Then, as in the proof of Lemma 5, the fact that $K \leqslant 0$ implies $f'(r) \geqslant 1$, $f(r) \geqslant r > 0$ for $r > 0$. Let $f(\rho) = a$, $f'(\rho) = b$, and

$$h(r) = a\cosh\alpha(r-\rho) + \frac{b}{\alpha}\sinh\alpha(r-\rho).$$

Then $h(\rho) = a$, $h'(\rho) = b$, and $h(r) > h(\rho) > 0$ for $r > \rho$. The argument of Lemma 7 then gives $f(r) \geqslant h(r) > 0$ for $r \geqslant \rho$, and

$$\frac{f'(r)}{f(r)} \geqslant \frac{h'(r)}{h(r)} = \alpha\frac{\alpha a + b\coth\alpha(r-\rho)}{b + \alpha a\coth\alpha(r-\rho)} \to \alpha \quad \text{as } r \to \infty.$$

Thus, for $0 < \varepsilon < \alpha$, we may choose $\rho_0 > \rho$ such that $f'(r) \geqslant (\alpha - \varepsilon)f(r)$ for $r \geqslant \rho_0$. The value of ρ_0 depends on θ, but its explicit dependence on the initial values $f(\rho), f'(\rho)$ shows that it is continuous in θ, and a single ρ_0 may be chosen that works for all θ. Then integrating with respect to θ gives

$$\frac{d}{dr}L_r \geqslant (\alpha - \varepsilon)L_r, \qquad r \geqslant \rho.$$

Then, as in the proof of (102), we have

$$\frac{d^2}{dr^2}L_r \leqslant \beta^2 L_r \leqslant \frac{\beta^2}{\alpha - \varepsilon}\frac{d}{dr}L_r, \qquad r \geqslant \rho_0.$$

Thus (99) is satisfied, and a direct application of Theorem 10 gives (104).

Corollary 1 follows immediately, since for ρ sufficiently large, D_ρ includes the compact set outside of which (105) holds. Letting $\rho_1 \to \infty$ in (104) gives $\lambda_1(S) \leqslant (\beta^2/2(\alpha-\varepsilon))^2$, and letting $\varepsilon \to 0$ gives (106).

Corollary 2 follows directly from Corollary 1.

Cheng ([35], p. 290 and 294) has obtained an upper bound for $\lambda_1(D_\rho)$ using just the right half of (101). When $\beta = 1$, his inequality is

$$\lambda_1(D_\rho) \leqslant \frac{1}{4} + 4\left(\frac{\pi}{\rho}\right)^2.$$

Whether this or (102) is a better bound depends on the relationship of ρ and α. However the interest in Theorem 11 is much more in (104) and the Corollaries, since they give an upper bound for λ_1 depending only on the curvature of the surface in a neighbourhood of infinity. More precisely, this is not quite true of (104), since changing the metric on a compact set changes the entire geodesic coordinate system, including the functions $g(r,\theta)$ and L_r. (In fact the polar distance r itself depends

on the metric near the origin). However, the formulation of the corollaries is independent of any system of coordinates and is independent of the metric on any compact set.

C. Higher dimensions. If we denote by ω_n the volume of the unit ball in $\mathbf{R}^n$, then the ball of radius r will have volume

$$V_r = \omega_n r^n, \tag{108}$$

and surface area

$$S_r = n\omega_n r^{n-1}. \tag{109}$$

Thus

$$(S_r)^n = n^n \omega_n (V_r)^{n-1}.$$

The statement that of all domains with given surface area S the maximum volume V is attained by the sphere translates into the isoperimetric inequality

$$S^n \geqslant n^n \omega_n V^{n-1}, \tag{110}$$

or equivalently

$$\left(\frac{S}{S_r}\right)^n \geqslant \left(\frac{V}{V_r}\right)^{n-1} \tag{111}$$

for any $r > 0$. A Bonnesen inequality would be of the form

$$\left(\frac{S}{S_r}\right)^n - \left(\frac{V}{V_r}\right)^{n-1} \geqslant B \tag{112}$$

or

$$\left(\frac{S}{S_r}\right)^{\frac{n}{n-1}} - \frac{V}{V_r} \geqslant B' \tag{113}$$

where B and B' are non-negative and vanish only for a sphere.

Bonnesen himself proved several inequalities of the form (113) ([**22**, p. 135 and p. 144]), but he was not able to obtain direct generalizations of his two-dimensional results. This was done much later, first by Hadwiger [**43**] for $n = 3$, and then by Dinghas for arbitrary n. Their result is

$$\left(\frac{S}{S_\rho}\right)^{\frac{n}{n-1}} - \frac{V}{V_\rho} \geqslant \left(\left(\frac{S}{S_\rho}\right)^{\frac{1}{n-1}} - 1\right)^n \tag{114}$$

for a convex domain D in $\mathbf{R}^n$ with inradius ρ. Since D includes a ball of radius ρ, one has $S \geqslant S_\rho$, with equality if and only if D coincides with the ball. Thus the right-hand side of (114) is non-negative, and (114) implies the isoperimetric inequality (111) together with the fact that equality holds only for a sphere. Using (114) together with the elementary inequality $a^k - b^k \geqslant (a-b)^k$ for $a \geqslant b \geqslant 0$, one finds (Hadwiger [**44**, p. 270])

$$\begin{aligned}
\left(\frac{S}{S_\rho}\right)^n - \left(\frac{V}{V_\rho}\right)^{n-1} &= \left[\left(\frac{S}{S_\rho}\right)^{\frac{n}{n-1}}\right]^{n-1} - \left[\frac{V}{V_\rho}\right]^{n-1} \\
&\geqslant \left[\left(\frac{S}{S_\rho}\right)^{\frac{n}{n-1}} - \frac{V}{V_\rho}\right]^{n-1} \\
&\geqslant \left[\left(\frac{S}{S_\rho}\right)^{\frac{1}{n-1}} - 1\right]^{n(n-1)}
\end{aligned}$$

or

$$\left(\frac{S}{S_\rho}\right)^n-\left(\frac{V}{V_\rho}\right)^{n-1}\geqslant\left[\left(\frac{S}{S_\rho}\right)^{\frac{1}{n-1}}-1\right]^{n(n-1)}, \tag{115}$$

a Bonnesen-style inequality of the form (112). Although, by its derivation, (115) is generally weaker than (114), for $n=2$ both inequalities reduce to Bonnesen's original inequality (11). One may ask if there is also an n-dimensional analog of (14). Recently, Wills conjectured that one should have

$$\rho S\geqslant V+(n-1)V_\rho \tag{116}$$

which for $n=2$ reduces to (14). It turns out that an even stronger result is true.

THEOREM 12. *Inequality* (114) *implies*

$$\rho S\geqslant V+(n-1)V_\rho\left(\frac{S}{S_\rho}\right)^{\frac{n-2}{n-1}}. \tag{117}$$

COROLLARY 1. *Inequality* (116) *holds for convex bodies in* $\mathbf{R}^n$, *and for* $n\geqslant 3$, *equality holds only for the sphere.*

COROLLARY 2. *For convex bodies in* $\mathbf{R}^n$, $n\geqslant 2$, *one has*

$$\frac{S}{V}>\frac{1}{\rho}. \tag{118}$$

COROLLARY 3. *Inequality* (114) *is not valid for arbitrary domains in* $\mathbf{R}^n$ *homeomorphic to a ball.*

Proof of Corollary 3. Starting with a sphere, and deforming it by a number of inward pointing spikes, one can change S and V by an arbitrarily small amount while making ρ arbitrarily small. Thus (118) cannot hold in general, and neither can (116) or (114) from which it is derived.

Proof of Corollary 1. Since $S\geqslant S_\rho$, with equality only for the sphere, (116) follows from (117), and strict inequality holds in (116) whenever $S>S_\rho$ and $n>2$.

Proof of Theorem 12. We use the inequality

$$(1-x)^n\geqslant 1-nx+(n-1)x^2 \tag{119}$$

which holds for $x\leqslant 1$ and all positive integers n, as follows easily by induction on n. Setting

$$y=\left(\frac{S}{S_\rho}\right)^{\frac{1}{n-1}}\geqslant 1,$$

(114) takes the form

$$\begin{aligned}\frac{V}{V_\rho}&\leqslant y^n-(y-1)^n=y^n\left[1-\left(1-\frac{1}{y}\right)^n\right]\\&\leqslant ny^{n-1}-(n-1)y^{n-2}=n\frac{S}{S_\rho}-(n-1)\left(\frac{S}{S_\rho}\right)^{\frac{n-2}{n-1}}\end{aligned}$$

where we have used (119) with $x=1/y$. Substituting the values of V_ρ and S_ρ from (108) and (109) gives (117).

It may be worth noting that Schmidt proved an inequality similar to (116) ([**68**, p. 783]):

$$rS\geqslant(n-1)V+V_r \tag{120}$$

for a suitably defined r; the isoperimetric inequality (111) is a simple consequence, as one sees by replacing r in (120) by the value that maximizes $rS-V_r$. His paper thus contains a complete proof of the isoperimetric inequality for domains in $\mathbf{R}^n$. Schmidt went on to give analogous proofs in spaces of constant curvature, positive or negative, first for rotationally symmetric bodies ([**69**, **70**]) and then in

general, using symmetrization [**72**]. He also proves n-dimensional analogs of (39) for constant negative curvature ([**69**, p. 221]) and constant positive curvature ([**70**, p. 780]).

We also note that the method of Yau used in proving Theorem 8(c) above works equally in any number of dimensions and gives the inequality

$$\frac{S}{V} \geqslant (n-1)\alpha \coth \alpha R \tag{121}$$

for a domain D of circumradius R on a complete simply connected manifold M whose sectional curvature is bounded above by $-\alpha^2$, $\alpha > 0$. Since Cheeger's theorem is also valid for n-dimensional manifolds, it follows that under the same hypotheses,

$$\lambda_1(D) \geqslant \frac{1}{4}[(n-1)\alpha \coth \alpha R]^2 > \frac{(n-1)^2\alpha^2}{4}, \tag{122}$$

which implies the n-dimensional theorem of McKean.

Next, we apply (118) to extend (89) to higher dimensions.

THEOREM 13. *Let D be a convex body in $\mathbf{R}^n$, with inradius ρ. Then*

$$\lambda_1(D) \geqslant \frac{1}{4\rho^2}. \tag{123}$$

Proof. Cheeger's proof of (84) needs the inequality

$$S' \geqslant hV' \tag{124}$$

only for subdomains D' of D bounded by level surfaces of the first eigenfunction of D. It follows immediately from recent work of Brascamp and Lieb ([**24**, Th. 1.13] or [**25**, Th. 6.1]) that if D is convex, then those subdomains D' are also convex. If ρ' is the inradius of D', then by (118),

$$\frac{S'}{V'} > \frac{1}{\rho'} > \frac{1}{\rho}.$$

Thus (124) holds with $h = 1/\rho$, and (123) follows from Cheeger's Theorem.

There are several remarks to be made concerning Theorem 13.

First, it is sufficient to have weak inequality in (118), a result obtained earlier by Wills.

Second, an inequality like (123) but with a much weaker constant, depending on the dimension, was proved by Hayman. Hayman's proof works not only for convex domains, but for more general domains characterized as follows: given any point on the boundary of D and any sphere centered at that point, a "sufficiently large" part of the sphere lies outside of D. Hayman points out that some such restriction is necessary when $n > 2$, since an inequality like (123) cannot hold in general. In fact, starting with a sphere in $\mathbf{R}^n$, $n \geqslant 3$; and introducing a number of inward-pointing spikes, one can produce a domain D for which λ_1 is arbitrarily close to its value for the ball bounded by the original sphere but for which ρ can be made arbitrarily small.

Finally, we come to the extensions of Theorems 10 and 11. The proof of Theorem 10 goes through unchanged in any number of dimensions. One may state the result as follows.

THEOREM 14. *Let M be a complete n-dimensional Riemannian manifold such that for some point p on M the exponential map at p is a diffeomorphism onto M. Let D_r be the geodesic ball of radius r with center at p; let $V(r)$ be the volume of D_r and $S(r)$ the $(n-1)$-dimensional measure of its boundary. Suppose there exists a constant H such that*

$$S''(r) \leqslant HV''(r), \qquad r_0 \leqslant r \leqslant r_1, \tag{125}$$

or equivalently,

$$S''(r) \leqslant HS'(r), \qquad r_0 \leqslant r \leqslant r_1. \tag{126}$$

Then

$$\lambda_1(D_{r_1}) \leqslant \frac{H^2}{4} + \left(\frac{\pi}{r_1 - r_0}\right)^2. \tag{127}$$

COROLLARY. *If* (125) *holds for all* $r \geqslant r_0$, *then*

$$\lambda_1(M) \leqslant \frac{H^2}{4}. \tag{128}$$

In contrast to this, the proof of Theorem 11 does not generalize without going further afield from our central theme. We therefore conclude by referring once more to the paper of Cheng [35] for geometrically inspired bounds for λ_1 that hold in any number of dimensions.

Note (added Feb. 22, 1978). I should like to thank Yu. D. Burago for calling to my attention two additional papers of relevance to our discussion. They are:

B. V. Dekster, An inequality of isoperimetric type for a domain in a Riemannian space, Mat. Sbornik 90 (1973) 258–274; English translation: Math. USSR Sbornik 19 (1973) 257–274.

V. I. Diskant, A generalization of Bonnesen's inequalities, Dokl. Akad. Nauk SSSR 213 (1973) 873–877; English translation: Soviet Math. Dokl. 14 (1973) 1728–1731.

Dekster proves, among other things, that the inequality (118) is valid for strictly convex domains on n-dimensional Riemannian manifolds of positive sectional curvature. Diskant gives a proof of Wills's conjecture (116) using an argument similar to the one given here. However, he does not get the stronger inequality (117).

Note (added in proof: July 14, 1978). Since this was written, Michael Gage has proved sharpened versions of Theorem 11 and its corollaries both in two and higher dimensions. (Ph.D. Thesis, Stanford University 1978.) I should also like to thank Mr. Gage for a number of useful comments that have been incorporated in the present paper.

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THE "WHY-DON'T-YOU-JUST . . . ?" BARRIER IN DISCRETE ALGORITHMS

HERBERT S. WILF

One of the big differences between teaching discrete algorithms and teaching a more conventional mathematics course is that students will almost always see some way to construct an algorithm, while much less often will they be able to prove a theorem without assistance.

The first thought that we have about how to solve an algorithmic problem is, however, often very far from an efficient or elegant solution. The idea that the problem is trivial, because "why-don't-you-just- . . . [insert here the thought which comes first to mind]?" constitutes a formidable barrier to the presentation of the subject, because one is often not ready to listen to a more difficult solution before being convinced that the simpler solution is undesirable. (I must hasten to add that I have taught extremely bright students whose first thoughts were very desirable indeed!)

For this reason it seems that the best strategy for the teacher is to pose the problem, and then to listen to the various suggestions that will come from the students, discussing with them the execution time, storage requirements, elegance, compactness of program, etc., of their proposals in some detail, before proceeding with the presentation of a difficult algorithm.

I would like to offer a few examples of this kind of situation—where *some* solution is immediately evident, but the *best* solution is some distance away. These examples arose in connection with teaching from a book [1] and illustrate the point fairly typically.

As a small introductory tidbit, consider the calculation of Fibonacci numbers $f_1, f_2, f_3, \cdots$. When I teach an introductory computer course and assign this for homework, I find that most students will program the recurrence exactly as it appears in its written form, namely

```
   :
┌→ F(N) = F(N-1) + F(N-2)
│  PRINT
│  :
└──
```

Some students, of course, recognize that a lot of storage is wasted by using an array, when really only three locations are needed. Hence a good program for the job will usually look like this:

```
   :
┌→ C = A + B
│  PRINT C
│  A = B
└─ B = C
```

Hence we now use just three locations, and we pay a price of two "moves."

Is there a way to use just two locations *and* to avoid the moves? A few moments' thought shows that the following program will work:

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```
      :
   ┌→A = A + B
   │ PRINT A
   │ B = A + B
   │ PRINT B
   └────
```

This pattern—an obvious but inefficient solution, a fairly immediate improvement, and a rather sneaky sequence of further refinements which save time or storage or both—is typical of work in this field.

Now to our next problem. How shall we choose, at random, 15 objects out of 97 given objects? More precisely, suppose we have a machine which will produce, when its button is pressed, an integer chosen at random from $\{1,2,\ldots,97\}$. How shall we select a subset of 15 integers from this set?

"Why don't you just press the button 15 times?"

Because some integer might have been selected twice, and we would not, then, end up with a subset of cardinality 15.

"In that case, why don't you just press the button repeatedly until 15 distinct integers are available?"

This is indeed a possible algorithm, but how much work is involved in it? Consider the moment when we press the button on our random-integer generator for the twelfth time, and it presents us with the integer 53. How do we know if 53 has previously been chosen? Obviously(?) by comparing it with the previous 11 integers.

How much work is involved in checking whether an integer is or is not contained in a set of $j-1$ given integers? If we do the checking by comparing the new one with each of the old, we do $j-1$ comparisons. The total amount of labor involved in selecting k integers will then be at least

$$\sum_{j=1}^{k}(j-1)=\frac{k^2}{2}-\frac{k}{2}$$

comparisons.

Are we willing to pay a price of about $\frac{1}{2}k^2$ units of labor just to select k integers? Surely there must be a way to do it with only about k units of labor.

It is usually at this moment that the audience is convinced that a problem of some substance has been posed, and therefore that a solution of some complexity is justified.

The next thought might be to keep a master list of the integers $1,2,\ldots,97$. Each time we select a random integer we could check it off on the master list. Then a duplication could be spotted immediately by the presence of a check mark.

The difficulty here is that in order to keep the master list we need to write the integers $1,2,\ldots,97$ on a piece of paper (or in a computer, to initialize an array of length 97). We begin, therefore, by paying a price of 97 units of labor before selecting a single integer! A further problem is that the number of words of storage required for choosing k integers from among n will depend on n instead of only on k. To choose 50 lucky people from the New York telephone directory would require several million storage registers, and so we shall continue the search for a better way.

Let us pause to state the question a little more clearly. Given integers $1 \leqslant k \leqslant n$, and given a machine (subroutine) which will, on demand, produce an integer in the range $[1,n]$, devise an algorithm which yields a subset $\{a_1,\ldots,a_k\}$ of $\{1,\ldots,n\}$ subject to the conditions:

(a) each subset S, $|S|=k$, is to have equal a priori probability of being selected;

(b) the method is to require just $k+c$ words of storage for its execution;

(c) the elements of the output set are to be presented in ascending order of size $a_1 < a_2 < \cdots < a_k$;

(d) the execution time of the method is to be proportional to k.

Now we give another method of handling the problem. It will result in a method whose execution time is proportional to $k^{3/2}$ and which otherwise satisfies conditions (a), (b), (c). It is therefore close to optimal, but is not as good as the final method, which indeed satisfies all four conditions.

Why would we bother to explain a method when we know a better one? Because the techniques have a number of other fruitful applications.

So, once again, imagine that we have chosen 11 elements of our output subset from $[1,97]$, namely the elements $37,2,15,18,91,13,22,7,73,10,29,$ and consider the situation when we press the button and are given the number 53. We have to find out if 53 is on the list of 11. Suppose we store the elements of the list in a binary tree, instead of only as a linear list. This means that each element in the list is the parent of at most two later elements of the list, one smaller than it and one bigger.

As the first 11 elements of our set are chosen successively, the tree grows as in Figure 1.

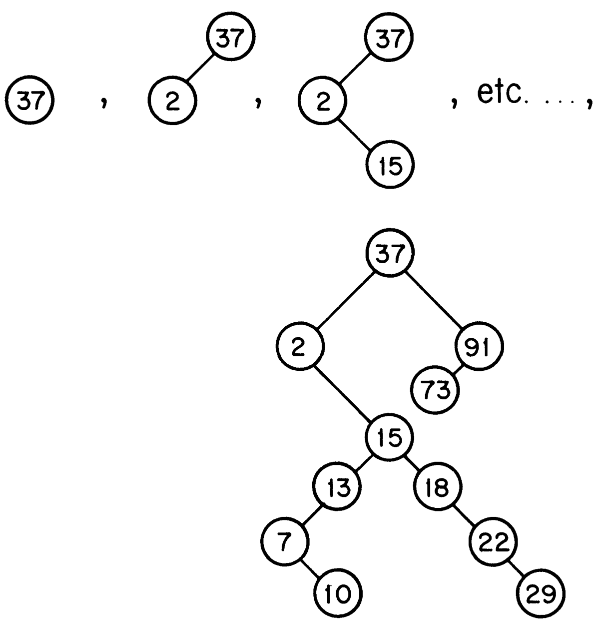

FIG. 1

The general rule is that we take the newest integer and bring it to the top of the tree. If it is smaller than the element there, we move southwest; if larger, southeast. Thus, carrying the newest integer 53 in our hand, we go to the top of the last tree in Figure 1, where we meet 37. Since $53 > 37$ we go down to the 91. Since $53 < 91$ we go down to the 73. Since $53 < 73$, finally, our tree sprouts a new leaf, southwest of the 73, which holds the 53.

This leaves just two questions: Why do such a thing? How? The reason for the binary tree is that in order to discover whether 53 is already in the list we no longer have to compare it with *every* item on the list. We need only compare it with the elements which we encounter as we go down the path in the tree to insert the 53 in its place. (Why?)

Thus, we compared 53 with $37,91,73$ only, and we were then quite sure that 53 was not an element of the current tree. For a binary tree with $m \geqslant 1$ vertices it is not hard to show (see Appendix) that the

average length of a path from the root to a terminal vertex is

$$\frac{4^{m-1}}{\binom{2m-2}{m-1}}-1\sim\sqrt{\pi m}\ .$$

It follows that the average number of comparisons needed to choose k integers is $Ck^{3/2}$, an improvement over Ck^2, but far from optimal.

As regards the "How?" a binary tree can be stored in a computer by storing the ith element in the ith position of an array, while storing in two other arrays elements left(i), right(i), the indices of the left-child and right-child of this ith element. The last tree in Figure 1 might appear in memory as shown in Table 1.

TABLE 1

i	Array (i)	Left (i)	Right (i)
1	37	2	5
2	2	—	3
3	15	6	4
4	18	—	7
5	91	9	—
6	13	8	—
7	22	—	11
8	7	—	10
9	73	—	—
10	10	—	—
11	29	—	—

The complexity of this calculation has now been brought down from (time, memory)$=(Ck^2,k)$ to $(Ck^{3/2},Ck)$. In the first edition of [**1**] there appears another algorithm whose complexity is (Ck,Ck), a considerable improvement. It fails to satisfy conditions (c) and (d) of our desiderata; namely, the output list is not presented in ascending order, and the storage requirement is somewhat more than $k+c$ words.

Hence we were interested in finding an algorithm which would deliver its output sorted, while preserving the linearity in k of the time and storage.

An undergraduate student in a course I taught at Swarthmore College in the spring of 1977, Mr. David Bayer, produced a "why-don't-you-just . . . " which improved things still further.

He suggested that in order to select k integers from $[1,n]$ we should divide up the interval into k "bins," each containing a block of about n/k consecutive integers. As we select a new integer, we first identify its bin number, which is easy, and then we compare the new integer with each occupant of that bin, inserting it into the bin if indeed it is new. The virtue of this idea is that on the average a bin will contain just one element of the output subset, because there are as many bins as there are elements of that set. Hence in order to compare the newly chosen integer with the elements of a single bin, we do an average of just one comparison.

Unfortunately, in precisely the above form, the implementation of this algorithm required extra array storage, and so we would have gained linear time and sorted output at the cost of more than one array of length k.

The coup de grâce was administered by A. Nijenhuis, who showed how the method could operate with no additional array space. His idea was that after dividing up the interval $[1,n]$ into k equal bins, as before, we should choose *how many* elements of our output subset will live in each bin, and then, in a separate operation, decide *which* elements those are to be.

This elegant idea is carried out as follows. First, the lth bin is the range

$$\lfloor (l-1)n/k \rfloor < m \leqslant \lfloor ln/k \rfloor \qquad (l=1,\ldots,k).$$

Next, we choose an integer x at random by pressing the button on our generator. We know at once that x lies in bin l, where

$$l = 1 + \lfloor (kx-1)/n \rfloor.$$

Now we decide whether x duplicates some element which already is in the lth bin, a stunning feat of prestidigitation, since we don't even know what is in the lth bin, only *how many* elements are in the bin.

Suppose, for instance, that 2 elements have been assigned to the bin [7, 12], and that we now select the integer $x=8$. Then with probability $4/6=2/3$, x is not one of the two already-chosen (but invisible) elements in the bin, and with probability $1/3$ it is. Hence with probability $1/3$ we simply reject x (without comparing it with anything!) and with probability $2/3$ we "keep" x. To keep x simply means that its bin now contains 3 elements while the actual value $x=8$ is discarded.

When we are finished playing this little game, we end up with the occupancy numbers of each bin. In the next phase (see **[1]** for details) we go through each bin in turn and choose the elements at random in the bin, to match the now-known occupancy numbers of the bin. This is a simpler problem, although it sounds the same as the original question, because each bin has an average occupancy of one element, and so we can be rather crude in choosing the representatives. Finally, a good deal of delicacy is needed in order to accomplish all of this in one array of k words, but it can be done, and all of our conditions have now been met. The distance between the original "why-don't-you-just- . . ." and the final algorithm is considerable.

Here is one more example of the process, albeit a much simpler one. The question now is, How do we list *all* of the 3-subsets of {1, 2, 3, 4, 5}? Many would respond by instinctively writing the following list:

$$\{1,2,3\},\{1,2,4\},\{1,2,5\},\{1,3,4\},\{1,3,5\},\{1,4,5\},\{2,3,4\},\{2,3,5\},\{2,4,5\},\{3,4,5\}.$$

Precisely how is this list constructed? That is, how do we go from a subset to its immediate successor?

If the last element of a subset, a_3, is <5, we simply increase it by 1. If $a_3=5$ but $a_2<4$, we increase a_2 by 1 and set a_3 to $1+\text{new } a_2$. In general let

$$a_1 < a_2 < \cdots < a_k$$

be a k-subset of $\{1,2,\ldots,n\}$. We find the least h such that

$$a_{k+1-h} < n+1-h.$$

To find the successor of the input subset, we then (a) increase a_{k+1-h} by 1, and (b) make all following elements a consecutive block of integers. For example, after

$$\{3,5,14,27,28,29,30\}$$

the next 7-subset of $\{1,2,3,\ldots,30\}$ is

$$\{3,5,15,16,17,18,19\}.$$

The algorithm is therefore split into two parts: a backward pass in which we search for h, and a pass to the right in which we modify all subset elements $a_j (j \geqslant k+1-h)$.

What more is there to say? Just this: it turns out that half of the work, namely the backward pass, can be eliminated by a simple observation. The main point of this observation is that by remembering the new value of $t=a_{k+1-h}$ of the leftmost element that was changed, we can predict what the value of h will be on the next execution, and therefore will not have to search for it.

In fact, there are just two cases. If the new $a_{k+1-h} < n+1-h$, then the next value of h will be 1, i.e.,

only the rightmost element a_k will change next time; if, however, $a_{k+1-h}=n+1-h$ now, then on the next call, the new h will be one more than it is now. Briefly, the only two possibilities for the next h are 1 or $h+1$, and we distinguish these by comparing a_{k+1-h} with $n+1-h$.

Thus, in addition to the output set $a_1,\cdots,a_k$, each time, we remember h also, and the speed of our original "why-don't-you-just . . ." suggestion is doubled.

Appendix. Here we prove that the average length of a path from the root to a terminal vertex of a binary tree of m vertices is

$$4^{m-1}\binom{2m-2}{m-1}^{-1}-1, \qquad (m\geqslant 1). \tag{1}$$

If b_m is the number of binary trees on m vertices, we will use the fact (see [2], vol. 2, p. 389) that

$$\sum_{m\geqslant 0} b_m x^m=\frac{1}{2x}(1-\sqrt{1-4x}\,) \overset{\text{def.}}{=\!=} B(x). \tag{2}$$

Let $b(m,r,s)$ denote the number of binary trees on m vertices which have exactly r maximal paths of length s. Now, all binary trees on $m+1$ vertices with r maximal paths of length s are obtained by choosing for the left subtree at the root a tree on j vertices with μ paths of length $s-1$ and for the right subtree a tree on $m-j$ vertices with $r-\mu$ paths of length $s-1$, and so

$$b(m+1,r,s)=\sum_{0\leqslant j\leqslant m}\ \sum_{0\leqslant \mu\leqslant r} b(j,\mu,s-1)b(m-j,r-\mu,s-1). \tag{3}$$

If we put

$$B_s(x,t)=\sum_{r,m\geqslant 0} b(m,r,s)t^r x^m \tag{4}$$

then (3) yields

$$\begin{cases} B_s(x,t)=xB_{s-1}(x,t)^2+1, & (s\geqslant 1)\\ B_0(x,t)=xt+B(x)-x. \end{cases} \tag{5}$$

Next, let

$$C_s(x)=\frac{\partial}{\partial t}B_s(x,t)\Big]_{t=1}=\sum_{r,m\geqslant 0} rb(m,r,s)x^m. \tag{6}$$

Then partial differentiation of (5) yields

$$\begin{aligned} C_s(x)&=2xB_{s-1}(x,1)C_{s-1}(x), && (s\geqslant 1)\\ &=2xB(x)C_{s-1}(x), && (s\geqslant 1)\\ C_0(x)&=x \end{aligned} \tag{7}$$

because $B_s(x,1)=B(x)$, independent of s (see (4), or (5) with $t=1$).

It follows that $C_s(x)=x(1-\sqrt{1-4x}\,)^s$, $(s\geqslant 0)$, and so

$$\sum_{s\geqslant 0} C_s(x)t^s=x/\big(1-t(1-\sqrt{1-4x}\,)\big)$$

from which

$$\sum sC_s(x)=x(1-\sqrt{1-4x}\,)/(1-4x) \tag{8}$$

follows by differentiation.

Comparing (8) with (6), the total *length* of all maximal paths in all binary trees of m vertices is the coefficient of x^m in (8), namely,

$$4^{m-1}-\binom{2m-2}{m-1} \qquad (m\geqslant 1).$$

The *number* of such maximal paths is

$$\sum_{r,s} rb(m,r,s)$$

which, after (6) is the coefficient of x^m in

$$\sum_s C_s(x) = x/\sqrt{1-4x}\ ,$$

namely,

$$\binom{2m-2}{m-1} \qquad (m \geqslant 1)$$

and the result follows.

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TONE PERCEPTION AND DECOMPOSITION OF PERIODIC FUNCTIONS

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1. Introduction. In this paper we present two results about periodic functions which answer some rather natural questions in the subject, although as far as I know they have not previously been considered. Before presenting the precise formulation of the theorems, I will describe briefly the physical-psychological facts which suggested the problem.

It is a familiar fact of experience that, with rare exceptions, people have the ability to hear two or more musical tones simultaneously. The whole phenomenon of harmony is a consequence of this fact. Although we tend to take this phenomenon for granted, it is not clear a priori that one should be able to "decompose" sounds in this way. For example, the analogous situation for vision is color perception and here the situation is quite different. When red and blue lights are superimposed, the original components are lost and one experiences the entirely new sensation of purple.

This study was motivated by the fact that among musical sounds there are two different types which will be referred to as *simple tones* and *composite tones*, the latter corresponding to the above-mentioned sounds in which the listener is aware of two or more pitches simultaneously. A rigorous classification of tones as simple or composite would require performing careful psychological experiments. However, without resorting to an acoustical laboratory, I hope to illustrate by means of examples the desired distinction with sufficient precision for present purposes. Typical examples of composite tones are chords played on a piano, a choral group singing in harmony, the usual sounds produced by an orchestra. Typical examples of simple tones are those produced by most of the orchestra instruments or by a single human voice. However, the criterion for simplicity or compositeness does not depend on how a sound is produced. Thus, most woodwind instruments can be made to produce a composite tone by using certain fingerings and "over blowing." (For example, this can be

achieved on a recorder by covering holes 2, 4, 5, 7, and 8. The composite tone is approximately a minor tenth.) In the other direction, when all members of a mixed chorus sing the same melody, the tones they produce are simple even though the women are singing an octave above the men. In general, when two notes an octave apart (i.e., one having twice the frequency of the other) are heard simultaneously, the resulting sound is simple rather than composite. (A striking example of this is the "drone" sound produced by bagpipes, which sounds like a single note though it is actually produced by pipes tuned an octave apart.) For the reader who does not perceive the clear qualitative difference between an octave (simple) and, say, a fifth (composite), which is the sound produced by two notes whose frequency ratio is 3 to 2, the theorems to follow are still valid though probably less interesting than they will be to those who can make this auditory distinction. For this latter group there are in fact some rather surprising empirical consequences of our theoretical results which can be easily verified by anyone with access to a piano. We will mention them at the end of this section.

For purposes of the analysis to follow, a tone will be identified with the periodic function corresponding to the sound wave it produces. We will say a function f of period p is *decomposable* if there exist functions $f_1,\dots,f_n$ of periods $p_1,\dots,p_n$ where each p_i is commensurable with p and such that

$$f=\sum_{i=1}^{n} f_i, \tag{1.1}$$

$$p \text{ does not divide } p_i \text{ for any } i. \tag{1.2}$$

A function is called *indecomposable* if no such decomposition exists. Our empirical assertion is that, roughly speaking, simple tones are produced by indecomposable sound waves, composite tones by decomposable ones. The mathematical content of this paper will consist in verifying that this classification makes sense, that is, first that there actually exist indecomposable functions. The proof turns out to be rather tricky and is presented in the final section. (I am indebted to Professors P. Chernoff and R. M. Robinson for showing me how to make the final step of the argument.) Second, we will establish a uniqueness theorem for decomposable functions to the effect that, although the functions f_i which make up the decomposition (1) are not unique, their periods are (e.g., a function of period 30 cannot be expressed both as the sum of two indecomposable functions of periods 5 and 6 and also as the sum of two such functions of periods 3 and 10). The proof is not difficult once the problem has been properly formulated and will be given in the next section. Before proceeding to these formal matters, we discuss briefly some of their empirical implications.

Note first that condition (1.2) above is crucial, for without it every function would be decomposable. Namely, if f is any function of period p, and g any function of a commensurable period q, then one has

$$f=(f-g)+g,$$

but the above is not a legitimate decomposition in our sense because $f-g$ will in general have period l.c.m. (p,q) and this would violate condition (1.2).

The earlier observation about octaves is a consequence of our definition, for suppose f and g are indecomposable and have periods p and $p/2$ and let $h=f+g$. Now h clearly has period p and if h were decomposable, we would have

$$h=f+g=f_1+\cdots+f_n$$

where f_i has period p_i and p does not divide any of the p_i. But then solving (1.3) for f would give the decomposition

$$f=f_1+\cdots+f_n-g$$

which contradicts the assumption that f is indecomposable. The above argument applies as well if g has period p/k for any positive integer k. In general, if two simple tones have frequencies one of which is a multiple of the other, then superimposing them will preserve simplicity. The case $k=3$, which is probably less familiar than that of the octave, is easily verified with a piano. For example

playing middle C together with G above high C (the frequencies are approximately 261.6 and 785 cycles per second, respectively) produces a tone which is clearly simple.

Because of condition (1.2), our law of decomposition is rather different from those in other branches of mathematics. For example, if one adds an indecomposable function to a decomposable function, the result may be indecomposable. As an illustration, let g_1 and g_2 be indecomposable of periods $p/2$ and $p/3$ and let f be indecomposable of period p. Then $g=g_1+g_2$ is by definition decomposable, but if we now add in the indecomposable function f of period p the sum $h=f+g_1+g_2$ will be indecomposable. To see this, suppose h were a sum of functions h_i with periods p_i not divisible by p. Then one would get a decomposition

$$f=\sum_{i=1}^{n} h_i-g_1-g_2$$

contrary to the assumed indecomposability of f.

The above observation has an interesting empirical consequence referred to earlier. Here is a simple piano experiment. First play a fifth, say, middle C and middle G simultaneously. Since these notes produce tones whose frequencies are in the ratios of 2 to 3, the sound will be composite. Now, however, the argument presented in the preceding paragraph "predicts" that if in addition one plays along with middle C and G the note low C (C below middle C) the resulting tone should be simple, and indeed this turns out to be the case. It is as though adding in the additional low C "erased" the multi-pitch effect of the other two notes. To make the effect even more striking, one should first play a scale in fifths, that is, start with C and G, then play D and A, E and B, etc. Each tone will be composite. Now repeat the exercise adding in the low notes, thus low C, middle C, middle G, then low D, middle D, middle A, etc. If one stays on the white keys, the tones produced will be simple, except at the seventh step, low B, middle B, high F, where the sound is clearly composite, the reason being that the frequency of high F is not three times that of low B. The same game can be played with other combinations. Thus, middle G and high C produce the composite tone of a fourth (frequency ratio 4 to 3). If one now adds in low C, the resulting sound again becomes simple. (I would be very grateful for any information regarding references to this phenomenon in the literature.)

We must now express some rather obvious reservations so that the reader does not think we are overstating the empirical case. The mathematical notion we have defined is clear-cut. Either a function is decomposable or it is not. The corresponding psychological notions are not and cannot be made so precise. Our claim is merely that some sounds are clearly composite, others clearly simple; but, as in almost all psychological phenomena, there is an ambiguous "gray area" in the middle. The situation is typical of psychological phenomena generally. Thus, the sky is clearly blue and the grass clearly green, but as one moves continuously through the color spectrum there is bound to be an ambiguous region of frequencies somewhere in the middle. The ear responses described here must also be understood in this rough, qualitative sense. When I speak of hearing three notes simultaneously, it is to be understood that the notes should be played with roughly equal volume. Further, the piano is a tempered instrument so the frequency ratios produced by the various note combinations are never exactly those I have specified. In other words, almost anything said in this section requires qualification. To make them explicitly in each case, however, would extend the text of this note without adding anything to its content. We request the reader's indulgence, therefore, for our deliberate imprecision.

Finally, I want to say a word concerning the use or rather the non-use of Fourier analysis in this paper. If one prefers to think in terms of sines and cosines, much of what we have said can be rephrased in those terms. For example, the reason an octave sounds simple rather than composite could be explained by saying that the ear has no way of knowing whether the octave sound is being produced by two sources or by a single source in which more than the normal amount of energy is in the even harmonics. But, in our view, bringing in sines and cosines is somewhat of a red herring. The fundamental argument is embodied in our condition (1.2). A function f of period p cannot be

distinguished from the sum of two functions f' and f'' of periods p and $p/2$, because in fact f is *already* expressible as the sum of two such functions in infinitely many ways (again, choose g of period $p/2$; then $f=(f-g)+g$) and neither the ear nor any other detection device one might invent would "know" how to choose one particular decomposition out of the infinitude of possibilities. In the next section, the formal results will continue to be completely general and algebraic, requiring no assumption of continuity or even measurability of the functions involved.

2. Uniqueness. The theorem of this section holds for any function between commutative groups and will therefore be presented in that more general setting.

Let G and H be commutative groups and let f map G into H. An element p of G is called a *period* of f if $f(v+p)=f(x)$ for all x in G. Let $P(f)$ be the set of all periods of f.

LEMMA 1. *The set $P(f)$ is a subgroup of G.*

The proof is immediate.

LEMMA 2. *If f and g are functions from G to H, then $P(f+g)\supset P(f)\cap P(g)$.*

This is also obvious.

Note that if $P(f)=G$ then f is a constant and if $P(f)=\{0\}$ then f has no nontrivial period.

DEFINITION. The function f is called *decomposable* if there exist functions $f_1,\ldots,f_n$ such that

$$f=f_1+\cdots+f_n \tag{2.1}$$

and

$$P(f_i) \text{ is not contained in } P(f) \text{ for any } i. \tag{2.2}$$

Observe that condition (2.1) of the previous section implies condition (2.2) for the case where G is the reals and the groups $P(f)$ and $P(f_i)$ are cyclic.

Let us call the decomposition (2.1) *admissible* if the subgroups $P(f_i)$ have the property that no one of them is contained in another.

LEMMA 3. *If f has a decomposition, it has an admissible decomposition.*

■By induction on n. For $n=1$ there is nothing to prove. Now suppose f has some non-admissible decomposition

$$f=f_1+\cdots+f_n+f_{n+1}$$

where, say, $P(f_n)\subset P(f_{n+1})$. Let $\tilde{f}_n=f_n+f_{n+1}$. From Lemma 2 we have $P(\tilde{f}_n)\supset P(f_n)$. It follows that $P(\tilde{f}_n)$ is not contained in $P(f)$ since, if it were, then $P(f_n)$ would be contained in $P(f)$ contrary to (2.2). Therefore, we have a new decomposition with fewer summands

$$f=f_1+\cdots+f_{n-1}+\tilde{f}_n$$

so by the induction hypothesis f must have an admissible decomposition.■

UNIQUENESS THEOREM. *Let*

$$f_1+\cdots+f_n=\tilde{f}_1+\cdots+\tilde{f}_m \tag{2.3}$$

be two admissible decompositions of some function f where all the f_i and $\tilde{f}_i$ are indecomposable. Then the families $\{P(f_1),\ldots,P(f_n)\}$ and $\{P(f_1),\ldots,P(\tilde{f}_m)\}$ are identical (so, in particular, $m=n$).

■ Suppose, say, $P(f_1)$ is not among the $P(\tilde{f}_i)$.

CASE I. $P(f_1)$ does not contain any $P(\tilde{f}_i)$. Then solving (3.3) for f_1 gives decomposition of f_1 contrary to assumption that f_1 is indecomposable.

CASE II. $P(f_1)\supset P(\tilde{f}_i)$ for some i, say $i=1$. Then $P(\tilde{f}_1)$ does not contain any $P(f_i)$ for any i, for if it did we would have $P(f_1)\supset P(f_i)$ contradicting admissibility. But then (3.3) can be solved for $\tilde{f}_1$ contradicting the indecomposability of $\tilde{f}_1$. ■

3. Existence of indecomposable periodic functions. The uniqueness theorem above would be vacuous unless one could show that there exist indecomposable functions. In fact, over some groups all functions are decomposable. For example, take G to be the Klein four group with members 0, a, b, c. We claim that any function from G to the rationals is decomposable. It will be enough to show that f is decomposable where

$$f(0)=1$$
$$f(x)=0 \quad \text{for} \quad x=a,b,c.$$

One verifies that

$$f=f_a+f_b+f_c \tag{3.1}$$

where

$$\begin{array}{ll} f_a(0)=f_a(a)=0 & f_a(b)=f_a(c)=-1/2 \\ f_b(0)=f_b(b)=1/2 & f_b(a)=f_b(c)=0 \\ f_c(0)=f_c(c)=1/2, & f_c(a)=f_c(b)=0. \end{array}$$

One verifies that f_a has period a, f_b has period b, and f_c has period c, while f has period 0, so (3.1) is a legitimate (in fact even admissible) decomposition.

We will now show that if G is the real numbers, then indecomposable functions exist. The function whose indecomposability we shall establish is probably the simplest periodic function one could think of, namely the characteristic function of the integers, which we will denote by χ. This is not a very natural function to choose from the point of view of applicability to tone perception. However, knowing χ is indecomposable enables one to establish the indecomposability of other, more musical functions, in particular $\cos x$. The argument runs as follows:

Suppose there were a decomposition of $\cos x$, thus

$$\cos x=\sum_{i=1}^{k} f_i(x) \tag{3.2}$$

where the periods of the f_i are not divisible by 2π. Now define the function q by

$$\begin{aligned} q(x)&=1 \quad \text{if } x=2n\pi \\ &=-1 \quad \text{if } x=(2n+1)\pi \\ &=0 \quad \text{otherwise.} \end{aligned}$$

Next define function $\tilde{f}_i$, $i=1,\ldots,k$ by

$$\begin{aligned} \tilde{f}_i(x)&=f_i(x) \quad \text{if } x=n\pi \\ &=0 \quad \text{otherwise.} \end{aligned}$$

from (3.2) we see that

$$q(x)=\sum_{i=1}^{k} \tilde{f}_i(x). \tag{3.3}$$

Now $q(x)=r(x)-s(x)$ where

$$\begin{aligned} r(x)&=2 \quad \text{for } x=2n\pi \\ &=0 \quad \text{otherwise,} \\ s(x)&=1 \quad \text{for } x=n\pi \\ &=0 \quad \text{otherwise.} \end{aligned}$$

Now from (3.3) we have

$$r(x)=\sum_{i=1}^{k} \tilde{f}_i(x)+s(x).$$

Furthermore, since $s(x)$ has period π and $r(x)$ has period 2π, this is a legitimate decomposition of

$r(x)$. But since $r(x)=2\chi(x/2\pi)$ it follows that $r(x)$ is indecomposable, giving the desired contradiction. It now also follows that $\sin x(=\cos(x-(\pi/2)))$ is also indecomposable (trivial exercise for the reader).

We should remark that if one were to consider only decompositions in which the summands are continuous or even merely integrable functions then the indecomposability of sine and cosine would follow from elementary Fourier analysis. For the present application this would certainly be adequate (it would be a remarkable instrument indeed that could produce a non-measurable tone). The interest in the present stronger result is therefore algebraic rather than musical.

EXISTENCE THEOREM. *The function χ from R to R defined by*

$$\chi(x)=1 \quad \text{for } x\in Z$$
$$=0 \quad \text{otherwise}$$

is indecomposable.

Suppose χ is decomposable so that

$$\chi=f_1+\cdots+f_k+g_1+\cdots+gl \tag{3.2}$$

where the f_i have rational (non-integral) periods and the g_j have irrational periods. We first concentrate on the f_i. By change of variable we may suppose that χ and all f_i have integral periods p and p_i where of course p does not divide any p_i. Let m be any common multiple of $p_1,\ldots,p_k$ so that $m=p_iq_i$ for some integer q_i and let $n=mp$ so that $n=p_iq_ip$ for each i. Let ζ be a primitive pth root of 1. Then we assert

$$\sum_{r=1}^{n} f_i(r)\zeta^r=m \tag{3.3}$$

because $\chi(r)=0$ except for $r=p,\ 2p,\ldots,mp$ and $\zeta^r=1$ for $r=kp$ since ζ is a pth root of 1.

Next, for any real number γ and $i=1,\ldots,k$ we claim

$$\sum_{r=1}^{n} f_i(\gamma+r)\zeta^r=0 \tag{3.4}$$

because

$$\sum_{r=1}^{n} f_i(\gamma+r)\zeta^r=\sum_{r=1}^{p_i} f_i(\gamma+r)(\zeta^r+\zeta^{r+p_i}+\cdots+\zeta^{r+(q_ip-1)p_i})$$
$$=\sum_{r=1}^{p_i}(f_i(\gamma+r)\zeta^r)(1+\zeta^{p_i}+\cdots+(\zeta^{p_i})^{q_ip-1})$$
$$=\sum_{r=1}^{p_i}(f_i(\gamma+r)\zeta^r)(1-(\zeta^{p_i})^{q_ip})/(1-\zeta^{p_i})=0$$

because the numerator above is zero since ζ is a pth root of 1, and the denominator is not zero because ζ is a primitive pth root.

We now consider the functions g_j with irrational periods $p'_1,\ldots,p'_l$. We need some notation. Let $L=\{1,2,\ldots,l\}$ and let $m_1,\ldots,m_l$ be positive integers. Then for each subset S of L define

$$p'_S=\sum_{i\in S} m_ip'_i \quad \text{if } S\neq\emptyset$$
$$p'_\emptyset=0.$$

Let $|S|$ denote the cardinality of S.

LEMMA 4. *There exist integers $m_1,\ldots,m_l$ such that p_S is irrational for all non-empty S in L.*

Induction on l. For $l=1$ take any m_1. Now suppose $m_1,\ldots,m_l$ have been chosen. For any $S\subset L$ we

claim $p_S + mp'_{l+1}$ is rational for at most one integer m for if $p'_S + mp'_{l+1} = r$ and $p'_S + m'p'_{l+1} = r'$ then $p'_{l+1} = (r-r')/(m-m')$ contradicting irrationality of p'. Thus the set $m_1, \ldots, m_l, m_{l+1}$ will satisfy the Lemma provided m_{l+1} avoids at most 2^l values.

We continue the proof of the Theorem by choosing integers $m_1, \ldots, m_l$ as in the Lemma and note that

$$\sum_{S \subset L} \chi(r + p'_S)(-1)^{|S|} = 1 \tag{3.5}$$

where r is some integer because $r + p_S$ is irrational, hence $\chi(r + p_S) = 0$ except for $S = \varnothing$.

Now we claim for all j and any real γ

$$\sum_{S \subset L} g_j(\gamma + p'_S)(-1)^{|S|} = 0 \quad \text{for all } j. \tag{3.6}$$

To see this, note that (3.6) can be rewritten

$$\sum_{S \subset L - \{j\}} \left(g_j(\gamma + p'_S)(-1)^{|S|} + g_j(\gamma + p'_{S \cup \{j\}})(-1)^{|S \cup \{j\}|} \right),$$

but each term in the above sum is zero because

$$g_j(\gamma + p'_{S \cup \{j\}}) = g_j(\gamma + p'_S + m_j p'_j) = g_j(\gamma + p'_S)$$

since g_j has period p'_j, but also $(-1)^{|S|} + (-1)^{|S \cup \{j\}|} = 0$, which proves the assertion. To prove the theorem observe that

$$\begin{aligned} \sum_{r=1}^{n} \sum_{S \subset L} \chi(r + p'_S)(-1)^{|S|} \zeta^r &= \sum_{r=1}^{n} \chi(r)(-1)^{|\varnothing|} \zeta^r \text{ from (3.5)} \\ &= m \text{ from (3.3).} \end{aligned} \tag{3.7}$$

On the other hand, from (3.6) we have

$$\sum_{r=1}^{n} \sum_{S \subset L} g_j(r + p'_s)(-1)^{|s|} \zeta^r = \sum_{r=1}^{n} 0 = 0 \tag{3.8}$$

and

$$\sum_{r=1}^{n} \sum_{S \subset L} f_i(r + p'_s)(-1)^{|s|} \zeta^r = \sum_{S \subset L} (-1)^{|s|} \sum_{r=1}^{n} f_i(r + p'_s) \zeta^r = \sum_{S \subset L} 0 = 0 \tag{3.9}$$

from (3.4). Combining (3.7) with (3.8) and (3.9) gives the contradiction $m = 0$. ■

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PROGRESS REPORTS

EDITED BY P. R. HALMOS

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It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

Progress Reports is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses,

what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal: usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

ZETA ZEROS ON THE CRITICAL LINE

HUGH L. MONTGOMERY

The celebrated Riemann zeta function is intimately connected with the study of prime numbers. It is easy to define $\zeta(s)$ for those complex numbers s whose real part is greater than 1; in that case the series

$$\sum_{n=1}^{\infty} \frac{1}{n^s}$$

converges and its sum is, by definition, $\zeta(s)$. To get $\zeta(s)$ for other values of s, use analytic continuation. It turns out that there exists a unique analytic function whose domain is the entire complex plane (except for the point 1) that agrees with what the series gives when $\operatorname{Re} s > 1$. At 1 the function has a simple pole, and, in fact, Riemann showed that the difference $\zeta(s) - 1/(s-1)$ defines an entire function.

Riemann proved also that ζ satisfies a useful *functional equation*. To state it, write

$$\xi(s) = \zeta(s)\Gamma(s/2)\pi^{-s/2}$$

(where Γ is the gamma function, the extension of factorials, satisfying $\Gamma(n+1) = n!$ for positive integers n); in this notation the functional equation says that

$$\xi(s) = \xi(1-s).$$

From it and known properties of the gamma function it follows that ζ has zeros at the negative even integers. These are the so-called *trivial zeros* of the zeta function.

The unique factorization of integers into prime numbers now yields the first connection between ζ and primes, namely the *Euler product formula*

$$\zeta(s) = \prod_p \left(1 + \frac{1}{p^s} + \frac{1}{p^{2s}} + \frac{1}{p^{3s}} + \cdots\right).$$

Here the product is extended over all primes; the equation holds in the half-plane defined by $\operatorname{Re} s > 1$. In that half-plane the product is absolutely convergent so that $\zeta(s) \neq 0$ there. It follows from the functional equation that ζ has no zeros, other than trivial ones, in the half-plane defined by $\operatorname{Re} s < 0$. The zeta function does, however, have infinitely many zeros in the *critical strip* between these two half-planes, i.e., in the strip defined by $0 \leqslant \operatorname{Re} s \leqslant 1$.

The intimate connection between the zeta function and prime numbers has to do with the location of the zeros of ζ. For each positive real number x, let $\pi(x)$ be the number of primes not exceeding x. When a number theorist says he wants to understand the distribution of prime numbers, he means that he wants to know all about the function π. Relatively easy "experimental" work soon reveals that π is in some sense near to the logarithmic integral defined by

$$\operatorname{li}(x) = \int_2^x \frac{dt}{\log t}.$$

How near? Could it possibly be true, for instance, that, for each real number θ, the difference between $\pi(x)$ and $\operatorname{li}(x)$ is bounded by a constant multiple of x^θ (for large x)? In standard notation, could it be

true that

$$\pi(x) = \mathrm{li}\,x + O(x^{\theta})?$$

What is known is that if that were true, then $\zeta(s)$ would have no zeros in the half-plane defined by Re $s > \theta$. Observe: an assumption about the distribution of primes gives information about the zeros of zeta. In the converse direction, it is known that if $\zeta(s) \neq 0$ for Re $s > \theta$, then

$$\pi(x) = \mathrm{li}(x) + O(x^{\theta}(\log x)^2).$$

To know more about π, we need to know more about the zeros of ζ. Riemann demonstrated that the first few zeros of zeta have real part equal to $\frac{1}{2}$, and he conjectured that all non-trivial zeros lie on the *critical line* defined by Re $s = \frac{1}{2}$. This assertion has become known as the Riemann Hypothesis.

Our present knowledge concerning the non-trivial zeros points toward the truth of the Riemann Hypothesis in various ways. Empirical evidence is encouraging: recent calculations of Richard Brent have revealed that all of the first 70,000,000 zeros lie on the critical line. We know that no zero can lie too close to the edge of the critical strip, in the sense that if $\zeta(\beta + i\gamma) = 0$, then $|\gamma| > 14$ and $1/\log|\gamma| < \beta < 1 - (1/\log|\gamma|)$. We know also that most zeros lie very close to the critical line, if not actually on it. For example, if $\delta > 0$, then there is a $\theta = \theta(\delta) < 1$ such that the number of zeros $\beta + i\gamma$ with $0 < \gamma \leqslant T$, $|\beta - \frac{1}{2}| > \delta$ is $< T^{\theta}$. This estimate is much smaller than the total number $N(T)$ of zeros $\beta + i\gamma$ for which $0 < \gamma \leqslant T$, since we know that

$$N(T) \sim \frac{T}{2\pi}\log T$$

(i.e., the ratio tends to 1) as T tends to infinity. Yet another approach to the Riemann Hypothesis was initiated by G. H. Hardy in 1914, when he announced that he had shown that infinitely many zeros lie on the critical line. If we let $N_0(T)$ denote the number of zeros of the form $\frac{1}{2} + i\gamma, 0 < \gamma \leqslant T$, then Hardy's theorem asserts that $N_0(T) \to \infty$ as $T \to \infty$. Subsequently, Hardy and Littlewood put this in sharper quantitative form, $N_0(T) > cT$, and still later A. Selberg introduced new ideas in showing that $N_0(T) > cT\log T$. Consequence: "a positive proportion" of the zeros lie on the critical line. Selberg did not specify the value of the constant c in his inequality, but it is clearly rather small.

In [9] Norman Levinson showed that

$$N_0(T) > \frac{1}{3}N(T)$$

for all large T, so that at least one-third of the zeros are on the critical line. Levinson's approach employs some of Selberg's techniques, but the basic device used to detect the zeros on the critical line is completely different. This is particularly evident from the fact that Selberg's argument succeeds without the need to calculate constants, while Levinson's method cannot be seen in advance to succeed; it is only at the end of many detailed calculations that it is found that the constants work out favorably. Levinson's attack is audacious in that an enormous amount of work (filling 50 printed pages) is required before one can see whether the resulting lower bound for $N_0(T)$ is positive. Another difference between Selberg's and Levinson's methods is in the manner in which they count multiple zeros. Selberg's method detects changes of sign of $\xi(\frac{1}{2} + it)$, so that one can say that a positive proportion of the zeros are of odd multiplicity and on the critical line. Levinson's discussion of multiple zeros is inefficient, but Selberg and D. R. Heath-Brown have independently observed that with small changes the method counts only simple zeros. Thus at least one-third of the zeros are simple and on the critical line.

It remains to be seen how much further Levinson's ideas can be pushed. Several possibilities need

to be explored, but the heavy technical details make progress difficult. Though mortally ill, Levinson himself energetically investigated some of the further ramifications of his work. Certainly his work will demand closer scrutiny in the years to come.

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MATHEMATICAL NOTES

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

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Advice to prospective authors: The editors have recently been receiving about **ten times** as many Mathematical Notes as can be used. It will simplify our work if authors will submit only especially interesting manuscripts. Mathematical Notes should be short papers of one to four printed pages which give new insights, new and improved proofs of old theorems, brief bits of mathematical folklore that have not found a home in the literature, or (occasionally!) new results that are not too technical. The topics should be of wide current interest. Papers that have already been rejected by a research journal are only very rarely suitable as Mathematical Notes.

R.P.B.

THE EXISTENCE OF NON-MEASURABLE SETS

R. DANIEL MAULDIN

Let X be a set and τ a locally compact T_2 topology of subsets of X. Let I be a nonnegative linear functional on $C_{00}(X)$ and let ι be the outer measure on X induced by I. These notations are taken from Hewitt and Stromberg [**1**]. In their book, on pages 134 and 135, it is stated that no general facts concerning the existence of non-ι-measurable sets or methods of constructing such sets are known. In this note, we hope to clear up this situation using only techniques which fall within the scope of those developed in [**1**].

In order to give the general solution, we will use the construction given in the following lemma.

LEMMA. *Let F be a compact subset of X such that $\iota(F)=k>0$ and $\iota(\{x\})=0$, for each x in F. Then there is a compact subset H of F such that $\iota(H)\geqslant k/2$ and a continuous map f of H onto the Cantor set, C, such that $\iota(f^{-1}(t))=0$, for each t in C.*

Proof. Let $H(0)$ and $H(1)$ be disjoint closed subsets of F such that $k/4<\iota(H(t))<k/2$, for $t=0$ or 1. An induction argument reveals that there is a system, $\{H(t_1,\dots,t_n)\colon (t_1,\dots,t_n)$ is a finite sequence of zeroes and ones$\}$, such that for each n, the sets $H(t_1,\dots,t_n)$ are pairwise disjoint closed sets with $k/2^{n+1}<\iota(H(t_1,\dots,t_n))<k/2^n$ and such that $H(t_1,\dots,t_n)$ contains $H(t_1,\dots,t_n,t)$, for $t=0$ or 1.

Now, let $H(n)=\cup\{H(t_1,\ldots,t_n)\colon (t_1,\ldots,t_n)\in\{0,1\}^n\}$. Notice that each set $H(n)$ is a compact subset of F, $H(n)\supseteq H(n+1)$ and $\iota(H(n))>k/2$. Also notice that for each x in $H=\cap H(n)$, there is exactly one sequence $t=(t_1,t_2,t_3,\ldots)$ in $\{0,1\}^N$ such that $x\in\cap\{H(t_1,\ldots,t_n)\colon n\in N\}$. So, define $f\colon H\to\{0,1\}^N$, by $f(x)=t$. It can be easily checked that f is onto $C=\{0,1\}^N$. (We identify the Cantor set with $\{0,1\}^N$ under the product topology.) It is easy to see that f is a continuous map and if $t=(t_1,t_2,t_3,\ldots)$ is a point of the Cantor set, then $f^{-1}(t)=\cap\{H(t_1,\ldots,t_n)\colon n\in N\}$, which means that $\iota(f^{-1}(t))=0$. Finally, $\iota(H)\geqslant k/2$.

We can now state some conditions under which every subset of X is measurable.

THEOREM. *The following two statements are equivalent:*

(1) *every subset of X is ι-measurable*; *and* (2) *there is a subset D of X such that $X-D$ is a locally ι-null set and such that $D\cap F$ is countable, for each compact subset F of X.*

Proof. First, let us assume (1) holds. Let $D=\{x:\iota\{x\})>0\}$. Since ι is locally bounded (Theorem 9.5 of [**1**]), $D\cap F$ is countable, for each compact set F.

Suppose that $X-D$ is not locally null. Then there is a compact set T such that $\iota(T\cap(X-D))>0$. Since every subset of X is measurable, there is a compact set F contained in $T\cap(X-D)$ which has positive measure. Let H be a compact subset of F with positive measure and f a continuous map of H onto the Cantor set, C, as constructed in the lemma. Let W be a subset of C such that W and $C-W$ intersect every closed uncountable subset of C (Exercise 10.54 of [**1**]). Let $E_1=f^{-1}(W)$ and $E_2=f^{-1}(C-W)$. If H_1 is a compact set contained in E_1, then $f(H_1)$ is a compact set contained in W. Thus, $f(H_1)$ is countable. Since $f^{-1}(f(H_1))$ has measure zero, H_1 has measure zero. Since E_1 is measurable, $\iota(E_1)=0$. Similarly, $\iota(E_2)=0$. Therefore, $\iota(H)=0$. This contradiction establishes the fact that (1) implies (2).

Now, let us assume that (2) holds. If A is a subset of X, then $A\cap(X-D)$ is locally null and therefore $A\cap(X-D)$ is measurable (Corollary 10.32 of [**1**]). Also, $A\cap D$ is measurable, since $(A\cap D)\cap F$ is measurable for every compact set F (Theorem 10.31 of [**1**]). Therefore, A is measurable. Thus, (2) implies (1). Q.E.D.

COROLLARY. *If $\iota(\{x\})=0$, for each x in X and ι is not the zero measure, then there are nonmeasurable sets.*

Let us remark that if E is a subset of X such that $X-E$ is locally null, then $X-\bar{E}$ is of measure zero, and if X is σ-compact, then obviously $X-E$ is of measure zero.

The following example shows that it is possible that (2) holds and $\iota(X-D)=\infty$.

Example: Let $Y=\{0\}\cup\{1/n:n\in N\}$ and let Y have the relative topology as a subset of R, the reals. Then Y is a compact T_2 space. Define $\mu(A)=\Sigma\{1/n^2:1/n\in A\}$, for every subset A of Y. Let $X=R\times Y$, where R is the reals provided with the discrete topology and X has the product topology. Then X is a locally compact T_2 space. If f is in $C_{00}(X)$, then there are only finitely many t in R so that f_t is not the zero function on Y, where $f_t(y)=f(t,y)$. Define I on $C_{00}(X)$ by $I(f)=\Sigma\{\int f_t\,d\mu:t\in R\}$. It can be easily seen that I is a nonnegative linear functional on X and that every subset of X is measurable with respect to the induced measure ι. The set D of statement (2) of the theorem is $\{(t,y)\colon y\neq 0\}$. Finally, it can be checked that $\iota(X-D)=\infty$.

The author wishes to thank Edwin Hewitt for his helpful comments.

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COMPLEX REPRESENTATIONS OF METACYCLIC GROUPS

B. G. BASMAJI

In this note we study the representations of the metacyclic groups by applying, to the knowledge acquired in an undergraduate algebra course, a few definitions and a main theorem. First we mention these. All representations in this paper are over the complex field C.

A representation T of degree n of a finite group G is a homomorphism

$$T: G \to GL(n, C),$$

where $GL(n,C)$ is the multiplicative group of the non-singular $n \times n$ matrices over C. A representation T (of degree n) of G is irreducible if there exists no $S \in GL(n,C)$ such that

$$S^{-1}T(g)S = \left(\begin{array}{c|c} T_1(g) & L(g) \\ \hline 0 & T_2(g) \end{array}\right)$$

for all $g \in G$. Equivalently T is irreducible if for any non-zero vector $v \in V$, V an n-dimensional column vector space over C, the subspace spanned by $\{T(g)v \mid g \in G\}$ is V. Two representations T and T' (both of degree n) of G are equivalent if there exists $S \in GL(n,C)$ such that $S^{-1}T(g)S = T'(g)$ for all $g \in G$. The irreducible representations for most groups have not been explicitly constructed. However, knowing the characters of these representations is sufficient for most purposes, and these are usually much easier to compute. (The character χ of a representation T is the function $\chi: G \to C$ defined by $\chi(g) = \operatorname{trace} T(g)$ for all $g \in G$.) We need a theorem that is usually introduced early in texts on this subject. (See Dornhoff [**2**, p. 17], or Feit [**3**, p. 14].) If $T_1, T_2, \ldots, T_f$ are representatives of all the equivalent classes of irreducible representations of G and if T_i is of degree n_i, $i = 1, 2, \ldots, f$, then $n_1^2 + n_2^2 + \cdots + n_f^2 = |G|$, the order of G. The results that follow are proved in [**1**, p. 104] using properties of induced representations.

A finite group G is called a metacyclic group if G has a cyclic normal subgroup A such that G/A is also cyclic. It is easy to show that such a group is generated by two elements a and b with the defining relations,

$$a^n = b^m = 1, a^k = b^t, b^{-1}ab = a^r$$

with $t \mid m$, $r^t \equiv 1 \pmod n$, and $kr \equiv k \pmod n$. We fix the above presentation of G and state a few notations. Let U be the set of all nth roots of unity in C and let $\sigma: U \to U$ be defined by $\sigma(\zeta) = \zeta^r$ for all $\zeta \in U$. For $\zeta \in U$ let $t(\zeta)$ be the size of the orbit of ζ under $\langle \sigma \rangle$. Let $s(\zeta)$ be the order of ζ in U. Then $s(\zeta) \mid n$. Since $\zeta^{r^{t(\zeta)}} = \zeta$, it follows that $r^{t(\zeta)} \equiv 1 \pmod{s(\zeta)}$. But $t(\zeta)$ is the smallest such positive integer, hence $t(\zeta) \mid t$. Let θ be a solution of $\theta^{t/t(\zeta)} = \zeta^k$ and define the map $T_{\zeta,\theta}: G \to GL(t(\zeta), C)$ by

$$T_{\zeta,\theta}(a) = \begin{bmatrix} \zeta & & & & \\ & \zeta^r & & & \\ & & \zeta^{r^2} & & \\ & & & \ddots & \\ & & & & \zeta^{r^{t(\zeta)-1}} \end{bmatrix}, T_{\zeta,\theta}(b) = \begin{bmatrix} 0 & & & & \theta \\ 1 & 0 & & & \\ & & \ddots & & \\ & & & 1 & 0 \\ & & & & 1 \quad 0 \end{bmatrix}.$$

Below, we shall prove that $T_{\zeta,\theta}$ is a representation of G by showing that the matrices $T_{\zeta,\theta}(a)$ and $T_{\zeta,\theta}(b)$ satisfy the defining relations of G.

PROPOSITION 1. $T_{\zeta,\theta}$ is an irreducible representation of G.

Proof. To simplify notations let $T = T_{\zeta,\theta}$ and $\eta_h = \zeta^{r^{h-1}}$. We first prove that T is a representation of G by showing that $T(a)$ and $T(b)$ satisfy the defining relations of G. Since ζ is an nth root of unity it follows that $T(a)^n$ is the $t(\zeta) \times t(\zeta)$ identity matrix. Since $kr \equiv k \pmod n$ we have $T(a)^k =$

$\mathrm{diag}(\zeta^k,\ldots,\zeta^k)$. But $T(b)^{t(\zeta)}=\mathrm{diag}(\theta,\ldots,\theta)$, and since $\theta^{t/t(\zeta)}=\zeta^k$, we have $T(a)^k=T(b)^t$. A short computation shows that $T(a)T(b)=T(b)T(a)^r$. Hence T is a representation of G.

To prove T is irreducible, let V be a $t(\zeta)$-dimensional column vector space over C, and let $v\in V, v\neq 0$. Then $v={}^t(v_1,v_2,\ldots,v_{t(\zeta)})$, the transpose of the row vector $(v_1,v_2,\ldots,v_{t(\zeta)})$, and $v_j\neq 0$ for some j. If $i\leqslant j$, let $B=T(b)^{t(\zeta)-j+1}$ which has θ in the (i,j)th position. If $i>j$ let $B=T(b)^{i-j}$ which has 1 in the (i,j)th position. Hence $Bv=w={}^t(w_1,w_2,\ldots,w_{t(\zeta)})$, where $w_i\neq 0$, since $w_i=\theta v_j$ or $w_i=v_j$. Now $T(a^x)w={}^t(\eta_1^x w_1,\ldots,\eta_{t(\zeta)}^x w_{t(\zeta)})$ and

$$\eta_i^{-x}T(a^x)w={}^t(\eta_1^x\eta_i^{-x}w_1,\ldots,w_i,\ldots,\eta_{t(\zeta)}^x\eta_i^{-x}w_{t(\zeta)}).$$

But for $h\neq i, \eta_h\eta_i^{-1}\neq 1$ is an nth root of unity and thus $\sum_{x=0}^{n-1}\eta_h^x\eta_i^{-x}=0$. Hence $\sum_{x=0}^{n-1}\eta_i^x T(a^x)w=(0,\ldots,0,nw_i,0,\ldots,0)$. Since $v(\neq 0)\in V$ and i were arbitrarily chosen, it follows that $T=T_{\zeta,\theta}$ is an irreducible representation of G.

PROPOSITION 2. *$T_{\zeta,\theta}$ and $T_{\zeta',\theta'}$ are equivalent if and only if $\theta=\theta'$ and ζ and ζ' are in the same orbit under $\langle\sigma\rangle$.*

Proof. Assume $t(\zeta)=t(\zeta')$ for otherwise there is nothing to prove. Now assume $\theta\neq\theta'$. Then $T_{\zeta,\theta}(b^{t(\zeta)})=\mathrm{diag}\ (\theta,\ldots,\theta)$ and $T_{\zeta',\theta'}(b^{t(\zeta)})=\mathrm{diag}\ (\theta',\ldots,\theta')$. Let $S=(s_{pq})$ be a $t(\zeta)\times t(\zeta)$ matrix and assume $ST_{\zeta,\theta}(b^{t(\zeta)})=T_{\zeta',\theta'}(b^{t(\zeta)})S$. Then $s_{pq}\theta=\theta'\ s_{pq}$ or $s_{pq}=0$ and $S=0$.

Now assume ζ' is not in the orbit $\{\zeta,\zeta^r,\ldots,\zeta^{r^{t(\zeta)-1}}\}$ of ζ under $\langle\sigma\rangle$ If the matrix S is given as above and $ST_{\zeta,\theta}(a)=T_{\zeta',\theta'}(a)S$, then $s_{pq}\zeta^{r^{q-1}}=(\zeta')^{r^{p-1}}s_{pq}$. Thus $s_{pq}=0$ or $S=0$. Hence, in the above two cases, $T_{\zeta,\theta}$ and $T_{\zeta',\theta'}$ are not equivalent.

Now assume $\theta=\theta'$ and ζ and ζ' are in the same orbit under $\langle\sigma\rangle$. Then $\zeta'=\zeta^{r^h}$, $0\leqslant h<t(\zeta)$. Define

$$S=\left(\begin{array}{c|c}0 & \theta I\\ \hline I' & 0\end{array}\right)$$

where I is the identity $(t(\zeta)-h)\times(t(\zeta)-h)$ matrix and I' is the identity $h\times h$ matrix. Then $ST_{\zeta,\theta}(a)=T_{\zeta',\theta'}(a)\ S$ and $ST_{\zeta,\theta}(b)=T_{\zeta',\theta'}(b)\ S$. This implies that $T_{\zeta,\theta}$ and $T_{\zeta',\theta'}$ are equivalent, proving the proposition.

PROPOSITION 3. *The set of all irreducible inequivalent representations (over C) of G is given by the set of all $T_{\zeta,\theta}$ where ζ runs over a set of representatives of the orbits of U under $\langle\sigma\rangle$ and θ runs over all solutions of $\theta^{t/t(\zeta)}=\zeta^k$.*

Proof. Each $T_{\zeta,\theta}$ is counted (up to equivalence) $t(\zeta)$ times as ζ runs over all the elements of U. For each ζ there are $t/t(\zeta)$ different θ's giving inequivalent representations $T_{\zeta,\theta}$. Summing the squares of the distinct $T_{\zeta,\theta}$'s we get

$$\sum_{\zeta\in U} t(\zeta)^2\frac{t}{t(\zeta)}\cdot\frac{1}{t(\zeta)}=\sum_{\zeta\in U} t=nt,$$

the order of G, proving the proposition.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

HOW MANY DEGENERATE SIMPLICES ARE GENERATED BY $n+1$ VERTICES OF THE UNIT n-CUBE?

B. LEO RAKTOE

Let C^n be the unit n-cube in Euclidean space E^n with vertices $(x_1,x_2,\ldots,x_n)$, where x_i is 0 or 1. Select a set of $n+1$ vertices of C^n and denote the resulting convex hull by S_n. If the n-dimensional volume of S_n is zero, then it is termed a degenerate simplex, and otherwise it is a non-degenerate simplex. An unsolved problem is to determine the precise number, say $\Delta_{n,0}$, of degenerate simplices when all possible selections of $n+1$ vertices of C^n are considered.

Computer enumerations by Wells [7] lead to the following values of $\Delta_{n,0}$ and the corresponding proportions for $n \leqslant 7$:

n	2	3	4	5	6	7
$\Delta_{n,0}$:	0	12	1360	350000	255036992	571462430224
$\Delta_{n,0}/\binom{2^n}{n+1}$:	0	.1714	.3114	.3862	.4105	.3997

The same unsolved problem in a slightly different setting is implied by Ryser [6]. An unsolved problem of greater difficulty is to find the exact distribution of the n-dimensional volume of S_n. Some work related to these problems has, among others, been done by Komlos [3], Metropolis and Stein [4], Raktoe [5], Anderson and Federer [1], and Huang and Raktoe [2].

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CLASSROOM NOTES

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

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SPLINE NOTATION APPLIED TO A VOLUME PROBLEM

D. L. BARROW AND P. W. SMITH

Recently, we were asked by Professor H. O. Hartley of the Institute of Statistics at Texas A & M University if we knew of a formula for the volume of the intersection of a half-space with a hypercube in R^n. Being unaware of such a formula, we derived the desired expression. This note presents the formula and its proof in a particularly simple form by appealing to a notation which is quite familiar to specialists studying splines, but is perhaps less familiar to the entire mathematical community.

Let $\boldsymbol{\alpha}=(\alpha_1,\ldots,\alpha_n)$ be a nonzero vector in R^n, and let γ be a real number. The set $H(\gamma)=\{\mathbf{x}\in R^n:\boldsymbol{\alpha}\cdot\mathbf{x}\leqslant\gamma\}$ is called a half-space in R^n. The problem is to find an expression for the volume of $H(\gamma)\cap C$ where $C=\{\mathbf{x}\in R^n:0\leqslant x_i\leqslant 1 \text{ for } i=1,\ldots,n\}$.

The "plus function" is the crucial ingredient in obtaining the volume formula. For $k=0,1,\ldots$ let

$$(x)_+^k=\begin{cases} x^k & \text{if } x\geqslant 0,\\ 0 & \text{if } x<0.\end{cases}$$

It is easy to verify that

$$\int (r-ax)_+^k\,dx=\frac{-(r-ax)_+^{k+1}}{a(k+1)}+K \tag{1}$$

for $k=0,1,\ldots$ and $a\neq 0$. This, of course, means that the plus function integrates formally just like a polynomial.

We are now in a position to prove

THEOREM. *Let $\boldsymbol{\alpha}$ be a vector in R^n with $\alpha_i\neq 0$ for $i=1,\ldots,n$. Then*

$$\operatorname{vol}(H(\gamma)\cap C)=\frac{\sum_{\mathbf{v}\in C}(\operatorname{sgn}\mathbf{v})(\gamma-\boldsymbol{\alpha}\cdot\mathbf{v})_+^n}{n!\left(\prod_{i=1}^n\alpha_i\right)} \tag{2}$$

with the summation taken over the 2^n vertices of C and $\operatorname{sgn}\mathbf{v}\equiv(-1)^m$ *where* $m=\sum_{i=1}^n v_i$.

REMARK. The right side of (2) is a spline of degree n with simple knots, and hence it has $n-1$ continuous derivatives as a function of γ; see Curry and Schoenberg [1] for more details relating splines to volumes.

Proof.

$$\operatorname{vol}(H(\gamma)\cap C)=\int_{x_n=0}^1\cdots\int_{x_1=0}^1(\gamma-\boldsymbol{\alpha}\cdot\mathbf{x})_+^0\,dx_1\cdots dx_n$$

$$=\frac{1}{\alpha_1}\int_{x_n=0}^1\cdots\int_{x_2=0}^1\left[-\left(\gamma-\sum_{i=1}^n\alpha_i x_i\right)_+^1\right]_{x_1=0}^{x_1=1}dx_2\cdots dx_n$$

$$= \frac{1}{\alpha_1} \int_{x_n=0}^{1} \cdots \int_{x_2=0}^{1} \left[\left(\gamma - \sum_{i=2}^{n} \alpha_i x_i \right)_+^1 - \left(\gamma - \alpha_1 - \sum_{i=2}^{n} \alpha_i x_i \right)_+^1 \right] dx_2 \cdots dx_n.$$

Continued application of (1) increases the power on the "plus function" and produces (2).

There are two simple generalizations of the Theorem: first, to the case when some of the $\alpha_i = 0$; and second, to the case when C is replaced by a parallelopiped of the form $C' = \{\mathbf{u} \in R^n : \mathbf{u} = \sum_{i=1}^{n} \lambda_i \mathbf{u}_i, 0 \leqslant \lambda_i \leqslant 1\}$ where $\{\mathbf{u}_i\}_{i=1}^{n}$ are linearly independent. In the former case (2) must be modified to ignore those dimensions corresponding to $\alpha_i = 0$ (with a corresponding reduction in the degree and the smoothness of the spline). The latter case may be transformed to the former by a linear change of variables.

These results have a probabilistic interpretation. Suppose $X_1, \ldots, X_n$ are random variables which are independent and uniformly distributed on $[0, 1]$. Then formula (2) yields the cumulative distribution function of the random variable $Y \equiv \sum_{i=1}^{n} \alpha_i X_i$.

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MATHEMATICAL EDUCATION

Edited by W. E. Mastrocola

Material for this Department should be sent to W. E. Mastrocola, Department of Mathematics, Colgate University, Hamilton, NY 13346.

AN EXPERIMENTAL EVALUATION OF RETESTING

Gary A. Deatsman

The CRIMEL (Curriculum Revision and Instruction in Mathematics at the Elementary Level) program was a large-scale effort by the Ohio State University Mathematics Department to improve instruction through individualization. Riner [2], Waits [3], and Waits and Riner [4] have reported extensively the many features of the program. The CRIMEL project as such was terminated after the 1975–76 academic year, but multiple pacing of classes has been retained and the instructional resources center is still available for student use.

One feature of CRIMEL was retesting. A student not satisfied with his performance on an exam was allowed to take a similar exam a short time later. If he demonstrated increased learning on this retest, his grade might be raised. It was felt that this would encourage students to correct any deficiencies which were detected by the first exam. A detailed account of testing in the CRIMEL program was given by Elbrink [1]. Retesting was not continued after the CRIMEL project was terminated; according to Waits (interview, September 9, 1977), it was discontinued primarily for financial reasons, although there were also some pedagogical reservations.

Certainly students liked retesting. Waits [3, p. 309] observed: "The flexible retesting program was, as anticipated, a great success. As a result of the retesting aspect of our program, many students felt

that the Mathematics Department was actually trying to help them, not flunk them out!" However, in listing difficulties with CRIMEL Waits stated: "We also feel that the availability of retests created a false sense of security in the minds of the students. In some instances this false sense of security resulted in a student's postponing study until it was too late for success in the course" [3, pp. 309–310].

Some features of the CRIMEL program, including retesting, were introduced in modified form at the Ohio State University Mansfield Campus.

Certain problems with retesting soon became apparent. Some students probably used the retests as intended and benefited, but students' exam scores seemed to indicate that many were abusing the system by not preparing adequately for the first test and instead relying on retesting to raise their grades to an acceptable level. Also, some students appeared to take retests without significant additional preparation, hoping that luck alone would increase their scores. Some of our colleagues at the Columbus Campus shared our reservations and expressed the feeling that too many students viewed the initial tests as mere "rehearsals." Elbrink mentioned these problems and included among possible solutions plans to allow only one retest per exam and to give quizzes before the initial exams [2, p. 690].

At the Mansfield Campus we had always allowed only one retest per exam, due to limited personnel. We had also been giving frequent short quizzes between exams to students in slower-paced classes. Various other policies were tried to encourage students to make proper use of retesting. We hoped to motivate students to make a real effort to increase their knowledge before a retest by giving them the grade which was earned on the retest, whether higher or lower than their original grade. This probably did discourage retesting with no preparation, but it did not encourage students to do their best on initial exams. Since this policy generated considerable hostility on the part of students and discouraged less confident students from retesting, the practice was dropped. We tried restricting retesting to students who scored below 80% on the first test and limiting the maximum to which a score might be raised by retesting to 80%. We hoped that students would then make a whole-hearted effort on initial exams since very good grades could only be earned on the first attempt.

Despite our efforts we were left with the feeling that there was no way to avoid the tendency of retesting to encourage procrastination and that perhaps all of the labor involved was doing more harm than good, or at best accomplishing nothing. Some data had been gathered which indicated that CRIMEL, as a whole, was indeed effective in lowering dropout rate and that the majority of students liked the program [2, p. 86]. However, no controlled experimentation had been done to determine the effect on learning of the total CRIMEL program or its various features. We resolved to perform a controlled experiment during autumn 1975 to determine the effect of retesting on learning.

Experimental Procedure. Before the experiment began, it was described to the students, and each student had the opportunity to decline participation. None did decline, and all students in the course became experimental subjects. There were two separate classes involved in the experiment. About half of the students were in a faster-paced class, which completed five hours of credit during the quarter. The others were in a class which proceeded at a more leisurely rate, completing only three hours. The experiment was conducted in the same way with both faster- and slower-paced classes. In each class, students were randomly assigned to one of two groups. These groups were not separated in any way. They attended the same classes and were in all respects treated the same, except that one group was allowed to take retests and the other was not.

The experiment proceeded without any unforeseen difficulties. Students in the retesting group were allowed to take retests on any exam on which they had scored below 80%; they could increase their grade to a maximum of 80%. The grade assigned was either the initial test score or the retest score, whichever was higher. No retests were given for the end-of-quarter comprehensive final exams.

Results and Conclusions. We compared final exam scores and attrition of the students who were allowed to take retests with those of the students who were not.

A statistical analysis of final exam scores was done to answer the question: "Does retesting have an effect on the mean final exam scores of freshman algebra students?" (We assume for the purpose of analysis that the experimental subjects are equivalent to a random sample of a large population of similar freshman algebra students.) The null hypothesis tested was: "There is no difference between the mean final exam scores of freshman algebra students who are allowed to take retests and the scores of those who are not."

This null hypothesis was treated with a two-tailed t test. The results of the analysis are summarized in the table below:

Final Exam Scores

Faster Class	Retesters $n=39$ mean 151.4 S.D. $=23.5$	Non-retesters $n=35$ mean 152.7 S.D. $=30.6$
	$t=0.211$	
Slower Class	Retesters $n=30$ mean $=71.7$ S.D. $=16.3$	Non-retesters $n=30$ mean $=76.2$ S.D. $=17.9$
	$t=1.019$	

The means for the faster class were almost identical, and the value of t obtained was, of course, not sufficient to reject the null hypothesis at any reasonable level of confidence. The probability of obtaining a value of t as large or larger than the one obtained for the slower class was between 0.3 and 0.4. Thus the null hypothesis could be rejected at the .60 level of confidence, and we may conclude that there is at least a 60% probability that the mean final exam scores of the slower group were actually decreased by allowing retests.

Eight faster-class retesting students dropped out compared with nine non-retesters. In the slower class, two retesters and one non-retester dropped out. Because of the small numbers involved, no statistical analyses of these figures were made, but we saw no reason to conclude that allowing retests made any difference of practical importance in attrition.

We concluded from our experiment that retesting was certainly not benefiting our students and that it would actually have been harming the slower students. The following quarter retesting was discontinued in mathematics courses at the Ohio State University Mansfield Campus.

Perhaps more controlled experimentation early in the development of major curriculum revision projects would be prudent. Various innovations could be evaluated individually before incorporating them into the program. Resources could then be concentrated on the most promising techniques.

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1. L. C. Elbrink, A flexible and forgiving testing program, School Science and Mathematics, 73 (1973) 686–690.

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4. B. K. Waits and J. W. Riner, Brief history and description of CRIMEL, School Science and Mathematics, 75 (1975) 684–690.

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PROBLEMS AND SOLUTIONS

EDITED BY A. P. HILLMAN

The editors receive more problems than they can use, but they are always on the lookout for interesting and novel problems. The task of selection will be much easier if proposers will keep the following guidelines in mind:

Send all problems (both elementary and advanced) to A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131, in duplicate if possible. The editors urge proposers to include any solutions or information that will help the editors, including reasons why the problem is interesting. Problems in well-known textbooks and results that appear in generally accessible sources are not acceptable.

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution. If you submit a problem without a solution, you should tell the editors whether you know (or somebody else knows) how to solve the problem. If you are uncertain, the problem might be more suitable for the Research Problems section; see the instructions given there.

Proposers are asked to aim for the same audience as for the rest of the MONTHLY: *a rule of thumb is to think of people who have had at least a year of graduate work in Mathematics, very likely some years ago. Proposals that can be both understood and solved by high school students, or by those in the first two years of college, are usually more appropriate for the Mathematics Magazine or the Two-Year College Mathematics Journal. No precise line can be drawn between elementary and advanced problems, but in general an elementary problem will involve notions no more difficult than those encountered in the first year of graduate work and should be solvable by methods common at the same level. An advanced problem may deal with more advanced concepts or require more sophisticated techniques for its solution.*

A problem that takes half a page or more to state, or that involves several ad hoc definitions, is usually unsuitable. The statement of a problem should avoid specialized jargon and symbols whenever the English language is more readily understood: for example, "f is a continuous function" is preferable to "$f \in C$."

Before submitting a problem, be sure that it says exactly what you intend it to say. Solvers are not telepathic and will usually solve what was proposed, not what you had in mind.

PROBLEMS DEDICATED TO EMORY P. STARKE

Solutions of the problems in this issue dedicated to Professor Emory P. Starke should be mailed to Prof. A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, N.M. 87131 (U.S.A.) before May 31, 1979. To facilitate consideration, solutions should be typed (with double spacing).

S 1. *Proposed by George Pólya, Stanford University*

Consider the integer n, $n>2$, the three sequences

$$\binom{n}{1},\binom{n}{2},\binom{n}{3},\ldots,\binom{n}{n-1}, \tag{1}$$

$$s_2^n, s_3^n, \ldots, s_{n-1}^n, \tag{2}$$

$$S_2^n, S_3^n, \ldots, S_{n-1}^n \tag{3}$$

(binomial coefficients, Stirling numbers of the first and second kind, respectively), and the two statements:

(I) If n is prime, all terms of the sequence are divisible by n.

(II) If n is composite, there is in the sequence a term (there may be several terms) non-divisible by n.

Statement (I) has been proved; prove statement (II) for all three sequences.

(For the notation used, see G. Pólya and G. Szegö, *Problems and Theorems in Analysis*, vol. 2, problem VIII 247.1, p. 153, also its solution on p. 358 and the passages there quoted.)

S 2. *Proposed by H. S. M. Coxeter, University of Toronto*

In the hyperbolic plane, the locus of a point at constant distance δ from a fixed line (on one side of the line) is one branch of an "equidistant curve" (or "hypercycle"). In Poincaré's half-plane model, this curve can be represented by a ray making a certain angle with the bounding line of the half-plane. Show that this angle is equal to $\Pi(\delta)$, the angle of parallelism for the distance δ.

S 3. *Proposed by Albert A. Mullin, Huntsville, Alabama*

Prove that *any* strictly positive real-valued arithmetical function f satisfying the functional equation

$$(f(n+1)/(n+1))+n=(n+1)f(n)/f(n+1)$$

for every integer n exceeding some prescribed positive integer m is *necessarily* asymptotic to $\pi(n)$, the number of prime numbers not exceeding n; i.e., $f(n)\sim\pi(n)$.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA (USA) 94303. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (in duplicate, with double spacing) and should be mailed before May 31, 1979. Please enclose a self-addressed label or card (for acknowledgment).

E 2709 [1978, 276] **(correction)**. *Proposed by R.M.Norton, The College of Charleston, South Carolina*

The entry a_{ij} for $i + j$ even should be the binomial coefficient

$$\binom{i+j}{\frac{i+j}{2}}.$$

E 2749. *Proposed by Leo J. Alex, SUNY at Oneonta*

(i) Show that neither of the equations

$$3^a+1=5^b+7^c, \quad 5^a+1=3^b+7^c$$

has a solution in integers a, b, c other than $a=b=c=0$.

(ii) Show that the only solutions to the equation

$$7^a+1=3^b+5^c$$

in integers a, b, c are $(a, b, c)=(0, 0, 0)$ or $(1, 1, 1)$.

E 2750. *Proposed by the editors (based on* E 2749)

Find all solutions in integers a, b, c of the equation

$$9+5^a=3^b+7^c.$$

E 2751. *Proposed by Paul Monsky, Brandeis University*

Let X be a conic section. Through what points in space do there pass three mutually perpendicular lines, all meeting X?

E 2752*. *Proposed by Clark Kimberling, University of Evansville, Indiana*

Suppose a, b, c, d are real numbers satisfying $[an]+[bn]=[cn]+[dn]$ for $n=1,2,\dots,$ where $[x]$ is the greatest integer $\leqslant x$.

Prove or disprove: $a-c=d-b=$ an integer.

E 2753. *Proposed by Haim Rose, Kiriat Shmonah, Israel*

Let p be a prime and $g=\{r_1,r_2,\dots,r_k\}$ be any group under multiplication modulo p, where the r_i are integers with $0<r_i<p$. Let P be the product of all the r_i and Q be the product of those r_i satisfying $0<r_i<p/2$. Prove:

(i) $P\equiv-(-1)^k \pmod p$. [An extension of Wilson's Theorem.]
(ii) If $k=2h$, with h an odd integer, then $Q\equiv\pm 1 \pmod p$.
(iii) If $1\leqslant r_i\leqslant (p-1)/2$ for $1\leqslant i\leqslant k$, then $P\equiv 1 \pmod p$. Can this situation actually occur?
(iv) If $k=2h$, $h\geqslant 2$, then p^2 is an integral divisor of the numerator of the sum

$$\frac{1}{r_1}+\frac{1}{r_2}+\cdots+\frac{1}{r_k}.$$

[An extension of Wolstenholme's Theorem.]

E 2754. *Proposed by Jim Fickett, University of Colorado*

Given n arbitrary lines $k_1,\dots,k_n$ in the plane, need there exist another n lines $h_1,\dots,h_n$ having the same intersection pattern but with all intersection points rational? The first condition means that for every subset S of $\{1,\dots,n\}$ we have

$$\bigcap_{i\in S} k_i\neq\varnothing \Leftrightarrow \bigcap_{i\in S} h_i\neq\varnothing.$$

SOLUTIONS OF ELEMENTARY PROBLEMS

Orientation and Vertex-Coloring of Complete Graphs

E 2672 [1977, 651]. *Proposed by Marianne Gardner, North Carolina State University*

Each of the $\binom{m}{2}$ edges of the complete graph K_m is assigned a direction and each vertex is assigned one of n colors in such a way that there is no directed path of length k ($k<m$) whose vertices are all of the same color. How large can m be in terms of n and k? (This problem is similar to E 2562 [1977, 218].)

Solution by David M. Bloom, Brooklyn College of CUNY. The maximum value of m is nk (the restriction $k<m$ is redundant). We use the fact (which can easily be proved by induction on m) (*) for every orientation of K_m there exists a directed path of length $m-1$ going through all the vertices.

Let S_i be the set of vertices of K_m which are assigned the ith color. It follows from (*) that $|S_i|\leqslant k$ and so $m\leqslant nk$. If $m=nk$, let $1,2,\dots,m$ label the vertices of K_m and let every edge $\{i,j\}$ be directed from i to j if $i<j$. Choose S_i $(1\leqslant i\leqslant n)$ arbitrary subject only to $|S_i|=k$. Then clearly K_m contains no directed path of length k having all vertices of the same color.

Also solved by Roger Eggleton & Katherine Heinrich (Australia), S. C. Locke (Canada), L. E. Mattics, Robert Singleton, David Witte, and the proposer.

n-Residues Modulo a Prime $6n+1$

E 2673 [1977, 652]. *Proposed by Haim Rose, Kiriat Schmonah, Israel*

Let $p=6n+1$ be a prime number, n a positive integer. An *n-residue* mod p is an integer a such that $0<a<p$ and $a\equiv b^n$ mod p for some integer b. Prove that the product of all n-residues mod p which are less than $p/2$ is congruent to -1 mod p.

Solution by Emma Lehmer, Berkeley, California. Since $a\equiv b^n$ (mod p) is equivalent to $a^6\equiv 1$ (mod p), there are exactly six n-residues modulo p. These can be written as

$$1, w, w+1, p-w-1, p-w, p-1,$$

where $w^2+w+1\equiv 0$ (mod p) and further $1<w<(p-1)/2$ (since $ww'\equiv 1$, $w+w'\equiv -1$). The n-residues less than $p/2$ are the first three listed above; hence the result.

Also solved by Mangho Ahuja, Kenneth Bernstein, Robert Breusch, Stephen Bronn, Bronx Community College Problem Solving Group, Paul Bruckman, Leonard Carlitz, Jeffrey Cohen, Michael Filaseta, Emily Fine, Lorraine Foster, Steven Galovich, Marguerite Gerstell, M. G. Greening (Australia), Emil Grosswald, G. A. Heuer & C. V. Heuer, A. A. Jagers (Netherlands), L. Kuipers (Switzerland), Mordechai Lewin (Israel), L. E. Mattics, Hugh Noland, Leonard Palmer, Reinhard Razen, Blair Spearman, Allen Stenger, B. M. Stewart, David Sumner, Ken Yocom, and the proposer.

One Regular n-simplex Inscribed in Another

E 2674* [1977, 652]. *Proposed by G. Tsintsifas, Thessaloniki, Greece*

Let $S=\{A_0,A_1,\ldots,A_n\}$ and $S'=\{A'_0,A'_1,\ldots,A'_n\}$ be regular n-simplices such that A'_i lies on the face $\{A_0,\ldots,A_{i-1},A_{i+1},\ldots,A_n\}$ of S $(0\leqslant i\leqslant n)$. Is it true that the centroids of S and S' coincide?

Solution by D. Ž. Djoković (Associate Editor). We show that the answer is negative for $n\geqslant 3$.

Let $e\in\mathbf{R}^{n+1}$ be the column vector with all coordinates equal to 1. It is not difficult to show that for $n\geqslant 3$ there exists a real orthogonal $(n+1)\times(n+1)$ matrix $P=(p_{ij})$ such that $Pe=e$, $p_{ii}<0$ for all i, $p_{ij}>0$ for $i\neq j$, and $p_{11}\neq p_{12}$.

Let p_k be the kth column of such a matrix P, and let p be the column vector whose ith coordinate is p_{ii}. Since $Pe=e$ implies $P'e=P^{-1}e=e$ (prime denotes matrix transpose), we have $e'P=e'$, i.e., $e'p_k=1$ for all k. Further, let $\lambda=(1-e'p)^{-1}$ and $v_k=\lambda(p_k-p)$ for $1\leqslant k\leqslant n+1$.

Let S be the regular n-simplex whose vertices are the standard basic vectors $e_1,\ldots,e_{n+1}$ of $\mathbf{R}^{n+1}$. Let S' be the n-simplex whose vertices are the vectors $v_1,\ldots,v_{n+1}$. S' is regular because

$$|v_i-v_j|^2=\lambda^2|p_i-p_j|^2=2\lambda^2 \quad \text{for} \quad i\neq j.$$

Since $e'v_k=\lambda(e'p_k-e'p)=\lambda(1-e'p)=1$, both simplices lie in the (affine) hyperplane $H=\{x\in\mathbf{R}^{n+1}|e'x=1\}$. Since all coordinates of p_k are positive except the kth which is zero, it follows that v_k lies in the interior of the kth face of S.

The centroid of S is $c=(n+1)^{-1}e$ and the centroid of S' is

$$c'=\frac{\lambda}{n+1}\sum_{k=1}^{n+1}(p_k-p)=\frac{\lambda}{n+1}(Pe-(n+1)p)=\lambda(c-p).$$

Since $p_{11}\neq p_{12}$, we have $c'\neq c$.

Comment. Surprisingly, no solutions were submitted for this problem.

Behavior Of A Series

E 2675 [1977, 652]. *Proposed by R. P. Boas, Northwestern University*

If n is a positive integer, let $f(n)$ be the number of zeros in the decimal representation of n. For which values of $a>0$ is the following series convergent

$$\sum_{n\geqslant 1}\frac{a^{f(n)}}{n^2}?$$

Solution by Allen Stenger, American Express Company. Consider just the part of the sum over the $(m+1)$-digit n $(10^m\leqslant n<10^{m+1})$. For each k, there are $\binom{m}{k}9^{m+1-k}$ $(m+1)$-digits n with $f(n)=k$, and for each such n we have

$$\frac{1}{10^{2m+2}}<\frac{1}{n^2}<\frac{1}{10^{2m}}.$$

Hence this portion of our sum is bounded above by

$$\frac{1}{10^{2m}}\sum_{k=0}^{m}\binom{m}{k}9^{m+1-k}a^k=9\left(\frac{9+a}{100}\right)^m$$

and below by $1/100$ of this quantity. Thus the question of convergence is the same as for $\Sigma((9+a)/100)^m$ which converges for $0<a<91$ and diverges for $a\geqslant 91$.

Also solved by M. T. Bird, David Bloom, Robert Breusch, Bronx Community College Problem Group, Paul Chernoff, Jeffrey Cohen, Roger Cooke, N. J. Fine, Thomas Foregger, Daniel Gallin, Emil Grosswald, University of Guelph Problems Group (Canada), Martin Helling, Michael Herschorn (Canada), G. A. Heuer, Carl Hurd, O. P. Lossers (Netherlands), L. E. Mattics, Walter Maxey, Greg Moore, William Myers, G. W. Peck, Ken Roberts, T. Šalát (Czechoslovakia), Robert Shafer, L. A. Shepp, Michael Skalsky, David Sumner, William Wong, Ken Yocom, and the proposer.

Editor's comment. A number of solvers considered the series $\Sigma_{n\geqslant 1}a^{f(n)}n^{-p}$ where $f(n)$ is now the number of zeros in the representation of n to base b. By the above argument it may be shown that if $a_0=b^p-b+1>0$, then this series converges for $0<a<a_0$ and diverges for $a\geqslant a_0$, whereas if $a_0\leqslant 0$ then it diverges for all $a>0$.

Ideals in Matrix Rings

E 2676 [1977, 652]. *Proposed by Robert Gilmer, University of Texas at Austin.*

Let R be a ring (not necessarily with identity). We denote by R_n the ring of n by n matrices over R. Show that the following are equivalent.

(i) Every ideal of R_n is of the form I_n where I is an ideal of R,
(ii) $I=IR=RI$ holds for every ideal I of R.

Solution by A. A. Jagers, Technische Hogeschool Twente, Enschede, Netherlands.

(i)$\Rightarrow$(ii). Let I be an ideal of R. Then IR is an ideal too and $IR\subset I$. Let S be the set of all matrices in I_n having elements of IR in their first column. Then S is an ideal of R_n, hence by (i) of the form J_n for some ideal J of R. Thus $J=I=IR$ for $n>1$. In a similar way one proves that $RI=I$ for $n>1$.

(ii)$\Rightarrow$(i). Let M be an ideal of R_n, and let M_{ij} be the set of all entries in position ij of matrices in M. Then M_{ij} is an ideal of R. Denote by r_{pq} a matrix in R_n with entry r in position pq and zeros elsewhere. Then for all $r,s\in R$, $r_{pi}Ms_{jq}\subset M$. That is, $RM_{ij}R\subset M_{pq}$, and so, by (ii) and symmetry reasons, all M_{ij} are equal, $M_{ij}=J$, say. Then $M\subset J_n$. On the other hand $r_{ii}Ms_{jj}\subset M$ also implies that $m_{ij}\in M$ for all $m\in RM_{ij}R=M_{ij}=J$. Thus also $J_n\subset M$, and finally $J_n=M$.

Also solved by J. Arnold, Stephen Bronn & Gilbert Orr, Thomas Foregger, Lorraine Foster, Robert Geist, Ralph Grimaldi, W. G. Leavitt, John O'Neill, Dalton Orr, Earl Taft, Paul Vojta, and the proposer.

A Composite Number

E 2679 [1977, 738]. *Proposed by Solomon W. Golomb, University of Southern California*

If a positive integer m has a prime factor greater than 3, show that 4^m-2^m+1 is composite.

Solution by D. Borwein and K. L. Roberts, University of Western Ontario (revised by the editor). Let r,s be relatively prime positive integers. Then the polynomial $P(x)=(x^{rs}-1)/(x^s-1)$ is divisible by the polynomial $Q(x)=(x^r-1)/(x-1)$. Indeed, all zeros of Q are simple and if $Q(\alpha)=0$ then $\alpha^r=1$, $\alpha\neq 1$ and it follows that $\alpha^s\neq 1$, $\alpha^{rs}=1$, i.e., $P(\alpha)=0$. By Gauss' Lemma we have $P(x)=Q(x)R(x)$ where $R(x)$ is also a polynomial with integral coefficients.

Now let $m=ns$ where s is a prime >3 and let $r=3$. Then $P(x)=x^{2s}+x^s+1$ and $Q(x)=x^2+x+1$ and so $Q(-2^n)=4^n-2^n+1$ divides $P(-2^n)=4^m-2^m+1$.

Also solved by Jerry Bergum & Ken Yocom, Kenneth Bernstein, D. M. Bloom, Robert Breusch, Bronx Community College Problem Group, Paul Bruckman, Michael Chandler, Columbia University Problem Group, Lorraine Foster, Owen Fraser, Irving Gerst, Marguerite Gerstell, Sidney Heller, Carl Hurd, A. A. Jagers (Netherlands), Free Jamison, Allan Johnson, Jr., Elgin Johnston, S. C. Locke (Canada), O. P. Lossers (Netherlands), Carolyn MacDonald, Helen Marston, L. E. Mattics, R. G. Nath, G. W. Peck, David Penney, Barry Powell, I. A. Sakmar (Canada), Robert Shafer, George Shulman, Allen Stenger, Volitae Xylachrais, and the proposer.

Editor's comments. Powell proves that $(2^{mn}-1)/(2^m-1)$ is composite if $m>1,n>1$, and provided that the case $n=p,m=p^s,p$ a prime, is excluded.

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to Professor Roger C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109. To facilitate their consideration, solutions of Advanced Problems in this issue should be typed (in duplicate, with double spacing) and should be mailed before May 31, 1979.

6246. *Proposed by L. Washington, University of Maryland and W. Parry, Rutgers University*

Let G be a compact Hausdorff topological group. Show that the only group homomorphism (not assumed continuous) from G to the integers is the trivial one.

6247. *Proposed by Bencze Miha'ly, University Babes-Bolyai, Cluj-Napoca, Romania*

Let $\alpha>1$, $m>1$, and $n>1$ with m and n integers. Also let $[x]$ denote the greatest integer in x. Prove that:

$$\sum_{k=1}^{n^m-1}\alpha^k[\sqrt[m]{k}]\leqslant(n-1)\frac{\alpha^{n^m}-\alpha^{(n/2)^m}}{\alpha-1}.$$

6248*. *Proposed by Milton P. Eisner, J. Sargeant Reynolds Community College, Richmond, Va.*

Let the set $S=\{1,2,\ldots,mn\}$, where m and n are positive integers, be partitioned randomly into n subsets each with m elements. For $0\leqslant k\leqslant n$, what is the probability $P(m,n,k)$ that exactly k of these subsets have the property of consisting of m consecutive integers?

6249. *Proposed by H. Kestelman, University College, London, England*

The norm $\|A\|$ of a real 2×2 matrix A is by definition the maximum of $\|A\hat{x}\|$ when $\|\hat{x}\|=1$; if $\|x\|$ is the euclidean norm $(x^Tx)^{\frac{1}{2}}$, then $\|A\|\leqslant\||A|\|$ where $|A|$ is the matrix whose elements are the absolute magnitudes of those of A. Find necessary and sufficient conditions on an invertible 2×2 matrix N in order that $\|A\|\leqslant\||A|\|$ for all A when $\|x\|$ is defined as the euclidean norm of Nx.

(One tends to use the inequality $\|A\| \leqslant |||A|||$ automatically in matrix analysis and might "naturally" assume it, when $\|x\|$ is the euclidean norm of Nx, for all N.)

6250. *Proposed by Harold Shapiro, Kungl. Tekniska Högskolan, Stockholm, Sweden*

Let $w=f(z)$ be a continuous complex-valued function on the closed unit disc $|z| \leqslant 1$ which is one-to-one on the open disc $|z|<1$. Show that the set of boundary points of the image which have three or more distinct pre-images under the map f is at most countably infinite.

6251. *Proposed by William P. Wardlaw, U.S. Naval Academy*

Let m and n be positive integers. What pairs of matrices C and D, over any field K, have the property that if A is an $m \times n$ matrix over K and B is an $n \times m$ matrix over K such that $AB=C$, then $BA=D$?

SOLUTIONS OF ADVANCED PROBLEMS

Subspaces of a Normal, Separable Space

6147 [1977, 300]. *Proposed by Richard Johnsonbaugh, Chicago State University*

Can a normal, separable space possess a closed, uncountable, discrete subspace?

Solution by F. S. Cater, Portland State University. Since the solution to this problem is given in Theorem 3 of [**1**], I offer the following generalization.

THEOREM. *Let A and B be subsets of a normal topological space X such that $A \subseteq \bar{B}$, and A is closed and discrete.* Then $2^{|A|} \leqslant 2^{|B|}$. Thus, in particular, the continuum hypothesis $\aleph_1 = 2^{\aleph_0}$ implies that any uncountable closed subset of a normal separable space cannot be discrete.

Proof. Each subset of A is closed in X. By the choice axiom we choose, for each subset P of A, an open set U_P such that $P \subseteq U_P$ and $\bar{U}_P \cap (A \setminus P) = \varnothing$. This construction is possible because P and $A \setminus P$ are disjoint closed sets and X is normal. Thus

$$k: P \to U_P \cap B$$

is a mapping of 2^A into 2^B. It remains only to prove that k is $1-1$.

Suppose P and Q are distinct subsets of A. We suppose, without loss of generality, that some $x \in P \setminus Q$. Then $x \in (X \setminus \bar{U}_Q) \cap U_P \cap A$. Since $A \subseteq \bar{B}$, it follows that $x \in \bar{B}$, and the open set $(X \setminus \bar{U}_Q) \cap U_P$ must meet B. Then

$$k(P) \supseteq \left(X \setminus \bar{U}_Q\right) \cap U_P \cap B \neq \varnothing, \qquad \left(X \setminus \bar{U}_Q\right) \cap U_P \cap B \nsubseteq k(Q)$$

and $k(P) \neq k(Q)$. Thus k is $1-1$.

REMARK. The equation $2^{\aleph_0} < 2^{\aleph_1}$ or some other hypothesis is essential to problem 6147. Otherwise the argument in [**1**] would imply that any normal separable Moore space must be metrizable, and this need not hold [**2**].

References

1. F. B. Jones, Concerning normal and completely normal spaces, AMS Bulletin 43 (1937) 671–677.

2. L. A. Steen, Conjectures and counterexamples in metrization theory, this MONTHLY 79, no. 2 (1972) p. 122.

Also solved by W. K. Dwyer, Michael Ecker, James Howard, A. A. Jagers (Netherlands), Carl Kohls, Ronnie Levy, Elliot Marshall & Eugene Walbridge & Dorothee Blum & Robert Krystock & Cheong-Chon Tong, L. E. Mattics, Don Mattson, James Munkres, Adam Mysior (Poland), David Neu, Stephen Noltie, A. V. Satyanarayana, Arthur Solomon, A. J. Vince, Joseph Woo, and the proposer.

Notes. (1) Besides the reference to Jones (above), a solution also appears in A. Wilansky, *Topology for Analysis*, Example 5.3.3, p. 78. For the case where the continuum hypothesis does not hold, Mysior refers us to R. W. Heath, *Separability and $\aleph_1$-compactness*, Colloq. Math. v. 12 (1964), pp. 11 ff.

(2) Noltie refers to Examples 1.L and 3.N in Kelley, *General Topology*, and notes that the subspace with the stated properties is possible if the space is taken to be only completely regular instead of normal.

Sum of Squares (mod n)

6148 [1977, 300]. *Proposed by Charles Small, Queen's University, Kingston, Canada*

Let $s(n)$ denote the smallest r such that -1 is a sum of r squares mod n. (Thus $s(n)$ is the *Stufe*, or "level", of a ring $Z/(n)$.) Show that $s(n)$ is computed as follows:

$$s(n)=1 \text{ if } 4\nmid n \text{ and } p\nmid n \text{ for all primes } p\equiv 3 \pmod 4;$$
$$s(n)=2 \text{ if } 4\nmid n \text{ and } p\mid n \text{ for some prime } p\equiv 3 \pmod 4;$$
$$s(n)=3 \text{ if } 4\mid n \text{ and } 8\nmid n;$$
$$s(n)=4 \text{ if } 8\mid n.$$

Solution by R. L. McFarland, Wright State University. Remember that -1 is a quadratic residue of primes of the form $4k+1$ and a quadratic nonresidue of primes of the form $4k+3$. Furthermore, if a is a quadratic residue of an odd prime p, then a is a quadratic residue of all powers of p.

CASE I. $4\nmid n$ and $p\nmid n$ for all primes $p\equiv 3 \pmod 4$. Let $n=p_1^{e_1}\cdots p_t^{e_t}$, where $p_1=2$, $e_1=0$ or 1 and $p_2,\ldots,p_t$ are distinct primes of the form $4k+1$. There are integers x_i such that $x_i^2\equiv -1 \pmod{p_i^{e_i}}$. Assume n is not 1 or a prime power; for otherwise it is obvious that $s(n)=1$. Let $q_i=n/p_i^{e_i}$ and let q_i' be defined by $q_iq_i'\equiv 1 \pmod{p_i^{e_i}}$, $i=1,2,\ldots,t$. Let $x=\Sigma_{i=1}^{t}x_iq_iq_i'$. Since $p_i^{e_i}|q_j$ for $i\neq j$, $x^2\equiv x_i^2(q_iq_i')^2\equiv -1 \pmod{p_i^{e_i}}$. Therefore $x^2\equiv -1 \pmod n$. So $s(n)=1$.

CASE II. $4\nmid n$ and $p|n$ for some prime $p\equiv 3 \pmod 4$. Then $x^2\equiv -1 \pmod n$ has no solution modulo the prime $p\equiv 3 \pmod 4$. Thus $s(n)>1$. Let m be the largest odd divisor of n. By a theorem of Dirichlet the arithmetic progression $-1+2m+4mk$, $(k=0,1,\ldots)$, contains a prime q. Since $q\equiv 1 \pmod 4$, there are integers x and y such that $x^2+y^2=q$. Then $x^2+y^2\equiv -1 \pmod n$. Therefore $s(n)=2$.

CASE III. $4|n$ but $8\nmid n$. Since $x^2+y^2\equiv -1 \pmod n$ has no solution mod 4, $s(n)>2$. The only positive integers not expressible as the sum of three squares are those of the form $4^k(8l+7)$. Since $n-1$ is not of this form, $s(n)=3$.

CASE IV. $8|n$. Since $x^2+y^2+z^2\equiv -1 \pmod n$ has no solution mod 8, $s(n)>3$. But every positive integer can be expressed as the sum of four squares, so $s(n)=4$.

Also solved by Leonard Carlitz, L. E. Clarke (England), F. J. Flanigan, Lorraine Foster, Enzo Gentile (Argentina), M. G. Greening (Australia), A. A. Jagers (Netherlands), Carl Kohls, L. Kuipers (Switzerland), L. E. Mattics, E. W. Trost (Switzerland), and the proposer.

Walk on the Edges of a Dodecahedron

6149 [1977, 301]. *Proposed by Gérard Letac, Université Paul-Sabatier, Toulouse, France*

A bug runs along the edges of a regular dodecahedron with constant speed: one edge per unit of time. At time 0 the bug is on some vertex A; at time n (n an integer) it chooses randomly one of the three possible edges. If p_n is the probability that the bug is on A at time n, then it is trivial to compute $p_0=1$, $p_1=0$, $p_2=\frac{1}{3}$, $p_3=0$, $p_4=5/27,\ldots$. Determine the generating function $\Sigma_{n=0}^{\infty}p_ns^n$ of the sequence $\{p_n\}$.

Solution by Lajos Takács, Case Western Reserve University. We define the distance between two vertices as the number of edges in the shortest connecting path. The distance between two vertices may be 0, 1, 2, 3, 4, 5. Define the random variables $X_0,X_1,X_2,\ldots$ as follows: $X_n=i$ if at time n the

bug is on a vertex whose distance is i from A. Then $\{X_n: n \geqslant 0\}$ is a homogeneous Markov chain with state space $I=\{0,1,2,3,4,5\}$ and transition probabilities $p_{ij}=P\{X_{n+1}=j|X_n=i\}$ given by $p_{01}=1$, $p_{10}=1/3$, $p_{12}=2/3$, $p_{21}=p_{22}=p_{23}=p_{32}=p_{33}=p_{34}=1/3$, $p_{43}=2/3$, $p_{45}=1/3$, $p_{54}=1$ and $p_{ij}=0$ otherwise.

Now $p_n=P\{X_n=0|X_0=0\}$ for $n \geqslant 0$ and the problem is to determine the generating function

$$P(s)=\sum_{n=0}^{\infty} p_n s^n$$

for $|s|<1$. Let us define

$$f_i(n)=P\{X_n=0, X_r \neq 0 \quad \text{for } 0<r<n|X_0=i\}$$

for $n \geqslant 1$ and $i \in I$, and

$$F_i(s)=\sum_{n=0}^{\infty} f_i(n)s^n$$

for $|s| \leqslant 1$ and $i \in I$.

From $P(s)=1+F_0(s)P(s)$, we get $P(s)=[1-F_0(s)]^{-1}$ for $|s|<1$, and determine $F_0(s)$ by solving the following system of linear equations:

$$F_i(s)=sp_{i0}+s\sum_{j=1}^{5} p_{ij}F_j(s) \qquad (i \in I),$$

that is

$$\begin{aligned}
&F_0(s)=sF_1(s), \qquad 3F_1(s)=s[1+2F_2(s)],\\
&3F_2(s)=s[F_1(s)+F_2(s)+F_3(s)],\\
&3F_3(s)=s[F_2(s)+F_3(s)+F_4(s)],\\
&3F_4(s)=s[2F_3(s)+F_5(s)], \qquad F_5(s)=sF_4(s).
\end{aligned}$$

Expressing $F_1(s),\ldots,F_5(s)$ in terms of $F_0(s)$, the last equation yields

$$F_0(s)=\frac{27s^2-18s^3-15s^4+8s^5}{81-54s-63s^2+30s^3+10s^4-2s^5}$$

for $|s| \leqslant 1$. Thus

$$\begin{aligned}
P(s)&=\frac{81-54s-63s^2+30s^3+10s^4-2s^5}{81-54s-90s^2+48s^3+25s^4-10s^5}\\
&=\frac{1}{5}+\frac{1}{20(1-s)}+\frac{3}{4(3-s)}+\frac{3}{5(3+2s)}+\frac{9}{20(3-\sqrt{5}\,s)}+\frac{9}{20(3+\sqrt{5}\,s)}
\end{aligned}$$

for $|s|<1$ and we have explicitly

$$p_n=\frac{1}{20}+\frac{1}{4}\left(\frac{1}{3}\right)^n+\frac{1}{5}\left(\frac{-2}{3}\right)^n+\frac{3[1+(-1)^n]}{20}\left(\frac{\sqrt{5}}{3}\right)^n$$

for $n \geqslant 1$ and $p_0=1$.

Notes. 1. The derivatives $F_i'(1)$ $(i \in I)$ give the expectations in Problem E 1752 [1965, 75; 1966, 200]. In particular, $F_0'(1)=20$.

2. In Problem E 1897 [1966, 665; 1967, 1008] it was assumed that each time the bug comes to a vertex it chooses with equal probability one of the two edges which end in that vertex distinct from the one by which it reached that vertex. If we consider the present problem with this modification and p_n^* denotes the probability that the bug is on A at time n, then the generating function

$$p^*(s)=\sum_{n=0}^{\infty} p_n^* s^n$$

is given by

$$p^*(s)=\frac{64-16s^3-4s^4-12s^5-4s^6-4s^7-3s^8-2s^9-s^{10}}{64-16s^3-4s^4-20s^5-4s^6-6s^7-5s^8-4s^9-2s^{10}-2s^{11}-s^{12}}$$

for $|s|<1$.

Also solved by Daniel Asimov, Frank Bernhart, Paul Bruckman, Carl Bumiller, L. E. Clarke (England), A. Hartman (Australia), John Hoyt, Jonathan Kane, J. R. Kuttler, J. C. Lagarias, Joel Levy, Roger Lyndon, L. E. Mattics, J. G. Mauldon, Lennart Råde (Sweden), Allen Schwenk, Denmead Smith (England), Walter Taylor, Robert Ward, David Wright, and the proposer.

Editor's Comments. (i) Schwenk's solution proceeds from the adjacency matrix A for the dodecahedron, obtaining the formula $P_n=\operatorname{tr}A^n/(3^n\cdot 20)$. See also Frank Harary and Allen Schwenk, *The spectral approach to determining the number of walks in a graph*, Pacific J. of Math., 1978.

(ii) Lyndon points out that there is a slight connection between this problem and one for continuous groups considered by H. Kesten, *Symmetric random walks in groups*, Trans. A.M.S., 92 (1959) 336 ff. See also J. Milnor, *A note on curvature and fundamental groups*, J. Diff. Geom., 2 (1968), 1 ff.

(iii) With his solution for the given problem, Hoyt submits also solutions for some other solids:

(1) Tetrahedron: $P_n=(\frac{3}{4})(-\frac{1}{3})^n+\frac{1}{4}$
(2) Hexahedron: $P_n=\frac{1}{8}[(-1)^n+(\frac{1}{3})^{n-1}+(-\frac{1}{3})^{n-1}+1]$
(3) Octahedron: $P_n=\frac{1}{6}[1+(-\frac{1}{2})^{n-1}]$, $n>1$
(4) Icosahedron: $P_n=\frac{1}{12}[1+3(\frac{1}{5})^{n-1}+4(-\frac{1}{5})^{n-1}]$, $n>1$.

Near Identities

6150 [1977, 391]. *Proposed by Albert A. Mullin, Redstone Arsenal, Alabama*

Let G be any groupoid. Call $e\in G$ a *near identity* of G if e is idempotent and $ex=xe=x$ fails for at most one $x\in G$. It is well known that G can have at most one identity. (1) Show that if G is a semigroup then it can have at most two near-identity elements. Every group has precisely one near-identity element. (2) Give an example of an uncountably infinite semigroup with precisely two near identities which contains a countably infinite semigroup with precisely two near identities.

Solution by Emeric Deutsch, Polytechnic Institute of New York. If G is a groupoid, e is a near identity of G and $ex=xe=x$ $(x\in G)$, then we shall say that e *works* for x.

It is easy to see that even in a groupoid, if we have two distinct near identities, e and f, it cannot happen that each of them works for the other. Indeed, this would lead to $e=fe=ef=f\neq e$.

Let G be a semigroup and assume that G has three distinct near identities, e, f, and g. From the definition of a near identity it follows at once that each of e, f, and g must work for at least one of the other two. We can assume, without loss of generality, that e works for f. Then, according to what has been said above, f must work for g and g must work for e. But then

$$e=eg=e(gf)=(eg)f=ef=f\neq e.$$

Thus, G cannot have more than two near identities.

A group G cannot have two near identities since it cannot have two idempotents.

An example of an uncountable semigroup with two near identities is

$$G=\{(1,1)\}\cup\{(0,x):x\in\mathbf{R}\}.$$

R denotes the set of real numbers, the operation being coordinatewise multiplication. The near identities are $(0,1)$ and $(1,1)$, the latter being an identity. A countably infinite subsemigroup of G is

$$\{(1,1)\}\cup\{(0,n):n=1,2,\dots\}$$

having the same two near identities as G.

Also solved by George Akst, Floyd Barger, J. H. Carruth, Gregory Cook, Dryden Cope, S. W. Davis & J. E.

Cruthirds, Michael Ecker, Marguerite Gerstell, Ellen Hertz, David Hammer, James Howard, A. A. Jagers (Netherlands), Bruce Jensen, Jordan Levy, L. E. Mattics, Mark Merriman, Jerry Metzger, Mark Meyerson, Colin Missel, William Myers, Ralph Seifert, Jr., John Shafer, Blair Spearman, Daniel Symancyk, Walter Taylor, T. M. Thompson, and the proposer.

Note. Pursuing the traditional goal of generalizing, Hammer has not let us down. He proves the following: Let S be a semigroup, and k a cardinal (finite or otherwise). Define: $a \in S$ is a k-identity, if $a^2 = a$, and the set $\{x \in S : ax \neq x \text{ or } xa \neq x\}$ has less than k elements. (So, perversely, identities are 1-identities, and near identities are 2-identities.) Then

(i) For finite k, a semigroup has at most k, k-identities.

(ii) A semigroup has at most ω_0, ω_0-identities.

(iii) A semigroup has at most 2^k, k^+-identities for infinite k.

For the second part the easiest example is surely this: Take the first uncountable ordinal, ω_1, and two additional elements a and b. Define a semigroup operation as follows: $xy = \max\{x,y\}$ on ω_1, $ab = ba = 0$, and $ax = xa = bx = xb = x$ for x in ω_1. Then a and b are near identities. A countably infinite subsemigroup lies at hand.

Partitions of Finite Sets

6151 [1977, 391]. *Proposed by Clarence H. Best, Central Missouri State University*

A two-dimensional array is defined according to the following rule:

$$a_{11} = 1, a_{i1} = a_{1,i-1} (i > 1), a_{ij} = a_{i+1,j-1} + a_{i,j-1} (j > 1).$$

(A) Prove that a_{1j} equals the number of distinct partitions of a j-element set.

(B) Choose an nth order determinant D_n from the upper left corner of the array and prove

$$D_n = \prod_{0 \leqslant i \leqslant n-1} i!$$

Solution of (A) *by Jerry Griggs and of* (B) *by Ira Gessel, both of M.I.T. Combinatorics Class.* Part (A): One sees that the number $B(j)$ of distinct permutations of a j-element set satisfies the recursion

$$B(n+1) = \sum_{k=0}^{n} \binom{n}{k} B(k)$$

by distinguishing one element of the set and splitting the partitions by the total number k of elements in all those subsets of the partition not containing the distinguished element. (Here, we let $B(0) = 1$.) Using the defining relations and induction one shows that the a_{1j} have the same recursion formula as the $B(j)$ if we let $a_{1,0} = 1$. This implies $B(j) = a_{1j}$ for all j, as desired.

Part (B): We use generating functions. Given power series

$$F_i = \sum_{k=0}^{\infty} f_i(k) \frac{x^k}{k!}$$

for $i = 0, 1, \ldots, n$, we define $[F_0, F_1, \ldots, F_n]$ to be the determinant

$$\begin{vmatrix} f_0(0) & \cdots & f_0(n) \\ f_1(0) & \cdots & f_1(n) \\ \cdots & & \cdots \\ f_n(0) & \cdots & f_n(n) \end{vmatrix}.$$

Then we can transform $[F_0, \ldots, F_n]$ as follows without changing its value:

Rule 1: If G is a power series with constant term 1 then

$$[F_0, \ldots, F_n] = [F_0 G, \ldots, F_n G].$$

Rule 2: If $\bar{F}_i = F_i + \sum_{j \neq i} a_j F_j$ for arbitrary a_j, then

$$[F_0, \ldots, F_i, \ldots, F_n] = [F_0, \ldots, \bar{F}_i, \ldots, F_n].$$

Rule 1 corresponds to an elementary column operation and Rule 2 to an elementary row operation.

Now let $F=\sum_{k=0}^{\infty}B(k)\frac{x^k}{k!}$. We want to evaluate $D_{n+1}=[F,F',F'',\dots,F^{(n)}]$ where the prime indicates the derivative with respect to x, since

$$F^{(i)}=\sum_{k=0}^{\infty}B(k+i)\frac{x^k}{k!}.$$

From the well-known relation $B(n+1)=\sum_{k=0}^{n}\binom{n}{k}B(k)$ we have $F'=e^xF$. Then

$$\begin{aligned}F''&=e^xF+e^xF'=(e^x+e^{2x})F\\ F'''&=(e^x+2e^{2x})F+(e^x+e^{2x})F'\\ &=(e^x+3e^{2x}+e^{3x})F,\end{aligned}$$

and in general, $F^{(k)}=\phi_k(e^x)F$ where ϕ_k is a monic polynomial of degree k. (In fact ϕ_k is the so-called "exponential polynomial" whose coefficients are Stirling numbers of the second kind.)

Thus, $D_{n+1}=[\phi_0(e^x)F,\phi_1(e^x)F,\dots,\phi_n(e^x)F]$. Since F has constant term 1, by Rule 1 we have

$$D_{n+1}=[\phi_0(e^x),\phi_1(e^x),\dots,\phi_n(e^x)].$$

Now since ϕ_k is monic of degree k, repeated application of Rule 2 gives

$$D_{n+1}=[1,e^x,e^{2x},\dots,e^{nx}].$$

Further application of Rule 2 gives

$$D_{n+1}=\left[1,e^x-1,(e^x-1)^2,\dots,(e^x-1)^n\right].$$

But since $(e^x-1)^k=k!\frac{x^k}{k!}+$ higher powers of x, the determinant for this last expression is upper triangular; multiplying the diagonal entries gives $D_{n+1}=0!\cdot1!\cdot2!\cdots n!$

Also solved by Clark Givens & Otto Ruehr and F.T. Howard.

Notes. Givens & Ruehr offer other solutions for the stated problem and a generalization.

(1) If we formally define

$$F(x,y)=\sum_{j=1}^{\infty}\sum_{i=1}^{\infty}a_{ij}x^iy^j,\qquad Q(y)=\sum_{j=1}^{\infty}a_{1j}y^j,$$

it follows that

$$\text{(a) } F(x,y)=\frac{Q(y)-xQ(x)-x}{1+(1/x)-(1/y)},\qquad \text{(b) } Q\left(\frac{x}{x+1}\right)=x+Q(x).$$

Setting $x=1/s$ in (b) and taking the inverse Laplace transform gives

$$a_{ij}=\sum_{k=0}^{j-1}a_{1,i+k-1}\binom{j-1}{k}$$

using properties of the binomial coefficients and recursive relations for the exponential numbers in the derivation.

(2) The evaluation of the determinant in Part B is started by reducing to the Hankel matrix, whose elements are $a_{1,i+j-2}$, and then proceeding by means of Touchard polynomial orthogonality using results that may be found in G.C. Rota, *Finite Operator Calculus*, and A. Erdélyi, et al., *Higher Transcendental Functions*, vol. 2.

(3) In the generalization, the condition $a_{i1}=a_{1,i-1}, i>1$, is replaced by the condition $a_{i1}=ta_{1,i-1}$ and the corresponding determinant becomes $\prod_{i=1}^{n}(i-1)!t^{i-1}$.

Products of Ideals

6152 [1977, 391]. *Proposed by R. Raphael, Université de Poitiers, France*

In some rings one has unique factorization for ideals. Show that the following limited form of factorization holds in all rings: if $I_j, j=1,\cdots,n,$ are distinct non-zero ideals in a ring R, and if a_j and b_j are positive integers with $a_j<b_j$ for each j then

$$\prod_{j=1}^{n} I_j^{a_j}=\prod_{j=1}^{n} I_j^{b_j} \text{ implies } \prod_{j=1}^{n} I_j^{a_j}=\prod_{j=1}^{n} I_j^{c_j},$$

where $c_j, j=1,\ldots,n,$ are any integers satisfying $a_j\leqslant c_j\leqslant b_j$. In particular $\Pi I_j^{a_j}=\Pi I_j^{a_j+1}$. Show by an example that this is best possible, that is, show that one can have the products equal when the exponents are not.

Solution by Hwa Tang, California State University, Hayward. Let $a_j\leqslant c_j\leqslant b_j$. Since each I_j is an ideal, we have

$$(I_j)^{a_j}\supseteq(I_j)^{c_j}\supseteq(I_j)^{b_j}$$

for each j, and hence

$$\prod_{j=1}^{n}(I_j)^{a_j}\supseteq\prod_{j=1}^{n}(I_j)^{c_j}\supseteq\prod_{j=1}^{n}(I_j)^{b_j}.$$

If the first and last terms are equal, then all three are equal.

Example: Let $R=Z_4=\{0,1,2,3\}$ be the integers modulo 4. Let I be the ideal $\{0,2\}$. Then $I^2=I^3=\{0\}$, but $2\neq 3$.

Also solved by Robert Gilmer, R. P. Grimaldi, Mark Merriman, Blair Spearman, and the proposer.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR., AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf and Carleton Colleges
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

The Mathematical Work of Charles Babbage. By J. M. Dubbey. Cambridge University Press, New York, 1978. viii+235 pp. $26.50. (Telegraphic Review, June–July 1978.)

This is a well-written and interesting book about the mathematical side of Charles Babbage and about mathematics in Great Britain in the first part of the nineteenth century. The author has used unpublished mathematical manuscripts by Babbage in the British Museum.

Babbage's mathematical career can be divided into three parts: the reform of British mathematics in the period 1800–30, contributions to pure mathematics, and the invention and development of computers. With regard to reform, the book discusses the low state of British mathematics in this period. Little mathematics was taught in the schools or universities of Great Britain and there was a belief that mathematics was not useful other than for training in logical thinking. The author traces

the low level of mathematical research to Newton. British and continental mathematics developed along diverging lines, the British following the fluxional notions of Newton and the continentals following the differential notions of Leibniz. This divergence appeared to isolate the British from the great European developments in analysis. Babbage, as a Cambridge undergraduate in 1812, "became convinced that the notation of fluxions must ultimately prove a strong impediment to the progress of English science." This reviewer is uncertain as to why a notation based on dots instead of d's was an impediment. Kline [3] points out that Leibniz was more algorithmic in his development of the calculus, Newton more physical. The Leibniz development proved superior, and Babbage, with other undergraduates, such as George Peacock and J. F. W. Herschel, waged a battle for differentials and won. Nevertheless, according to Hardy [2], in the years 1840–50 the continental analysts were still not understood in England.

Babbage's contributions to pure mathematics were principally centered on functional equations in a single variable. If $f\colon [0,1]\to[0,1]$, the equation $f(f\cdots(f(x)\cdots)=x$, where n f's are composed, is called Babbage's equation. According to the author, "there is no evidence that ... modern mathematicians have interested themselves in [that] equation." However, the recent book of Kuczma [4] discusses Babbage's equation and some of the other functional equations considered by Babbage and gives many references to papers since Babbage's on these functional equations. Dubbey also discusses less interesting contributions of Babbage to analysis, probability, and geometry.

All the above would not have made him the towering figure in mathematics he became. It is the invention of the general purpose computer for which Babbage is remembered. (He inherited a fortune, which is a good thing if one is to spend fifty years trying to bring to fruition with little support a general purpose computer in Victorian England.) He invented two computers: the difference machine to solve difference equations and the analytical engine, which might now be called a universal Turing machine. (Some of the source material is reprinted in [1].) Only a small part of the analytical engine was ever built, but there is little doubt that it would have worked as proposed. An irony of Babbage's continued lack of government support is that difference engines suggested by the Babbage design were subsequently built in Sweden and one was bought by the British government to calculate life tables based on statistical principles developed by Babbage many years before. The first analytical engine was not built until seventy years after his death and it is difficult to understand even now why it took so long.

In sum, Babbage should be an important person in the history of science, and this book makes a contribution to developing this history.

References

1. P. and E. Morrison, eds., Charles Babbage and His Calculating Engines, Dover, 1961.
2. G. H. Hardy, Divergent Series, Oxford University Press, 1949, pp. 18 ff.
3. M. Kline, Mathematical Thought from Ancient to Modern Times, Oxford University Press, 1972, pp. 378 ff.
4. M. Kuczma, Functional Equations in a Single Variable, Polish Scientific Publishers, 1968.

W. A. BEYER, Los Alamos Scientific Laboratory

Studies in Harmonic Analysis. Edited by J. M. Ash. Studies in Mathematics, Vol. 13, Mathematical Association of America, 1976. xv + 319 pp. $10.00. (Telegraphic Review, December 1976.)

The MAA's series *Studies in Mathematics*, begun in 1962, has grown at the rate of nearly one volume per year. Topics covered range from algebraic logic to applied mathematics, and each of Studies 1 through 12 has proved to be remarkably useful. The newest Study to be released continues this distinguished, if young, tradition. In 1974, DePaul University in Chicago celebrated its seventy-fifth birthday. As part of the proceedings, the university sponsored a conference on harmonic analysis, featuring eleven speakers and a large group of participants. The volume under review

contains the addresses of these eleven speakers, eight of them written up by the authors and three transcribed fairly closely from videotapes. Lecture 1 is *Notes on the History of Fourier Series* by Antoni Zygmund. This lecture is such that one wishes it were 190 pages long instead of 19. Zygmund knows his subject intimately, he was present at the creation of much that has occurred since 1920, and he permits himself an occasional personal observation.

The remaining lectures are technical, deal with very difficult matters, and are at the cutting edge of the profession. Nevertheless, all of them are accessible to the nonspecialist. Every author takes up classical and well-known cases first, leads the reader gently through thickets of computation, and ends up with a view of the unexplored country that lies beyond. The ten other authors and their titles follow.

Lecture 2: Richard A. Hunt, *Developments Related to the a.e. Convergence of Fourier Series*;
Lecture 3: Charles L. Fefferman, *Harmonic Analysis and H^p Spaces*;
Lecture 4: J. Marshall Ash, *Multiple Trigonometric Series*;
Lecture 5: Elias M. Stein, *Harmonic Analysis on R^n*;
Lecture 6: D. L. Burkholder, *Harmonic Analysis and Probability*;
Lecture 7: Yves Meyer, *Harmonic Analysis of Mean-Periodic Functions*;
Lecture 8: Colin C. Graham, *Harmonic Analysis and LCA Groups*;
Lecture 9: Guido L. Weiss, *Harmonic Analysis on Compact Groups*;
Lecture 10: Paul J. Sally, Jr., *Harmonic Analysis and Group Representations*;
Lecture 11: Stephen Vági, *Harmonic Analysis on Cartan and Siegel Domains*.

Naturally not all of the lectures are of equal quality, and one's judgment depends in part upon one's own interests. Still, everyone interested in contemporary analysis will find much of interest in some of the lectures, and everyone interested in contemporary harmonic analysis will find much of interest in every lecture.

It is well known that much mathematical research proceeds through the agency of the Invisible University, which functions by letters, telephone calls, and personal visits among what is often a very small group of technical specialists. The Invisible University is nothing new, of course; if mathematical history were known completely, we should probably find it functioning at the time of Euclid. Certainly it was in full operation in Newton's day. A charming example of its work is to be found in the gracious references by Hermann Weyl to Hardy and Littlewood in his great paper *Über die Gleichverteilung von Zahlen mod. Eins* (Math. Ann. 77 (1916) 313–352; also in *Gesammelte Abhandlungen,* Springer–Verlag, 1968, Band I, pp. 563–599). Signs of the workings of the Invisible University are plentiful throughout the book, and it is interesting for the public to be vouchsafed a view of its current operations.

One could wish that some of the other fields of interest in harmonic analysis had been treated; but the number of invited speakers at any conference is limited, the organizers have their own tastes, and one does not know whether this or that expert was uninvited or had to decline. A principal omission noted by the reviewer is the burgeoning and at the same time venerable theory of $M(G)$, the algebra of measures on a locally compact Abelian group, under convolution. The index reveals a reference to Brook Taylor but none to Joseph L. Taylor, and a reference to the Brown of Brownian motion but none to Gavin Brown. It would have been immensely useful to have a survey of the current state of this interesting and applicable field.

This book will undoubtedly be a gold mine for masters' theses: every lecture gives enough hints for a dozen. It should be used with caution as a source for doctoral dissertations, however; a thesis director or a candidate looking independently for a topic should not rely even on the specialists who have written this Study, without personal investigation.

I wonder what subsequent Studies will contain: they will certainly have a hard act to follow.

EDWIN HEWITT, University of Washington

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook | P = professional reading
S = supplementary reading | L = undergraduate library purchase
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1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S, L? *Formulaire de mathématiques*. L. Chambadal. Dunod (US Distr: SMPF, 14 E. 60th St., NY 10022), 1978, v + 186 pp, (P). [ISBN: 2-04-010152-7] A small encyclopedia of undergraduate mathematics. Topics included are fundamental structures, polynomials, linear algebra, euclidean and affine geometry, topology, vector calculus, series and differential equations. It includes an index but no bibliography. A handy reference. CEC

GENERAL, S*(16-18), P, L***, *The Chauvenet Papers: A Collection of Prize-Winning Expository Papers in Mathematics*. Ed: J.C. Abbott. MAA, 1978. *V. I*, xviii + 323 pp [ISBN: 0-88385-425-2]; *V. II*, viii + 282 pp, $27 set. [ISBN: 0-88385-427-9] This reprinting of all 27 Chauvenet prize-winning papers (with author biographies and some appendices to supplement older articles) yields a superb survey of mathematics as well as a unique exemplar of mathematical exposition. Readings from these volumes would make excellent independent study projects; the volumes themselves would make unique club or prize awards to senior mathematics majors. LAS

GENERAL, T(13-16: 1), *Mathematics: Ideas and Applications*. Daniel D. Benice. Acad Pr, 1978, x + 430 pp, $13. [ISBN: 0-12-088250-7] Covers the usual topics, using pattern recognition as its main theme. Gives many examples of patterns, both within mathematics and in a variety of applications. The most successful, in my mind, are those in linguistics and cryptology. Finishes with a chapter on computers, emphasizing their role in recognizing patterns. Many good exercises. TLS

GENERAL, L, *McGraw-Hill Dictionary of Physics and Mathematics*. Daniel N. Lapedes. McGraw, 1978, xvi + 1120 pp, $22.50. [ISBN: 0-07-045480-9] 20,000 terms, approximately 10% from mathematics, drawn largely from McGraw's *Dictionary of Scientific and Technical Terms*. The mathematical definitions are fairly comprehensive in applied classical analysis, but are not very useful in more modern areas. The lack of references severely limits the usefulness of the dictionary: there is too little context to learn the properties of concepts being defined, nor any hints of where to look for further information. Few mathematicians will find this volume useful for its mathematics, although they may use it for its physics; conversely, some physicists may find its mathematics a useful refresher. LAS

PRECALCULUS, *Fundamentals of College Algebra, Fourth Edition*. Earl W. Swokowski. Prindle, 1978, vi + 410 pp, $13.50. [ISBN: 0-87150-253-2] Revisions in this new edition (*Third Edition*, TR, January 1976) reflect the author's attempt to be less formal. There are more problems, examples, and figures, more emphasis on graphing, and some optional exercises for hand-held calculators. LLK

PRECALCULUS, T(13: 1), *Fundamental Mathematics*. Louis J. Nachman. Wiley, 1978, x + 657 pp, $14.95 (P). [ISBN: 0-471-62815-8] Good coverage of graphing, including conics and intuitive treatment of asymptotes. Exponential, logarithmic and trigonometric functions covered in detail but very short on algebra. Many practical applications. Blanks in text which force students to read with pencil and paper may cause some confusion for weaker students. MW

EDUCATION, P, *Computers and Communication: Implications for Education*. Ed: Robert J. Seidel, Martin Rubin. Acad Pr, 1977, xx + 409 pp, $15.50. [ISBN: 0-12-635050-7] 37 papers from an invitational conference. They predict the influence that the next decade's technological changes in computer and communication systems will have on education. Sessions were on large scale integration, storage, communications, artificial intelligence and man-machine interface. RWN

HISTORY, L, *Early Editions of Euclid's Elements*. Charles Thomas-Stanford. Alan Wofsy Fine Arts, 1977, ix + 67 pp, $30. [ISBN: 0-915346-29-X] This book was reviewed in October 1978 with incorrect price information. The normal price is $30; the price of $65 listed in the earlier review is for a special collector's edition which includes an original leaf from the first edition of *Euclid's Elements* in 1492. LAS

HISTORY, P, L, *The Arithmetic of Al-Uqlīdisī*. Trans. and Annotated: A.S. Saidan. Reidel, 1978, xvii + 492 pp, $78. [ISBN: 90-277-0752-9] Annotated translation of earliest extant Arabic text of Hindu arithmetic. Translator has brought the text alive, conveying the lively spirit and insightful pedagogy of the ancient author. Scholarly introduction and commentary traces development of arithmetic in the Arabic world through this and other texts, suggesting numerous re-evaluations of Arabic contributions. Interesting reading even for nonspecialists. GHM

FOUNDATIONS, T(14-16), *Logic for Mathematicians*. A.G. Hamilton. Cambridge U Pr, 1978, viii + 224 pp, $31; $9.50 (P). [ISBN: 0-521-21838-1; 0-521-29291-3] Introduction to basic topics in propositional and predicate calculus, Gödel's incompleteness results, decidability and computability (via Turing machines) omitting model theory, set theory. Conscientiously aimed at general mathematics students. Well structured narrative shows great sensitivity to difficulties encountered by beginners. Avoids high powered professionalia in favor of well motivated treatment of fundamental topics. Seems to cover chosen topics well, without loss of rigor or accuracy. GHM

COMBINATORICS, T(17-18: 2), S, P, *Extremal Graph Theory*. Béla Bollobás. Acad Pr, 1978, xx + 488 pp, £19.50. [ISBN: 0-12-111750-2] A well-written, advanced and extensive book. Topics covered are extremal problems in connectivity, matchings, cycles, diameter, colorings, complexity, and complete and topological subgraphs. Each topic is treated in depth with recent and important results. Most proofs included. Exercises and conjectures. JPH

NUMBER THEORY, T(17: 2), S, P, L? *A Classical Invitation to Algebraic Numbers and Class Fields*. Harvey Cohn. Springer-Verlag, 1978, xiii + 328 pp, $14.80 (P). [ISBN: 0-387-90345-3; 3-540-90345-3] A substantial one year graduate-level course in classical algebraic number theory which stresses rational number theory but uses class field theory as the unifying motivation. The second half of the book is an introduction to class field theory. Includes two appendices (written by Olga Taussky), problems and a bibliography. CEC

LINEAR ALGEBRA, T(14: 1), *Linear Algebra with Applications*. Jeanne Agnew, Robert C. Knapp. Brooks/Cole, 1978, xi + 465 pp, $14.95. [ISBN: 0-8185-0256-8] Excellent--applications and computer programs included in every chapter. Linear space of polynomials introduced separately from linear transformations. Eigenvalues and eigenvectors introduced in Chapter 3 following square matrices, determinants and inverses. Chapter 7 returns to eigenvalues for a more thorough study and Chapter 8 deals with symmetric matrices and quadratic forms. LLK

LINEAR ALGEBRA, T(15-18), S, P, L*, *Permanents*. Henryk Minc. Ency. Math. and its Appl., V. 6. A-W, 1978, xviii + 205 pp, $21.50. [ISBN: 0-201-13505-1] The author presents, as volume 6 of the *Encyclopedia of Mathematics and Its Applications*, edited by Gian-Carlo Rota, a remarkably complete history, exposition, and bibliography of this old but only recently active area of linear algebra. Good exposition of a currently lively subject; problem sets with each chapter and good indices should make this an interesting (and rather unique) book for anything from an undergraduate seminar to general reading for a professional. JAS

ALGEBRA, S(17-18), P, *Symplectic Groups*. O.T. O'Meara. Math. Surveys, No. 16. AMS, 1978, xi + 122 pp, $22.80. [ISBN: 0-8218-1516-4] A sequel to the author's *Lectures on Linear Groups* (TR, April 1975). Treats the isomorphism theory of symplectic groups over arbitrary fields and integral domains. JD-B

ALGEBRA, T(16-17: 1), S, L, *A Course on Group Theory*. John S. Rose. Cambridge U Pr, 1978, ix + 310 pp, $29.50; $9.95 (P). [ISBN: 0-521-21409-2; 0-521-29142-9] A text intended for students with some knowledge of groups. Four chapters on the normal structure of groups, three on arithmetical structure, and two more on the interplay between these structures. Emphasis on group actions on sets and groups, including a chapter on H. Wielandt's approach to transfer and splitting. Representation theory omitted. 697 exercises. Bibliography. TRS

ALGEBRA, S(18), P, *Topics in Group Rings*. Sudarshan K. Sehgal. Pure and Appl. Math., V. 50. Dekker, 1978, vi + 251 pp, Sfr. 60. [ISBN: 0-8247-6755-1] Intended for advanced graduate and professional reading. Topics include: the unit group; Lie properties; the idempotent problem; the isomorphism problems. The last chapter lists unsolved problems. JEG

CALCULUS, T(13: 2-3), *Calculus, Second Edition*. Leonard Gillman, Robert H. McDowell. Norton, 1978, xiv + 899 pp, $19.95. [ISBN: 0-393-09051-5] Complete revision of first edition (TR, March 1974; ER, December 1974). There are two chapters on numerical methods, and a new chapter on line and surface integrals. Format makes use of marginal notes, selected answers at end of sections rather than end of book, and indication of problems involving hand-held calculators. LLK

CALCULUS, T*(13: 1), *Understanding Basic Calculus with Applications from the Managerial, Social and Life Sciences*. Monte B. Boisen, Jr., Max D. Larsen. Merrill, 1978, ix + 438 pp, $13.95. [ISBN: 0-675-08430-X] An easy-to-read text with an almost conversational style, attractively printed in 2 colors with careful intuitive and graphical descriptions of concepts and numerous examples. Begins with a chapter on functions and includes sections on partial differentiation and improper integration. An appendix reviews basic algebra and an annotated bibliography offers a wealth of further resources. JNC

CALCULUS, T(15-16: 2), S, L, *Advanced Calculus*. William F. Trench. Har-Row, 1978, xi + 754 pp, $19.95. [ISBN: 0-06-046665-0] Designed for a two-semester course. Topics include the real numbers, differential and integral calculus for functions of one variable, infinite sequences and series, real and vector valued functions of several variables, multiple integration and line and surface integrals. The large number of good exercises is the outstanding feature of this text. CEC

NUMERICAL ANALYSIS, T(17-18: 1), P, *Numerical Analysis of the Finite Element Method*. Philippe G. Ciarlet. Pr U Montreal, 1976, 295 pp, $9.25 (P). Lecture notes. Preliminary form of the author's *The Finite Element Method of Elliptic Problems*. General properties of finite elements. Interpolation in Sobolev spaces. Conforming and nonconforming methods. Applications to plates and shells. Relevant theory. Problems. RWN

NUMERICAL ANALYSIS, S(18), P, *Contractors and Contractor Directions Theory and Applications: A New Approach to Solving Equations*. Mieczyslaw Altman. Lect. Notes in Pure and Appl. Math., V. 32. Dekker, 1977, x + 290 pp, $24.75 (P). [ISBN: 0-8247-6672-5] This monograph presents a unification of iterative methods for solving general equations, usually in a Banach space setting. The main concepts introduced are contractors (similar to Kantorovich's notion of majorants) and contractor directions. Includes extension to nonlinear functionals. Presumes functional analysis. RWN

NUMERICAL ANALYSIS, P, *Méthodes Numériques en Mathématiques Appliquées*. J.F.G. Auchmuty, *et al.* Pr U Montreal (US Distr: Intern. Schol. Book Serv., 2130 Pacific Ave., Forest Grove, OR 97116), 1977, 206 pp, $8 (P). [ISBN: 0-8405-0361-X] Eight papers by six authors presented in a seminar at Montreal.

Includes stability and convergence methods for parabolic equations and finite element methods. Applications to rotating stars, transport theory, and elasto-plasticity. RWN

NUMERICAL ANALYSIS, P, *Algorithms for the Computation of Mathematical Functions.* Yudell L. Luke. Acad Pr, 1977, xiii + 284 pp, $15. [ISBN: 0-12-459940-6] Extends the work in the author's previous volumes on approximating functions of hypergeometric type by providing programs for obtaining coefficients in their Chebyshev and Padé approximations. RWN

FUNCTIONAL ANALYSIS, T(16-18: 2), P, *Integrals and Operators, Second Revised and Enlarged Edition.* Irving E. Segal, Ray A. Kunze. Grund. math. Wissenschaften, B. 228. Springer-Verlag, 1978, xiv + 371 pp, $37. [ISBN: 0-387-08323-5; 3-540-08323-5] Four new chapters (on semigroups, operator rings, C*-algebras, and trace operators) extend the well-known first edition (TR, May 1968; ER, May 1969) to current aspects of higher physical mathematics. Errors and "expository lapses" in the first edition have also been corrected. LAS

FUNCTIONAL ANALYSIS, P*, *Some Classes of Singular Equations.* Siegfried Prössdorf. Math. Lib., V. 17. North-Holland, 1978, xiv + 417 pp, $[illegible]. [ISBN: 0-7204-0501-7] Translation of the author's *Einege klassen Singulärer Gleichungen*, Akademie-Verlag, 1974. Systematic treatment of one-dimensional singular integral equations, emphasizing the Weiner-Hopf equations and those equations with Cauchy or Hilbert kernel. Two new chapters include recent work on equations with discontinuous coefficients and on approximation methods. TRS

FUNCTIONAL ANALYSIS, T(18: 1), P, *Functional Analysis.* Carl L. DeVito. Pure and Appl. Math., V. 81. Acad Pr, 1978, ix + 166 pp, $16. [ISBN: 0-12-213250-5] An introduction to normed linear space theory that covers the basics through a format of questions: How do infinite dimensional spaces differ from finite dimensional spaces? Does every subspace have a complement? Is every space a dual? Includes the Eberlein-Smulian theorem, some distribution theory, and a little on compact operators. Exercises. TRS

OPTIMIZATION, T(17-18: 1), P, *Stochastic Linear Programming.* Alan Gleit. Lect. Notes Ser., No. 49. Aarhus U, 1977, 200 pp, (P). Lecture notes. Inventory problems with recourse, distribution problems, decision making under uncertainty, and chance constraints. Examples and theory. Includes some preliminary theory on decomposition. RWN

OPTIMIZATION, T(14: 1), *Introduction to Operations Research Models.* Leon Cooper, U. Narayan Bhat, Larry J. LeBlanc. Saunders, 1977, ix + 404 pp, $14.95 [0-7216-2688-2]; *Instructor's Manual,* iii + 196 pp, (P). A modeling approach to introductory operations research. Linear programming, allocation, networks, location, scheduling, decisions, queueing, inventory. Case studies. Little concern for theory or analysis; mostly examples and discussion. RWN

OPTIMIZATION, P, *Multicriteria Decision Making.* Ed: G. Leitmann, A. Marzollo. Springer-Verlag, 1975, 386 pp, $22.50 (P). [ISBN: 0-387-81340-3; 3-211-81340-3] Eight papers. The general approach is to reduce the problem involving several, possibly conflicting, criteria to some sort of standard problem. Differential and quantitative games. Pareto optimality. Allocation problems. RWN

OPTIMIZATION, T(16-17: 1, 2), *Mathematical Programming Methods.* G. Zoutendijk. North-Holland, 1976, xiv + 500 pp, $34.75. [ISBN: 0-444-11069-0] Linear programming, common techniques, extensions. Constrained and unconstrained nonlinear programming; techniques include feasible directness and quasi-Newton. Emphasis is on the methods; some relevant theory. Examples. No exercises. RWN

OPTIMIZATION, T(15-16: 1), S, L, *An Introduction to Quantitative Methods for Decision Making, Second Edition.* Richard E. Trueman. HR&W, 1977, xx + 725 pp, $15. [ISBN: 0-03-018391-X] Mathematical methods. Probability theory, decision making under uncertainty using expected monetary value and utility. Linear programming, networks, dynamic programming, Markov analysis, inventory and queueing models. The approach is through examples, graphs and business applications. RWN

GEOMETRY, S? *The Geometry of Art and Life.* Matila Ghyka. Dover, 1977, xvii + 174 pp, $2.75 (P). [ISBN: 0-486-23542-4] Using ideas from Plato and others, a theory of proportions is developed and used in an examination of the "golden section," polyhedra, works of art and living organisms; 80 plates. (A slightly corrected republication of a 1946 edition.) JNC

GEOMETRY, S(15-17), P, L, *Hilbert's Third Problem.* Vladimir G. Boltianskiĭ. Trans: Richard A. Silverman. V.H. Winston, 1978, x + 228 pp, $19.95. [ISBN: 0-470-26289-3] Hilbert's third problem, solved by M. Dehn in the same year it was posed, was to prove that polyhedra of equal volume, unlike polygons of equal area, need not be equidecomposable. This proves that a theory of volumes for polyhedra cannot avoid limit processes. Dehn's confusing proof has been vastly simplified and extended but no comprehensive treatment in English has appeared. This translation of a recent Russian monograph fills this void. LAS

GEOMETRY, S(13), *Deductive Systems: Finite and Non-Euclidean Geometries.* Garth E. Runion, James R. Lockwood. NCTM, 1978, vi + 90 pp, $4 (P). [ISBN: 0-87353-129-9] Following an introductory discussion of axiom systems, this little gem introduces basic theorems and models of Riemannian and Lobachevskian geometries in a manner easily understood by present and future high school teachers as well as capable high school students. JNC

TOPOLOGY, T, S*, P*, L*, *Counterexamples in Topology, Second Edition.* Lynn Arthur Steen, J. Arthur Seebach, Jr. Springer-Verlag, 1978, xi + 244 pp, $12.80 (P). [ISBN: 0-387-90312-7; 3-540-90312-7] This useful book contains 34 pages of definitions about topological and metric spaces, 141 pages of examples, 17 pages of reference charts, and 149 exercises, with notes, bibliography, and index. The first edition has been brought up to date and extended by a revised version of Steen's article on metrization (this *Monthly* 79 (1972) 113-132). (First Edition, TR, March 1971; ER, August-September 1971). RPB

TOPOLOGY, P, *Hyperspaces of Sets: A Text with Research Questions.* Sam B. Nadler, Jr. Pure and Appl. Math., V. 49. Dekker, 1978, xvi + 707 pp, SF 118. [ISBN: 0-8247-6768-3] An exhaustive text on the theory which fails to discuss any of the applications. Does define the McDonald cheeseburger as a mathematical object. Many exercises. TLS

PROBABILITY, S(17-18), P, *Geometric Probability.* Herbert Solomon. CBMS Reg. Conf. in Appl. Math., No. 28. SIAM, 1978, vi + 174 pp, $14.50 (P). An excellent introduction. Six lectures cover six basic topics, such as the Buffon needle problem, Sylvester's problem, coverings of circles and spheres. Full details of the calculations are included which is quite helpful in getting a feel for the subject. Each chapter ends with a discussion of many applications and extensions. TLS

STATISTICS, T(13: 1), S, *Interactive Data Analysis, A Practical Primer.* Donald R. McNeil. Wiley, 1977, xii + 186 pp, $9.95 (P). [ISBN: 0-471-02631-X] Developed from class notes for introductory course. Originally intended to supplement *Exploratory Data Analysis* by John Tukey (TR, August-September 1978); later expanded. Assumes reader has access to interactive computer with appropriate programs. Short APL and Fortran routines included in text. TH

STATISTICS, S(15-18), P, L, *Prediction and Improved Estimation in Linear Models.* John Bibby, Helge Toutenburg. Wiley, 1977, xiii + 188 pp, $18.50. [ISBN: 0-471-01656-X] Recent developments about some biased estimation and prediction methods that are often better than the usual unbiased ones in a least squares sense. Ridge regression, Stein estimation, Bayesian methods, and Box-Jenkins methods.FLW

STATISTICS, S*(13-16), L*, *Statistics: A Guide to Political and Social Issues.* Ed: Judith M. Tanur, *et al.* Holden-Day, 1977, x + 141 pp, $4 (P). [ISBN: 0-8162-8574-8] Twelve essays taken from the editors' well-known 1972 collection *Statistics: A Guide to the Unknown* (TR, January 1973; ER, April 1974), plus one new essay, describing applications of statistics to political and social issues. Third of the "mini-SAGTU's" to appear, the first being *Statistics: A Guide to Business and Economics* (TR, November 1976), and the second *Statistics: A Guide to the Study of the Biological and Health Sciences* (TR, October 1977). RSK

COMPUTER PROGRAMMING, T(14-16), P, L, *Informal Introduction to Algol 68, Revised Edition.* C.H. Lindsey, S.G. van der Meulen. North-Holland, 1977, 361 pp, $28.75; $14.50 (P). [ISBN: 0-7204-0726-5] *Second Edition* of a very useful companion to the formal report. Proceeds from a "very informal" introduction to rather thorough discussions and examples of most constructs of the language. RWN

COMPUTER PROGRAMMING, T(14-15: 1), L, *A Practical Guide to Algol 68.* Frank G. Pagan. Wiley, 1976, ix + 213 pp, $18.95. [ISBN: 0-471-65746-8] A well-organized primer intended for an introductory course. Numerical and non-numerical examples. Problems. An appendix on the syntax. RWN

COMPUTER SCIENCE, T(16-17: 1), P*, L, *Data Base Organization for Data Management.* Sakti P. Ghosh. Comp. Sci. and Appl. Math. Acad Pr, 1977, xi + 376 pp, $29.50. [ISBN: 0-12-281850-4] Mathematically, the most pleasing book to appear in this recently expanding field. Collects and presents theoretical results and techniques. Data descriptions, the logical structure of queues, balanced filing schemes, retrieval, and organization. Examples, exercises, references. RWN

COMPUTER SCIENCE, S(16-18), P, *Beiträge zur Theorie der Polyeder.* Walter Nef. Herbert Lang & Cie, 1978, 297 pp, $16.50 (P). [ISBN: 3-261-05000-4] A concise presentation of the theory of polyhedra designed specifically to support the second half of the book which is "applications to computer graphics." This latter half builds a bridge from theory to flowcharts. JAS

COMPUTER SCIENCE, T(14-15), S, L, *Database Processing.* David Kroenke. SRA, 1977, viii + 408 pp, $13.56. [ISBN: 0-574-21100-4] A non-mathematical introduction. Presents data handling techniques, database modeling concepts, and implementation options. Compares several commercial systems. Case studies. RWN

COMPUTER SCIENCE, T(16-17: 1), P, *Digital Image Processing.* Rafael C. Gonzalez, Paul Wintz. Appl. Math. and Comp., No. 13. A-W (Adv. Bk. Prog.), 1977, xvi + 431 pp, $19.50 (P); $29.50. [ISBN: 0-201-02597-3; 0-201-02596-5] Introduction to basic concepts and techniques. Topics include image segmentation, encoding, restoration and enhancement. Employs various transforms (Fourier, Walsh, Hadamard and Hotelling), linear algebra and probability. Interesting examples. References. RWN

SYSTEMS THEORY, P, *The Qualitative Theory of Optimal Processes.* R. Gabasov, F. Kirillova. Trans: John L. Casti. Dekker, 1976, xlvi + 640 pp, $55. Dynamical systems and their controllability, observability and identification. The existence problem. The maximal principle; necessary and sufficient conditions. Computational problems. Theory of discrete systems. Bibliography. RWN

APPLICATIONS (ENGINEERING), T(17-18), S*, P**, L, *Variational Methods in Mathematics, Science and Engineering.* Karel Rektorys. Reidel, 1977, 571 pp, $49. [ISBN: 90-277-0488-0] A comprehensive treatment of variational methods used in solving boundary value problems in ordinary and partial differential equations. The theory of L_2 relevant to differential equations is carefully motivated and developed to provide the context for the numerous methods (e.g., Ritz, Galerkin, Courant, steepest descent, finite element). Written especially for scientists with an interest in elasticity. TRS

Reviewers Whose Initials Appear Above

Ralph P. Boas, Northwestern University; Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Jay E. Goldfeather, Carleton; Timothy Hoel, St. Olaf; Joan P. Hutchinson, Carleton; Lorraine L. Keller, St. Olaf; Richard S. Kleber, St. Olaf; George H. Mills, St. Olaf; R.W. Nau, Carleton; Thomas R. Savage, St. Olaf; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Timothy L. Strotman, St. Olaf; Martha Wallace, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1225 Connecticut Avenue, N. W., Washington, D.C. 20036.

PERSONAL ITEMS

Syracuse University: Professor Donald E. Kibbey has retired with the title of Professor Emeritus. Dr. Nozar Azarnia, University of Cincinnati, has been appointed Assistant Professor. Assistant Professor Philip Parker is a Visiting Professor at SUNY, Buffalo, New York.

Monroe Community College, Rochester, New York: Assistant Professor Vincent Motto has accepted an appointment at Asnuntuck Community College. Associate Professor Thomas Dellaquila has been promoted to Professor. Assistant Professor Lawrence Gilligan has been promoted to Chairperson, Department of Mathematics.

University of Alabama, Huntsville: Associate Professor J. Hoomani has been promoted to Professor. Instructor Helen H. James has been promoted to Assistant Professor.

College of Charleston, South Carolina: Assisting Professor George Pothering is a Visiting Assistant Professor at Tulane University. Assistant Professor Brian Wesselink has been promoted to Associate Professor.

Wayne State University, Detroit: Dr. Anita Solow, former Lecturer at Dartmouth College, has been appointed Assistant Professor. Associate Professor Bertram Schreiber has been promoted to Professor.

Purdue University: Professor Merrill E. Shanks has retired with the title of Professor Emeritus. Dr. Carl C. Cowen, Jr., University of Illinois, has been appointed Assistant Professor.

Trinity College, Hartford, Connecticut: Associate Professor David A. Robbins has been appointed Chairman of the Department of Mathematics. Mrs. Lucy Deephouse has been appointed Technician. Dr. Gene Sogliero of the University of Connecticut has been appointed Assistant Professor.

Loras College, Dubuque, Iowa: Dr. Donald Marxen has been appointed Associate Professor. Assistant Professor Marvin C. Papenfuss has been promoted to Associate Professor and elected Chairman of the Mathematics Department.

Oakland University, Rochester, Michigan: Dr. Stuart S. Wang of Texas Tech University has been appointed Visiting Assistant Professor. Maria M. Klawe is on leave at the University of Toronto.

U. S. Naval Academy: Dr. Marlene E. Gewand, SUNY at Buffalo; Dr. Charles C. Hanna, Louisiana State University; and Dr. Mark D. Meyerson, University of Illinois, have been appointed Assistant Professors. Assistant Professor Thomas J. Sanders has been promoted to Associate Professor.

Dr. Laurence Boxer, North Carolina A. and T. University, is a Visiting Assistant Professor.

Dr. Charles M. Chambers, Associate Professor of Mathematics and Associate Dean, Office of Planning and Development of the George Washington University, has been appointed as Staff Associate of the Council of Postsecondary Accreditation. He was an Associate Professor of Mathematics for five years at the University of Alabama.

Professor Merrill E. Shanks, Purdue University, has retired with the title of Professor Emeritus.

Instructor Dolores M. Riollano, Post College, Waterbury, Connecticut, has been promoted to Assistant Professor.

Professor Rufus Isaacs, Johns Hopkins University, has retired with the title of Professor Emeritus.

Associate Professor Edward J. Wugman, University of North Carolina, has been appointed Director of the Statistics and Probability Program, Office of Naval Research, Arlington, Virginia.

Associate Professor Meredith W. Potter, Rockford College, Rockford, Illinois, has been promoted to Professor.

Dr. Alan H. Schoenfeld, Lecturer and Research Mathematician, Berkeley, California, has been appointed Assistant Professor at Hamilton College.

Associate Professor Derick Wood, McMaster University, Hamilton, Ontario, has been promoted to Professor.

Dr. Peter Flusser, Ottawa University, Kansas, has been appointed Assistant Professor at Fort Hays State University, Hays, Kansas.

Associate Professor Elliott Bird, C. W. Post Center, Long Island University, has been appointed Chairman of the Mathematics Department.

Dr. Edward Pekarek, Drake University, has been appointed Assistant Professor at Appalachian State University, Boone, North Carolina.

Associate Professors Caroll Riecke and Francis B. Hajek, Cameron University, Lawton, Oklahoma, have been promoted to Professors.

Professor A. Bruce Clarke, Chairman of the Mathematics Department at Western Michigan University, has been named Dean of the College of Arts and Sciences.

Dr. Robert M. Tardiff, University of Minnesota, Morris, has been appointed Assistant Professor at Franklin and Marshall College, Lancaster, Pennsylvania.

Dr. Thomas A. Keagy, Wayland Baptist College, has been appointed Assistant Professor at Texas Eastern University, Tyler, Texas.

Dr. Alan Adamson, Oxford University, has been appointed Assistant Professor at the University of Waterloo, Ontario.

Professor H. D. Block, Cornell University, is on sabbatical leave for the academic year 1978-79. The Japan Society for the Promotion of Science has invited him to visit Japan, and he will be at Tokyo University from October 1978 until February 1979. He was elected Visiting Scholar at Corpus Christi College, Cambridge University, England, for the Easter term 1979.

Associate Professor Emeric Deutsch, Polytechnic Institute of New York, has been promoted to Professor.

Professor Alice King, formerly Associate University Dean for Academic Affairs, has been appointed to the post of Acting University Dean for Programs and Planning at Cal Poly, Pomona.

Professor Edwin G. Warner, Mohawk Valley Community College, Utica, New York, has retired with the title of Professor and Dean Emeritus.

Adjunct Professor Julius Hlavaty, Iona College, died on September 20, 1978. He was a member of the Association for forty years.

Professor Rufus Bowen, University of California, Berkeley, died on July 30, 1978, at the age of thirty-one.

Associate Professor Paul J. Hessler, Wittenberg University, died on November 4, 1978 at the age of forty-one. He was Chairman of the Department of Mathematics.

Professor E. K. McLachlan, Oklahoma State University, died on October 27, 1978, at the age of fifty-four. He was a member of the Association for thirty years. He was Secretary of the Oklahoma-Arkansas Section and a member of the Committee on Sections.

Professor Emeritus Free Jamison, San Jose State University, died on July 4, 1978, at the age of seventy-three.

HELP NEEDED IN IDENTIFYING PUERTO RICAN SCIENTISTS

The advisory committee of Puerto Rican scientists to the American Association for the Advancement of Science requests assistance in identifying Puerto Ricans in the U.S. involved in the fields of science or engineering. The committee will compile by December 15, 1978, a directory of Puerto Ricans in the natural, biomedical, social and physical sciences and in engineering. Names received after that date will be save for a supplement.

The information gathered will be used for the purpose of communication among those listed and for organizing a meeting of such individuals during the Summer of 1979. The directory will include names of Puerto Rican students enrolled in science and engineering programs, individuals with advanced, doctoral or postdoctoral degrees, and persons working in science and engineering careers in all sectors—private and public.

Persons who can provide any names are urged to contact Michele Aldrich, (202) 467-5431 before 5 p.m. weekdays, or Yvonne Benner, (703) 750-3945 after 7 p.m. weekdays and on weekends. Letters can be sent to AAAS/Office of Opportunities in Sciences, 1776 Massachusetts Avenue., N.W., Washington, D.C. 20036.

MATHEMATICIAN IS PHI BETA KAPPA VISITING SCHOLAR

Professor Mark Kac, a member of the faculty in mathematics and theoretical physics at Rockefeller University, has been appointed a Phi Beta Kappa Visiting Scholar for 1978-79. He is a Fellow of the Mathematical Association of America.

As a participant in the Visiting Scholar Program, Dr. Kac will travel to six institutions: Oberlin College and the College of the Holy Cross in October; Duke University, Goucher College, University of the South, St. Lawrence University in March. During his two-day stay at each institution, he will meet with students and faculty in a variety of formal and informal encounters, which usually include classroom discussions, seminars, and a public lecture. His lectures will cover such topics as: themes and trends in mathematics; chance and regularity; universality of some mathematical concepts.

FACULTY EXCHANGE CENTER

The Faculty Exchange Center, a non-profit, faculty-administered program, helps to arrange college and university faculty exchanges within the United States, and abroad where the language of instruction is English. After registration, a faculty member will receive gratis a list of the members in his discipline as it appears in the fall 1978 catalog. For more information write to: Faculty Exchange Center, Franklin and Marshall College, Lancaster, PA 17604.

APPLIED MATHEMATICS SYMPOSIUM

A symposium in Applied Mathematics will be held at Oklahoma State University on March 29, 30, 1979, immediately preceding the meeting of the Oklahoma-Arkansas section of the MAA. The purpose of the symposium is to provide a forum for an exchange of ideas related to various aspects of applied mathematics. It will include lectures by Henry O. Pollak, Bell Telephone Laboratories, and Philip R. Davis, Brown University, and a discussion on the topic "Some Industrial Views on the Mathematics Curriculum" by representatives of regional industries and government agencies. In addition there will be sessions for contributed papers. Enquiries can be directed to Marvin Keener or Jeanne Agnew, Department of Mathematics, Oklahoma State University, Stillwater, Oklahoma 74074.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

OCTOBER MEETING OF THE OHIO SECTION

The Ohio Section of MAA held its Fall meeting at Ohio Northern University, Ada, Ohio, October 20 and 21, 1978. Approximately one hundred and fifty people were in attendance. Section Chairperson M.D. Wetzel presided; C.A. Long was the Program Chairman.

Invited addresses included: *Numerical Instabilities and Their Cure*, by W.S. Dorn, University of Denver; and *Numerical Analysis—The State of the Art*, by R.S. Varga, Kent State University.

The following contributed papers were also presented:

Hilbert Polynomials, D.D. Berry, Wittenberg College
Numerical Solution to Differential Equations, J.C. Hintz, University of Akron
On Normal Sylow Subgroups, D.J. Horwath, John Carroll University
Computing Output Distributions of Stochastic Systems, C. Looney, University of Toledo
An Algorithm for the Characteristic Polynomial of a Matrix, W.A. McWorter, Jr., Ohio State U.
Minimum Two-Impulse Velocity Correction for a Space Probe, J.R. Michel, Marietta College
Orthogonal Polynomials for Non-Classical Weight Functions, T.E. Price, Jr., University Akron
Calculator Calculus at Ohio State, J. Riedl and H. Allen, Ohio State University
A Very Elementary Computer Model Based on the Brachistochrone Problem, L.J. Schneider, John Carroll University
Conjugate Direction Algorithms in Numerical Analysis and Optimization, J. Stein, Univ. Toledo
Numerical Methods in Finding Optimal Experimental Designs, J-Y Tsay, University of Cincinnati
What Can a Pocket Programmable Do for You in Numerical Analysis? W.C. Weber, Bowling Green U.

The meeting agenda also included meetings of the Executive Committee, of MAA Ohio Section Campus Representatives, of mathematics department chairpersons for state supported institutions, and of *ad hoc* committees: Committee on Co-operation Among Colleges and Universities, Committee on Curriculum, and Committee on Teacher Training and Certification.

Meeting highlights included discussion sessions. *A Panel Discussion on Numerical Methods in Mathematics—Teaching, Research, and Industry* was led by W.S. Dorn, University of Denver; D.O. Norris, Ohio University (moderator); I. Shavitt, Battelle Memorial Institute, Columbus; and R.S. Varga, Kent State University. 'Swap' sessions included: *Numerical Methods in Teaching Calculus*, led by J.C. Hintz, University of Akron; and *Collective Bargaining and Its Effect on Teaching and Research*, led by M. Levine, University of Cincinnati.

Section officers for academic year 1978-79 are: M.D. Wetzel, Dennison University, Chairperson; D.O. Koehler, Miami University, Chairman-Elect; W.H. Byer, University of Akron, Past Chairman; G. Mavrigian, Youngstown State University, Secretary-Treasurer; C.A. Long, Bowling Green State U., Program Chairman; H.W. Vayo, University of Toledo and D.J. Horwath, John Carroll University, Program Committee Members. Also, R.L. Wilson, Ohio Wesleyan University, serves as Sectional Governor; S.W. Hahn, Wittenberg University, as MAA representative on the Committee-On-Sections; L.J. Schneider, John Carroll University, as Supervisor of the MAA High School Mathematics Competition; C.F. Yang, Miami University, Middletown Campus, as representative to the Two-Year College Mathematics Journal; and R.A. Little, Baldwin-Wallace College, as Section Newsletter Editor.

GUS MAVRIGIAN, *Secretary-Treasurer*

OCTOBER MEETING OF THE INDIANA SECTION

The fall meeting of the Indiana Section of the MAA was held at Marian College at Indianapolis on Saturday, October 28, 1978, with approximately 50 persons in attendance. The chairman of the Section, M. Jerison of Purdue University, presided.

The following papers were presented:

The Goldbach conjecture, U. Dudley, DePauw University
Fractals - objects with curious dimensions, M. Jerison, Purdue University
If the axiom of choice were false, A. Blass, Universities of Michigan and Wisconsin
Panel discussion on how calculators will change the school mathematics curriculum. Panelists: G. Wheatley, Purdue University; J. McIntosh, Indiana University

At the business meeting it was decided to continue the tradition of one year of holding the spring meeting in conjunction with the Friendly Math Competition.

D. WILSON, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Sixty-second Annual Meeting, Biloxi, Mississippi, January 26–28, 1979.

Fifty-ninth Summer Meeting, University of Minnesota, Duluth, August 21-23, 1979.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, last weekend in April or first weekend in May. Deadline for papers 6 weeks before meeting.

FLORIDA, early March. Deadline for paper titles 2 weeks before meeting.

ILLINOIS, first Friday/Saturday in May.

INDIANA

INTERMOUNTAIN

IOWA, third weekend in April. Deadline for papers February 1.

KANSAS, Johnson County Community College, Overland Park, April 7, 1979.

KENTUCKY, early April. Deadline for papers 6 weeks before meeting.

LOUISIANA–MISSISSIPPI, Friday-Saturday before February 20. Deadline for papers 3 months before meeting.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK, Adelphi University, May 5, 1979.

MICHIGAN, first Friday and Saturday in May. Deadline for papers 6 weeks before meeting.

MISSOURI, late March/early April. Deadline for papers January 31.

NEBRASKA, April.

NEW JERSEY, early November and early May.

NORTH CENTRAL, end of October and April. Deadline for papers October 1 and April 1.

NORTHEASTERN, Saturday after Thanksgiving, and third week in June in odd-numbered years.

NORTHERN CALIFORNIA, Sonoma State University, Rohnert Park, February 24, 1979.

OHIO, Miami University, Middletown, April 20–21, 1979.

OKLAHOMA–ARKANSAS, Oklahoma State University, Stillwater, March 30–31, 1979.

PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 15–16, 1979.

PHILADELPHIA, Saturday before Thanksgiving.

ROCKY MOUNTAIN, University of Denver, Denver, April 27–28, 1979.

SEAWAY, first Saturday in November and Saturday in late April. Deadline for papers 6 weeks before meeting.

SOUTHEASTERN, University of Tennessee, Chattanooga, April 6–7, 1979.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, usually in April. Deadline for papers 2 weeks before meeting.

TEXAS, Friday and Saturday in early April. Deadline for papers March 1.

WISCONSIN, Friday and Saturday between mid-April and first week in May. Deadline for papers 6 weeks before meeting.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, Houston, Texas, January 3–8, 1979.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES

AMERICAN MATHEMATICAL SOCIETY, Biloxi, Mississippi, January 24–27, 1979.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION

ASSOCIATION FOR COMPUTING MACHINERY, Plaza Hotel, Detroit, Michigan, October 29–31, 1979.

ASSOCIATION FOR SYMBOLIC LOGIC, Biloxi, Mississippi, January 24–25, 1979.

ASSOCIATION FOR WOMEN IN MATHEMATICS

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS, Washington, D.C., August 13–16, 1979.

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Boston, Massachusetts, April 18–21, 1979.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Hyatt Regency Hotel, New Orleans, Louisiana, April 29–May 1, 1979.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Radisson Muehlbach, Kansas City, Missouri, November 8–10, 1979.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

New additions from the Leader in Mathematics

Calculus and Analytic Geometry,
Fifth Edition

by George B. Thomas, Jr. Massachusetts Institute of Technology and Ross L. Finney University of Illinois, Urbana-Champaign

This fifth edition retains the distinguishing features which made the book one of the most widely-used calculus texts, and offers many improvements that will enhance its already considerable teaching effectiveness.

Throughout the new edition a more concise treatment of topics, several organizational changes, and a general redesign foster a clearer understanding of the calculus. A few of the new features include: unique applications from business, economics, and the life sciences; trigonometric functions and graphs appear in chapter one; L'Hôpital's Rule is presented in Chapter three and used throughout the book; a reorganized presentation of vectors, as well as new material on vector analysis; a section on Lagrange multipliers; a rewritten chapter on infinite series; hand-held calculator exercises; and more examples and exercises. In addition, the authors have made it easier for students to work with limits throughout the text.

The complete supplementary package includes a Student Supplement, a Solutions Manual, a Self-Study Guide, and a Pocket Calculator Supplement for Calculus.

07540 complete Hardbound Part I 07541
Part II 07542

Pocket Supplement for Calculu:

by J. Barkley Rosser and Carl de Boor, bot' of University of Wisconsin-Madison

Ideally-suited to *Calculus and Analytic Geometry, Fifth Edition*, this supplement provides exercises for the scientific hand-held calculator. It is organized into chapters by calculator topics.

06502 Paper

Calculus of Several Variables,
Second Edition

by Serge Lang, Yale University

Meticulously revised and reorganized, this second-year calculus text emphasizes applications, particularly to maxima-minim and the Taylor formula. Special attention is focused on 2- and 3- dimensions througho while linear algebra topics are introduced as they are needed. Additional worked-out examples and detailed solutions to exercises help students understand concepts and problems. Flexible chapter organization allows the text to be used in either a one semester or a full year course.

04299 Hardbound

Analytic Geometry,
Fifth Edition

by Gordon Fuller, Texas Tech University

Strengthened and expanded, this revised edition introduces students to analytic geometry and highlights the basic

Prindle, Weber & Schmidt, through the gracious cooperation of Professor Howard Eves, University of Maine, Machias, Maine is pleased to announce the

Prindle, Weber & Schmidt Undergraduate Mathematics Competition III

The judges will select one grand prize winner plus the 25 "best" papers devoted to the solutions of the five mathematical problems below, subject to the rules of the competition. The grand prize winner will receive Morris Kline's HISTORY OF MATHEMATICS (published by Oxford Press). The 25 other winning entries will be awarded a set of three "circle" books by Howard Eves (IN MATHEMATICAL CIRCLES, MATHEMATICAL CIRCLES REVISITED, MATHEMATICAL CIRCLES SQUARED). Every effort will be made to notify the winners by June 30, 1979. Winners and their institutions will be announced in a 1979 fall issue of the AMERICAN MATHEMATICAL MONTHLY.

Rules of the Competition

1 The competition is restricted to individual undergraduate students (e.g., a mathematics club, or a mathematics class, may participate as a single unit). One set of solutions constitutes one entry.

2 Each set of solutions must be neatly typed (double-spaced) and signed by the solver or group of solvers.

3 Below the signature(s) of the solver(s) must appear the statement: "To the best of my knowledge the solutions of this paper were found by the above undergraduate student(s)," signed by the Chairperson of the Mathematics Department of the institution.

4 Each set of solutions must be postmarked not later than April 15, 1979. Mail to: PWS Mathematics Competition / Mathematics Department / University of Maine at Machias / Machias, Maine 04654.

The Problems

1 Find all positive integers m and n for which $m + n$ is a factor of mn.

2 What conditions must be satisfied by the coefficients of the cubic equation $x^3 - px^2 + qx - r = 0$ in order that its roots, considered as lengths, shall form a triangle, and what then is the area of the triangle?

3 Two planar areas are said to be *Cavalieri congruent* if they can be placed so as to cut off equal lengths on each member of a family of parallel lines. Show that any two equiareal triangles are Cavalieri congruent.

4 A cow is tethered to a 100-foot rope fastened to the side of a circular barn of 40-foot diameter. Find the cow's grazing area.

5 Two weights A and B rest on a smooth sphere, being supported by an inextensible string of length p that passes through a small smooth ring at R, a fixed point vertically over the center of the sphere. When in equilibrium, how is the string divided by the ring at R?

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THE MATHEMATICAL ASSOCIATION OF AMERICA
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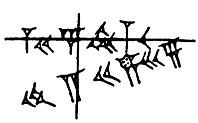
FEBRUARY

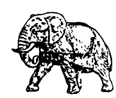
THE AMERICAN MATHEMATICAL MONTHLY

Volume 86, Number 2

Zbl. MR

Award for Distinguished Service: Otto Neugebauer





Prisoners' Dilemma and Professional Sports Drafts

Ramanujan's "Lost" Notebook

Finite Anti-Plane Shear

Detailed contents on cover 4

1979

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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VOLUME 86 **NUMBER 2**

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NOTICE TO AUTHORS

Research papers per se are unsuitable; see statement of policy (Vol. 85, p. 1).

Please follow the format in current issues of the MONTHLY. Manuscripts must be legibly typewritten or reproduced from typewritten copy, double spaced with wide margins, and on one side of the paper. Three copies should be submitted to the appropriate editor and one kept by the author as protection against loss. The author's full address *must* appear at the end of the manuscript.

Estimated time from submission to publication: Main Articles 20 months, Progress Reports 6 months, Math. Notes 15 months, Research Problems 6 months, Classroom Notes 15 months, Math. Education 9 months.

EDITORIAL CORRESPONDENCE AND MAIN ARTICLES: to R. P. BOAS, Department of Mathematics, Northwestern University, Evanston, IL 60201; NOTES, etc.: to the corresponding Associate Editor; REPRINT PERMISSION: to LEONARD GILLMAN, Mathematical Association of America, University of Texas, Austin, Texas 78712 (see also the copyright notice below); ADVERTISING CORRESPONDENCE: to RAOUL HAILPERN, Mathematical Association of America, SUNY at Buffalo, Buffalo, N.Y. 14214; CHANGE OF ADDRESS and SUBSCRIPTIONS: to A. B. WILLCOX, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036; BACK ISSUES: Contact P. and H. Bliss Co., Middletown, CT 06457.

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AWARD FOR DISTINGUISHED SERVICE TO OTTO NEUGEBAUER

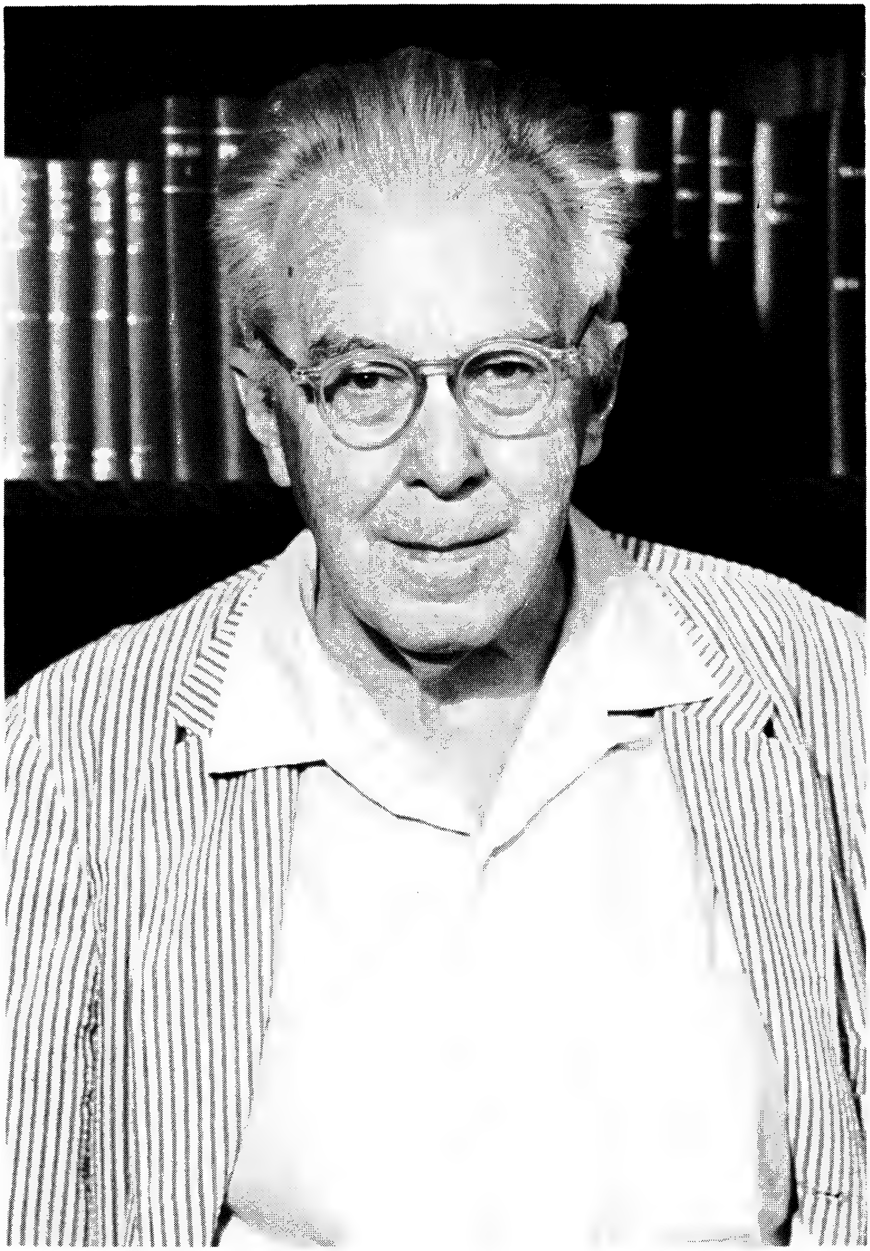
This year's recipient of the Award for Distinguished Service to Mathematics is a man who, although he has achieved the utmost distinction in his own field, is little known to the mathematical community at large; and yet, at the same time, he has had more influence on the daily work of mathematicians and on the progress of mathematics in this century than any other individual I can think of.

Otto Neugebauer was born in Innsbruck, Austria, May 26, 1899. In 1926 he received his Ph.D. from Göttingen, and the next year he became a Privatdocent there. He assisted Courant in forming the Göttingen Mathematical Institute, which he and Courant directed together until 1932. He moved to Copenhagen in 1933 and became Professor of Mathematics there in 1934; in 1939 he went to Brown University as Professor of Mathematics. Since 1947 he has been Professor of History of Mathematics at Brown.

This brief outline of his career does not yet explain why his name is before you today. Neugebauer's professional interest has always been the history of mathematics and astronomy, ancient and medieval, with emphasis on Egypt, Babylonia, and Greece. He has published very many papers and books, has contributed enormously to our understanding of his specialties, and has torn down much previously cherished nonsense about them. The breadth and depth of his scholarly work are astonishing. It is for this work that he has already received so many prizes, honorary degrees, and memberships in the most prestigious academies, including our own National Academy of Sciences, that we might well say of his honors, as was said of those of the title character in Booth Tarkington's *Monsieur Beaucaire*, "It takes a strong man two days to say all of them."

Neugebauer is a man of inspiring integrity, both personal and professional. I am sure he has never failed to verify a quotation or to go back to original sources. He has, and does not conceal, a low tolerance for stupidity. He opposed the National Socialists in Germany from the beginning and was forced out of his academic position in consequence. I cannot resist telling you a story to illustrate his common-sense approach and his abomination of cant. In the 1940s a certain mathematician complained that Neugebauer had written to him in English instead of in his "Muttersprache." Neugebauer's reply was that it was not a question of his mother's language, but of his secretary's. You will perhaps appreciate this more if I tell you, first, that his correspondent was not himself a native German; second, that the Neugebauer family spoke, on principle, Danish from the time they moved to Denmark, and then English from the time they came to the United States. There was never a less nostalgic émigré—I think he never had time for nostalgia.

As impressive as Neugebauer's scholarship has been, and however inspiring a personal example he has set us, these are still not our primary reasons for honoring him today. What we wish to recognize, and what I have only hinted at so far, is that he founded, and for many years edited, first, the *Zentralblatt für Mathematik* (an extension of Springer's reviewing journals that already existed in other sciences), and later, *Mathematical Reviews*, and so gave mathematics the essential tool of a working abstracting service. It must have been in the late 1920's that he became concerned about the (then) "information explosion" in Mathematics, amounting to some 2000 papers a year! The *Jahrbuch*, which tried to collect reviews of all the mathematics for each consecutive year into one volume, was falling further and further behind as mathematical progress accelerated, and had ceased to be of much current use. Neugebauer and Springer had the eminently practical idea of publishing the reviews of mathematics as soon as possible after the mathematics appeared. This seems natural to us now, but it was a radical idea at the time. Not only did Neugebauer have the idea of extending current reviewing service to Mathematics, he was able to convince Springer to back it, and the first issue of the *Zentralblatt* appeared on April 14, 1931, with Neugebauer as editor. It is noteworthy that both *Mathematical Reviews* and the Russian *Referativnyi Zhurnal* still operate on much the same plan as the *Zentralblatt*.



OTTO NEUGEBAUER

When Neugebauer moved from Göttingen to Copenhagen, the editorial office of the *Zentralblatt* went with him. In 1938, German official meddling with the scientific policies of the *Zentralblatt* became intolerable. Neugebauer, most of the editorial board, and most of the international reviewers resigned. Neugebauer destroyed the records of the *Zentralblatt*, keeping only the cumulative author index.

Many influential mathematicians, especially Bohr in Copenhagen, Hardy in Cambridge, England, and Veblen in Princeton, foresaw the coming catastrophe and the consequent problem of sustaining the flow of information in Mathematics. Efforts were made both to find a position in the United States for Neugebauer (who was already an eminent historian of mathematics and astronomy) and to find a way to continue the service that the *Zentralblatt* had provided. Under the influence of Veblen and of R. G. D. Richardson, then Dean of the Graduate School at Brown and Secretary of the American Mathematical Society, the A.M.S. undertook to publish *Mathematical Reviews*, and President Wriston of Brown provided both working space and a professorship for Neugebauer. This appointment showed an appreciation for scholarship and for Mathematics that was not common even then; Wriston went ahead even though he knew little about mathematics and even less about Neugebauer. I doubt whether there is a university president today who would or could do anything comparable. The index to the *Zentralblatt* came with Neugebauer, although the U.S. Customs almost confiscated it as potentially subversive, and it survives to this day.

The first issue of *Mathematical Reviews* appeared in January 1940 (coverage officially started with July 1939) with Neugebauer, Tamarkin, and Veblen as editors (although no editors' names appeared anywhere in the journal until September 1944). In the beginning, Neugebauer and Tamarkin did all the actual editorial work on their own time. Fairly soon, they brought in Feller to work half-time at the *Reviews* and half-time at Brown; but Neugebauer continued to play an active role in the daily work until well after 1945. By then there was a full-time editor (who was incidentally the first mathematician to be employed full-time by the American Mathematical Society). The enterprise may seem quite modest if we recall that *Mathematical Reviews* was then publishing per year about what a single issue contains today and that the editorial office contained a total of one editor and one secretary. But it did cover almost the whole world's production of mathematics—not without difficulty, especially during and immediately after the war. The production of an abstracting journal is considerably more complicated than it looks from the outside.

It is worth mentioning that Neugebauer also had the idea of starting the series *Ergebnisse der Mathematik* as a substitute for the *Enzyklopädie der mathematischen Wissenschaften*, which (like the *Jahrbuch*) had become an impossible enterprise as Mathematics expanded.

Things may be due for a change now: we need a new Neugebauer to show us how to cope with the rising tide of mathematics. For my generation, at least, the *Zentralblatt* and *Mathematical Reviews* have been central to all mathematical activity. They gave us a way, and the only practical way, to keep up both with what was going on and with what had recently happened. I suppose that something like 90 percent of the mathematics I know, I know (directly or indirectly) because of these two journals. Consequently I have always felt profoundly grateful to Otto Neugebauer for having the idea of a current reviewing service for mathematics, for maintaining its scientific standards against many—far from negligible—pressures, and for leaving it to us as a going concern. I hope that all of you will feel the same gratitude. This is principally why we are honoring Otto Neugebauer today.

R. P. Boas

AWARD OF THE CHAUVENET PRIZE TO DR. NEIL J. A. SLOANE

The Board of Governors of the Mathematical Association of America voted to award the 1979 Chauvenet Prize to Dr. Neil J. A. Sloane for his paper "Error-Correcting Codes and Invariant Theory: New Applications of a Nineteenth-Century Technique," which appeared in this MONTHLY, 84 (1977) 82–107.

A certificate and monetary award in the amount of five-hundred dollars were presented to Dr. Sloane at the Business Meeting of the Association on January 27, 1979.

The Chauvenet Prize is awarded for a noteworthy paper of an expository or survey nature published in English that comes within the range of profitable reading for members of the Association. The purpose of the prize is to stimulate the writing of expository and survey articles. The 1979 Prize, awarded for a paper published in the three-year period 1975–77, is the twenty-seventh award of the Chauvenet Prize since its institution by the Association in 1924. For the list of names of the previous winners, see this MONTHLY, 71 (1964) 589; 72 (1965) 2–3; 74 (1967) 3; 75 (1968) 3–4; 77 (1970) 117–118; 78 (1971) 112–113; 79 (1972) 112–113; 80 (1973) 120; 81 (1974) 113–114; 82 (1975) 108–109; 83 (1976) 84–85, 84 (1977) 417, and 85 (1978) 74–75. The award-winning papers are now available in the two-volume collection, *The Chauvenet Papers*, published by the MAA.

Neil J. A. Sloane was born in Beaumaris, Wales, on October 10, 1939. He received the B.E.E. and B.A. degrees (with Honors) from the University of Melbourne in 1959 and 1960, and the M.S. and Ph.D. degrees from Cornell University, Ithaca, N.Y., in 1964 and 1967.

From 1956 to 1961 Dr. Sloane worked for the Postmaster General's Department of the Commonwealth of Australia. From 1963 to 1965 he was a research assistant with the Cognitive Systems Research Program at Cornell University and was an instructor in Electrical Engineering at Cornell University from 1966 to 1967. In 1967 he became an assistant professor of Electrical Engineering at Cornell and remained at that post until 1969. Since 1969 he has been a Member of Technical Staff at Bell Laboratories, Murray Hill, N.J.

Dr. Sloane is engaged in research in coding theory, communication theory, and combinatorial mathematics. He is the author of four books: *A Handbook of Integer Sequences*, Academic Press, New York, 1973; *A Short Course on Error-Correcting Codes,* New York, Springer-Verlag, 1975; *The Theory of Error-Correcting Codes*, 2 vols., with F. J. MacWilliams, New York, Elsevier/North-Holland, 1977, and *Hadamard Encoding of Optical Instruments: Spectrometers and Imaging Devices*, with Martin Harwit, New York, Academic Press, in preparation.

Dr. Sloane is an editor of the *SIAM Journal on Applied Mathematics*, and the editor-in-chief of the *IEEE Transactions on Information Theory*. He is a fellow of the IEEE, and a member of the American Mathematical Society and the Mathematical Association of America.

Upon learning of the selection of his paper for the Chauvenet Prize, Dr. Sloane said that it was "good news and a very great honor."

David P. Roselle, *Secretary*

MISCELLANEA

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A MINIMAL SELF-PREDICATIVE BIBLIOGRAPHY[1]

WILLIAM F. ORR

1. W. F. Orr, A minimal self-predicative bibliography, this MONTHLY, 86 (1979) 79.

DEPARTMENT OF MATHEMATICS, HOFSTRA UNIVERSITY, HEMPSTEAD, NY 11550.

PRISONERS' DILEMMA AND PROFESSIONAL SPORTS DRAFTS

STEVEN J. BRAMS AND PHILIP D. STRAFFIN, JR.

In social situations in which each individual rationally pursues his self-interest, it is well known that the result may be unsatisfactory for everyone involved. Despite the familiarity of this phenomenon, it is difficult to escape the feeling of paradox if the situation is such that everyone, by acting differently (irrationally?), could do better. The most famous pure form of this phenomenon is the two-person, nonzero-sum game of Prisoners' Dilemma, due to A. W. Tucker [10] and illustrated in Figure 1. In this game, player A may select strategy A_1 or A_2, player B strategy B_1 or B_2. The choice of a strategy by each player leads to the payoffs shown in the corresponding cells of the payoff matrix in Figure 1, where the first number is the payoff to the row player (A) and the second number the payoff to the column player (B). In Tucker's original story, A and B are two prisoners held in separate cells and under interrogation about a common crime. A_1 and B_1 are "don't confess," A_2 and B_2 are "confess," and the payoffs are cleverly arranged by the district attorney.

		Player B	
		B_1	B_2
Player A	A_1	$(1,1)$	$(-2,2)$
	A_2	$(2,-2)$	$(-1,-1)$

FIG. 1. Prisoners' Dilemma

If player A in Prisoners' Dilemma rationally pursues his self-interest, he will choose strategy A_2, since A_2 yields higher payoffs for player A regardless of which strategy player B chooses. Similarly, player B will select strategy B_2, and the outcome will be at A_2B_2 (both prisoners confess), with each player receiving a payoff of -1. The problem is that A_2B_2 is not a so-called *Pareto optimal* outcome, since there is another outcome, namely A_1B_1 (both prisoners don't confess), which is better for both players.

In the last twenty-five years, Prisoners' Dilemma has been used by social scientists to model a wide variety of social phenomena. Applications, particularly to politics, are reviewed in [1], and experimental work in which Prisoners' Dilemma has been used to model human behavior in conflict situations is discussed in [9] and [8]. Much interesting recent work has focused on the appropriate generalizations of Prisoners' Dilemma to situations with more than two players, as presented, for instance, in [5].

In this paper, we will examine some game theoretic paradoxes similar to Prisoners' Dilemma which can arise in the player draft system now (1978) used in several professional sports in the United States. In particular, we will show that, if the number of teams drafting is at least three, individually optimal drafting strategies can produce an outcome which is not Pareto optimal. Furthermore, when teams use optimal drafting strategies an early position in the draft may be less desirable than a later position. We will explore some implications of these strange phenomena and discuss possible remedies for the present draft system.

Steven J. Brams is Professor of Politics at New York University and specializes in mathematical modeling. He received his Ph.D. in political science from Northwestern University and is the author of *Game Theory and Politics* (1975), *Paradoxes in Politics: An Introduction to the Nonobvious in Political Science* (1976), and *The Presidential Election Game* (1978). Philip Straffin received his Ph.D. in mathematics from the University of California at Berkeley under the direction of Emery Thomas and now teaches at Beloit College. His fields of interest include algebraic topology, mathematical ecology, game theory, and mathematical political science.—*Editors*

1. A model of professional sports drafts. In the draft system used in football, basketball, and other professional sports in the United States, the teams having the worst win-loss records in the previous season get first pick of the new players in the draft. The worst team gets first choice, the next-worst team second choice, and so on. After each team has made one draft choice, this procedure is repeated, round by round, until every team has exhausted its choices. Presumably, this system, by giving the worst teams priority in the selection process, makes the teams more competitive the next season, thereby fostering public interest in the sport.

To model this situation, suppose there are n teams, and for simplicity suppose the number of players is a multiple kn of n. The teams make choices in a specified order until each team has chosen k players. We make the following assumptions in our model:

1. *Strict preferences and partial ordering.* Each team X has a strict preference ordering on the players, which induces a partial ordering $\succ_X$ on the sets of k players it might receive in the draft. This partial ordering is by *pairwise comparison*. Suppose $\{a_1,\ldots,a_k\}$ and $\{b_1,\ldots,b_k\}$ are different sets of players, each listed in order of decreasing preference by team X. Then X will prefer the first set to the second set if either $a_i = b_i$ or X prefers a_i to b_i, for all $i=1,\ldots,k$. Thus, for example, if there are two teams and six players, and team X's preference ordering on the players is 123456, then we will have $\{1,4,5\}\succ_X\{2,4,6\}$. On the other hand, $\{1,2,6\}$ and $\{2,3,4\}$ will be incomparable in X's partial ordering.

   Of course, teams may occasionally have to make judgments between sets which are incomparable by pairwise comparison. However, the partial-ordering assumption is sufficient to establish our major results, and we henceforth assume that complete team preferences are compatible with it. This assumption does rule out more complicated criteria, such as "I want this quarterback only if I can also get this pass receiver."
2. *Self-interest.* Each team X judges entire *allocations* of the kn players to the n teams only by the subset of players which it receives in the allocation, according to the partial order defined in (1) above. In other words, we assume that each team's goal is to benefit itself, not to hurt other teams. This assumption rules out selections through spite: "I don't want this player, but I will choose him so that team Y can't get him."
3. *Independence.* Each team acts independently: there are no coalitions or hidden agreements.
4. *Complete information.* Each team knows the other teams' preference orderings. Although this might seem to be a strong assumption, we believe that it is not unrealistic, at least with respect to top-round choices of teams. Team preferences are based on judgments of both the potential performance of draft players and team needs. The fact that today's sports world includes computerized rating services, cooperative scouting reports, and careful analyses of other teams' strengths and weaknesses strongly supports the proposition that sports drafts take place in a context of high information.

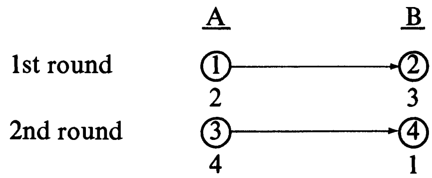
2. The case of two teams. Given the assumptions of section 1, what can we say about how the player draft might proceed? The simplest case of interest is that in which there are only two teams. For example, suppose there are two teams and four players, with preference orderings as follows:

A	B
1	2
2	3
3	4
4	1

Notice that it is not unreasonable to suppose such different preference orderings. One might think of player 1 as a quarterback, which team A needs while team B does not.

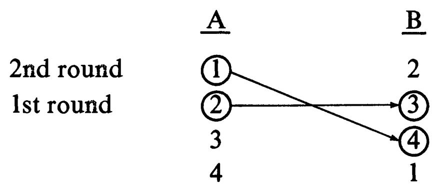
One obvious choice strategy is to choose *sincerely*: on each round, choose the player among

those as yet unchosen who is highest on your preference list. If each team follows its sincere strategy, choices will be as follows:



Sincere Choices

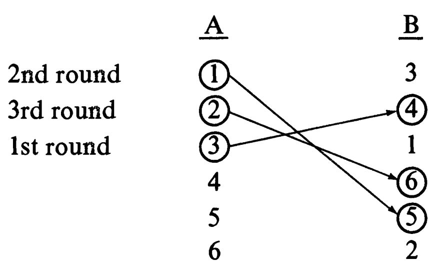
However, given assumption (4)—that each team knows the other teams' preference orderings—the sincere choice strategy may not be optimal and, in fact, is not in this example. To see this, note that since team A knows that team B does not want player 1, A can choose 2 in the first round and not fear loss of 1. Team B, whose goal is to benefit itself by assumption (2), has no defense against this strategy and can do no better than choose player 3 on the first round. The resulting choices then are:



Sophisticated Choices

It is easy to check that these choices are in fact "optimal" for both team A and team B: by making different choices, neither team could assure itself of a better assignment, given the assumptions of our model. We call this the *sophisticated outcome*, which is the product of *optimal play* by both teams.

For the general situation of two teams' making sequential choices with complete information about each other's preferences, there is an elegant algorithm for optimal play, due to Kohler and Chandrasekaran [6]. It works "from the bottom up" as follows. Team B's *last* choice will be the player ranked last in team A's preference ordering. Cross that player off both preference lists, thereby reducing each list by one player. Team A's last choice is the player who is last on B's reduced list. Cross that player off. Continue in this fashion, with each team's next-higher choice being the player who is last in the reduced list of the other team, until all players are chosen. Kohler and Chandrasekaran prove that the resulting choice strategy is optimal for both players. In the following example, it is instructive to check this algorithm—working backwards from team B's last choice of player 6—against one's intuition:



Sophisticated Choices

Returning to the problem raised by Prisoners' Dilemma, we now ask whether or not the sophisticated outcome—the outcome determined by individual optimal play—is Pareto optimal.

To answer this question, we need a precise notion of Pareto optimality, which we base on partial-ordering assumption (1) extended to allocations by assumption (2).

DEFINITION. *An allocation $\mathcal{A}$ of players to teams is Pareto optimal with respect to pairwise comparison if there is no different allocation $\mathcal{A}'$ such that each team either gets the same players in $\mathcal{A}$ and $\mathcal{A}'$, or prefers $\mathcal{A}'$ to $\mathcal{A}$ by pairwise comparison.*

When there are only two teams, A and B, this definition simply says that an allocation $\mathcal{A}$ is Pareto optimal if there is no other allocation $\mathcal{A}'$ such that $\mathcal{A}' \succ_A \mathcal{A}$ and $\mathcal{A}' \succ_B \mathcal{A}$. The following result ensures that individually optimal play is also socially desirable if there are only two teams:

THEOREM 1. *If there are two teams, the sophisticated outcome as given by the Kohler-Chandrasekaran algorithm is always Pareto optimal with respect to pairwise comparison.*

Proof. Let $\mathcal{A}$ be the sophisticated outcome. If it is not Pareto optimal, there is another allocation $\mathcal{A}'$ such that $\mathcal{A}' \succ_A \mathcal{A}$ and $\mathcal{A}' \succ_B \mathcal{A}$. Let i be the player highest on team A's preference list among the players which A receives in $\mathcal{A}'$ but not in $\mathcal{A}$. Similarly, let j be the player highest on team B's preference list among the players B receives in $\mathcal{A}'$ but not in $\mathcal{A}$. Then team A must prefer i to any player it receives in $\mathcal{A}$ but not in $\mathcal{A}'$, and hence prefers him to player j. Similarly, B must prefer j to i. But in the allocation $\mathcal{A}$, team A gets j and team B gets i.

Now consider the Kohler-Chandrasekaran algorithm which led to $\mathcal{A}$. Since A prefers i to j, the algorithm would have assigned j to B before assigning i to B, unless it had already assigned j to A. Hence it must have assigned j to A before assigning i to B. But symmetric reasoning shows that it must have assigned i to B before assigning j to A. This contradiction shows that no such allocation $\mathcal{A}'$ can exist, and hence that $\mathcal{A}$ is Pareto optimal with respect to pairwise comparison. Q.E.D.

The situation for two teams is thus quite pleasing: there is an efficient algorithm for calculating the outcome obtained by optimal play, and this outcome is always Pareto optimal by the criterion of pairwise comparison. No Prisoners' Dilemma can arise whereby both teams suffer if they play optimally. We will see that the situation changes as soon as there are more than two teams.

3. More than two teams. Consider the following situation with three teams and six players:

	A	B	C
1st round	1	5	3
2nd round	2	6	6
	3	2	5
	4	1	4
	5	4	1
	6	3	2

Sincere Choices

Sincere choices are shown. Teams A and B get their two most preferred players, and even C does not do badly. In general, sincere choices cannot be improved upon by the pairwise comparison criterion applied to all teams, as we next show.

THEOREM 2. *For any number of teams, the sincere outcome is always Pareto optimal with respect to pairwise comparison.*

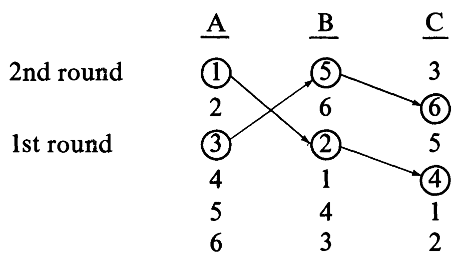
Proof. Let $\mathcal{A}$ be the sincere outcome, and let $\mathcal{A}'$ be any different allocation. Denote by P the (non-empty) set of players who are allocated to different teams by $\mathcal{A}$ and $\mathcal{A}'$. Now consider the sincere choice procedure which led to $\mathcal{A}$. Let i be the player in P who was chosen first among all

players in P by this procedure, and let X be the team which chose him. X thus receives i in $\mathcal{A}$, but not in $\mathcal{A}'$. Since X's choice was sincere, X prefers i to all other players in P, hence to all players which X receives in $\mathcal{A}'$ but not in $\mathcal{A}$. Hence X cannot prefer $\mathcal{A}'$ to $\mathcal{A}$ by pairwise comparison. Thus no allocation $\mathcal{A}'$ can satisfy the condition which would make $\mathcal{A}$ not Pareto optimal, so $\mathcal{A}$ is Pareto optimal by pairwise comparison. Q.E.D.

In this example, however, as in our first two-team example, sincere choices do not give optimal play. To see this, notice that if A chooses 1 and B chooses 5, C should choose 6, not 3; for choosing 6 results in the allocation $\left\{\begin{matrix} 1 & 5 & 6 \\ 2 & 4 & 3 \end{matrix}\right\}$, with C's getting his two most preferred players (6 and 3). Yet if C does this, B is hurt, ending up with $\{5,4\}$ instead of $\{5,6\}$.

Knowing that C will do this, B can ask if there is anything it can do to improve its assignment of $\{5,4\}$. Choosing 6 on the first round will not help, but choosing 2 will, for then C will have to choose 3 and the resulting allocation will be $\left\{\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{matrix}\right\}$, which is an improvement for B. Notice that the result of this optimal defensive action has been to hurt A, who now ends up with $\{1,4\}$.

Knowing that B will do this, A can ask if there is a different first choice which would improve its assignment. Choosing 2 will not help, but choosing 3 will. B can follow it by choosing either 5 or 2, but in either case the resulting allocation will be $\left\{\begin{matrix} 3 & 5 & 6 \\ 1 & 2 & 4 \end{matrix}\right\}$, an improvement for A. In fact, these choices are the optimal choices for all players, as can be seen by examining the "game tree" of possible sequential moves in the Appendix.



Sophisticated Choices

Comparing the sophisticated outcome shown above with the sincere outcome, we see that we have a Prisoners' Dilemma situation: the result of each team's following an individually optimal choice strategy is to make *all* teams worse off than if they had all chosen sincerely. This example proves

THEOREM 3. *If there are three or more teams, optimal play may lead to an outcome which is not Pareto optimal by pairwise comparison. In fact, such an outcome may be strictly worse than the sincere outcome for all teams.*

The bizarre fact that teams succeed only in hurting themselves by playing sophisticatedly we call the *paradox of player selection*. How can the paradox be avoided? It seems impossible to dictate to teams that they make sincere choices in player drafts to ensure Pareto optimality. Once innocence is lost, it is hard to regain. Notice that in the previous example it is only A's first choice which is insincere—the teams choose sincerely thereafter. Yet if A should make its first choice sincerely, B and C will be motivated to punish it for its sincerity.

One might respond that, after the draft, teams can always arrange trades to restore Pareto optimality. However, in our example the trade necessary to restore Pareto optimality is a three-way cyclical trade, with 3 going to C, 6 to B, and 2 to A. This cannot be effected by a sequence of mutually advantageous two-way trades, and in fact no mutually advantageous two-way trade is possible from the sophisticated outcome. It seems reasonable to believe that a

three-way cyclical trade would be difficult for sports teams to negotiate without outside help.

There is one way out of the paradox of player selection—at least in our example—which is surprising enough to warrant a separate statement. We call it the *paradox of team position*:

THEOREM 4. *If there are three or more teams, and teams follow optimal drafting strategies, a team may do better by occupying a later position in the draft.*

Proof. In the present example, team A will do better when it chooses second or third in the draft than when it chooses first. For instance, suppose A chooses third:

	B	C	A
1st round	5	3	1
2nd round	6	6	2
	2	5	3
	1	4	4
	4	1	5
	3	2	6

Sincere and Sophisticated Choices

Sincere choices are shown. We claim that these choices are also sophisticated. To see this, notice that the only team which might wish to deviate is C, and its only possibly profitable deviation would be to choose 6 in the first round. If A followed by choosing 1, the outcome would be $\begin{Bmatrix} 5 & 6 & 1 \\ 2 & 3 & 4 \end{Bmatrix}$, so C has benefited. However, in this situation A's best choice is not 1 but 3, which yields $\begin{Bmatrix} 5 & 6 & 3 \\ 2 & 4 & 1 \end{Bmatrix}$. Thus C would not benefit by choosing 6, so the sincere choices are optimal. Comparing this outcome with the sophisticated outcome when the order of choice is ABC proves the theorem.

In fact, even more is true—not only has A benefited by choosing last instead of first, but so have B and C. The change in order has in fact restored the Pareto-optimal outcome which sophisticated choices destroyed in the original order. We thus have the strange situation that the first-priority team—A in our example—may beg to be allowed to choose last, and it may be to the advantage of the other teams to support A's petition.

4. Other remedies and further difficulties. We showed in section 3 that in the transition from two to three teams, the player draft system becomes vulnerable to Prisoners' Dilemma, in which optimal play produces Pareto-inferior outcomes. Can we design a system which remedies this defect? One possible method would be to have teams submit preference lists to a referee, who would then effectively make sincere choices for the teams. With sincerity guaranteed, the resulting allocation would always be Pareto optimal by Theorem 2.

Unfortunately, sincerity is not so easily enforced. Under this procedure, teams could still effect insincere choices by the simple expedient of submitting insincere preference lists. For instance, in our first two-team example, team A would receive its top two choices if it submitted the insincere preference list 2134 and team B were sincere. It may be risky, however, for a team to submit an insincere list under the referee procedure, since whether or not a team gains by falsifying its preference list usually depends in a complicated way on whether or not other teams falsify theirs. The very fact that the referee scheme introduces these kinds of complex strategic considerations is surely an argument against it.

Alternatively, one might hope to design an algorithm which would use preference lists to produce an allocation which would be (i) at least as good as the sophisticated outcome for all teams, and (ii) Pareto optimal. In other words, such an algorithm would effectively take the sophisticated outcome and automatically arrange trades necessary to make it Pareto optimal. There are, however, several difficulties with such a program:

1. When there are more than two teams, no algorithm is known which will produce optimal play more efficiently than by checking many branches of the game tree of sequential choices. If there are n teams and $2n$ players, for example, n^n branches must be checked (see the argument in the Appendix for the three-team, six-player case), and this number is prohibitively large even for n of moderate size.
2. The partial ordering of k-sets of players by pairwise comparison may not be sufficient to produce a unique sophisticated outcome. One would sometimes have to solicit more complicated preference information from the teams, such as whether team A in our examples prefers $\{1,4\}$ to $\{2,3\}$, or vice versa.
3. Even if the problem in (2) does not occur, there still may be no unique sophisticated outcome. Consider the following three-team example:

A	B	C
1	3	6
2	5	5
3	6	2
4	4	1
5	1	3
6	2	4

Suppose A chooses 1. If B chooses 3, C should choose 2, and the allocation will be $\left\{\begin{matrix}1 & 3 & 2\\ 4 & 5 & 6\end{matrix}\right\}$. If B chooses 5, C should choose 6, and the allocation will be $\left\{\begin{matrix}1 & 5 & 6\\ 2 & 3 & 4\end{matrix}\right\}$. B will get the same players either way, but what A and C get depends on B's choice. Such indeterminacy makes unclear what optimal play would be. Should A take the gamble between getting assignment $\{1,2\}$ or $\{1,4\}$, or should it choose 3 on the first round, yielding the allocation $\left\{\begin{matrix}3 & 5 & 6\\ 1 & 4 & 2\end{matrix}\right\}$? Of these three possible allocations, the first two are Pareto optimal. The third is not, and its "repaired" version, $\left\{\begin{matrix}1 & 3 & 6\\ 2 & 4 & 5\end{matrix}\right\}$, is different from either of the first two.
4. Even if there is a unique sophisticated outcome, there may be several different ways in which it could be made Pareto optimal. For instance, consider the following example, in which there is a unique outcome of optimal play:

	A	B	C
2nd round	1	3	3
	2	5	6
1st round	3	6	5
	4	4	2
	5	1	1
	6	2	4

Sophisticated Choices

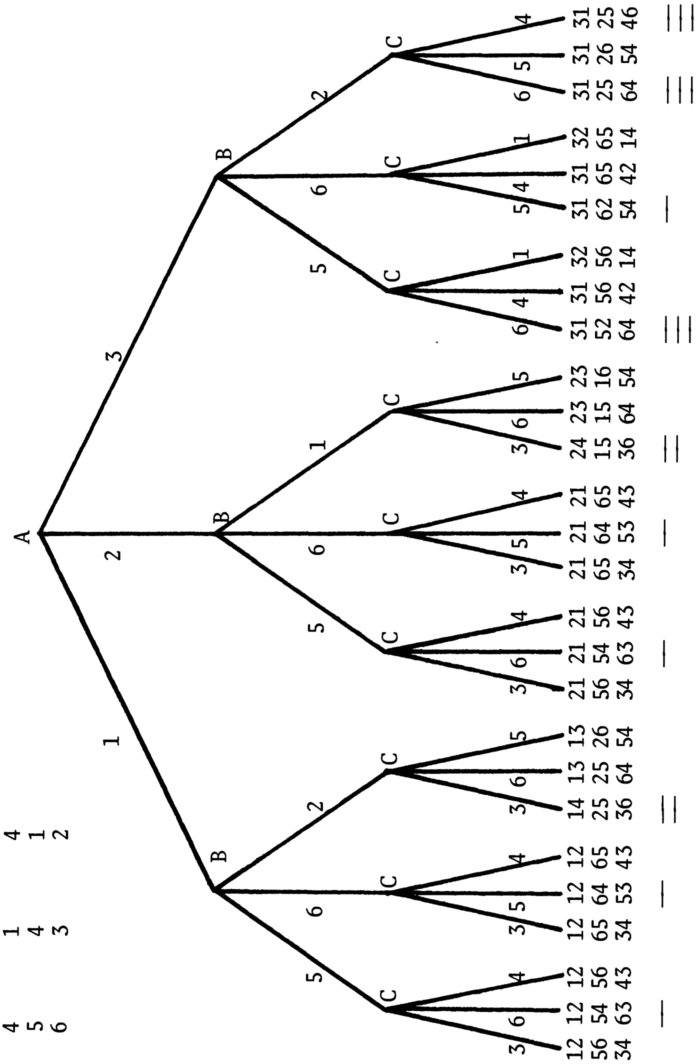
The sophisticated outcome, which is not Pareto optimal, can be "repaired" in two different ways. First, A and C could trade 3 and 2 to yield $\left\{\begin{matrix}1 & 5 & 3\\ 2 & 4 & 6\end{matrix}\right\}$. Alternatively, a three-way cyclical trade could give $\left\{\begin{matrix}1 & 3 & 6\\ 2 & 4 & 5\end{matrix}\right\}$. C would prefer the first trade, B the second, and A has no preference since he obtains the same assignment in either case. An algorithm which favors either allocation would seem arbitrary.

We conclude that there is no feasible way to eradicate the pathology of Prisoners' Dilemma

Preference orderings:

A	B	C
1	5	3
2	6	6
3	2	5
4	1	4
5	4	1
6	3	2

GAME TREE FOR SEQUENTIAL CHOICES



| C's preferred option

|| B's preferred option, assuming C will take his preferred option.

||| A's preferred option, assuming B will take his preferred option.

from the draft system. There are, however, many other schemes which might be used for allocating players to teams, including modified bidding schemes, fair division schemes based on bidding [2], and "marriage algorithms" which take into account player preferences for teams as well as team preferences for players [4, 11]. With draft systems now under challenge, some of these seem worthy of further investigation. The essential desiderata of any system, we believe, are that its allocations (i) foster team competition by favoring the weaker teams, and (ii) be Pareto optimal. The system should also be relatively easy to understand and as invulnerable as possible to strategic manipulation.

For other analyses of sports drafts and suggestions for alternatives, the interested reader might consult [3, 7].

Appendix. In this appendix we present the game tree of sequential choices which demonstrates that the sophisticated choices in our main three-team, six-player example are indeed optimal. Although there are $6!=720$ possible choice sequences, we do not have to investigate all of them. First, note that each team should make its last choice sincerely, for insincere last choices can only hurt. Second, in making its first choice, each team need consider only its top three players (among those as yet unchosen). Team A, for example, should never choose player 4 on the first round, for then the *best* assignment it could hope for would be $\{1,4\}$, whereas if it chose player 1 on the first round, $\{1,4\}$ would be the *worst* assignment it could obtain. In game theory terminology, A's choosing player 1 on the first round *dominates* his choosing 4, 5, or 6 on the first round, and similarly for B and C. We thus need only analyze the twenty-seven branches of the tree which are shown.

The tree is analyzed from the bottom up. When C chooses, he should always choose the player leading to the allocation he most prefers. These allocations are shown with a single underline. Knowing that C's choice will result in a single-underlined allocation, B should choose the player leading to the single-underlined allocation which is best for it. These allocations are double underlined. Similarly, knowing what B will do, A can determine his optimal choice, and the allocations that result are triple underlined. Notice that although there are three optimal choice sequences, they all lead to the same allocation of players.

Similar trees can be drawn for the other examples in the text.

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AN INTRODUCTION TO RAMANUJAN'S "LOST" NOTEBOOK

GEORGE E. ANDREWS

1. Introduction. In the spring of 1976, I visited the Trinity College Library at Cambridge University. Dr. Lucy Slater had suggested to me that there were materials deposited there from the estate of the late G. N. Watson which might contain some work on q-series. In one box of materials from Watson's estate I found a number of items written by the famous Indian mathematician S. Ramanujan (1887–1920). The most interesting item in this box was a manuscript of more than one hundred pages in Ramanujan's distinctive handwriting which contains over six hundred mathematical formulae listed one after the other without proof. *It is my contention that this manuscript, or notebook, was written during the last year of Ramanujan's life, after his return to India from England.* My evidence (given in Section 3) for this assertion is all indirect; in the words of Stephen Leacock, "It is what we call circumstantial evidence—the same thing that people are hanged for."

The fascinating story of Ramanujan's short life and brilliant career has been told several times, and the interested reader is referred to the accounts in Ramanujan's *Collected Papers* **[18]**, in *Ramanujan* by G. H. Hardy **[14]**, and in *Ramanujan the Man and the Mathematician* by S. R. Ranganathan **[20]**. There are three famous notebooks written by Ramanujan **[19]**. During the past 60 years, these have formed the basis for numerous papers by many mathematicians, including G. H. Hardy, G. N. Watson, L. J. Rogers, and many others. G. N. Watson **[26]** presented a nice survey of the notebooks in 1931, and B. Berndt **[10]** has written a new survey giving an up-to-date account of recent work related to the notebooks. Watson **[26**, p. 138] suggests that Ramanujan's work on the three famous notebooks concluded around 1913. A fully edited version of them was never completed, although G. N. Watson and B. M. Wilson initiated such a project. In 1957, the Tata Trust brought out a photostat edition of the three notebooks **[19]**.

We shall consider the origin of this "lost" notebook later. Before going further, let us state a few of the marvelous identities that appear in this work:

$$1+\sum_{n=1}^{\infty}\frac{q^j}{\prod_{j=1}^{n}(1+aq^j)\left(1+\frac{q^j}{a}\right)}$$

$$=(1+a)\sum_{n=0}^{\infty}a^{3n}q^{\frac{1}{2}n(3n+1)}(1-a^2q^{2n+1})-a\frac{\sum_{n=0}^{\infty}(-1)^n a^{2n}q^{n(n+1)/2}}{\prod_{j=1}^{\infty}(1+aq^j)\left(1+\frac{q^j}{a}\right)}; \tag{1.1}$$

$$\sum_{n=0}^{\infty}\frac{q^n}{(1+q)(1+q^3)\cdots(1+q^{2n+1})}=\sum_{n=0}^{\infty}(-1)^n q^{6n^2+4n}(1+q^{4n+2}); \tag{1.2}$$

Professor Andrews received his Ph.D. at the University of Pennsylvania in 1964. He was the late Professor Hans Rademacher's last student. He was a Fulbright scholar at Cambridge University in 1960–61, and has spent leaves of absence from Pennsylvania State University at M.I.T. and Wisconsin. He has published extensively on the theory of partitions and basic hypergeometric functions. His books are *Number Theory* (1971) and *The Theory of Partitions* (1976). He has recently edited *The Collected Papers of P. A. MacMahon.—Editors*

$$\left[\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{1+\cfrac{q^4}{\vdots}}}}}\right]^3=\frac{\sum_{n=0}^{\infty} q^{5n^2+4n}\frac{1+q^{5n+2}}{1-q^{5n+2}}-\sum_{n=0}^{\infty} q^{5n^2+6n+1}\frac{1+q^{5n+3}}{1-q^{5n+3}}}{\sum_{n=0}^{\infty} q^{5n^2+2n}\frac{1+q^{5n+1}}{1-q^{5n+1}}-\sum_{n=0}^{\infty} q^{5n^2+8n+3}\frac{1+q^{5n+4}}{1-q^{5n+4}}}; \tag{1.3}$$

$$\frac{G(aq,\lambda q)}{G(a,\lambda)}=\cfrac{1}{1+\cfrac{aq+\lambda q}{1+\cfrac{bq+\lambda q^2}{1+\cfrac{aq^2+\lambda q^3}{1+\cfrac{bq^2+\lambda q^4}{\vdots}}}}}, \tag{1.4}$$

where

$$G(a,\lambda)=G(a,\lambda;b;q)=1+\sum_{n=1}^{\infty}\frac{q^{n(n+1)/2}(a+\lambda)(a+\lambda q)\cdots(a+\lambda q^{n-1})}{(1-q)(1-q^2)\cdots(1-q^n)(1+bq)(1+bq^2)\cdots(1+bq^n)};$$

$$\frac{1}{1-\alpha}+\sum_{n=1}^{\infty}\frac{\beta^n}{(1-\alpha x^n)(1-\alpha x^{n-1}y)(1-\alpha x^{n-2}y^2)\cdots(1-\alpha y^n)}$$

$$=\frac{1}{1-\beta}+\sum_{n=1}^{\infty}\frac{\alpha^n}{(1-\beta x^n)(1-\beta x^{n-1}y)(1-\beta x^{n-2}y^2)\cdots(1-\beta y^n)}. \tag{1.5}$$

To my knowledge none of these five identities appears in the literature. Identity (1.1) was rediscovered by N. J. Fine in the early 1950's; however, he never published his proof.

Identity (1.2) is a "false" theta series identity. Results like (1.2) were studied by L. J. Rogers [22]; however, this elegant result appears to have escaped him. As we shall see, identity (1.2) implies a partition identity like that deduced from Euler's Pentagonal Number Theorem [5, p. 10].

Identity (1.3) is related to the famous Rogers–Ramanujan continued fraction [5, p. 104]:

$$\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\vdots}}}}=\prod_{n=0}^{\infty}\frac{(1-q^{5n+1})(1-q^{5n+4})}{(1-q^{5n+2})(1-q^{5n+3})}$$

$$=\frac{\sum_{n=0}^{\infty}(-1)^n q^{(5n^2+3n)/2}-\sum_{n=0}^{\infty}(-1)^n q^{(5n^2+7n+2)/2}}{\sum_{n=0}^{\infty}(-1)^n q^{(5n^2+n)/2}-\sum_{n=0}^{\infty}(-1)^n q^{(5n^2+9n+4)/2}}; \tag{1.6}$$

however, it lies somewhat deeper.

Identity (1.4) is a new Rogers–Ramanujan type continued fraction from which Ramanujan

deduces five corollaries (our equations (7.10)–(7.14)) in which continued fractions are represented by infinite products. The special case $a=0$ was given in Chapter 16 of Ramanujan's older notebooks [**19**, Vol. 2, Eq. (13), p. 195]. In the older notebooks Ramanujan gives a hint for proving the continued fraction identity by giving explicit formulae for the convergents. The proof we shall give relies on some q-series transformations due to Rogers.

Identity (1.5) looks disarmingly simple (and is). Its analytic proof is quite mundane; however, there is an amusing combinatorial proof.

I have chosen these five identities to give some flavor of Ramanujan's achievements in this "lost" notebook. Since there were over 600 to choose from, these results cannot really be called an accurate sample, but merely a tantalizing introduction. I plan to write a series of papers in which I shall organize these formulae into somewhat sensible groupings. I shall prove as many of Ramanujan's formulae as I can, and the rest I shall present for the consideration of others.

2. The mathematical setting for the "lost" notebook. The vast majority of the formulae in the lost notebook (including the results we have chosen) concern q-series and related theta functions. For example, we meet again and again the series

$$\sum_{n=0}^{\infty}\frac{(1-a)(1-aq)\cdots(1-aq^{n-1})(1-b)(1-bq)\cdots(1-bq^{n-1})t^n}{(1-q)(1-q^2)\cdots(1-q^n)(1-c)(1-cq)\cdots(1-cq^{n-1})}. \tag{2.1}$$

This is called the q-analog of the famous hypergeometric series studied by Euler, Gauss, and others:

$$\sum_{n=0}^{\infty}\frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)t^n}{n!\ \ c(c+1)\cdots(c+n-1)}. \tag{2.2}$$

It is called the q-analog because the rising factorial $a(a+1)\cdots(a+n-1)$ of the hypergeometric series has been uniformly replaced by the rising q-factorial $(1-a)(1-aq)\cdots(1-aq^{n-1})$. There are many, many results known about the hypergeometric series (2.2) (see [**25**, Chap. 1] for a nice introduction). For example, when $b=c$ in (2.2) we have the famous binomial series theorem:

$$\sum_{n=0}^{\infty}\frac{a(a+1)\cdots(a+n-1)t^n}{n!}=(1-t)^{-a},\qquad |t|<1. \tag{2.3}$$

In parallel we have the q-binomial series theorem if $b=c$ in (2.1):

$$\sum_{n=0}^{\infty}\frac{(1-a)(1-aq)\cdots(1-aq^{n-1})t^n}{(1-q)(1-q^2)\cdots(1-q^n)}=\prod_{n=0}^{\infty}\frac{(1-atq^n)}{(1-tq^n)}. \tag{2.4}$$

There is also an integral representation of the hypergeometric function (2.2) [**25**, p. 20]:

$$\sum_{n=0}^{\infty}\frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)t^n}{n!\ \ c(c+1)\cdots(c+n-1)}$$
$$=\frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)}\int_0^1 x^{b-1}(1-x)^{c-b-1}(1-tx)^{-a}\,dx. \tag{2.5}$$

The investigations of q-series have shown that the natural q-analog of (2.5) is not an integral representation for the series in (2.1), but rather an identity (due to Heine) between two such series given in this paper as identity (4.1). L. J. Rogers [**21**] noted that (4.1) could be applied to itself, and as a result he easily deduced the new and significant identities (4.6) and (7.1). Heine [**15**, p. 325] originally discovered (7.1), but he proved it in a more complicated way. Actually, (6.1) is a q-analog of Euler's famous identity for the hypergeometric function [**24**, p. 10].

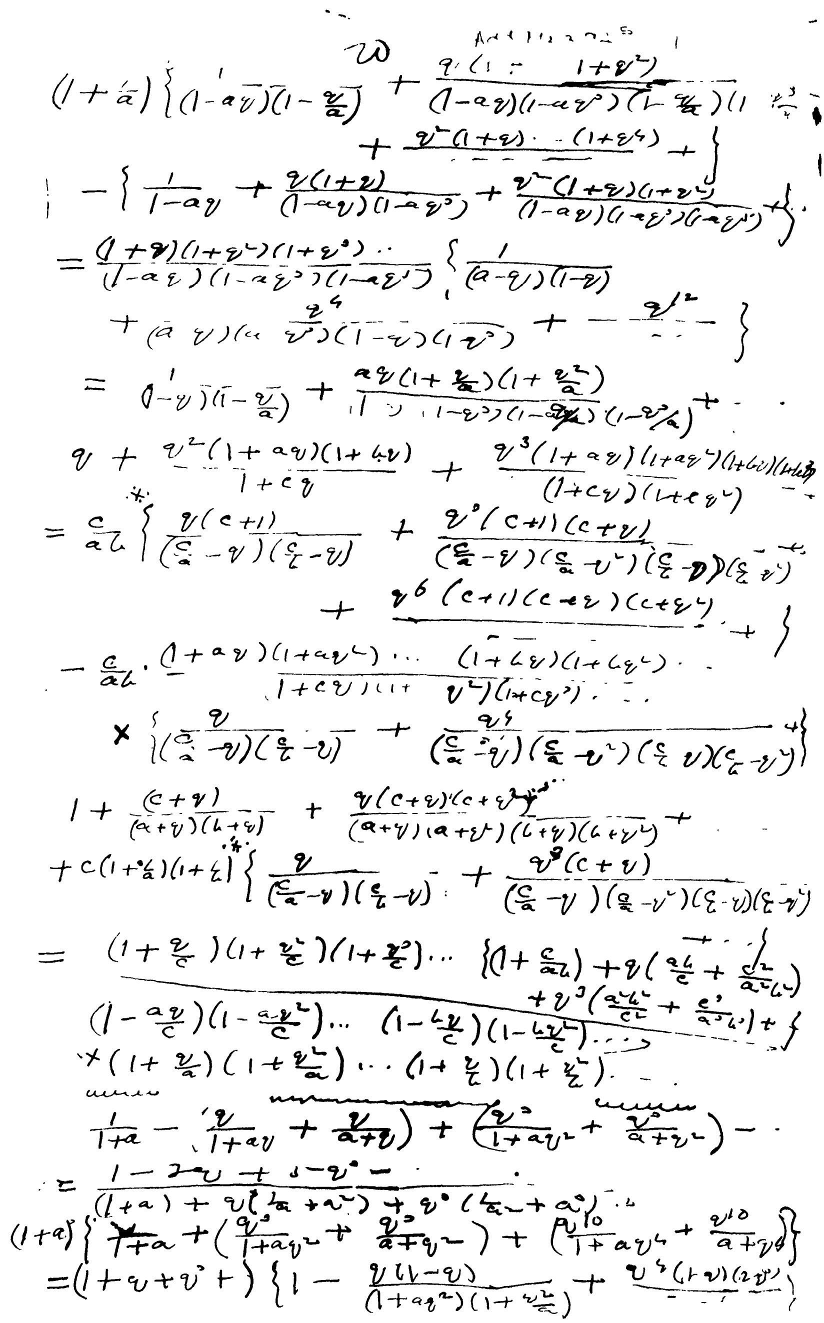

FIG. 1. Power series computations and comparisons from Ramanujan's "Lost" Notebook.

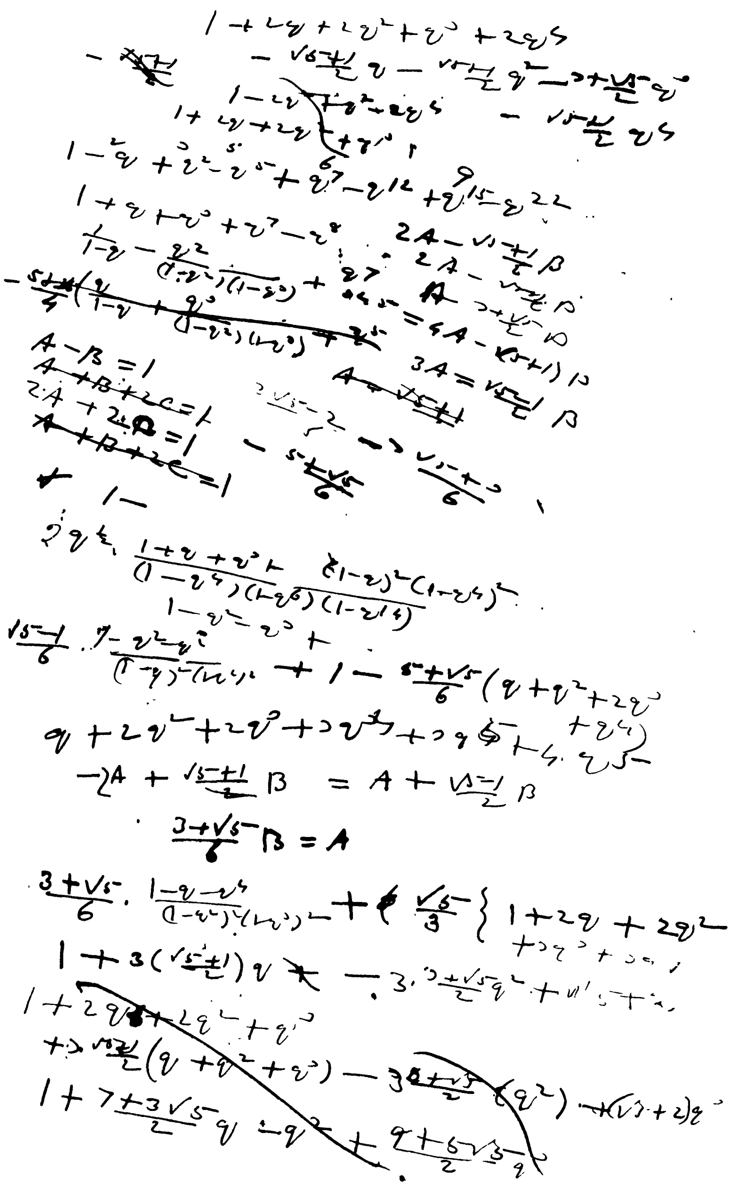

FIG. 2. A page of formulas from Ramanujan's "Lost" Notebook.

$$\sum_{n=0}^{\infty}\frac{a(a+1)\cdots(a+n-1)b(b+1)\cdots(b+n-1)t^n}{n!\ \ c(c+1)\cdots(c+n-1)}$$

$$=(1-t)^{c-a-b}\sum_{n=0}^{\infty}\frac{(c-a)(c-a+1)\cdots(c-a+n-1)(c-b)(c-b+1)\cdots(c-b+n-1)t^n}{n!\ \ c(c+1)\cdots(c+n-1)}. \tag{2.6}$$

To understand fully Ramanujan's use of the terms "theta series," "false theta series," and "mock theta series," we quote from Ramanujan's last letter to Hardy [27, pp. 57–61] (also mentioned in Section 3 below). After the quotation we shall make a few comments to clarify further some of Ramanujan's ideas.

If we consider a ϑ-function in the transformed Eulerian form, e.g.,

$$1+\frac{q}{(1-q)^2}+\frac{q^4}{(1-q)^2(1-q^2)^2}+\frac{q^9}{(1-q)^2(1-q^2)^2(1-q^3)^2}+\cdots, \tag{A}$$

$$1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\frac{q^9}{(1-q)(1-q^2)(1-q^3)}+\cdots, \tag{B}$$

and determine the nature of the singularities at the points

$$q=1, q^2=1, q^3=1, q^4=1, q^5=1,\ldots,$$

we know how beautifully the asymptotic form of the function can be expressed in a very neat and closed exponential form. For instance, when $q=e^{-t}$ and $t\to 0$,

$$(\mathrm{A})=\sqrt{\left(\frac{t}{2\pi}\right)}\exp\left(\frac{\pi^2}{6t}-\frac{t}{24}\right)+O(1),$$

$$(\mathrm{B})=\sqrt{\left(\frac{2}{5-\sqrt{5}}\right)}\exp\left(\frac{\pi^2}{15t}-\frac{t}{60}\right)+O(1),$$

and similar results at other singularities.

If we take a number of functions like (A) and (B), it is only in a limited number of cases the terms close as above; but in the majority of cases they never close as above. For instance, when $q=e^{-t}$ and $t\to 0$,

$$1+\frac{q}{(1-q)^2}+\frac{q^3}{(1-q)^2(1-q^2)^2}+\frac{q^6}{(1-q)^2(1-q^2)^2(1-q^3)^2}+\cdots$$

$$=\sqrt{\left(\frac{t}{2\pi\sqrt{5}}\right)}\exp\left[\frac{\pi^2}{5t}+a_1t+a_2t^2+\cdots+O(a_kt^k)\right], \tag{C}$$

where $a_1=1/8\sqrt{5}$, and so on. The function (C) is a simple example of a function behaving in an unclosed form at the singularities.

Now a very interesting question arises. Is the converse of the statements concerning the forms (A) and (B) true? That is to say: Suppose there is a function in the Eulerian form and suppose that all or an infinity of points are exponential singularities, and also suppose that at these points the asymptotic form of the function closes as neatly as in the cases of (A) and (B). The question is: Is the function taken the sum of two functions one of which is an ordinary ϑ-function and the other a (trivial) function which is $O(1)$ at all the points $e^{2m\pi i/n}$? The answer is it is not necessarily so. When it is not so, I call the function a Mock ϑ-function. I have not proved rigorously that it is not necessarily so. But I have constructed a number of examples in which it is inconceivable to construct a ϑ-function to cut out the singularities of the original function. Also I have shown that if it is necessarily so then it leads to the following assertion: viz., it is possible to construct two power series in x, namely, Σa_nx^n and Σb_nx^n, both of which have essential singularities on the unit circle, are convergent when $|x|<1$, and tend to finite limits at every point $x=e^{2r\pi i/s}$, and at the same time the limit of Σa_nx^n at the point $x=e^{2r\pi i/s}$ is equal to the limit of Σb_nx^n at the point $x=e^{-2r\pi i/s}$.

This assertion seems to be untrue. Anyhow, we shall go to the examples and see how far our assertions are true. I have proved that, if

$$f(q)=1+\frac{q}{(1+q)^2}+\frac{q^4}{(1+q)^2(1+q^2)^2}+\cdots,$$

then

$$f(q)+(1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-2q^9+\cdots)=O(1)$$

at all the points $q=-1, q^3=-1, q^5=-1, q^7=-1, \ldots$; and at the same time

$$f(q)-(1-q)(1-q^3)(1-q^5)\cdots(1-2q+2q^4-2q^9+\cdots)=O(1)$$

at all the points $q^2=-1, q^4=-1, q^6=-1, \ldots$. Also, obviously, $f(q)=O(1)$ at all the points $q=1, q^3=1, q^5=1, \ldots$. And so $f(q)$ is a Mock ϑ-function.

When $q=-e^{-t}$ and $t \to 0$,

$$f(q)+\sqrt{\left(\frac{\pi}{t}\right)}\exp\left(\frac{\pi^2}{24t}-\frac{t}{24}\right)\to 4.$$

The coefficient of q^n in $f(q)$ is

$$(-1)^{n-1}\frac{\exp\left\{\pi\sqrt{\left(\frac{1}{6}n-\frac{1}{144}\right)}\right\}}{2\sqrt{\left(n-\frac{1}{24}\right)}}+O\left[\frac{\exp\left\{\frac{1}{2}\pi\sqrt{\left(\frac{1}{6}n-\frac{1}{144}\right)}\right\}}{\sqrt{\left(n-\frac{1}{24}\right)}}\right].$$

It is inconceivable that a single ϑ-function could be found to cut out the singularities of $f(q)$.

When Ramanujan refers to a ϑ-function, he apparently means sums, products, and quotients of series of the form

$$\sum_{n=-\infty}^{\infty}(-1)^{\epsilon n}q^{an^2+bn},$$

where $\epsilon=0$ or 1. Note that the full assertions connected with his series (A) and (B) are:

$$1+\frac{q}{(1-q)^2}+\frac{q^4}{(1-q)^2(1-q^2)^2}+\cdots=\frac{1}{\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}} \qquad \text{(A)}'$$

(an identity of Euler) [5, Chaps. 1 and 2], and

$$1+\frac{q}{1-q}+\frac{q^4}{(1-q)(1-q^2)}+\cdots=\frac{\sum_{n=-\infty}^{\infty}(-1)^n q^{n(5n-1)/2}}{\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}} \qquad \text{(B)}'$$

(the first Rogers–Ramanujan identity [5, Chap. 7]). The mock theta function $f(q)$ described by Ramanujan is also discussed in Section 3.

Theta functions have had considerable impact on various branches of mathematics, extending from mathematical physics to number theory. R. Bellman [9] has given a charming introduction to the many facets of theta functions. The mock theta functions are related to problems in additive number theory (see [8] and [7]).

Finally there are the "false theta functions." These are simply theta series with the "wrong" signs. For example, both

$$\sum_{n=-\infty}^{\infty}(-1)^n q^{6n^2+4n}=1-q^2-q^{10}+q^{16}+q^{32}-\cdots$$

and

$$\sum_{n=-\infty}^{\infty}q^{6n^2+4n}=1+q^2+q^{10}+q^{16}+q^{32}+\cdots$$

are theta series; however, the series on the right-hand side of (1.2) is not, in that

$$\sum_{n=0}^{\infty}(-1)^n q^{6n^2+4n}(1+q^{4n+2})=1+q^2-q^{10}-q^{16}+q^{32}+\cdots.$$

These "false" theta series do not seem to possess the analytic interest that Ramanujan describes for the ϑ-series and mock ϑ-series; however, they do crop up in some elegant identities such as (1.2).

There are other results from the general theory of q-series that we shall require in our treatment of the "lost" notebook. However, the ideas covered in this section constitute the fundamental tools in our treatment.

3. The origin of the "lost" notebook. There is no introduction or covering letter with this manuscript. Indeed there are only a few words scattered here and there throughout the manuscript. Concerning this notebook, Miss Rosemary Graham, Manuscript Cataloguer of the Trinity College Library, says: "... the notebook and other material was discovered among Watson's papers by Dr. J. M. Whittaker, who wrote the obituary of Professor Watson for the Royal Society. He passed the papers to Professor R. A. Rankin of Glasgow University, who, in December 1968, offered them to Trinity College so that they might join the other Ramanujan manuscripts already given to us by Professor Rankin on behalf of Professor Watson's widow."

Ramanujan's wife gives the following description of Ramanujan's last year (April 1919–April 1920) before his death: "He returned from England only to die, as the saying goes. He lived for less than a year. Throughout this period I lived with him without break. He was only skin and bones. He often complained of severe pain. In spite of it he was always busy doing his Mathematics. That evidently helped him to forget the pain. I used to gather the sheets of paper which he filled up. I would also give the slate whenever he asked for it." (Quoted from [**20**, p. 91].)

Of this intense mathematical activity, we have up to now only known of the mock theta functions. These functions were described in Ramanujan's last letter to Hardy, written from the University of Madras and dated January 12, 1920 [**18**, p. xxxi]: "I am extremely sorry for not writing you a single letter up to now...I discovered very interesting functions recently which I call 'Mock' ϑ-functions. Unlike the 'False' ϑ-functions (studied partially by Prof. Rogers in his interesting paper) they enter into mathematics as beautifully as the ordinary theta functions. I am sending you with this letter some examples." Besides the material quoted in Section 2, Ramanujan also defines four third-order mock theta functions, ten fifth-order functions and three seventh-order functions. He also includes three identities satisfied by the third-order functions and five identities satisfied by his first five fifth-order functions. He states that the other five fifth-order functions also satisfy similar identities.

Subsequent authors ([**27**], [**28**], [**12**], [**1**], [**2**], [**3**], [**7**]) have studied the mock theta functions extensively. All of these papers study the work that Ramanujan described in his last letter. Now the "lost" notebook contains all of the formulae for the third- and fifth-order mock theta functions given in Ramanujan's last letter. Furthermore, it contains the five identities for the second family of fifth-order functions that were only mentioned but not stated in the letter. The "lost" notebook also contains one-parameter generalizations of the third-order identities that were rediscovered independently in 1965 [**2**].

It appears that either Watson did not possess the "lost" notebook in the late 1930's when he

worked on mock theta functions or he had filed it away and forgotten it. In any event, Watson [27, p. 61] says that he believes Ramanujan did not possess transformation formulae for the third-order mock theta functions such as

$$f(q)\prod_{n=1}^{\infty}(1-q^n)=1+4\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(3n+1)/2}}{1+q^n},\tag{3.1}$$

where $f(q)$ is one of Ramanujan's third-order mock theta functions given by

$$f(q)=1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1+q)^2(1+q^2)^2\cdots(1+q^n)^2}.\tag{3.2}$$

However in the "lost" notebook we find

$$\prod_{n=1}^{\infty}(1-abq^n)\left(1+\sum_{n=1}^{\infty}\frac{q^{n^2}}{(1+aq)(1+aq^2)\cdots(1+aq^n)(1+bq)(1+bq^2)\cdots(1+bq^n)}\right)$$
$$=(1+a)(1+b)\sum_{n=0}^{\infty}\frac{(-1)^n a^n b^n q^{n(3n+1)/2}(1-abq^{2n})}{(1+aq^n)(1+bq^n)(1-ab)}\prod_{j=1}^{n}\frac{(1-abq^{j-1})}{(1-q^j)},\tag{3.3}$$

which reduces to (3.1) for $a\to 1$, $b\to 1$. Indeed the master identity used by Watson to construct all of his mock theta function transformations is obtained directly from (3.3) by the substitutions $a\to e^{i\theta}$, $b\to e^{-i\theta}$.

Watson rightly suggests the importance of (3.1) by pointing out that he was only able to prove that $f(q)$ really was a mock theta function by utilizing (3.1). In a second paper [**28**], Watson studied the fifth-order mock theta functions and proved the five identities associated with each family. While Watson did succeed in proving that the fifth-order mock theta functions do indeed behave like theta functions [**28**, §6], he was unable to prove that they are not themselves theta functions. This was because he did not know a result comparable to (3.1) for the fifth-order functions. However, the "lost" notebook contains several such results. For example, $\phi_0(q)$ defined by

$$\phi_0(q)=1+\sum_{n=1}^{\infty}q^{n^2}(1+q)(1+q^3)\cdots(1+q^{2n-1})\tag{3.4}$$

is one of the fifth-order functions, and in the "lost" notebook we find a result equivalent to

$$\phi_0(-q)=\prod_{n=0}^{\infty}\frac{(1-q^{5n+5})(1+q^{5n+2})(1+q^{5n+3})}{(1-q^{10n+2})(1-q^{10n+8})}$$
$$+1-\sum_{n=0}^{\infty}\frac{q^{5n^2}}{(1-q)(1-q^6)\cdots(1-q^{5n+1})(1-q^4)(1-q^9)\cdots(1-q^{5n-1})}.\tag{3.5}$$

Hence by (3.3) with $q\to q^5$ and then $a\to -q$, $b\to -q^{-1}$, we see that

$$\phi_0(-q)=\prod_{n=0}^{\infty}\frac{(1-q^{5n+5})(1+q^{5n+2})(1+q^{5n+3})}{(1-q^{10n+2})(1-q^{10n+8})}$$
$$+1-\prod_{n=0}^{\infty}(1-q^{5n+5})^{-1}\left\{\frac{1}{1-q}+(1-q^{-1})\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(15n+5)/2}(1+q^{5n})}{(1-q^{5n+1})(1-q^{5n-1})}\right\}.\tag{3.6}$$

Now Watson was obviously totally unaware of (3.5) or he surely would have found (3.6). If he had found (3.6), the methods he used to prove that $f(q)$ is a mock theta function should have led him to a proof that the fifth-order functions are indeed true mock theta functions.

In a later paper I hope to examine all of the formulae in the "lost" notebook for the mock theta-functions. For now, I hope I have made the case for my assertion that this notebook was composed during the last year of Ramanujan's life, when, by his own words, he discovered the mock theta functions. I should add that while only a fraction (perhaps 5%) of the notebook is on the mock theta functions themselves, much of the rest of it involves related q-series expansions of theta functions and false theta functions. Thus it is not unreasonable to assume that the entire notebook was composed during Ramanujan's last year, especially since the results on the mock theta functions are scattered through it. Finally, the fact that its existence was never mentioned by anyone for over 55 years leads me to call it "lost." B. Birch [**11**] has found some other notes of Ramanujan's in the library of the Oxford Mathematical Institute; however, these notes comprise only 33 pages and, for the most part, apparently treat different formulae from those found in the "lost" notebook.

4. Proof of identity (1.1). In order to prove (1.1), we shall prove the following stronger result, which is also found in the "lost" notebook:

$$1+\sum_{n=1}^{\infty}\frac{q^n}{(1+aq)(1+aq^2)\cdots(1+aq^n)(1+bq)(1+bq^2)\cdots(1+bq^n)}$$

$$=(1+a^{-1})\left(1+\sum_{n=1}^{\infty}\frac{(-1)^n q^{n(n+1)/2}b^n a^{-n}}{(1+bq)(1+bq^2)\cdots(1+bq^n)}\right)-\frac{a^{-1}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}b^n a^{-n}}{\prod_{j=1}^{\infty}(1+aq^j)(1+bq^j)}. \tag{4.1}$$

To obtain (1.1) from (4.1), first interchange a and b in (4.1) and then set $b=a^{-1}$; identity (1.1) follows once we recall the identity of L. J. Rogers [**22**, p. 335, Eq. (3)]:

$$1+\sum_{n=1}^{\infty}\frac{(-1)^n y^{2n} q^{n(n+1)/2}}{(1-yq)(1-yq^2)\cdots(1-yq^n)}=\sum_{n=0}^{\infty}(-1)^n y^{3n} q^{n(3n+1)/2}(1-y^2q^{2n+1}). \tag{4.2}$$

To simplify working with these series, we introduce the following standard notation:

$$(a)_n=(a;q)_n=(1-a)(1-aq)\cdots(1-aq^{n-1}). \tag{4.3}$$

The symbol $(a)_n$ is defined for all real n by the relation

$$(a)_n=(a;q)_n=\prod_{m=0}^{\infty}\frac{(1-aq^m)}{(1-aq^{m+n})},\qquad |q|<1.$$

Finally, we define

$$(a)_\infty=(a;q)_\infty=\prod_{m=0}^{\infty}(1-aq^m),\qquad |q|<1. \tag{4.4}$$

To prove (4.1) let us recall an identity of Euler [**5**, p. 19]:

$$\sum_{m=0}^{\infty}\frac{A^m q^{m(m+1)/2}}{(q)_m}=(-Aq)_\infty, \tag{4.5}$$

and an identity of L. J. Rogers [**21**, p. 171]:

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n\tau^n}{(q)_n(\gamma)_n}=\frac{(\gamma/\beta)_\infty(\beta\tau)_\infty}{(\gamma)_\infty(\tau)_\infty}\sum_{n=0}^{\infty}\frac{(\beta)_m(\alpha\beta\tau/\gamma)_m\gamma^m\beta^{-m}}{(q)_m(\beta\tau)_m}. \tag{4.6}$$

Now multiply both sides of (3.1) by $(-bq)_\infty$ and apply (3.5) with $A=bq^n$:

$$\sum_{n=0}^{\infty}\frac{q^n}{(-aq)_n}\sum_{m=0}^{\infty}\frac{q^{mn+m(m+1)/2}b^m}{(q)_m}$$

$$=(1+a^{-1})\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}b^n a^{-n}\sum_{n=0}^{\infty}\frac{q^{mn+m(m+1)/2}b^m}{(q)_m}-\frac{a^{-1}\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}b^n a^{-n}}{(-aq)_\infty}. \tag{4.7}$$

If we compare coefficients of b^N on both sides of (4.7) and multiply the resulting identity by $q^{-N(N+1)/2}$, we find that (4.1) is equivalent to

$$\frac{1}{(q)_N}\sum_{n=0}^{\infty}\frac{q^{(N+1)n}}{(-aq)_n}=a^{-N}(1+a^{-1})\sum_{m=0}^{N}\frac{(-1)^{N+m}a^m}{(q)_m}-\frac{(-1)^N a^{-N-1}}{(-aq)_\infty}. \tag{4.8}$$

To prove (4.8) we set $\beta=q,\alpha=0,\gamma=-aq$, and $\tau=q^{N+1}$ in (4.6); hence after dividing by $(q)_N$ we find

$$\frac{1}{(q)_N}\sum_{n=0}^{\infty}\frac{q^{(N+1)n}}{(-aq)_n}=\frac{1}{(q)_N}\frac{(q^{N+2})_\infty(1+a)}{(q^{N+1})_\infty}\sum_{n=0}^{\infty}\frac{(-a)^n}{(q^{N+2})_n}$$

$$=(1+a)(-a)^{-N-1}\sum_{n=0}^{\infty}\frac{(-a)^{n+N+1}}{(q)_{n+N+1}}$$

$$=(1+a)(-a)^{-N-1}\left\{\sum_{n=0}^{\infty}\frac{(-a)^n}{(q)_n}-\sum_{n=0}^{N}\frac{(-a)^n}{(q)_n}\right\}$$

$$=(1+a)(-a)^{-N-1}\left\{\frac{1}{(-a)_\infty}-\sum_{n=0}^{N}\frac{(-a)^n}{(q)_n}\right\}$$

$$=\frac{-(-1)^N a^{-N-1}}{(-aq)_\infty}+a^{-N}(1+a^{-1})\sum_{n=0}^{N}\frac{(-1)^{n+N}a^n}{(q)_n}, \tag{4.9}$$

which is (4.8). Thus (4.1) is established. Note that in the penultimate step in (4.9) we used a second identity of Euler [5, p. 19]:

$$\sum_{n=0}^{\infty}\frac{A^n}{(q)_n}=\frac{1}{(A)_\infty}. \tag{4.10}$$

There exist generalizations of (4.1), and we shall discuss these later in this series of papers.

5. **Proof of identity (1.2).** To prove this result we shall require (4.2) and (4.6) as well as the fundamental transformation of E. Heine [15, p. 306]:

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n\tau^n}{(q)_n(\gamma)_n}=\frac{(\beta)_\infty(\alpha\tau)_\infty}{(\gamma)_\infty(\tau)_\infty}\sum_{m=0}^{\infty}\frac{(\gamma/\beta)_m(\tau)_m\beta^m}{(q)_m(\alpha\tau)_m}. \tag{5.1}$$

To begin we replace q by q^2 in (5.1) and then set $\alpha=0,\beta=q^2,\gamma=q^3$ and $\tau=-q$; after dividing both sides of the resulting identity by $(1-q)$ we find that

$$\sum_{n=0}^{\infty}\frac{(-1)^n q^n}{(q;q^2)_{n+1}}=\frac{(q^2;q^2)_\infty}{(q;q^2)_\infty(-q;q^2)_\infty}\sum_{m=0}^{\infty}\frac{(q;q^2)_m(-q;q^2)_m q^{2m}}{(q^2;q^2)_m}$$

$$
\begin{aligned}
&=(q^2;q^2)_\infty \sum_{m=0}^{\infty} \frac{q^{2m}}{(q^2;q^2)_m} \frac{1}{(q^{4m+2};q^4)_\infty} \\
&=(q^2;q^2)_\infty \sum_{m=0}^{\infty} \frac{q^{2m}}{(q^2;q^2)_m} \sum_{n=0}^{\infty} \frac{q^{n(4m+2)}}{(q^4;q^4)_n} \quad \text{(by (4.10))} \\
&=(q^2;q^2)_\infty \sum_{n=0}^{\infty} \frac{q^{2n}}{(q^4;q^4)_n} \frac{1}{(q^{4n+2};q^2)_\infty} \quad \text{(by (4.10))} \\
&=\sum_{n=0}^{\infty} \frac{(q^2;q^2)_{2n} q^{2n}}{(q^4;q^4)_n} \\
&=\sum_{n=0}^{\infty} (q^2;q^4)_n q^{2n}. \qquad (5.2)
\end{aligned}
$$

Now in (4.6) replace q by q^4, then set $\beta=q^4, \alpha=q^2, \tau=q^2$, and let $c \to 0$; this yields

$$
\begin{aligned}
\sum_{n=0}^{\infty} (q^2;q^4)_n q^{2n} &= \sum_{m=0}^{\infty} \frac{(-1)^m q^{2m^2+2m}}{(q^2;q^4)_{m+1}} \\
&= -\sum_{m=1}^{\infty} \frac{(-1)^m q^{2m^2-2m}}{(q^2;q^4)_m} \\
&= 1-\sum_{n=0}^{\infty} (-1)^n q^{6n^2-4n}(1-q^{8n}) \quad \text{by (4.2)} \\
&= \sum_{n=0}^{\infty} (-1)^n q^{6n+4n}(1+q^{4n+2}). \qquad (5.3)
\end{aligned}
$$

Now if we equate the first member of (5.2) with the final member of (5.3), we deduce (1.2) upon replacing q by $-q$.

The following partition theorem is easily deduced from (1.2); it would be nice to have a combinatorial proof of this result:

THEOREM. *Let $R_i(n)$ denote the number of partitions of n into odd parts wherein the largest part is congruent to i (mod 4) and appears an odd number of times while all other parts appear an even number of times. Then*

$$
R_1(n)-R_3(n)=\begin{cases} (-1)^j & \text{if } n=12j^2+8j+1 \quad \text{or} \quad 12j^2+16j+5 \\ 0 & \text{otherwise.} \end{cases} \qquad (5.4)
$$

Proof. We note that

$$
1+\sum_{n=1}^{\infty} (R_1(n)-R_3(n))q^n = \sum_{n=0}^{\infty} \frac{(-1)^n q^{2n+1}}{(1-q^2)(1-q^6)\cdots(1-q^{4n+2})}. \qquad (5.5)
$$

since

$$
\frac{1}{(1-q^{1+1})(1-q^{3+3})\cdots(1-q^{(2n+1)+(2n+1)})} \qquad (5.6)
$$

is the generating function for partitions with odd parts each $\leqslant 2n+1$ and each appearing an even number of times. Our theorem now follows if we replace q by $-q^2$ in (1.2) and multiply the resulting identity by q.

For example, when $n=5$ the partitions enumerated by $R_1(5)$ are 5 and $1+1+1+1+1$ while

that enumerated by $R_3(5)$ is $3+1+1$; hence $R_1(5)-R_3(5)=2-1=1$ as predicted. When $n=11$, $R_1(11)$ enumerates $9+1+1, 5+3+3, 5+1+1+1+1+1+1$ and $1+1+1+1+1+1+1+1+1+1+1$ while $R_3(11)$ enumerates $11, 7+1+1+1+1, 3+3+3+1+1, 3+1+1+1+1+1+1+1+1$; hence $R_1(11)-R_3(11)=4-4=0$ as predicted.

Later in this series we shall present further partition theorems implied by Ramanujan's work in this "lost" notebook.

6. Proof of identity (1.3). This continued fraction identity depends both on the well-known Rogers–Ramanujan continued fraction (1.6) and on what has become known as Ramanujan's ${}_1\psi_1$ summation:

$$\begin{aligned}{}_1\psi_1\left[\begin{matrix} a; q, t \\ b \end{matrix}\right] &= \sum_{n=-\infty}^{\infty} \frac{(a)_n}{(b)_n} t^n \\ &= \frac{(b/a)_\infty (az)_\infty (q/az)_\infty (q)_\infty}{(q/a)_\infty (b/az)_\infty (b)_\infty (z)_\infty}.\end{aligned} \tag{6.1}$$

Proofs of (6.1) can be found in **[4]**, **[6]**, **[7]**.

We start on (1.3) by proving an elementary series identity: for $i=1$, 2, 3, or 4

$$\sum_{n=0}^{\infty} q^{5n^2+2in} \frac{1+q^{5n+i}}{1-q^{5n+i}} = \sum_{n=0}^{\infty} \frac{q^{in}}{1-q^{5n+i}}. \tag{6.2}$$

This identity holds because

$$\begin{aligned}\sum_{n=0}^{\infty} \frac{q^{in}}{1-q^{5n+i}} &= \sum_{n=0}^{\infty} \sum_{n=0}^{\infty} q^{in+5nm+mi} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} + \sum_{m=n+1}^{\infty} \right) q^{in+5nm+im} \\ &= \sum_{m=0}^{\infty} \sum_{n=0}^{\infty} q^{i(n+m)+5(n+m)n+im} \\ &\quad + \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q^{in+5n(m+n+1)+i(m+n+1)} \\ &= \sum_{m=0}^{\infty} \frac{q^{5m^2+2im}}{1-q^{5m+i}} + \sum_{n=0}^{\infty} \frac{q^{5n^2+5n+2in+i}}{1-q^{5n+i}} \\ &= \sum_{n=0}^{\infty} q^{5n^2+2in} \frac{1+q^{5n+i}}{1-q^{5n+i}}.\end{aligned} \tag{6.3}$$

Next we see that by (6.1) for $i=1$, 2, 3, or 4

$$\begin{aligned}\sum_{n=-\infty}^{\infty} \frac{q^{in}}{1-q^{5n+i}} &= \frac{1}{1-q^i} {}_1\psi_1\left[\begin{matrix} q^i; q^5, q^i \\ q^{5+i} \end{matrix}\right] \\ &= \frac{(q^5;q^5)_\infty^2 (q^{2i};q^5)_\infty (q^{5-2i};q^5)_\infty}{(a^{5-i};q^5)_\infty^2 (q^i;q^5)_\infty^2},\end{aligned} \tag{6.4}$$

Hence

$$\frac{\sum_{n=0}^{\infty} q^{5n^2+4n}\frac{1+q^{5n+2}}{1-q^{5n+2}} - \sum_{n=0}^{\infty} q^{5n^2+6n+1}\frac{1+q^{5n+3}}{1-q^{5n+3}}}{\sum_{n=0}^{\infty} q^{5n^2+2n}\frac{1+q^{5n+1}}{1-q^{5n+1}} - \sum_{n=0}^{\infty} q^{5n^2+8n+3}\frac{1+q^{5n+4}}{1-q^{5n+4}}}$$

$$= \frac{\sum_{n=0}^{\infty}\frac{q^{2n}}{1-q^{5n+2}} - \sum_{n=0}^{\infty}\frac{q^{3n+1}}{1-q^{5n+3}}}{\sum_{n=0}^{\infty}\frac{q^{n}}{1-q^{5n+1}} - \sum_{n=0}^{\infty}\frac{q^{4n+3}}{1-q^{5n+4}}} \quad \text{(by (6.3))}$$

$$= \frac{\sum_{n=-\infty}^{\infty}\frac{q^{2n}}{1-q^{5n+2}}}{\sum_{n=-\infty}^{\infty}\frac{q^{n}}{1-q^{5n+1}}}$$

$$= \frac{(q^5;q^5)_\infty^2(q^4;q^5)_\infty(q;q^5)_\infty}{(q^3;q^5)_\infty^2(q^2;q^5)_\infty^2}\cdot\frac{(q^4;q^5)_\infty^2(q;q^5)_\infty^2}{(q^5;q^5)_\infty^2(q^2;q^5)_\infty(q^3;q^5)_\infty}$$

$$= \left(\frac{(q^4;q^5)_\infty(q;q^5)_\infty}{(q^2;q^5)_\infty(q^3;q^5)_\infty}\right)^3$$

$$= \left[\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2}{1+\cfrac{q^3}{\vdots}}}}\right]^3 \quad \text{(by (1.6))}.$$

Also in the "lost" notebook are more than a dozen identities deduced from special cases of (6.4) combined with (1.6).

7. Proof of identity (1.4). Instead of determining the convergents of this continued fraction, we deduce this identity from a family of relations. To obtain this family, we require a further identity of E. Heine [**15**, p. 325] (see also L. J. Rogers [**21**, p. 171]):

$$\sum_{n=0}^{\infty}\frac{(\alpha)_n(\beta)_n\tau^n}{(q)_n(\gamma)_n} = \frac{(\alpha\beta\tau/\gamma)_\infty}{(\tau)_\infty}\sum_{n=0}^{\infty}\frac{(\gamma/\alpha)_n(\gamma/\beta)_n}{(q)_n(\gamma)_n}\left(\frac{\alpha\beta\tau}{\gamma}\right)^n. \tag{7.1}$$

Next we define for nonnegative integral i:

$$F_i(a,\lambda;b;q) = \sum_{n=0}^{\infty}\frac{q^{(n+2i+1)n/2}(-\lambda a^{-1}q^i)_n a^n}{(q)_n(-bq)_{n+i}}, \tag{7.2}$$

$$H_i(a,\lambda;b;q) = \sum_{n=0}^{\infty}\frac{q^{(n+2i+3)n/2}(-\lambda a^{-1}q^i)_n a^n}{(q)_n(-bq)_{n+i}}. \tag{7.3}$$

Immediately from (7.2) and (7.3) we obtain our first relation:

$$
\begin{aligned}
F_i(a,\lambda;b;q)-H_i(a,\lambda;b;q) &= \sum_{n=0}^{\infty} \frac{q^{(n+2i+1)n/2}(-\lambda a^{-1}q^i)_n a^n}{(q)_n(-bq)_{n+i}}(1-q^n)\\
&= \sum_{n=0}^{\infty} \frac{q^{(n+2i+2)(n+1)/2}(-\lambda a^{-1}q^i)_{n+1} a^{n+1}}{(q)_n(-bq)_{n+i+1}}\\
&= (aq^{i+1}+\lambda q^{2i+1})\sum_{n=0}^{\infty} \frac{q^{(n+2(i+1)+1)n/2}(-\lambda a^{-1}q^{i+1})_n a^n}{(q)_n(-bq)_{n+i+1}}\\
&= (aq^{i+1}+\lambda q^{2i+1})F_{i+1}(a,\lambda;b;q). \qquad (7.4)
\end{aligned}
$$

Next we transform $F_i(a,\lambda;b;q)$ and $H_i(a,\lambda;b;q)$ utilizing (6.1). First in (7.1) we set

$$\gamma=-bq^{i+1},\quad \tau=-\frac{q^{i+2}a}{\alpha},\quad \beta=-\gamma a^{-1}q^i$$

and let $\alpha\to\infty$:

$$H_i(a,\lambda;b;q)=(-\lambda b^{-1}q^{i+1})_\infty \sum_{n=0}^{\infty}\frac{(ba\lambda^{-1})_n(-\lambda b^{-1}q^{i+1})^n}{(q)_n(-bq)_{n+i}}; \qquad (7.5)$$

next we set

$$\gamma=-bq^{i+1},\tau=-\frac{q^{i+1}a}{\alpha},\beta=-\lambda a^{-1}q^i$$

and let $\alpha\to\infty$:

$$F_i(a,\lambda;b;q)=(-\lambda b^{-1}q^{i})_\infty \sum_{n=0}^{\infty}\frac{(ba\lambda^{-1})_n(-\lambda b^{-1}q^{i})^n}{(q)_n(-bq)_{n+i}}. \qquad (7.6)$$

From (7.5) and (7.6) we obtain our second relation:

$H_i(a,\lambda;b;q)-F_{i+1}(a,\lambda;b;q)$

$$
\begin{aligned}
&=(-\lambda b^{-1}q^{i+1})_\infty \sum_{n=0}^{\infty}\frac{(ba\lambda^{-1})_n(-\lambda b^{-1}q^{i+1})^n}{(q)_n(-bq)_{n+i+1}}(1+bq^{n+i+1}-1)\\
&=(bq^{i+1}+\lambda q^{2i+2})(-\lambda b^{-1}q^{i+2})_\infty \sum_{n=0}^{\infty}\frac{(ba\lambda^{-1})_n(-\lambda b^{-1}q^{i+2})^n}{(q)_n(-bq)_{n+i+1}}\\
&=(bq^{i+1}+\lambda q^{2i+2})H_{i+1}(a,\lambda;b;q). \qquad (7.7)
\end{aligned}
$$

Finally we note that from the definition of $G(a,\lambda)$ given after (1.4), we have

$$G(a,\lambda)=F_0(a,\lambda;b;q), \qquad (7.8)$$

and

$$G(aq,\lambda q)=H_0(a,\lambda;b;q). \qquad (7.9)$$

We now deduce (1.4) from iterated application of (7.7) and (7.4):

$$\frac{G(aq,\lambda q)}{G(a,\lambda)}=\frac{1}{\dfrac{F_0(a,\lambda;b;q)}{H_0(a,\lambda;b;q)}} \quad \text{(by (7.8) and (7.9))}$$

$$=\cfrac{1}{1+\cfrac{aq+\lambda q}{\cfrac{H_0(a,\lambda;b;q)}{F_1(a,\lambda;b;q)}}}\quad\text{(by (7.4))}$$

$$=\cfrac{1}{1+\cfrac{aq+\lambda q}{1+\cfrac{bq+\lambda q^2}{\cfrac{F_1(a,\lambda;b;q)}{H_1(a,\lambda;b;q)}}}}\quad\text{(by (7.7))}$$

$$=\cfrac{1}{1+\cfrac{aq+\lambda q}{1+\cfrac{bq+\lambda q^2}{1+\cfrac{aq^2+\lambda q^3}{\cfrac{H_1(a,\lambda;b;q)}{F_2(a,\lambda;b;q)}}}}}\quad\text{(by (7.4))}$$

$$\vdots \quad .$$

Convergence of the continued fraction is assured by the fact that both H_i/F_{i+1} and F_i/H_i are analytic in $|q|<1$ with each of the form $1+O(q^{i+1})$.

Having proved (1.4), let us now examine the corollaries discovered by Ramanujan:

$$\prod_{n=0}^{\infty}\frac{(1-q^{2n+1})}{(1-q^{4n+2})^2}=\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2+q}{1+\cfrac{q^3}{1+\cfrac{q^4+q^2}{1+\vdots}}}}}\ . \tag{7.10}$$

Identity (7.10) follows if we set $a=0$, $b=1$, $\lambda=1$ in (1.4) and recall (4.5), which implies

$$G(0,\lambda)=\sum_{n=0}^{\infty}\frac{q^{n^2}\lambda^n}{(q^2;q^2)_n}$$
$$=(-\lambda q;q^2)_\infty,$$

so

$$\frac{G(0,q)}{G(0,1)}=\frac{(-q^2;q^2)_\infty}{(-q;q^2)_\infty}=\frac{1}{(q^2;q^4)_\infty(-q;q^2)_\infty}\quad\text{(by [5, p. 5, Eq. (1.2.5)])}\quad=\frac{(q;q^2)_\infty}{(q^2;q^4)_\infty^2}.$$

Next we have an identity originally discovered by Eisenstein [**13**, p. 36]:

$$\sum_{n=0}^{\infty}(-1)^n q^{n(n+1)/2}=\cfrac{1}{1+\cfrac{q}{1+\cfrac{q^2-q}{1+\cfrac{q^3}{1+\cfrac{q^4-q^2}{\vdots}}}}}\ . \tag{7.11}$$

This result follows by our setting $a=0$, $\lambda=1$, $b=-1$ in (1.4) and noting

$$\sum_{n=0}^{\infty} \frac{q^{n^2-n}\tau^n}{(q)_n(\gamma)_n} = \frac{1}{(\gamma)_\infty} \sum_{m=0}^{\infty} \frac{(\tau/\gamma)_m}{(q)_m}(-1)^m \gamma^m q^{m(m-1)/2},$$

which follows from (3.6) if we replace τ by $\tau\alpha^{-1}\beta^{-1}$ and let α and β tend to infinity. Hence

$$\frac{G(0,q;-1;q)}{G(0,1;-1;q)} = \frac{\displaystyle\sum_{m=0}^{\infty} \frac{q^{m^2+m}}{(q)_m^2}}{\displaystyle\sum_{m=0}^{\infty} \frac{q^{m^2}}{(q)_m^2}} = \frac{\displaystyle\frac{1}{(q)_\infty}\sum_{m=0}^{\infty}(-1)^m q^{m(m+1)/2}}{1/(q)_\infty} = \sum_{m=0}^{\infty}(-1)^m q^{m(m+1)/2}.$$

$$\prod_{n=0}^{\infty} \frac{(1-q^{6n+1})(1-q^{6n+5})}{(1-q^{6n+3})^2} = \cfrac{1}{1+\cfrac{q+q^2}{1+\cfrac{q^2+q^4}{1+\cfrac{q^3+q^6}{1+\vdots}}}}\,. \tag{7.12}$$

To obtain this result we replace q by q^2 in (1.4), then we set $a=q^{-1}$, $b=1$, $\lambda=1$. Then we observe

$$G(q^{-1},1;1;q^2) = \sum_{n=0}^{\infty} \frac{q^{n^2}(-q;q^2)_n}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty(q^3;q^3)_\infty(q^3;q^6)_\infty}{(q^2;q^2)_\infty},$$

a result given by Slater [24, p. 154, Eq. (25)], and finally

$$G(q,q^2;1;q^2) = \sum_{n=0}^{\infty} \frac{q^{n^2+2n}(-q;q^2)_n}{(q^4;q^4)_n} = \frac{(-q;q^2)_\infty(q^6;q^6)_\infty(q;q^6)_\infty(q^5;q^6)_\infty}{(q^2;q^2)_\infty}.$$

This last formula has apparently not been stated in the literature before; however, it is easily deduced if we set $y=-q^{1/2}$ and let $z\to\infty$ in Slater's identity E(4) [23, p. 469].

$$\prod_{n=0}^{\infty} \frac{(1-q^{8n+1})(1-q^{8n+7})}{(1-q^{8n+3})(1-q^{8n+5})} = \cfrac{1}{1+\cfrac{q^2+q}{1+\cfrac{q^4}{1+\cfrac{q^6+q^3}{\vdots}}}}\,. \tag{7.13}$$

Here we replace q by q^2 in (1.4); then we set $a=q^{-1}$, $b=0$, $\lambda=1$. Under these substitutions

$$\frac{G(q,q^2;0;q^2)}{G(q^{-1},1;0;q^2)}=\frac{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2+2n}(-q;q^2)_n}{(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{q^{n^2}(-q;q^2)_n}{(q^2;q^2)_n}}$$

$$=\frac{(q;q^8)_\infty(q^7;q^8)_\infty}{(q^3;q^8)_\infty(q^5;q^8)_\infty},$$

where the last equation follows from two identities of Slater [**24**, p. 155, Eqs. (34) and (36)].

Besides the case $a=b=0,\lambda=1$ which yields (1.6), Ramanujan presented one other corollary which involves a false theta function.

$$\sum_{n=0}^{\infty}(-1)^n q^{3n^2+2n}(1+q^{2n+1})=\cfrac{1}{1+\cfrac{q^2-q}{1+\cfrac{q^4-q^2}{1+\cfrac{q^6-q^3}{\vdots}}}}\;. \tag{7.14}$$

To obtain this result we replace q by q^2 in (1.4); then we set $a=-q^{-1}$, $b=-1$, $\lambda=1$. Thus

$$\frac{G(-q,q^2;-1;q^2)}{G(-q^{-1},1;-1;q^2)}=\lim_{\tau\to\infty}\frac{\displaystyle\sum_{n=0}^{\infty}\frac{(q/\tau;q^2)_n(q;q^2)_n\tau^n q^{2n}}{(q^2;q^2)_n(q^2;q^2)_n}}{\displaystyle\sum_{n=0}^{\infty}\frac{(q/\tau;q^2)_n(q;q^2)_n\tau^n}{(q^2;q^2)_n(q^2;q^2)_n}}$$

$$=\frac{\dfrac{(q^3;q^2)}{(q^2;q^2)}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n q^{n^2+n}}{(q^3;q^2)_n}}{\dfrac{(q;q^2)_\infty}{(q^2;q^2)_\infty}} \quad \text{(by (4.6))}$$

$$=\sum_{n=0}^{\infty}(-1)^n\frac{q^{n^2+n}}{(q;q^2)_{n+1}}$$

$$=\sum_{n=0}^{\infty}(-1)^n q^{3n^2+2n}(1+q^{2n+1}),$$

where the last equation is due to L. J. Rogers [**22**, p. 333, Eq. (4)].

8. Proof of identity (1.5). This result is a good deal easier than the others we have considered. First we give an analytic proof. For this proof we require a special case of the q-analog of the binomial series (2.4) [**5**, p. 36, Eq. (3.37)]:

$$\frac{1}{(z)_{n+1}}=\sum_{m=0}^{\infty}\frac{(q)_{n+m}}{(q)_n(q)_m}z^m. \tag{8.1}$$

Thus the left-hand side of (1.5) is

$$\sum_{n=0}^{\infty} \frac{\beta^n}{(\alpha x^n)_{n+1}}$$

where $q = y/x$. Hence

$$\sum_{n=0}^{\infty} \frac{\beta^n}{(\alpha x^n)_{n+1}} = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \beta^n \alpha^m \frac{(q)_{n+m}}{(q)_n (q)_m} x^{nm},$$

and the symmetry in α and β is now obvious. Hence (1.5) is established.

Combinatorially we see that the coefficient of $\alpha^m \beta^n$ in

$$\sum_{n=0}^{\infty} \frac{\beta^n}{(1-\alpha x^n)(1-\alpha x^{n-1}y)\cdots(1-\alpha y^n)} \tag{8.2}$$

is a homogeneous polynomial in x and y of degree nm. In this polynomial the coefficient of $y^j x^{nm-j}$ is the number of partitions of j into at most m parts each at most n, say $j = \alpha_1 + \cdots + \alpha_r$. Then the exponent on x is

$$nm - j = (n-\alpha_1) + (n-\alpha_2) + \cdots + (n-\alpha_r) + \underbrace{n + \cdots + n}_{m-r \text{ terms}}$$

which is another partition of at most m parts each $\leqslant n$ and it is uniquely determined by $\alpha_1 + \cdots + \alpha_r$, n and m. Now if we conjugate all the partitions under consideration we merely interchange the roles of n and m. Hence (8.2) is symmetric in α and β.

9. Conclusion. This is the first of several papers that will treat Ramanujan's "lost" notebook. I hope that the results discussed here indicate that Ramanujan discovered many results of interest in the last year of his life. I wish to close by quoting the last two paragraphs of G. N. Watson's presidential address to the London Mathematical Society in 1935 [**27**, p. 80]. In this address Watson has just finished an extensive discussion of the mock theta functions based on the last letter of Ramanujan, which we quoted in Sections 2 and 3. Watson concludes as follows:

The study of Ramanujan's work and of the problems to which it gives rise inevitably recalls to mind Lamé's remark that, when reading Hermite's papers on modular functions, "on a la chair de poule." I would express my own attitude with more prolixity by saying that such a formula as

$$\int_0^{\infty} e^{-3\pi x^2} \frac{\sinh \pi x}{\sinh 3\pi x}\, dx = \frac{1}{e^{2\pi/3}\sqrt{3}} \sum_{n=0}^{\infty} \frac{e^{-2n(n+1)\pi}}{(1+e^{-\pi})^2(1+e^{-3\pi})^2 \cdots (1+e^{-(2n+1)\pi})^2}$$

gives me a thrill which is indistinguishable from the thrill which I feel when I enter the Sagrestia Nuova of the Capelle Medicee and see before me the austere beauty of the four statues representing Day, Night, Evening, and Dawn which Michelangelo has set over the tombs of Guiliano de'Medici and Lorenzo de'Medici.

Ramanujan's discovery of the mock theta functions makes it obvious that his skill and ingenuity did not desert him at the oncoming of his untimely end. As much as any of his earlier work, the mock theta functions are an achievement sufficient to cause his name to be held in lasting remembrance. To his students such discoveries will be a source of delight and wonder until the time shall come when we too shall make our journey to that Garden of Proserpine where

"Pale, beyond porch and portal,
Crowned with calm leaves, she stands
Who gathers all things mortal
With cold immortal hands."

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UNIVERSAL STATES OF FINITE ANTI-PLANE SHEAR: ERICKSEN'S PROBLEM IN MINIATURE

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1. Introduction. A solid body which in its undeformed state occupies a cylindrical region is said to undergo a deformation corresponding to *anti-plane shear* if each particle of the body is displaced parallel to the generators of the cylinder by an amount which is independent of the axial position of the particle. The displacement vector field thus has a nonvanishing component u only in the axial direction, and u is a function of position on a cross-section $\mathfrak{D}$ of the cylinder. Problems involving such deformations are ordinarily simpler than those in which the displacement vector has a more elaborate character, and for this reason they often serve usefully as pilot problems for the analysis of qualitative effects, especially when nonlinearity is involved. (Examples may be found in [1] and in the references cited in [2].) The present paper is intended to illustrate anti-plane shear in its role as exemplar in the setting of finite elasticity theory and with particular reference to an issue which has come to be called *Ericksen's problem*.

Prompted in part by the analytical and experimental work on large deformations of rubber by Rivlin and others, J. L. Ericksen raised the question of determining the class of *all* deformations which are possible according to the theory of finite elastostatics in *every* homogeneous, isotropic elastic material in the absence of body forces. (A presentation of the foundations of the theory of finite elasticity, including definitions of the terms *homogeneous*, *isotropic*, and *body force*, may be found in [3]. All materials considered in the present paper will be assumed to be homogeneous and isotropic.) A universal deformation of this kind can be maintained in each of two bodies identical in shape—but consisting of different elastic materials—solely by the application of suitably distributed forces to the surfaces of each body. (By imposing suitable *body* forces (as distinguished from forces applied to the surface) a given deformation can be brought about in *any* elastic material.) Quite apart from its theoretical interest, Ericksen's problem acquires additional significance because of its connection with the task of determining experimentally the stress-deformation relation characterizing the mechanical behavior of a given elastic material. A very brief account of this aspect of the matter may be found in the prefaces to Ericksen's articles as they appear reprinted in [4]. (See papers 10 and 11 in [4].)

In his first paper [5] on this subject, Ericksen posed the question for *incompressible* elastic materials—those such as rubber which admit only locally volume-preserving deformations. For materials of this kind, Ericksen's problem in its full generality is a rather difficult one, largely because of the complexity of the nonlinear system of partial differential equations involved. Although the problem was very nearly solved in full in [5], there remain some open questions which continue to attract attention. (See, for example, [6] and the references cited there.)

When the constraint of incompressibility is relinquished, the problem is easier and the class of universal deformations (which is smaller than in the incompressible case) was completely determined by Ericksen in [7]. A simpler and somewhat more direct proof of the main result established in [7] for the compressible case was given recently by Shield [8].

The miniature version of Ericksen's problem to be considered here may be roughly posed as follows: What is the most general finite *anti-plane shear* of a cylindrical body which is sustainable without body forces in every homogeneous, isotropic elastic material? Although the question is but a trivial special case of Ericksen's problem, it is analyzed here on an ad hoc basis rather than by appeal to the general results of [5] and [7]. The analysis illuminates clearly the distinction between compressible and incompressible materials and at the same time furnishes

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an entertaining exercise in the elementary theory of analytic functions of a complex variable. The result has alternative interpretations in the theory of minimal surfaces as well as in the subject of steady flows of an inviscid fluid.

2. Finite anti-plane shear. Let $\mathfrak{D}$ be an open, bounded, simply connected cross-section of a cylindrical region occupied by a homogeneous, isotropic elastic solid in its undeformed state. Take x_1, x_2, x_3 to be rectangular cartesian coordinates with the origin in $\mathfrak{D}$ and the x_3-axis parallel to the generators of the cylinder. Suppose the body is deformed to an equilibrium state in which a particle originally located at (x_1, x_2, x_3) moves to the point (x_1, x_2, x_3+u), where $u=u(x_1, x_2)$ is the out-of-plane displacement associated with the anti-plane shear. It is shown in [2] that for such a deformation the field equations of finite elastostatics for an *incompressible* material can be reduced in the absence of body forces to the following system of three differential equations in two unknowns, u and q:

$$\left[F(|\nabla u|^2)u_{,\beta}\right]_{,\beta}=0 \text{ on } \mathfrak{D}, \tag{2.1}$$

$$q_{,\alpha}+\left[G(|\nabla u|^2)u_{,\alpha}u_{,\beta}\right]_{,\beta}=0 \text{ on } \mathfrak{D}. \tag{2.2}$$

(The functions q and u are assumed to be once and twice continuously differentiable on $\mathfrak{D}$, respectively.) In these equations, subscripts take the values 1, 2, a subscript preceded by a comma indicates partial differentiation with respect to the corresponding cartesian coordinate, and repeated subscripts are summed. Thus, for example,

$$|\nabla u|^2=u_{,1}^2+u_{,2}^2=u_{,\beta}u_{,\beta}. \tag{2.3}$$

The functions F and G of $|\nabla u|^2$ which appear in (2.1), (2.2) are given and depend on the particular material under consideration. (F and G are related to the strain energy density—or elastic potential— associated with the material.) The unknown u in (2.1), (2.2) is the out-of-plane displacement introduced above, while q is related to the arbitrary hydrostatic pressure which occurs in the stress-deformation law because of the constraint of incompressibility. It should be emphasized that no linearization is involved in the derivation of (2.1), (2.2).

For a *compressible* elastic material, it is again possible in the case of anti-plane shear to reduce the field equations of the equilibrium theory of finite elasticity (this reduction is carried out in detail in [9]) to a system of the form (2.1), (2.2), except that now q is related directly to u by means of the additional equation

$$q=H(|\nabla u|^2), \tag{2.4}$$

where H is a third given function characteristic of the material at hand.

The functions F, G and H are assumed to be continuously differentiable, and F is required to be positive for all values of its argument.

There is no loss of generality involved in assuming, as will be done here, that

$$u(0, 0)=0, \tag{2.5}$$

since this may always be arranged by adding a constant, if necessary, to any solution u of (2.1), (2.2) or (2.1), (2.2), (2.3). This corresponds to the superposition of a rigid body translation in the axial direction.

In the incompressible case, the basic system (2.1), (2.2) consists of three differential equations in the two unknowns u, q; for compressible materials, the system (2.1), (2.2), (2.4) leads immediately to *three* differential equations for the single unknown u. The relevant set of equations is thus over-determined for either type of material, the more severe over-determination occurring in the compressible case. (This over-determination raises a question as to which, if any, elastic materials will actually sustain nontrivial states of anti-plane shear. This matter is studied in [2] for incompressible materials and in [9] for compressible ones.) To see that there are at least *some* solutions, one notes that in the incompressible case

$$u=k_\beta x_\beta, \qquad q=q_0, \tag{2.6}$$

where k_1, k_2, and q_0 are arbitrary constants, furnishes a solution of (2.1), (2.2) for any choice of

F and G and hence for any incompressible material. An anti-plane shear in which u has the form appearing in (2.5) is called a *simple shear*. Such a deformation subjects each cross-section $\mathfrak{D}$ of the cylinder to a rotation about an axis in the plane of $\mathfrak{D}$. The axis and the angle of rotation are determined by k_1, k_2 and are thus the same for all cross-sections.

For a compressible material, (2.6) also supplies a solution of (2.1), (2.2), (2.4), but now only if q_0 is chosen to be

$$q_0 = H(k_\beta k_\beta), \tag{2.7}$$

so as to assure that (2.4) holds.

3. Universal states. The solution (2.6) furnishes a universal state of anti-plane shear for the incompressible case in the sense that it fulfills (2.1), (2.2) for every choice of F, G and hence for all incompressible materials. The same may be said of the displacement field u given by the first of (2.6) in the compressible case. The present section is devoted to the determination of all such universal states of finite anti-plane shear for each of the two classes of materials.

Suppose $u \not\equiv 0$ is the out-of-plane displacement associated with any such universal state, for either the incompressible or the compressible case. Then u is a universal solution of (2.1) in the sense that the latter equation is satisfied by u for every positive, continuously differentiable function F. If in (2.1) the indicated differentiations are carried out, one finds

$$F(|\nabla u|^2)u_{,\beta\beta} + 2F'(|\nabla u|^2)u_{,\beta\gamma}u_{,\beta}u_{,\gamma} = 0 \quad \text{on } \mathfrak{D}, \tag{3.1}$$

where the prime indicates differentiation. In particular (3.1) must hold for the special choice $F \equiv 1$, so that u must satisfy Laplace's equation:

$$u_{,\beta\beta} = 0 \quad \text{on } \mathfrak{D}. \tag{3.2}$$

(This choice of F corresponds in the incompressible case to the so-called Mooney–Rivlin model for the mechanical behavior of certain rubberlike materials. See chapter 10 of **[10]**.) On the other hand, for any material for which $F'(t) \neq 0$ for all $t \geqslant 0$, (3.1) and (3.2) show that u must also satisfy

$$u_{,\beta\gamma}u_{,\beta}u_{,\gamma} = 0 \quad \text{on } \mathfrak{D}. \tag{3.3}$$

Thus (3.2), (3.3) are necessary if u is to be a universal solution of (2.1). It is clear from (3.1) that they are also sufficient.

In order to determine all functions u which are twice continuously differentiable on $\mathfrak{D}$ and satisfy (3.2) and (3.3), set $z = x_1 + ix_2$ and let $f(z)$ be the function analytic on $\mathfrak{D}$ whose real part is the harmonic function u and for which

$$f(0) = 0. \tag{3.4}$$

(Recall from (2.5) that $u(0, 0) = 0$.) It is easy to show that (3.3) is then equivalent to

$$\operatorname{Re}\left\{[f'(z)]^2\overline{f''(z)}\right\} = 0, \qquad z \text{ in } \mathfrak{D}, \tag{3.5}$$

where the bar denotes complex conjugate and Re stands for real part. It is not possible for $f'(z)$ to vanish identically on $\mathfrak{D}$; if it did, (3.4) would lead to $f \equiv 0$ and hence to $u \equiv 0$. As a result, there are a point z_1 in $\mathfrak{D}$ and a neighborhood $\mathfrak{N}$ of z_1 lying entirely in $\mathfrak{D}$ such that $f'(z) \neq 0$ for z in $\mathfrak{N}$. One may therefore define

$$g(z) = \frac{1}{f'(z)}, \qquad z \text{ in } \mathfrak{N}; \tag{3.6}$$

g is analytic on $\mathfrak{N}$. Direct calculation shows that (3.5) is equivalent to

$$\operatorname{Re}[g'(z)] = 0, \qquad z \text{ in } \mathfrak{N}. \tag{3.7}$$

It follows from the Cauchy-Riemann equations applied to g' that

$$g'(z) = ic, \qquad z \text{ in } \mathfrak{N} \tag{3.8}$$

where c is a real constant.

It is now useful to distinguish two cases: $c = 0$ and $c \neq 0$. Suppose first that $c = 0$. Then (3.8) implies that g is constant on $\mathfrak{N}$, say

$$g(z)=\frac{1}{k}, \qquad z \text{ in } \mathfrak{N}, \qquad k=k_1-ik_2, \tag{3.9}$$

where k_1 and k_2 are real constants. From (3.9), (3.6) one concludes that $f'(z)=k$ for z in $\mathfrak{N}$ and hence, by the identity theorem for analytic functions, that $f'(z)=k$ for all z in $\mathfrak{D}$ as well. Thus

$$f(z)=kz, \qquad z \text{ in } \mathfrak{D}. \tag{3.10}$$

Next, suppose that $c\neq 0$ in (3.8). Then

$$g(z)=ic(z-z_0), \qquad z \text{ in } \mathfrak{N}, \tag{3.11}$$

where z_0 is an arbitrary complex constant. Since, by (3.6) and the analyticity of f, g cannot vanish in $\mathfrak{N}$, one concludes that z_0 lies outside $\mathfrak{N}$. Substitution from (3.11) into (3.6) leads to

$$f(z)=\frac{1}{ic}\log(z-z_0)+d, \qquad z \text{ in } \mathfrak{N}, \tag{3.12}$$

where d is a constant of integration. Here the logarithm is to be single-valued and analytic in the z-plane cut along a ray from z_0 to infinity. Since f must be analytic in all of $\mathfrak{D}$, another argument based on the identity theorem shows that z_0, as well as the cut, must in fact lie outside of $\mathfrak{D}$. (Note that the simple connectivity of $\mathfrak{D}$ plays a role here.) Finally, invoking the normalization (3.4) one has

$$f(z)=\frac{1}{ic}[\log(z-z_0)-\log(-z_0)], \qquad z \text{ in } \mathfrak{D}. \tag{3.13}$$

Collecting the results (3.10) and (3.13) and taking real parts provides two families of universal solutions of (2.1):

$$u=k_\beta x_\beta \tag{3.14}$$

$$u=\frac{1}{c}(\theta-\theta_0), \qquad c\neq 0, \tag{3.15}$$

where

$$\theta=\theta(x_1,x_2)=\arg(z-z_0), \qquad \theta_0=\arg(-z_0). \tag{3.16}$$

The solution (3.14) is the simple shear which appears in (2.6). Equation (3.15), however, describes a more complicated deformation in which the collection of particles which occupies the domain $\mathfrak{D}$ *before* deformation is carried into a portion of a right helicoid of pitch $2\pi/c$ whose axis passes through z_0 and is parallel to the generators of the cylinder. (To see quickly that (3.15) indeed supplies a universal solution of (2.1), one need only write (2.1) in polar coordinates centered at z_0.)

It was observed in the preceding section that the simple shear (3.14) is a universal state of anti-plane shear for either incompressible or compressible materials. To determine the status in this respect of the second universal solution of (2.1), it is necessary to see whether u as given by (3.15) survives when subjected to the additional restriction (2.2) in the incompressible case or to (2.2), (2.4) in the compressible case. To this end, suppose that u is *any* universal solution of (2.1), so that (3.2) and (3.3) hold. It is easy to show that (2.2) may then be written in the form

$$\left[q+K(|\nabla u|^2)\right]_{,\alpha}=0 \quad \text{on } \mathfrak{D}, \tag{3.17}$$

where

$$K(t)=\frac{1}{2}\int_0^t G(\tau)\,d\tau, \qquad t\geqslant 0. \tag{3.18}$$

It follows immediately that any universal solution of (2.1) will supply a universal state of anti-plane shear for incompressible materials, since it is possible to choose $q=-K(|\nabla u|^2)$ and thus to satisfy (2.2). One concludes that $u=u(x_1,x_2)$ corresponds to a universal state of finite anti-plane shear for homogeneous, isotropic, *incompressible* elastic materials if and only if it is either a simple shear (3.14) or a deformation of the form (3.15) which carries a typical cross-section of the cylinder into a portion of a right helicoid.

For compressible materials, on the other hand, a universal solution of (2.1) must also satisfy

(2.4) as well as (3.17) if it is to furnish a universal state of anti-plane shear. This means that u must be such that

$$H(|\nabla u|^2)+K(|\nabla u|^2)=\text{constant on } \mathcal{D} \tag{3.19}$$

for every choice of the functions H and G. Choosing $H(t)\equiv 1$, $G(t)\equiv 1$, corresponding to $K(t)=\frac{1}{2}t$, one finds from (3.19) that $|\nabla u|$ is necessarily constant on $\mathcal{D}$. While this is of course true for u as given by (3.14), it fails when u is of the form (3.15). As a consequence, the only universal state of finite anti-plane shear for homogeneous, isotropic, compressible elastic materials is the simple shear (3.14).

These results are indeed special cases of the universal deformations found by Ericksen in [5] and [7].

4. Other interpretations. The fact that all universal solutions of (2.1) are given by (3.14) and (3.15) has simple interpretations in areas other than finite elastostatics. If, for example, F is chosen to be

$$F(|\nabla u|^2)=(1+|\nabla u|^2)^{-\frac{1}{2}}, \tag{4.1}$$

then (2.1) reduces to the differential equation for minimal surfaces (see [11], pp. 182–187) when the latter are represented in the form $x_3=u(x_1, x_2)$. The results of the preceding section show that the only minimal surfaces for which u is a harmonic function of x_1, x_2 are the plane and the right helicoid.

Still another interpretation of the universal solutions (3.14), (3.15) may be given in terms of fluid flow. The field equations for plane, irrotational, steady flow of an inviscid fluid may be reduced (see [12] chapter 1) to a single equation of the form (2.1), where u is the velocity potential, ∇u the fluid particle velocity, and $F(|\nabla u|^2)$ the fluid density at a particle whose speed is $|\nabla u|$. The special case $F\equiv$constant corresponds to an *incompressible* fluid, for which the velocity potential is a harmonic function. The universal solutions (3.14), (3.15) thus describe the only plane, steady, irrotational flows which are possible for *both* compressible and incompressible inviscid fluids. The flow associated with (3.14) is uniform with a velocity vector whose component in the x_α-direction is k_α. A velocity potential of the form (3.15) corresponds to a vortex flow with constant angular speed $1/c$ about an axis through the point z_0 and perpendicular to the x_1, x_2-plane.

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PROGRESS REPORTS

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It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

Progress Reports is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses, what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal: usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

BOREL DETERMINACY

PETER G. HINMAN

Let $\mathbf{R}$ be the set of all infinite sequences of natural numbers. With any set $A \subset \mathbf{R}$ is associated an infinite game for two players, I and II. Player I begins by choosing a natural number n_0, then II chooses n_1, I chooses n_2, and thus, playing alternately, the players generate a sequence $\{n_0, n_1, n_2, \ldots\}$. Player I wins if that sequence belongs to A, otherwise II wins. A *winning strategy* for I is a function F on finite sequences, such that regardless of what numbers $n_1, n_3, \ldots$ II plays, if I always plays $n_{2i} = F((n_0, n_1, \ldots, n_{2i-1}))$ at his ith turn $(i = 1, 2, 3, \ldots)$, he wins. A winning strategy for II is defined similarly. The set (game) A is *determined* just in case there exists a winning strategy for one of the players. Examples: the set of all unbounded sequences (I can always win); the set of all sequences with only finitely many even terms (II can always win).

The basic question is: which sets are determined? In 1953 Gale and Stewart proved that all open or closed games are determined—the topology here is that generated by sets of the form $\{\alpha : \alpha(0) = m_0, \ldots, \alpha(k) = m_k\}$. This was extended in 1955 to G_δ and F_σ sets, in 1964 to $G_{\delta\sigma}$ and $F_{\sigma\delta}$ sets, and in 1972 to $G_{\delta\sigma\delta}$ and $F_{\sigma\delta\sigma}$ sets. Many made the obvious conjecture—that all Borel sets are determined—but the methods used for these special cases did not seem to go further.

The reason for the difficulty of the problem was discovered before its solution. Everyone knows that all of ordinary mathematics can be formulated and proved in any standard axiomatic system of set theory. In particular, any acceptable proof that all A in some class $\mathcal{C}$ are determined should be formalizable in set theory. It is an empirical fact that *most* of ordinary mathematics does not require the full strength of set theory. The axiom of power sets (**P**) guarantees the existence of $\mathbf{P}(\mathbf{R}), \mathbf{P}^2(\mathbf{R}) = \mathbf{P}(\mathbf{P}(\mathbf{R})), \ldots, \mathbf{P}^n(\mathbf{R}), \ldots$, and the axiom of replacement allows these sets to be collected and the process continued with $\mathbf{P}^\omega(\mathbf{R}) = \mathbf{P}(\cup_{n \in \omega} \mathbf{P}^n(\mathbf{R}))$, etc. Most classical results about $\mathbf{R}$ (or any other structure) require only finitely many levels of this hierarchy for their statement *and* for their proof, and thus they are not dependent on the axiom of replacement. The determinacy of all Borel sets requires only $\mathbf{P}^4(\mathbf{R})$ for its statement, but H. Friedman showed in 1971 that it cannot be proved without use of $\mathbf{P}^\sigma(\mathbf{R})$ for all countable ordinals σ and thus without essential use of replacement.

The determinacy of all Borel sets was finally proved in 1975 by D. A. Martin. A key element in the proof is a method for reducing a Borel game A to a "simpler" game A^*, in which instead of numbers the players play at each turn objects which are something like strategies for A. If A is at level σ in the Borel hierarchy, σ iterations of this method reduce A to a game A^σ, in which the players play sets in $\mathbf{P}^\sigma(\mathbf{R})$. In the appropriate topology A^σ is open, hence determined by the proof of Gale and Stewart, and winning strategies for A^σ translate to winning strategies for A.

In one sense this settles the question. The next natural conjecture would be that all analytic (projections of two-dimensional Borel) sets are determined, but this contradicts Gödel's Axiom of Constructibility, which is known to be consistent, and thus cannot be proved in set theory. It seems very likely, however, that the determinacy of all analytic sets (and even of all projective or more sets) is also consistent with set theory, so that these assertions are natural candidates for new axioms. To many current researchers they seem much more natural than the Axiom of Constructibility, and indeed they lead to very deep and elegant theories of definable sets of reals, which have been intensively developed over the last decade. Interest in this aspect of determinacy began in 1962 with the suggestion of Mycielski and Steinhaus that we adopt as an axiom the determinacy of *all* subsets of $\mathbf{R}$. Although this has also many elegant consequences, it is generally regarded as too radical. For one thing it contradicts the Axiom of Choice—a simple diagonal argument based on a well-ordering of $\mathbf{R}$ yields a non-determined set. Which, if any, of these assertions of determinacy will eventually be seen as true of the intuitive universe of sets remains a question of the mathematical taste of the future.

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MATHEMATICAL NOTES

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DIVISIBILITY OF $\binom{m}{k}$ BY $m(m-1)\cdots(m-h+1)$

HEIKO HARBORTH

For fixed $h\geqslant 1$ we consider the number $S(n)$ of binomial coefficients $\binom{m}{k}$, $0\leqslant k\leqslant m\leqslant n$, which are not divisible by $m(m-1)\cdots(m-h+1)$. It is one purpose of this note to prove

$$\lim_{n\to\infty} S(n)\Big/\binom{n+2}{2}=0, \tag{1}$$

that is, in other words:

THEOREM. *For fixed $h\geqslant 1$ almost all binomial coefficients $\binom{m}{k}$ are divisible by $m(m-1)\cdots(m-h+1)$.*

In a weaker form we get:

COROLLARY. *For fixed $h\geqslant 1$ almost all binomial coefficients $\binom{m}{k}$ are divisible by $\binom{m}{h}$.*

The case $h=1$ was treated already in [4]; however, there are errors in the proof. We are thankful to R. J. Evans who informed us about these errors. So it is the second purpose of this note to correct [4]. This may be done by proving the Theorem above, or by the following changes in [4]:

In (5) add: "..., and $\binom{m}{k}\equiv 0 \pmod t$." The rightmost denominator in (7) should be $it+j$. In the last seven lines of page 36 use t only for prime powers p^α, and write "...there are at most $Z([n/t])$..." In (9) write "$2Z(\cdots)$" instead of "$Z(\cdots)$." [Note that in (9) it suffices to subtract $2Z(\cdots)$ for all $t=p^\alpha$.]

To prove the Theorem above we define

$$(m(m-1)\cdots(m-h+1),k(k-1)\cdots(k-h+1))=(M,K)=P. \tag{2}$$

Then for $h\leqslant k$ we obtain, by a simple argument,

$$\binom{m}{k}=\frac{M}{K}\binom{m-h}{k-h}\equiv 0 \pmod M \quad \text{if and only if} \tag{3}$$

$$\binom{m}{k}\equiv\binom{m-h}{k-h}\equiv 0 \pmod{p^\alpha}$$

for all prime powers p^α with $p^\alpha\| P$.

If a product of h consecutive numbers is divisible by p^α, then at least one of the numbers is divisible by $p^{\alpha-\beta}$ where β is determined by

$$p^\beta\| h! \qquad (\alpha\geqslant\beta). \tag{4}$$

Thus every pair m,k with $p^\alpha | P$ is covered by one of at most h^2 cases (pairs r,s)

$$m=vp^{\alpha-\beta}+r, \qquad k=wp^{\alpha-\beta}+s, \tag{5}$$

$$0\leqslant r,s<\min(h,p^{\alpha-\beta}),\ 0\leqslant w\leqslant v\leqslant n/p^{\alpha-\beta}.$$

Using the abbreviations

$$t=p^{\alpha-\beta},\quad P_1=\prod_{j=1}^{r}(vt+j),\quad P_2=\prod_{j=1}^{t-r+s-1}((v-w+1)t-j),\quad P_3=\prod_{j=1}^{s}(wt+j), \tag{6}$$

we can write

$$\binom{m}{k}=\binom{vt+r}{wt+s}=\binom{v}{w}\frac{P_1P_2}{P_3(t-1)!}\prod_{i=0}^{w-2}\prod_{j=1}^{t-1}\frac{(v-i)t-j}{(w-i)t-j}. \tag{7}$$

By comparing the powers of p which divide numerator and denominator in (7) we conclude

$$\binom{m}{k}\equiv 0 \pmod{p^\alpha} \quad \text{if } \binom{v}{w}\equiv 0 \pmod{p^\alpha}. \tag{8}$$

We obtain (8) also by use of the result in [1] that the exponent of p in $\binom{m}{k}$ is the number of borrows in the p-ary subtraction $m-k$. This is clearly at least the number of borrows in the p-ary subtraction $v-w$. With

$$m-h=vt+r-h=v_1t+r_1, \qquad k-h=wt+s-h=w_1t+s_1, \tag{9}$$

$$v_1\leqslant v, \qquad w_1\leqslant w, \qquad 0\leqslant r_1,s_1<\min(h,p^{\alpha-\beta}),$$

we obtain correspondingly

$$\binom{m-h}{k-h}\equiv 0 \pmod{p^\alpha} \quad \text{if } \binom{v_1}{w_1}\equiv 0 \pmod{p^\alpha}. \tag{10}$$

Now $S(n)$ counts the coefficients $\binom{m}{k}\not\equiv 0 \pmod M$, hence, by (3), the coefficients such that $\binom{m}{k}\not\equiv 0 \pmod{p^\alpha}$ or $\binom{m-h}{k-h}\not\equiv 0 \pmod{p^\alpha}$. From (8), (10), (9), and (5), we get an upper bound for $S(n)$ by taking $2h^2$ times the number of coefficients $\binom{v}{w}\not\equiv 0 \pmod{p^\alpha}$, summed over all p^α with

$$p^\alpha\leqslant P\leqslant n(n-1)\cdots(n-h+1)\leqslant n^h. \tag{11}$$

We denote by $Z_d(l)$ the number of $\binom{m}{k}$, $0 \leqslant k \leqslant m \leqslant l$, which are not divisible by d. Then

$$0 \leqslant S(n)\Big/\binom{n+2}{2} < \frac{2}{n^2} S(n) \leqslant \frac{2}{n^2} \sum_{p^\alpha \leqslant n^h} 2h^2 Z_{p^\alpha}\left(\frac{n}{p^{\alpha-\beta}}\right). \tag{12}$$

Trivially we have $Z_d(l) \leqslant l^2$ for $l \geqslant 4$. We choose an arbitrary $\varepsilon > 0$. From [2] or [3] the existence of $L = L(\varepsilon, d)$ follows in such a way that $Z_d(l) \leqslant \varepsilon l^2$ is valid for all $l > L$. There are also constants $c_1 = c_1(\varepsilon)$ with

$$\sum_{p^\alpha > c_1} p^{-2\alpha} < \varepsilon,$$

and $c_2 = (h!)^2$ with $p^{2\beta} \leqslant c_2$ for any prime p (compare (4)). Let

$$c_3 = c_3(\varepsilon) = \max_{p^\alpha \leqslant c_1} L(\varepsilon, p^\alpha).$$

Now for any $n > c_1 c_3 = c_4(\varepsilon)$, we have that $p^\alpha \leqslant c_1$ implies $n/p^{\alpha-\beta} \geqslant c_3$, so

$$\sum_{p^\alpha \leqslant n^h} Z_{p^\alpha}\left(\frac{n}{p^{\alpha-\beta}}\right) \leqslant \sum_{p^\alpha \leqslant c_1} \varepsilon \frac{n^2}{p^{2(\alpha-\beta)}} + \sum_{p^\alpha > c_1} \frac{n^2}{p^{2(\alpha-\beta)}}$$

$$\leqslant c_2 n^2 \left(\varepsilon \sum_{p^\alpha \leqslant c_1} p^{-2\alpha} + \sum_{p^\alpha > c_1} p^{-2\alpha} \right) \leqslant c_5(h) \varepsilon n^2. \tag{13}$$

This together with (12) proves (1) and the Theorem.

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THE PIXLEY–ROY SPACE IS CONTINUOUSLY SEMIMETRIZABLE

S. A. KENTON AND S. D. SHORE

The class of Moore spaces is interesting, in part because many of the properties of metrizable spaces are valid (or, at least, have analogs) in the larger class of Moore spaces. A result for metrizable spaces which does not have a particularly strong analog for Moore spaces is the Completion Theorem (which states that each metrizable space is a dense subspace of a unique completely metrizable space). In particular, there is a nonmetrizable Moore space that is not a subspace of any complete Moore space (e.g., the Pixley–Roy space described below; see [6]). And worse, when "Moore completions" do exist for a Moore space, there is no assurance that this completion is unique. The reader is referred to [1] for a more complete discussion of the completeness properties of Moore spaces in their relationship to metrizable spaces.

In an attempt to find a stronger analog for the Completion Theorem one might consider the somewhat smaller class of continuously semimetrizable spaces.

A *semimetric* for (the set) X is a symmetric, non-negative, real-valued function d defined on $X \times X$ such that $d(x,y) = 0$ iff $x = y$; d is *continuous* if, in addition, d is "continuous at (a,b)" for any (a,b) in $X \times X$ (i.e., for any $\epsilon > 0$, there is $\delta > 0$ such that $\max\{d(x,a), d(y,b)\} < \delta$ implies that $|d(x,y) - d(a,b)| < \epsilon$). A topological space (X, T) is *continuously semimetrizable* iff there is a

continuous semimetric d for X such that the set of spheres $S_d(x, \alpha)$ determined by d is a base for T (where, of course, $S_d(x,\alpha)=\{y\in X|d(x,y)<\alpha\}$). Note that the set of spheres determined by any continuous semimetric for X is always a base for a topology for X (as is the case for metrics), even though this need not be the case for an arbitrary semimetric (see [3]).

Moore spaces are usually defined in terms of "developments" (see [7]). However, for our discussion it is more useful to observe that a Hausdorff topological space (X,T) is a Moore space iff there is a semimetric d for X which is "continuous at (x,x)" for each $x\in X$ and whose set of spheres is a base for T.

It is known that the class of continuously semimetrizable spaces *properly* contains the class of metrizable spaces and is *properly* contained in the class of Moore spaces (see [2]). Certainly, in some respects, the properties of metrizable spaces generalize more neatly to continuously semimetrizable spaces than to Moore spaces. For example, continuously semimetrizable spaces are always completely regular Hausdorff (although they need not be normal), whereas Moore spaces need only be regular Hausdorff. Less obvious is the property that pseudocompactness and compactness are equivalent for continuously semimetrizable spaces (as they are for metrizable spaces), whereas a pseudocompact Moore space need not be compact.

We now focus on the notions of completeness and completions for continuously semimetrizable spaces. First of all, there are several different notions of "completeness" for these spaces (see [5]). However, if completeness means that every Cauchy sequence converges, then continuously semimetrizable completions, *when they exist*, are unique. (One of the standard proofs for the metric analog uses only the continuity of the metric; see, for example, [4].) One can show, in this context of completeness, that there are continuously semimetrizable completions that are not metric completions. Still, however, it is again not true that every continuously semimetrizable space is the dense subspace (or even the subspace) of any complete continuously semimetrizable space. Indeed, the Pixley–Roy space is continuously semimetrizable, as we will show, and hence does not admit such a completion (since such a completion would be Moore complete).

The Pixley–Roy Space. Let Φ be the set of all finite subsets of the set $\mathbf{R}$ of real numbers. For each $p\in\Phi$ and each real number $\epsilon>0$, let $U(p,\epsilon) = \cup\{(x-\epsilon,x+\epsilon)|x\in p\}$ and $G(p,\epsilon)=\{q\in\Phi|p\subseteq q\subseteq U(p,\epsilon)\}$. It is not difficult to show that $\{G(p,\epsilon)|p\in\Phi,\epsilon>0\}$ is a base (of open-and-closed subsets) for a completely regular Hausdorff topology T_Φ for Φ (see [6]).

We claim that (Φ,T_Φ) is continuously semimetrizable. To this end, define a semimetric d for Φ by

$$d(p,q)=\inf\{r|(p\cup q)\subseteq U(p\cap q,r) \text{ or } r=1\}.$$

To show that d is continuous it suffices to show that, if x_n and y_n are sequences in Φ that converge to p and q, respectively, then $d(x_n,y_n)$ converges to $d(p,q)$.

Case 1. If $p\cap q=\phi$, then there is $k\in\mathbf{N}$ such that, for an integer $m\geqslant k, x_m\cap y_m=\phi$ so that $d(x_n,y_n)$ converges to $1=d(p,q)$.

Case 2. Otherwise, $p\cap q\neq\phi$ so that $d(p,q)=\min\{a,1\}$, where $a=\inf\{r|(p\cup q)\subseteq U(p\cap q,r)\}$.

Consider $\epsilon>0$. There is a real number $\delta>0$ such that $\delta<\epsilon$ and, for any $x,y\in(p\cup q)$, if $x\neq y$, then $(x-\delta,x+\delta)\cap(y-\delta,y+\delta)=\phi$. There is $k\in\mathbf{N}$ such that, for each integer $m\geqslant k, x_m\in G(p,\delta), y_m\in G(q,\delta)$ and both $x_m,y_m\in G(p\cap q,a+\epsilon)$.

Thus $(x_m\cap y_m)\neq\phi$ and

$$\inf\{r|(x_m\cup y_m)\subseteq U(x_m\cap y_m,r)\}<a+\epsilon$$

which completes the proof, if $a\leqslant\epsilon$.

If $a>\epsilon$, then

$$(x_m\cap y_m)\subseteq(U(p,\delta)\cap U(q,\delta))=U(p\cap q,\delta)$$

so that

$$U(x_m\cap y_m,a-\epsilon)\subseteq U(p\cap q,a-\epsilon+\delta);$$

there is

$$z\in(p\cup q)\setminus U(p\cap q,a-\epsilon+\delta)\subseteq(x_m\cup y_m)\setminus U(x_m\cap y_m,a-\epsilon)$$

so that

$$(x_m\cup y_m)\subseteq U(x_m\cap y_m,a-\epsilon)\text{ fails}$$

and

$$a-\epsilon<\inf\{r|(x_m\cup y_m)\subseteq U(x_m\cap y_m,r)\}.$$

Hence, d is a continuous semimetric for (the set) Φ.

Now, for each $p\in\Phi$, there is $\delta>0$ such that, for any $x,y\in p$, if $x\neq y$, then $(x-\delta,x+\delta)\cap(y-\delta,y+\delta)=\phi$. Consequently, for each positive $\epsilon<\delta, S_d(p,\epsilon)=G(p,\epsilon)$ from which it follows that the spheres determined by d form a base for the topology T_Φ.

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RESEARCH PROBLEMS

Edited by Richard Guy

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

WHEN IS THE PERMANENT FUNCTION CONVEX ON THE SET OF DOUBLY STOCHASTIC MATRICES?

Edward T. H. Wang

For an $n\times n$ matrix $A=(a_{ij})$ the permanent of A is defined to be per $A=\sum a_{1\sigma(1)}a_{2\sigma(2)}\cdots a_{n\sigma(n)}$, where the summation extends over all permutations σ of the integers $1,2,\ldots,n$. All matrices in this paper are **doubly stochastic** (d.s.), i.e., they have nonnegative entries and each row sum and column sum is 1. For the past two decades, there has been extensive research on the permanent of d.s. matrices in connection with the well-known van der Waerden conjecture (see [3] and the

references therein) which is still unsettled for $n \geqslant 6$. One natural question that has been asked is whether or not the permanent function is convex on the set of all $n \times n$ d.s. matrices; i.e., is it true that $\operatorname{per}(\alpha A+(1-\alpha)B) \leqslant \alpha \operatorname{per} A+(1-\alpha) \operatorname{per} B$ for all d.s. matrices A, B and for all $\alpha \in [0,1]$? For $n=2$, it can be verified easily that this is true. However, for $n \geqslant 3$ it is not as we see from the following counterexample of Marcus [4]. Let

$$M_1=\frac{1}{15}\begin{bmatrix} 0 & 6 & 9 \\ 8 & 7 & 0 \\ 7 & 2 & 6 \end{bmatrix}, \qquad M_2=\frac{1}{15}\begin{bmatrix} 6 & 0 & 9 \\ 7 & 8 & 0 \\ 2 & 7 & 6 \end{bmatrix}.$$

Then $\operatorname{per}((M_1+M_2)/2) > (\operatorname{per} M_1+\operatorname{per} M_2)/2$. Larger counterexamples are given by the direct sums $M_i \oplus I_{n-3}$, $i=1,2$, where I_{n-3} denotes the identity matrix of order $n-3$. In [2], Brualdi and Newman considered the inequality $\operatorname{per}(\alpha J_n+(1-\alpha)A) \leqslant \alpha \operatorname{per} J_n+(1-\alpha)\operatorname{per} A$ where J_n has all its entries equal to $1/n$, and showed that it did not always hold, e.g., when $A=(3J_3-I_3)/2$.

The first result on the convexity of the permanent function was obtained by Perfect [4] who showed that $\operatorname{per}((I_n+A)/2) \leqslant (1+\operatorname{per} A)/2$. Brualdi and Newman [2] improved this by showing that $\operatorname{per}(\alpha I_n+(1-\alpha)A) \leqslant \alpha+(1-\alpha) \operatorname{per} A$ for all A and for all $\alpha \in [0,1]$. They also investigated the problem of finding other d.s. matrices with this property and proved the following theorem.

Let $B=(b_{ij})$ be fixed. Then $\operatorname{per}(\alpha B+(1-\alpha)A) \leqslant \alpha \operatorname{per} B+(1-\alpha) \operatorname{per} A$ for all A and for all $\alpha \in [0,1]$, if and only if $\Sigma_{i,j} b_{ij} \operatorname{per}(A_{ij}) \leqslant \operatorname{per} B+(n-1) \operatorname{per} A$ for all A with equality exactly if $A=B$, where A_{ij} denotes the $(n-1)\times(n-1)$ matrix obtained by deleting the ith row and jth column of A.

A matrix B is called a **star** if $\operatorname{per}(\alpha B+(1-\alpha)A) \leqslant \alpha \operatorname{per} B+(1-\alpha) \operatorname{per} A$ for all A and for all $\alpha \in [0,1]$.

It follows from this definition that for $n=2$, every d.s. matrix is a star. In general, since the permanent is invariant under permuting rows and columns, we see from the theorem of Brualdi and Newman that all permutation matrices are stars.

Are there any other stars?

We believe not and hence the following conjecture.

CONJECTURE. *For $n \geqslant 3$, B is a star if and only if B is a permutation matrix.*

To prove the "only if" part of this conjecture there are two possible lines of attack. One could assume that B is not a permutation matrix and try to find some A and some $\alpha \in (0,1)$ such that $\operatorname{per}(\alpha B+(1-\alpha)A) > \alpha \operatorname{per} B+(1-\alpha) \operatorname{per} A$. There does not, however, seem to be any clue as to how this A and this α should be chosen. Alternatively, one could choose as many A and as many values of α as necessary to obtain a set of inequalities by applying the definition of a star to force the entries of B''. An easy way to apply this technique is to employ the theorem of Brualdi and Newman, since it does not involve the parameter α. For example, if we choose $A=J_n$, then since $\operatorname{per} A_{ij}=\operatorname{per} A=n!/n^n$ for all i,j, we get $(n!/n^n)\Sigma_{i,j} b_{ij} \leqslant \operatorname{per} B+(n-1)n!/n^n$, or $\operatorname{per} B \geqslant n!/n^n$ with equality if and only if $B=J_n$. This is nothing but the statement of the van der Waerden conjecture. The next theorem shows that we can do slightly better than this.

THEOREM. *If B is a star, then* $\operatorname{per} B \geqslant 1/2^{n-1}$.

Proof. Consider the d.s. matrix $A=(I_n+P_n)/2$ where P_n is the full cycle permutation matrix corresponding to the full cycle $(12\cdots n)$. Then it can be seen easily that $\operatorname{per}(A_{ij})=\operatorname{per} A=1/2^{n-1}$ for all i,j. Therefore, from the theorem of Brualdi and Newman, we get $(1/2^{n-1})\Sigma_{i,j} b_{ij} \leqslant \operatorname{per} B+(n-1)/2^{n-1}$, or $\operatorname{per} B \geqslant 1/2^{n-1}$.

The argument used in the above proof actually shows that if B is a star and if there exists A such that $\operatorname{per} A_{ij}=k$ for all i,j, where k is a constant, then $\operatorname{per} B \geqslant k$. However, matrices with constant permanental minors are difficult to find. For example, it was conjectured by Brualdi and Foregger [1] that if M is an $n \times n$ (0,1)-matrix with constant permanental minors, then $M=nJ_n$ or $M=I_n+Q_n$ where Q_n is any $n \times n$ full cycle permutation matrix.

Acknowledgments. This research was supported by the National Research Council of Canada under grant A-9121 and by a Wilfrid Laurier University Summer Research Fellowship.

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CLASSROOM NOTES

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SYMMETRIC AND STRONG DIFFERENTIATION

C. L. BELNA, M. J. EVANS, AND P. D. HUMKE

Throughout we let f denote a real valued function defined on the real line $\mathbf{R}$. The *lower symmetric derivate of f at $x\in\mathbf{R}$* is given by

$$D_s f(x) \equiv \liminf_{h\to 0}\,[f(x+h)-f(x-h)]/2h,$$

and the *lower strong derivate of f at x* is given by

$$D_* f(x) \equiv \liminf_{\substack{(\xi,\eta)\to(x,x)\\ \xi\neq\eta}}\,[f(\xi)-f(\eta)]/(\xi-\eta).$$

Taking limits superior rather than limits inferior, we obtain the corresponding *upper derivates* $D^s f(x)$ and $D^* f(x)$. If $D_s f(x) = D^s f(x)$, then this common value is called the *symmetric derivative of f at x* and is denoted $f^s(x)$; the *strong derivative $f^*(x)$ of f at x* is defined similarly.

M. Esser and O. Shisha [2] have established certain relationships between the strong derivative and the ordinary derivative. The following theorem is a direct consequence of their work.

THEOREM ES. *If f is differentiable on $\mathbf{R}$, then f is strongly differentiable at all but a first category set of points in $\mathbf{R}$.*

S. N. Mukhopadhyay [4] improved upon this result by showing that the hypothesis of f being differentiable on $\mathbf{R}$ can be weakened to that of f being continuous and symmetrically differentiable on $\mathbf{R}$. The main purpose of this note is to show that the hypothesis of f being continuous on $\mathbf{R}$ can be reduced to that of f having the Baire property. (f has the *Baire property* if there exists a residual subset Σ of $\mathbf{R}$ such that the restriction $f|\Sigma$ of f to Σ is continuous.) That is, we give the following improvement of Theorem ES.

THEOREM 1. *If f has the Baire property and is symmetrically differentiable on $\mathbf{R}$, then f is strongly differentiable at all but a first category set of points in $\mathbf{R}$.*

It is easily seen that Theorem 1 is an immediate consequence of the following theorem, which we shall prove.

THEOREM 2. *If f has the Baire property, then for all but a first category set of points x in $\mathbf{R}$ both of the following equalities hold*:

$$\text{(i) } D_s f(x) = D_* f(x) \quad \textit{and} \quad \text{(ii) } D^s f(x) = D^* f(x).$$

Proof. It suffices to prove (i); the truth of (ii) then follows from applying (i) to the function $-f$.

To begin, we observe that $D_s f(x) \geqslant D_* f(x)$ for each $x \in \mathbf{R}$. Consequently, we need only show that the set $A(f) = \{x : D_s f(x) > D_* f(x)\}$ is of the first category. Furthermore, as $A(f)$ is the countable union of all the sets

$$A(f,\alpha) \equiv \{x : D_s f(x) > \alpha > D_* f(x)\} \qquad (\alpha \text{ rational}),$$

it suffices to show that each $A(f,\alpha)$ is of the first category. Also, since $A(f,\alpha) = A(g,0)$ for $g(x) = f(x) - \alpha x$, it is sufficient to show that $A(f,0)$ is of the first category. Now, for each positive integer n, set

$$S_n = \{x : f(x-h) \leqslant f(x+h) \quad \text{for} \quad 0 < h < 1/n\}.$$

Then $A(f,0)$ clearly equals the countable union of all the sets $A_n(f,0) \equiv S_n \cap A(f,0)$. Thus, it suffices to show that each $A_n(f,0)$ is of the first category.

Suppose to the contrary that $A_n(f,0)$ is of the second category. Then $A_n(f,0)$ is of the second category on every open subinterval of some open interval I. But, according to the ensuing Lemma, this implies that $D_* f(x) \geqslant 0$ for each x in I; that is, $A(f,0) \cap I = \varnothing$. This is a contradiction; hence, by proving the following lemma, we complete the proof of Theorem 2.

LEMMA. *Let f have the Baire property. If S_n is of the second category on every open subinterval of the open interval I, then f is nondecreasing on I.*

Proof. We first observe that it suffices to give the proof under the added assumption that I has length less than $1/n$ and so we make this assumption.

Choose points a and b in I with $a < b$. Let Σ be a residual subset of $\mathbf{R}$ for which $f|\Sigma$ is continuous, and choose a point $c \in \Sigma \cap (a,b)$. Let $\varepsilon > 0$ be given and let J be an open subinterval of (a,b) with $c \in J$ and

$$f(x) > f(c) - \varepsilon \text{ for each } x \in \Sigma \cap J. \tag{*}$$

Now pick any open interval $K \subset (c,b)$ for which the corresponding open interval $K^* \equiv \{x : (x+b)/2 \in K\}$ is a subset of J. Since S_n is of the second category on K and since Σ is residual on K^*, the set of points $x \in \Sigma \cap K^*$ for which $(x+b)/2 \in S_n$ is of the second category and hence nonempty. So pick such an $\hat{x}$ in $\Sigma \cap K^*$. Since $(\hat{x}+b)/2 \in S_n$ we have $f(\hat{x}) \leqslant f(b)$; and by (*) we have $f(c) - \varepsilon < f(\hat{x})$. Because ε is arbitrary, it follows that $f(c) \leqslant f(b)$.

A similar argument shows that $f(a) \leqslant f(c)$; that is, f is nondecreasing on I and the proof is complete.

Although measurable functions need not have the Baire property, it follows from the work of A. Khintchine [3] that a measurable function which is symmetrically differentiable on $\mathbf{R}$ necessarily has the Baire property. This together with Theorem 1 yields the following somewhat surprising result.

THEOREM 3. *If f is measurable and symmetrically differentiable on $\mathbf{R}$, then f is strongly differentiable at all but a first category set of points in $\mathbf{R}$.*

In closing we note that the exceptional set in Theorem ES need not be of measure zero. Indeed, there exists a differentiable function f whose derivative f' is discontinuous almost

everywhere (see [1, p. 27]); while Esser and Shisha [2] have shown that if f is differentiable on $\mathbf{R}$, then $f^*(x)$ exists if and only if f' is continuous at x.

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MATHEMATICAL EDUCATION

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RECURSIVELY DEFINED FUNCTIONS AND THEIR PEDAGOGIC VALUE

DOUG MACKENZIE

One objective of first-year mathematics is to open the students' minds to the gamut of functions and to lead them to discard misconceptions they might hold on the basis of their previous oversimplified experience. In Australia most students come to university having learned to draw graphs of the powers of x and of polynomial, rational, simple algebraic, trigonometric, and exponential functions. They then seem to think that a rational function of these functions is the most complicated function that can exist. The functions they have met are nearly always infinitely differentiable, and they tend to have the idea that a zero derivative is a necessary condition for maxima and minima.

The first step in shocking students away from such misconceptions could be the discussion of saw-tooth functions and functions with finite discontinuities such as $[x]$, the greatest integer function. Then the function

$$s\colon R\to R \quad \text{given by} \quad s(x)=x-[x]$$

might be the first function that they meet which is periodic but not formed from the trigonometric functions. One might also use functions with different specifications over different parts of the domain, such as

$$t\colon R\to R \quad \text{where} \quad \begin{cases} t(x)=4-x^2 & \text{for } x<1 \\ t(x)=x+1 & \text{for } x\geqslant 1 \end{cases}$$

which has a minimum value at 1. Students tend to be a little unhappy about such functions, thinking that they are somehow quite artificial and arbitrary. Indeed, this reaction is so marked that even such counterexamples often fail to convince them.

Some years ago at the University of New South Wales this author introduced into the first-year teaching the consideration of some simple recursively defined functions. They may be considered as even more artificial than the function mentioned above, but they have the advantage of being more complete and satisfying once students have solved the puzzle of

sketching their graphs. We always discussed even functions, odd functions, and periodic functions; these are a useful preliminary to the introduction of other recursively defined functions. The following are four examples chosen from many different types used. Readers might enjoy sketching the graphs.

Example 1.

$$f\colon R \to R \quad \text{where} \quad \begin{cases} f(x) = -x & \text{for } -1 \leqslant x < 1 \\ f(x+2) = 2f(x) & \text{for all } x \end{cases}$$

This is a good example for discussing one-sided limits and integrals across discontinuities such as $\int_{-1}^{4} f(x)\,dx$. It is a case where there are infinitely many maxima but no minima, and no zero derivatives. It is easy to prove that $\lim_{x\to-\infty} f(x) = 0$. This proof focuses on the definition of limit, rather than on derived methods such as those that students use for rational functions.

Example 2.

$$g\colon R \to R \quad \text{where} \quad \begin{cases} g(x) = x^2 + 1 & \text{for } -1 < x \leqslant 1 \\ g(x+2) = \dfrac{1}{g(x)} & \text{for all } x \end{cases}$$

For this function there is a sequence of two maxima, two minima, and so on, as x increases. It is easy to prove that g has period 4. It helps students if they are asked to find the explicit formula for $g(x)$ in $(1,3]$, namely,

$$g(x) = \frac{1}{(x-2)^2 + 1}.$$

Example 3.

$$h\colon R \to R \quad \text{where} \quad \begin{cases} h(x) = x^3 & \text{for } -1 \leqslant x < 1 \\ h(x+2) = h(x) + 2 & \text{for all } x \end{cases}$$

This differentiable "staircase" function has an infinite number of zero derivatives but no maxima or minima. It can be compared with the simpler example $u(x) = x + \sin x$. At the inflections where x is an odd integer, $h''(x)$ does not exist. Proving formally that h is odd requires some care.

Example 4.

$$m\colon R \to (1,3] \quad \text{where} \quad \begin{cases} m(x) = |x| & \text{for } 1 < |x| \leqslant 3 \\ m(3x) = m(x) & \text{for all } x \\ m(0) = 3 \end{cases}$$

This can be compared with $v(x) = \sin(1/x)$ for $x \neq 0$ and helps to make that freak seem less lonely. Both are useful for reference in later years when one considers an essential singularity of a function of a complex variable, although a function like $\tan x$ in place of $|x|$ may then be more appropriate. Note that $\int_0^{3^k} m(x)\,dx$ is easy to evaluate.

Discussion. Student reaction is varied. Some pass-level students find these ideas hard to understand, and a few never master them. Quite a number, however, come to feel a real sense of achievement on obtaining the pattern of the sketch of a recursively defined function. This is an important point when we consider that many students tend to be rather bored with first-year mathematics that covers work they have already met, in however intuitive a fashion, in high school.

From the pedagogic point of view these functions have several advantages. They have interesting properties and also give, in a satisfying way, many counterexamples to what students expect from their previous experience; this helps to realize the goal mentioned in the first

paragraph. One function can display several attributes. The functions can be introduced before the students have a first meeting with the ideas of the calculus, but they are more effective if introduced afterwards.

The use of these functions gives students a graphical experience with a notation they sometimes find difficult: $f(x+\Delta x)$, $f(x+k\pi)$, $f(x-ct)$, $f(x+a)$, $f(-x)$, $f(1/x)$, $f(g(x))$. Recursively defined functions help to make the other work easier by putting it into perspective. In the same way, students become more knowledgeable and more at ease with odd, even, and periodic functions; this is especially useful when Fourier series are discussed. The benefit of the extra attention to the function notation is also felt in many other areas, including the shifting theorems for Laplace transforms, the chain rule for multivariable calculus, and general solutions of partial differential equations. An early exposure to recursively defined functions helps students when they meet recursive situations such as those that arise in work on sequences, logic, and computing.

Recursively defined functions have been used successfully at the University of New South Wales for many years. They are worth considering everywhere.

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THE MATHEMATICS OLYMPICS AT THE UNIVERSITY OF WEST FLORIDA

D. L. SHERRY AND J. R. WEAVER

The University of West Florida is one of nine members of the State University System of Florida. Four institutions in the system, including The University of West Florida, enroll only juniors, seniors, and graduate students. These institutions depend almost entirely on transfer students from the state's community college system. This dependence partially motivates what is described below.

In 1974 the Faculty of Mathematics and Statistics at The University of West Florida developed a mathematics contest for the community colleges. Teams of four students from six Florida community colleges participated in the 1974 Mathematics Olympics. In the following years invitations were extended to all community colleges in the four-state area of Florida, Alabama, Georgia, and Mississippi. Each year the participation increased; in April 1978, there were twenty teams from fifteen community colleges in the Fifth Mathematics Olympics.

The contest is a day-long affair consisting of two parts. In the morning each member of a team takes a two-hour examination on topics from college-level mathematics up to, but not including, multivariable calculus. Initially multivariable calculus was included in the examination, but it was dropped since small schools could not field an entire team which had completed the calculus sequence. The test has 50 questions, so a maximum of 200 points can be scored collectively by the four members of a team.

In the afternoon session the team works together solving problems. A problem is displayed on a screen. The team presenting the first correct answer receives 16 points, while the second and third teams receive 12 and 8 points, respectively. There are 12 problems, each with a time limit of 10 minutes. Once a team submits an incorrect solution, it is disqualified from that problem.

To give the flavor of the afternoon session, two typical problems from that session are presented.

(a) Two cars have wheels of diameters 27 inches and 30 inches, respectively. In what distance in feet would a smaller wheel make 50 more revolutions than a larger one?

(b) During a period of days, it was observed that when it rained in the afternoon it had been clear in the morning, and when it rained in the morning it was clear in the afternoon. It rained on 9 days and was clear on 6 afternoons and 7 mornings. How long was this period?

Both individual and team efforts are honored. The top three individual efforts on the morning examination are singled out. Team scores consist of the total of the individual morning points plus the points won in the afternoon team session. The top three teams are also honored.

There have been many benefits from the Mathematics Olympics. It has stimulated interest in mathematics at the community colleges. Some schools form their teams early and practice solving the type of problems presented in the afternoon session of the Olympics. These problems are not the standard algebra or calculus problems. References [1], [2], and [3] have served as sources for these problems.

It also affords the Faculty of Mathematics and Statistics a chance to get to know their counterparts from the community colleges. An informal articulation meeting between these two groups is held while the students are taking the written examination in the morning.

The students get a chance to see the university campus and meet some of the faculty. The opportunity to encourage these students to attend the university is most welcome.

The enthusiasm of the students is a pleasure to observe, particularly during the afternoon session. A number of teams wanted to know whether there were additional contests which they could enter. One Alabama university is currently making plans to host a contest next year for the Alabama community colleges.

The contest is very rewarding. It is an event that other universities throughout the country might consider sponsoring. The authors would be happy to provide additional information regarding the Olympics.

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PROBLEMS AND SOLUTIONS

Edited by A. P. Hillman

The editors receive more problems than they can use, but they are always on the lookout for interesting and novel problems. The task of selection will be much easier if proposers will keep the following guidelines in mind:

Send all problems (both elementary and advanced) to A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131, in duplicate if possible. The editors urge proposers to include any solutions or information that will help the editors, including reasons why the problem is interesting. Problems in well-known textbooks and results that appear in generally accessible sources are not acceptable.

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution. If you submit a problem without a solution, you should tell the editors whether you know (or somebody else knows) how to solve the problem. If you are uncertain, the problem might be more suitable for the Research Problems section; see the instructions given there.

Proposers are asked to aim for the same audience as for the rest of the MONTHLY: *a rule of thumb is to think of people who have had at least a year of graduate work in mathematics, very likely some years ago. Proposals that can be both understood and solved by high school students, or by those in the first two years of college, are usually more appropriate for the Mathematics Magazine or the Two-Year College Mathematics Journal. No precise line can be drawn between elementary and advanced problems, but in general an elementary problem will involve notions no more difficult than those encountered in the first year of graduate work and should be solvable by methods common at the same level. An advanced problem may deal with more advanced concepts or require more sophisticated techniques for its solution.*

A problem that takes half a page or more to state, or that involves several ad hoc definitions, is usually unsuitable. The statement of a problem should avoid specialized jargon and symbols whenever the English language is more readily understood: for example, "f is a continuous function" is preferable to "$f \in C$."

Before submitting a problem, be sure that it says exactly what you intend it to say. Solvers are not telepathic and will usually solve what was proposed, not what you had in mind.

PROBLEMS DEDICATED TO EMORY P. STARKE

Solutions of the problems in this issue dedicated to Professor Emory P. Starke should be mailed to Prof. A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, N.M. 87131 (USA) before June 30, 1979. To facilitate consideration, solutions should be typed (with double spacing).

S 4. *Proposed by Richard K. Guy, University of Calgary*

For Emory Starke, who welcomed and inspired a wandering amateur mathematician 19 years ago and gave him a copy of the Otto Dunkel Memorial Problem Book.

In order to store a given length L of paper tape in an accessible way, I choose a length, λ, and an even integer, $2n$, so that $2n\lambda = L$. I then screenfold the tape with n "odd" folds in one sense at distances $f_1, f_3, \ldots, f_{2n-1}$ along the tape, and $n-1$ "even" folds, in the other sense, at distances $f_2, f_4, \ldots, f_{2n-2}$. The ends of the tape are $f_0 = 0$ and $f_{2n} = L = 2n\lambda$. I try to arrange that the quantities $f_{i+1} - f_i = \lambda_i$, $0 \leqslant i \leqslant 2n-1$ are each equal to λ, but in practice this rarely happens, so I then endeavor to improve the situation by lining up the ends and the even folds, $f_0, f_2, \ldots, f_{2n}$ and recreasing the odd folds at $f_1', f_3', \ldots, f_{2n-1}'$, so that hopefully better approximations, λ_i' to λ_i are produced, namely $\lambda_i' = \lambda_{i+1}'[=(\lambda_i + \lambda_{i+1})/2]$ for $i = 0, 2, \ldots, 2n-2$. I then line up the odd folds and recrease the even ones, giving $\lambda_{i-1}'' = \lambda_i''[=(\lambda_{i-1}' + \lambda_i')/2]$ for $i = 2, 4, \ldots, 2n-2$. I then repeat the process. Does it terminate, or even converge?

S 5. *Proposed by R. L. Graham, Bell Laboratories, Murray Hill, N.J.*

For a finite set X of integers, let $|X|$ denote the cardinality of X and let $X - X$ denote $\{x - x' : x, x' \in X\}$. Show that if $A, B \subseteq \{1, 2, \ldots, n\}$ with $|A||B| \geqslant 2n-1$ then $(A-A) \cap (B-B)$ contains a positive element. Here $n > 1$.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA (USA) 94303. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (in duplicate, with double spacing) and should be mailed before June 30, 1979. Please enclose a self-addressed label or card (for acknowledgement).

E 2755. *Proposed by Michael Slater, University of Bristol, England*

Let $f \in C^\infty(R)$ and suppose that $f(x) = o(x^n)$ as $x \to \pm\infty$ for some integer $n \geqslant 0$. Show that $f^{(r)}$ has a zero for every $r \geqslant n+1$.

Is this conclusion the best possible?

E 2756. *Proposed by Michael Slater, University of Bristol, England*

Let $f\in C^{\infty}(R)$, $f(0)f'(0)\geqslant 0$, and $f(x)\to 0$ as $x\to\infty$. Show that there exists a sequence $\{x_n\}$ with $0\leqslant x_1<x_2<\cdots$ such that $f^{(n)}(x_n)=0$ for $n=1,2,\ldots$.

E 2757. *Proposed by Harry D. Ruderman, Hunter College Campus School, New York*

Let a, b, c be three lines in $\mathbf{R}^3$. Find points A, B, C on a, b, c, respectively, such that $AB+BC+CA$ is a minimum.

E 2758. *Proposed by Bruce C. Berndt, University of Illinois, Urbana, and Ronald J. Evans, University of California, San Diego.*

Let c and d be relatively prime positive integers of opposite parity and define

$$F(d,c)=\sum_{j=1}^{c-1}(-1)^{j+1+[dj/c]},$$

where $[x]$ denotes the integer with $[x]\leqslant x<[x]+1$. Prove that $F(d,c)+F(c,d)=1$.

E 2759. *Proposed by Hugh L. Montgomery, University of Michigan, Ann Arbor*

Suppose that $a^{-1}\leqslant f''(x)\leqslant 2a^{-1}$ for $0\leqslant x\leqslant a$, where $a\geqslant 8$. Prove that there exists a lattice point (m,n) such that $0\leqslant m\leqslant a$ and $|f(m)-n|\leqslant 2a^{-1/2}$.

E 2760. *Proposed by Kenneth S. Williams, Carleton University, Canada*

Let p be a prime. If $p\equiv 1 \pmod 4$ let a be the unique integer such that

$$p=a^2+b^2,\qquad a\equiv -1 (\mathrm{mod}\, 4),\ b \text{ even}.$$

Prove that

$$\sum_{i=0}^{p-1}\left(\frac{i^3+6i^2+i}{p}\right)=\begin{cases}2\left(\dfrac{2}{p}\right)a, & \text{if } p\equiv 1 (\mathrm{mod}\, 4)\\ 0, & \text{if } p\equiv 3 (\mathrm{mod}\, 4)\end{cases}$$

where $(\frac{n}{p})$ is the familiar Legendre symbol.

SOLUTIONS OF ELEMENTARY PROBLEMS

A Quadrilateral in the Hyperbolic Plane

E 2680 [1977, 738]. *Proposed by Jerrold W. Grossman, Oakland University, Michigan*

Let $ABCD$ be a convex quadrilateral in the hyperbolic plane. Assume that $AD=BC$ and that

$$\angle A+\angle B=\angle C+\angle D.$$

Does $AB=CD$ follow from the above hypotheses? (It does in the Euclidean plane.)

Solution by J. B. Wilker, The Australian National University, Canberra

Clearly $\angle C+\angle D<\pi$ and consequently $(\pi-\angle C)+(\pi-\angle D)>\pi$. This shows that the rays AD and BC do not meet and are not parallel. The same claim is valid for the rays DA and CB. Hence the lines AD and BC are ultraparallel and so they have a unique common perpendicular PQ. The half-turn σ about the midpoint of PQ interchanges the lines AD and BC.

Let $XYX'Y'$ be an equiangular quadrangle invariant under σ ($\sigma(X)=X'$, $\sigma(Y)=Y'$) and which is large enough that $AD\subset XY'$ and $BC\subset YX'$. Since $XY'=YX'$ and $AD=BC$, it follows that $XA+DY'=YB+CX'$. If $XA>CX'$ then $YB>DY'$ and quadrangle $XYBA$ properly contains the σ-image of quadrangle $X'Y'DC$. But this is impossible because angle considerations

show that these quadrangles have the same area. It follows that $XA=CX'$, $YB=DY'$ and σ maps CD onto AB. Thus $AB=CD$.

This result holds also in spherical geometry.

Also solved by Howard Eves, Mark Meyerson, Brian Peterson, Esther Portnoy, and Raymond Spaulding.

An Identity with Binomial Coefficients

E 2681 [1977, 738]. *Proposed by David Burman, Bell Laboratories, Holmdel, New Jersey*

If $x+y=1$ show that

$$\sum_{i=0}^{m-1}\binom{n+i-1}{i}x^iy^n+\sum_{j=0}^{n-1}\binom{m+j-1}{j}x^my^j=1.$$

I. *Solution by F. G. Schmitt, Jr., College of Marin, Kentfield, California*

If x and y denote the probabilities of success and failure, respectively, in Bernoulli trials, then the two sums represent the respective probabilities of the nth failure and mth success before trial $m+n$, events easily seen to be complementary. The extension to values of x and y outside the unit interval is immediate.

II. *Solution by A. A. Jagers, Technische Hogeschool Twente, Enschede, Netherlands*

The coefficients a_i, b_j in the partial fraction expansion

$$\frac{1}{x^my^n}=\sum_{i=0}^{m-1}\frac{a_i}{x^{m-i}}+\sum_{j=0}^{n-1}\frac{b_j}{y^{n-j}} \qquad (y=1-x) \tag{1}$$

are given by

$$a_i=\frac{1}{i!}\left(\frac{d}{dx}\right)^i\frac{1}{(1-x)^n}\bigg|_{x=0}=\binom{n+i-1}{i},$$

$$b_j=\frac{1}{j!}\left(\frac{d}{dy}\right)^j\frac{1}{(1-y)^m}\bigg|_{y=0}=\binom{m+j-1}{j}.$$

The desired result follows after multiplying (1) by x^my^n.

Also solved by 47 other solvers and the proposer.

Determinant of a Cyclic Matrix

E 2683 [1977, 820]. *Proposed by Ira Gessel, M.I.T.*

Let A be the cyclic matrix with $(a_0,a_1,\ldots,a_{p-1})$ as first row, p a prime. If a_i are integers show that $\det A\equiv a_0+a_1+\cdots a_{p-1} \pmod p$.

Solution and a generalization based on that of A. A. Jagers, Technische Hogeschool Twente, Enschede, Netherlands, and several other solvers.

Let W be the cyclic $p\times p$ matrix with first row $(0,1,0,\ldots,0)$. Assume that $a_0,a_1,\ldots,a_{p-1}\in F$ where F is a field of characteristic p. If $f(t)=a_0+a_1t+\cdots+a_{p-1}t^{p-1}$ then $A=f(W)$. Since $W^p=I_p$, the characteristic polynomial of W is $(t-1)^p$. Consequently the characteristic polynomial of A is $(t-f(1))^p$. Hence

$$\det A=f(1)^p=(a_0+\cdots+a_{p-1})^p=a_0^p+\cdots+a_{p-1}^p.$$

For the original problem, take $F=\mathbf{Z}/(p)$ and use that $k^p\equiv k \pmod p$ for $k\in\mathbf{Z}$.

Also solved by D. M. Bloom, California Poly Solution Group, R. J. Evans, Joe Fiedler, N. J. Fine, I. M. Isaacs, P. R. Kuttler, S. C. Locke (Canada), O. P. Lossers (Netherlands), Russell Lyons, L. E. Mattics, Robert Patenaude, John Sadowski, St. Olaf College Problem Group, D. B. Shapiro (Germany), F. B. Strauss, Gillian Valk, and the proposer.

Units of $Z/(n)$ in Arithmetic Progression

E 2684 [1977, 820]. *Proposed by Charles A. Nicol, University of South Carolina*

Let A_n be the set of positive integers which are less than n and relatively prime to n. For which n is A_n an arithmetic progression?

Solution by Graham Lord, Université Laval, Québec

Assume that A_n is an arithmetic progression. If n is odd ($n \geqslant 3$), then $A_n = \{1,2,\ldots,n-1\}$ and so n must be a prime. If n is even but $3 \nmid n$, then $A_n = \{1,3,5,\ldots,n-1\}$ and so n is a power of 2. Finally, if $6|n$ let p be the smallest prime not dividing n. Assume that $p \equiv 1 \pmod 6$. Since 1 and p are the first two elements of A_n and $n-1$ is the last, we have $1+(p-1)k = n-1$ for some k. This gives a contradiction $n \equiv 2 \pmod 6$. Therefore we must have $p \equiv 5 \pmod 6$. But then $3|(2p-1)$ and so $2p-1 \notin A_n$. Consequently $A_n = \{1,p\}$, $\phi(n) = 2$ and $n = 6$.

Hence A_n is an arithmetic progression if and only if n is six, a prime, or a power of two.

Also solved by 49 other readers.

A Congruence for a Sum of Binomial Coefficients

E 2685 [1977, 820]. *Proposed by Ronald Evans, La Jolla, California*

If p is an odd prime, show that

$$\sum_{i=0}^{p-1} (-1)^i \binom{p^2-p}{pi} \equiv p^{p-1} \qquad (\operatorname{mod} p^p).$$

Solution by H. F. Mattson, Jr., Syracuse University.

We use the binomial expansion

$$(1-x)^{p^2-p} = \sum_{k \geqslant 0} (-1)^k \binom{p^2-p}{k} x^k.$$

Replacing x by $1, z, z^2, \ldots, z^{p-1}$, where $z = \exp(2\pi i/p)$, and adding, we obtain, since p is odd,

$$pS = \sum_{n=1}^{p-1} (1-z^n)^{p^2-p},$$

S being the sum in the problem.

For simplicity we denote $1-z^n$ by a_n. The statement of the problem is equivalent to

$$\frac{1}{p^p} \sum_{n=1}^{p-1} a_n^{p(p-1)} \equiv 1 \qquad (\operatorname{mod} p).$$

Let $A = \mathbf{Z}[z]$ be the ring of algebraic integers of the cyclotomic field $\mathbf{Q}(z)$. Since $p = a_1 a_2 \cdots a_{p-1}$, and since a_m/a_n are units of A for all m,n we find that $a_n^{p-1}/p = u$ is a unit in A for all n. Therefore $u^p \equiv u \pmod P$ where $P = (1-z)A$ is the prime ideal of A containing p. Thus

$$p^{-p+1} S \equiv \sum_{n=1}^{p-1} \frac{1}{p} a_n^{p-1} \qquad (\operatorname{mod} P).$$

The binomial expansion of $(1-x)^{p-1}$, which has the form $1 + b_1 x + \cdots + b_{p-1} x^{p-1}$, takes the

value p when summed as x runs through $1, z, z^2, \ldots, z^{p-1}$. Thus

$$p^{-p+1}S \equiv \sum_{n=0}^{p-1} \frac{1}{p}(1-z^n)^{p-1} \equiv 1 \qquad (\operatorname{mod} P).$$

Since $p^{-p+1}S$ is a rational number, it follows that $S/p^{p-1} \equiv 1 \pmod{p}$.

Also solved by L. Carlitz, N. J. Fine, L. E. Mattics, N. Miku (Netherlands), and the proposer.

LCM of Binomial Coefficients

E 2686 [1977, 820]. *Proposed by Peter L. Montgomery, Huntsville, Alabama*

Show that

$$(n+1) \operatorname*{LCM}_{0 \leqslant k \leqslant n} \left\{ \binom{n}{k} \right\} = \operatorname{LCM}\{1, 2, \ldots, n+1\}.$$

Solution by Robert Breusch, Amherst College. Let p be a prime $\leqslant n+1$ and let α (resp. β) be the multiplicity of p in the prime factorization of the left-hand member (resp. right-hand member) of the above equality. Choose r so that $p^r \leqslant n+1 < p^{r+1}$. Then clearly $\beta = r$.

We claim that

$$\text{if } p^r \leqslant m < p^{r+1} \text{ then } p^{r+1} \nmid \binom{m}{k} \text{ for } 0 \leqslant k \leqslant m. \tag{*}$$

Indeed the multiplicity of p in $\binom{m}{k}$ is

$$\gamma = \sum_{s=1}^{r} \left\{ \left[\frac{m}{p^s} \right] - \left[\frac{k}{p^s} \right] - \left[\frac{m-k}{p^s} \right] \right\}.$$

Since each summand in this sum is 0 or 1, we have $\gamma \leqslant r$, i.e., (*) holds. For $0 \leqslant k \leqslant n$ we have

$$a_k = (n+1)\binom{n}{k} = (n-k+1)\binom{n+1}{k} = (k+1)\binom{n+1}{k+1}.$$

By (*), p^{r+1} does not divide any of the numbers $\binom{n}{k}$, $\binom{n+1}{k}$, $\binom{n+1}{k+1}$. Thus p^{r+1} could divide a_k only if p divides each of the numbers $n+1$, $n-k+1$, $k+1$. This implies that p divides $(n+1)-(n-k+1)-(k+1) = -1$, a contradiction. Therefore $p^{r+1} \nmid a_k$.

On the other hand, for $k = p^r - 1$ we have $k \leqslant n$ and $a_k = (k+1)\binom{n+1}{k+1}$ is divisible by p^r. Therefore $\beta = r = \alpha$. This establishes the equality stated in the problem.

Also solved by Michael Ecker, Ronald Evans, C. T. Giel, M. G. Greening (Australia), S. C. Locke (Canada), O. P. Lossers (Netherlands), L. E. Mattics, David Singmaster (England), Gillian Valk, and the proposer.

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to Professor R. C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109. Solutions of Advanced Problems in this issue should be typed (with double spacing) on separate sheets and should be mailed before June 30, 1979.

6252. *Proposed by Ioan Tomescu, University of Bucharest, Romania*

Let $f(n) = \sum_{i=1}^{n} \sum_{j=1}^{n} \binom{n}{i}\binom{n}{j} i^{n-j} j^{n-i}$ and show that

$$\lim_{n \to \infty} \frac{[f(n)]^{1/2n} \ln n}{n} = \frac{1}{e}.$$

6253. *Proposed by Maurice Machover, St. John's University, Jamaica, N. Y.*

If $0 \leqslant \theta_1 < \theta_2 < \theta_3 < \cdots < \theta_n < 2\pi$, are the functions

$$\exp[i\cos(\theta - \theta_j)] \quad (j = 1, 2, \ldots, n)$$

linearly independent over the complex numbers?

6254. *Proposed by Thomas E. Elsner, General Motors Institute, Flint, MI.*

For real numbers r_{ij} with $0 \leqslant r_{ij} \leqslant 1$ for $i = 1, 2, \ldots, m$ and $j = 1, 2, \ldots, n$ prove that

$$1 - \prod_{j=1}^{n}\left(1 - \prod_{i=1}^{m} r_{ij}\right) \leqslant \prod_{i=1}^{m}\left[1 - \prod_{j=1}^{n}(1 - r_{ij})\right].$$

6255. *Proposed by Adam Riese, Wright State University, Dayton, O.*

Let $f: R \to R$ be a function whose graph, considered as a subset of R^2, is both closed and connected. Prove f is continuous. What can be said when $f: R^m \to R^n$?

6256. *Proposed by A. Kussmaul and P. E. Kopp, University of Hull, England*

Prove or disprove the assertion in a recent text that every countably additive real-valued set function on a ring R of sets is of bounded variation. Is the assertion true if R is an algebra of sets?

6257. *Proposed by Jan Mycielski, University of Colorado, Boulder.*

Give a proof of the following theorem, announced without proof by S. Banach in the Annales de la Société Polonaise de Mathématique during the 1920's:

Let X be the space of continuous nondecreasing functions f: $[0,1] \to [0,1]$ having $f(0) = 0$ and $f(1) = 1$ and with the distance function $d(f,g) = \max|f(x) - g(x)|$ over $0 \leqslant x \leqslant 1$. Let Y be the subset of all f in X such that f is strictly increasing and the length of f is 2. Prove that $X - Y$ is meager in X. (Here the length of f is the supremum of the lengths of piecewise linear functions with finitely many vertices which agree with f on all vertices.)

SOLUTIONS OF ADVANCED PROBLEMS

Integral $n/\pi(n)$

6153 [1977, 392]. *Proposed by Bernardo Mz.-Recamán, University of Warwick, England*

Let $\pi(x)$ denote the number of primes $\leqslant x$. Are there infinitely many integers, such as 2, 4, 6, 8, 30, 33, 100, with the property that $\pi(n)$ divides n?

Solution by Jordan I. Levy, undergraduate, University of Delaware. The answer is "yes." Let $f(x) = x/\pi(x), x \geqslant 2$. Then $f(x)$ has discontinuities at and only at odd primes, is strictly increasing on each interval $[p, q)$ determined by successive primes, has "negative jumps" at odd primes, and is unbounded above (by virtue of the prime number theorem). It follows from these properties and $f(4) = 2$ that for every integer $k \geqslant 2$ there is an x_k with $f(x_k) = k$. Clearly, $x_k = k\pi(x_k)$ is an integer and the x_k form an infinite set of distinct integers n such that $\pi(n)|n$.

Also solved by Robert Breusch, Paul Erdös, J. C. Lagarias, A. Makowski (Poland), L.E. Mattics, Michael Mays, Mark Merriman, A.A. Mullin, Carl Pomerance, S. Porubský & Š. Znám (Czechoslovakia), Steve Ricci, Blair Spearman, and E.G. Strauss.

Notes. (1) Breusch, Lagarias, Makowski, Porubský & Znám, and Strauss show that the argument given proves the analogous result if the primes are replaced by any sequence of integers of zero asymptotic density.

(2) Solomon Golomb and also H. J. Ricardo point out that the result may be found in Golomb's paper "On the ratio of N to $\pi(N)$," this MONTHLY, 69 (1962) 36–37.

(3) Another example is $x_6=1008$ since $\pi(1008)=168$.

Iterating Reflections and Integrations

6154 [1977, 392]. *Proposed by Richard Stanley, Massachusetts Institute of Technology*

Define a sequence of polynomials (with rational coefficients) as follows: $p_0(x)=1, p_n(0)=0$ if $n>0$, and $p'_{n+1}=p_n(1-x)$ if $n>0$. Thus $p_1(x)=x, p_2(x)=x-\frac{1}{2}x^2, p_3(x)=\frac{1}{2}x-\frac{1}{6}x^3$, etc. Find $p_n(x)$. In particular, what is $p_n(1)$?

Solution by W. O. Egerland, Ballistic Research Laboratory, Aberdeen, Md. The example $p_1(x)=x$ shows that the third defining condition should be $p'_{n+1}(x)=p_n(1-x)$ for $n\geqslant 0$. Then the given conditions imply that $p''_{n+2}+p_n=0$ and that the generating function $G(x,t)=\Sigma_{n=0}^{\infty} p_n(x)t^n$ satisfies $G_{xx}+t^2G=0$ subject to $G(0,t)=1$ and $G_x(1,t)=t$. Hence

$$G(x,t)=\frac{\cos(1-x)t+\sin xt}{\cos t}$$

so that

$$p_{2n}(x)=(-1)^n\sum_{k=0}^{n}\frac{E_{2n-2k}}{(2k)!(2n-2k)!}(1-x)^{2k}, \qquad p_{2n}(1)=(-1)^n\frac{E_{2n}}{(2n)!},$$

$$p_{2n+1}(x)=(-1)^n\sum_{k=0}^{n}\frac{E_{2n-2k}}{(2k+1)!(2n-2k)!}x^{2k+1},$$

$$p_{2n+1}(1)=(-1)^n\frac{2^{2n+2}(2^{2n+2}-1)}{(2n+2)!}B_{2n+2},$$

where the E_{2n} and B_{2n} are the Euler and Bernoulli numbers, respectively.

Also solved by Theodore Bolis, P. S. Bruckman, Leonard Carlitz, Emeric Deutsch, Irving Gerst, Clark Givens, A. W. Johnson, Jr., J. C. Lagarias, J. I. Levy, O. P. Lossers (Netherlands), J. G. Mauldon, O. G. Ruehr, R. E. Stone, L. E. Ward, Sr., J. S. White, and the proposer.

Expectation of the Width of a Set

6155 [1977, 392]. *Proposed by Milton P. Eisner, Ball State University*

Let $\{x_1,x_2,\cdots,x_k\}$ be a set of numbers. Define the *width* of the set to be $\min_{i\neq j}\{|x_i-x_j|\}$. Suppose the k numbers are selected at random from the set $\{1,2,\cdots,n\}$. Find the expected value of the width of the resulting set if the numbers are chosen without replacement.

Solution by Richard Staum, City University of New York. The width is unchanged if the x_i are renumbered so that $x_1<x_2<\cdots<x_k$. There are C_k^n such subsets $\{x_1,\ldots,x_k\}$ in $\{1,\ldots,n\}$ and the width equals $\min\{x_{i+1}-x_i\}$ over $1\leqslant i\leqslant k-1$. The maximum possible width p of such a subset is the largest integer p satisfying $(k-1)p\leqslant n-1$.

We now demonstrate a one-to-one correspondence between the strictly increasing sequences $\{x_1,\ldots,x_k\}$ in $\{1,\ldots,n\}$ whose width exceeds 1, and the strictly increasing sequences $\{a_1,\ldots,a_k\}$ in $\{1,\ldots,n-k+1\}$; this correspondence is given by $a_i=x_i-i+1$ $(i=1,\ldots,k)$. Thus, the number of strictly increasing sequences in $\{1,\ldots,n\}$ whose width exceeds 1 is C_k^{n-k+1}; and hence, the number whose width equals 1 is $C_k^n-C_k^{n-k+1}$.

Similarly, for every integer j from 1 through $p-1$, there is a one-to-one correspondence between the strictly increasing sequences $\{x_1,\ldots,x_k\}$ in $\{1,\ldots,n\}$ whose width exceeds j and the strictly increasing sequences $\{a_1,\ldots,a_k\}$ in $\{1,\ldots,n-j(k-1)\}$; this correspondence is given by $a_i=x_i-j(i-1)$ $(i=1,\ldots,k)$. Thus, the number of strictly increasing sequences in $\{1,\ldots,n\}$

whose width exceeds j is $C_k^{n-j(k-1)}$; and hence, the number whose width equals j is $C_k^{n-(j-1)(k-1)} - C_k^{n-j(k-1)}$. For $j=p$, the number of sequences whose width equals j is $C_k^{n-(j-1)(k-1)}$.

Therefore, the expected value of the width of such a sequence is

$$[C_k^n]^{-1}\sum_{j=1}^{p} j[C_k^{n-(j-1)(k-1)} - C_k^{n-j(k-1)}],$$

it being understood that the second term in the summation equals zero when $j=p$. Simplifying, we obtain:

$$[C_k^n]^{-1}[C_k^n + C_k^{n-(k-1)} + C_k^{n-2(k-1)} + \cdots + C_k^{n-(p-1)(k-1)}].$$

Also solved by Theodore Bolis, Marguerite Gerstell, J. I. Levy, L. E. Mattics, Michael Skalsky, and Lajos Takács.

Note. Bolis in his solution develops a method for estimating the formula derived for the expectation. With n large the expectation for $k=2$ is $\omega(n,2)\backsim(n+1)/3$ and for $k=3$ is $\omega(n,3)\backsim n(n+2)/8(n-1)$ for n even and $\omega(n,3)\backsim(n+1)(n^2-3)/8n(n-2)$ for n odd.

Quadratic Residues

6156 [1977, 491]. *Proposed by Herbert Knothe, Bremen, Germany*

Prove: If a prime p has the form $8n+7$ then the number of even quadratic residues $>p/2$ equals $n+1$. If a prime p has the form $8n+3$ then the number of even quadratic residues $<p/2$ equals n. (Each residue r is restricted so that $0\leqslant r<p-1$.)

Solution by Emma Lehmer, Berkeley, California. If $p=8n+7$ the number of even quadratic residues between $p/2$ and p is equal to the total number of residues between $p/4$ and $p/2$, since 2 is a residue. As there are altogether $2n+2$ numbers in this range, the problem states that there are as many residues as non-residues in the second quarter for primes of the form $8n+7$. Similarly for primes $p=8n+3$ the number of even quadratic residues less than $p/2$ is equal to the total number of non-residues less than $p/4$, since in this case 2 is a non-residue. As there are $2n$ numbers less than $p/4$, the problem again asserts that there are as many residues as non-residues in the first quarter for primes of the form $8n+3$. Hence the problem can be restated as follows: Show that

$$\sum_{a=1}^{(p-3)/4}\left(\frac{a}{p}\right)\equiv 0(\mathrm{mod}\,p), \text{ if } p=8n+3;$$

$$\sum_{a=(p+1)/4}^{(p-1)/2}\left(\frac{a}{p}\right)\equiv 0(\mathrm{mod}\,p), \text{ if } p=8n+7.$$

The problem can now be generalized as follows: Show that

$$\sum_{a=1}^{(p-3)/4}\left(\frac{a}{p}\right)\equiv\begin{cases} 0(\mathrm{mod}\,p), & \text{if } p=8n+3\\ -2B_{(p+1)/2}(\mathrm{mod}\,p), & \text{if } p=8n+7\end{cases}$$

and

$$\sum_{a=(p+1)/4}^{(p-1)/2}\left(\frac{a}{p}\right)\equiv\begin{cases} -6B_{(p+1)/2}(\mathrm{mod}\,p), & \text{if } p=8n+3\\ 0(\mathrm{mod}\,p), & \text{if } p=8n+7\end{cases}$$

where B_k is the kth Bernoulli number. $(B_1=1/2, B_2=1/6, B_3=0,\cdots.)$

Proof. If $B_k(x)$ is the Bernoulli polynomial, then

$$\sum_{a=1}^{m} a^{k-1}=[B_k(m+1)-B_k]/k \qquad (*)$$

and

$$B_k(1/2)=(1-2^{k-1})B_k/2^{k-1}$$
$$B_k(1/4)=(1-2^{k-1})B_k/2^{2k-1} \qquad (k \text{ even}).$$

Letting $k=(p+1)/2$, we have

$$a^{k-1}\equiv\left(\frac{a}{p}\right)(\mathrm{mod}\,p).$$

Taking $m=(p-1)/2$ and $m=(p-3)/4$, we obtain

$$B_k[(p+1)/2]\equiv B_k(1/2)\equiv\left[\left(\frac{2}{p}\right)-1\right]B_k(\mathrm{mod}\,p)$$

and

$$B_k[(p+1)/4]\equiv B_k(1/4)\equiv\left[1-\left(\frac{2}{p}\right)\right]B_k/2(\mathrm{mod}\,p).$$

Substituting these into (*) we obtain

$$\sum_{a=1}^{(p-1)/2}\left(\frac{a}{p}\right)\equiv 2\left[\left(\frac{2}{p}\right)-2\right]B_{(p+1)/2}(\mathrm{mod}\,p)$$
$$\sum_{a=1}^{(p-3)/4}\left(\frac{a}{p}\right)\equiv -\left[\left(\frac{2}{p}\right)+1\right]B_{(p+1)/2}(\mathrm{mod}\,p).$$

Subtracting, we obtain

$$\sum_{a=(p+1)/4}^{(p-1)/2}\left(\frac{a}{p}\right)\equiv 3\left[\left(\frac{2}{p}\right)-1\right]B_{(p+1)/2}(\mathrm{mod}\,p).$$

The last two congruences are equivalent to the generalized statement of the problem.

Also solved by Robert Breusch, L. Carlitz, S. Chowla & J. & M. Cowles, Ronald Evans, J.S. Frame, M.G. Greening (Australia), O.P. Lossers (Netherlands), R.L. McFarland, L.E. Mattics, N. Miku (Netherlands), R.E. Shafer, Lawrence Somer, Blair Spearman, and the proposer.

The Maximum Number of Edges in a Graph Without Triangles

6159. [1977, 491]. *Proposed by Thomas E. Elsner, General Motors Institute*

It is well known that for a graph on k vertices with no triangles the maximum number of edges is $L(k)=mn$ where $m=[k/2]$ and $n=[(k+1)/2]$ and that this value occurs for the complete bigraph $K_{m,n}$. Express the maximum number of edges in case we add the restriction that the graph be (a) Hamiltonian; (b) Eulerian.

Solution by Allen J. Schwenk, U.S. Naval Academy

(a) If $k=2n$ is even, clearly $K_{n,n}$ remains the unique graph with as many as n^2 edges and no triangles. Consider the case $k=2n+1$. Let G be a Hamiltonian graph with no triangles and the maximum number of edges. Since G is known to have one odd cycle, it must have a shortest odd cycle, say C, of length $2s+1$ with $s\geqslant 2$. There can be no additional edges forming diagonals in C without creating a shorter odd cycle. Each of the $2n-2s$ vertices outside of C can be joined to at most 2 vertices in C, for any choice of 3 or more must form a shorter odd cycle. Finally, as noted in the background to this problem, these $2n-2s$ vertices can induce at most $(n-s)^2$ edges without forming any triangles. Thus, G has at most

$$(2s+1)+(2n-2s)(2)+(n-s)^2$$

edges. The largest possible value occurs when $s=2$ and is n^2+1. Many Hamiltonian graphs

attain this bound. One of then, H_k, is constructed from $K_{n,n}$ by inserting a vertex of degree 2 on any one edge.

(b) The solution depends on the congruence class modulo 4. If $k=4m$, then $K_{2m,2m}$ is Eulerian and gives the maximum number of edges. If $k=4m+2$, the graph $K_{2m+1,2m+1}$ has odd degrees, but $K_{2m,2m+2}$ has all even degrees and only 1 fewer edge.

No other Eulerian bigraph has as many edges; and a triangle-free nonbigraph, by an argument similar to the one in part (a), has at most $4m^2+2m+1<4m^2+4m$ edges. Thus, the solution graph is unique.

If $k=4m+1$, the approach used in (a) guarantees that a nonbigraph has at most $4m^2+1$ edges. Graph H is Eulerian and attains the bound. If the maximum could possibly be attained by a bigraph, its vertices would be partitioned into sets of size $2a$ and $2b+1$. The vertices in the even set have even degree at most $2b$, and so the number of edges $\leqslant(2a)(2b)\leqslant 4m^2$. The graph H_k is just one of many Eulerian nonbigraphs attaining the maximum of $4m^2+1$ edges.

If $k=4m+3$, the nonbigraph bound is $4m^2+4m+2$, but the construction that attains this bound must give at least $4m-2$ vertices of odd degree. However, the graph J_k obtained by bisecting any edge of $K_{2m,2m+2}$ is Eulerian with $4m^2+4m+1$ edges.

Similarly to the preceding case, we find that an Eulerian bigraph has at most $(2a)(2b)\leqslant 4m^2+4m$ edges. Graph J_k is just one of many Eulerian nonbigraphs attaining the maximum.

Also solved by Ira Gessel.

A Characterization of Irrationals by Distribution of Residues

6161* [1977, 491]. *Proposed by Clark Kimberling, University of Evansville.*

For $0<r<1$, let $S(r)$ be the set of integers n such that one and only one integer lies in the open interval $(nr, nr+r)$. Prove or disprove that r is irrational if and only *if*, for every positive integer M, the set $S(r)$ contains a complete residue system modulo M.

Solution by W. C. Waterhouse, Pennsylvania State University. If r is rational with denominator M, there is no k in the interval for $n\equiv 0$ (modulo M). Let r be irrational, and choose arbitrary integers M and m with $0\leqslant m<M$. Then the multiples of $1/r$ are dense in the reals modulo M, whence some multiple k/r has its image in the image of the interval $(m, m+1)$. Then, for some integer q, we have $qM+m<k/r<qM+m+1$. Setting $n=qM+m$, this gives $n\equiv m$ (modulo M) and $nr<k<nr+r$.

Also solved by Mangho Ahuja, Kenneth L. Bernstein, John Bryant & Robert Gilmer, Michael W. Ecker, Kwang-Chul Ha, L. Kuipers (Switzerland), Joel Levy, Jordan I. Levy, L. E. Mattics, Jerry Metzger, Lewis Pakula, and Stefan Purobský (Czechoslovakia).

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR., AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf and Carleton Colleges
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Man and His Technology. By the Engineering Concepts Curriculum Project. McGraw-Hill, New York, 1973. xvi + 332 pp. Paperback. $12.50

The Engineering Concepts Curriculum Project, a group funded mostly by the National Science Foundation, at the State University of New York at Stony Brook, has produced books intended for "technology appreciation" courses for liberal arts students. Most of *Man and His Technology*, however, can be best described as a course in "applied mathematics appreciation." Here are some of the topics covered: Decision Making, which includes algorithms, optimization, and dynamic programming; Optimization, i.e., probability, queueing, games, and linear programming; Modeling, using graphs, population models, and exponential growth; Feedback, with goal-seeking, self-regulation, and instability; and Stability, with the epidemic model, and the law of supply and demand. There are nine chapters, usually of eight sections each.

Many topics are approached with much more motivation, and in somewhat less mathematical detail, than the same topics might be given in a college algebra or mathematics for business decisions text. On the other hand, the treatment of some of these topics—graphing inequalities, exponential functions, linear programming, two-person games—is probably sufficient for the level of mastery nontechnical students typically are expected to achieve in such courses; and the added motivation and selection of additional topics, such as seven bridges, queueing, systems with feedback, is very attractive. The writing style is often more like that of a history or psychology text than that of a standard mathematics text. Indeed, the exercises at the chapter ends are in that tradition: "Discuss the differences between dynamic and static models." "What examples of queue formation in your school can you think of? . . . Is it a single-service queue or a multi-service queue?" "Draw a block diagram which describes the feedback during a political campaign." There are very few computational exercises, and none of the repetitive computational drill usual in mathematics texts.

This book would lend itself very well to an "applied mathematics appreciation" course for nontechnical students. It could be used without supplementary material for a semester course, if the instructor was content to place little emphasis on computational work, for example by giving essay tests. A somewhat more traditional course would require the instructor to provide supplementary examples and exercises and could cover a few chapters in one semester or much of the book in two; such a course would overlap considerably, but not completely, the usual courses in college algebra or mathematics for business-decisions courses. In particular, unlike most traditional mathematics texts, the book is highly appropriate for interdisciplinary courses and team-taught courses crossing departmental lines.

A predecessor volume, *The Man-Made World* (of which *Man and His Technology* is in large part a reprinting) has been used as the principal text in a year-long, six semester hour, team-taught course at Memphis State University. The course contained roughly two to three semester hours of college algebra and some other mathematical material. The students, who were well motivated but had weak mathematical backgrounds, received the book much better than they would have a traditional mathematics textbook.

EDWARD T. ORDMAN, New England College

A First Course in Probability. By Sheldon Ross. Macmillan, New York, 1976. x + 305 pp. $14.95. (Telegraphic Review, May 1976.)

This book was used in a one-semester probability course at the junior-senior level for students whose minimum background was a year of calculus and our sophomore sequence of linear algebra and differential equations. We covered the first eight chapters at a comfortable pace, treating, in order, counting principles, axioms of probability, conditional probability, discrete random variables, continuous random variables, joint random variables, expectation, and limit theorems.

I prefer a problem-solving approach to probability at the post-calculus level and selected

Ross's text because of its impressive and far-reaching collection of examples and exercises. I was not disappointed; indeed, it is safe to say that homework assignments took over the course! In all, I assigned and collected nearly 200 problems and graded over half of them. Homework accounted for almost 50 percent of the course grade, a much higher percentage than in any other undergraduate course I've taught.

My students found Ross's problems challenging. This may be understated; I received complaints throughout the semester about how time-consuming the homework assignments were. (With so many attractive problems available, I had difficulty cutting the assignments to reasonable size. Perhaps I didn't always succeed.) Still, the expected high quality of the exercises was confirmed, and students found them intriguing if difficult. The problems offer a blend of theory and applications and amply illustrate basic probability concepts and techniques. Further, they provide an exposure to counting arguments and a substantial review of calculus (notably, single and multiple integration with a few infinite series).

Two additional strong points of the text should be mentioned. First, Ross includes more material on conditional probability than comparable books. For instance, several problems ask for computation of a probability or expectation by conditioning; one first obtains a recursion formula and proceeds from there to prove a general formula (often difficult to derive using other methods) by mathematical induction. Second, the discussion of limit theorems is thorough and effective. Following the book, I presented detailed proofs of the Central Limit Theorem and the Strong Law of Large Numbers as the capstone of the semester's work. This unit, plus a few key words I added to the text discussion at appropriate points (e.g., Borel field, probability measure, Lebesgue integral), formed a preview of mathematical probability at the graduate level.

Ross does not have a chapter on Markov chains or stochastic processes, a drawback given the linear algebra background of my class. Yet both students and instructor were kept quite busy without including such topics.

In summary, I strongly recommend this text for consideration in a problem-oriented probability course for students with a calculus background.

CHARLES JEPSEN, Grinnell College

MISCELLANEA

20. St. Augustine and "mathematicians"

When I was a student a theological friend amused himself by quoting at me St. Augustine's alleged injunction to beware of mathematicians lest they lead one to damnation. I have seen this quoted again quite recently. Mathematics has a poor press, but this particular derogatory statement is a canard: when St. Augustine wrote "mathematician" he meant, like many classical authors, "astrologer." The actual text reads, "bono christiano sive mathematici sive quilibet inpie divinantium, maxime dicentes vera, cavendi sunt, ne consortio daemoniorum animam deceptam pacto quodam societatis inretiant." This may be rendered more or less as follows: a good Christian must beware of astrologers as well as of those soothsayers who make predictions by unholy methods, and most especially when their predictions come true; he must guard against their having arranged to ensnare his soul by deceiving him through association with demons.

(De genesi ad litteram, Book II, chapter xvii; in Corpus Scriptorum Ecclesiasticorum Latinorum, vol. 28, part 1 (Vol. 3, part 1 of Augustine's works), edited by J. Zycha, Prague-Vienna-Leipzig, 1894, pp. 61–62.)

—R. P. B.

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S**(14-16), L**, *Mathematical Morsels*. Ross Honsberger. Dolciani Math. Expos., No. 3. MAA, 1978, xii + 249 pp, $14. [ISBN: 0-88385-303-5] A personal selection of elementary problems and solutions, mostly from the *Monthly*, engagingly rewritten by a master expositor as a "showcase for some of mathematics' minor miracles." Concludes with a selection of 25 similar problems, with references but not solutions. LAS

GENERAL, P, *Nikola Obrechkoff, 1896-1963: Opera, Volume 1*. Birkhäuser, 1978, 431 pp, Sfr. 92. [ISBN: 3-7643-0988-1] This first of several proposed volumes presents those of Obrechkoff's papers which are related to the theory of polynomials and integralfunctions. JAS

GENERAL, S**(13-16), L**, *Catastrophe Theory*. Alexander Woodcock, Monte Davis. Dutton, 1978, viii + 152 pp, $9.95. [ISBN: 0-525-07812-6] A superbly written nontechnical exposition of elementary catastrophe theory: its geometry, its significance, its applications and its controversy. Written jointly by one of the leading practitioners of catastrophe theory and a professional science writer, this brief gem communicates an immense amount of information without using a single formula! LAS

GENERAL, P, L, *Index of Mathematical Papers, Volume 8: An Annual Index to Mathematical Reviews, Volume 51 and 52 (1976)*. AMS, 1978. *Part 1: Author/Key Index*, 532 pp; *Part 2: Subject Index*, 633 pp, $120 set (P). [ISBN: 0-8218-4011-8]

GENERAL, P, *Open Questions in Mathematics (II)*. Ed: Dagmar R. Henney (The George Washington U, Washington, D.C. 20006), (P). Second edition of a very rough and largely unedited collection of open problems submitted to the editor by various correspondents. First edition TR, February 1975. LAS

PRECALCULUS, T(13: 1), *Essential Precalculus*. Doris S. Stockton. HM, 1978, xi + 655 pp, $14.50 [ISBN: 0-395-25417-5]; *Instructor's Manual*, viii + 200 pp, $1.70 (P). [ISBN: 0-395-25418-3] Good format. Divided into four parts: algebra review, elementary functions and analytic geometry, precalculus trigonometry, and precalculus college algebra. *Instructor's Manual* gives diagnostic tests to help with placement of students. LLK

HISTORY, S(14-17), P, L**, *Why I Left Canada: Reflections on Science and Politics*. Leopold Infeld. Trans: Helen Infeld. McGill-Queen's U Pr, 1978, xii + 212 pp, $16.95. [ISBN: 0-7735-0272-6] An open, unusually frank interpretation of the politics of science in the context of Infeld's forced "exile" to Poland in 1950, written in masterly style by "Einstein's Boswell," co-author (with Einstein) of *The Evolution of Physics* and author of *Whom the Gods Love*, the popular biography of Galois. LAS

HISTORY, S(15-16), *Carl Friedrich Gauss*. Hans Wussing. B.G. Teubner, 1976, 100 pp, 4,70 M (P). An account of his life and work, written (in German) for the educated layman. JD-B

HISTORY, P, L, *Oscar Zariski: Collected Papers, Volume III*. Ed: M. Artin, B. Mazur. MIT Pr, 1978, xxvi + 786 pp, $35. [ISBN: 0-262-24021-1] Early papers on the topology of varieties, many from the period 1928-1937, together with a few others related to this theme. Includes a Preface by Zariski with a unique personal account of the history of algebraic geometry, an Introduction by the editors explaining the context and significance of papers in this volume, and a comprehensive Bibliography keyed to this and earlier volumes of *Collected Papers*. LAS

FOUNDATIONS, P, L, *The Infinite in Mathematics: Logico-mathematical writings*. Felix Kaufmann. Reidel, 1978, xvii + 235 pp, $39.50. [ISBN: 90-277-0847-9] Detailed critique of Cantorian notions of "actually infinite sets." Translation of 1930 essays (pre-Gödel) bearing imprint of logical positivist and phenomenological influences. Traces Cantor's error to confusion in meaning between empirical vs. nonempirical universal statements. Challenging and insightful analysis by an acute thinker, much of it still valid today. GHM

FOUNDATIONS, P, *Lecture Notes in Mathematics-661: Indexed Categories and Their Applications*. Ed: P.T. Johnstone, R. Paré. Springer-Verlag, 1978, vii + 260 pp, $12 (P). [ISBN: 0-387-08914-4; 3-540-08914-4] An indexed category is a category whose morphisms lie in a topos rather than in a family of sets. This volume consists of four papers whose collective purpose is to describe varied developments of indexed categories, to apply these ideas to appropriate versions of the adjoint functor theorem and to give partial results showing that categories of algebras have colimits. JAS

FOUNDATIONS, P, L, *Multiple-conclusion Logic*. D.J. Shoesmith, T.J. Smiley. Cambridge U Pr, 1978, xiii + 396 pp, $35. [ISBN: 0-521-21765-2] Study of logical calculi which allow sets of conclusions rather than single conclusions, thus establishing (and exploiting) symmetry between premises and conclusions. Generalizes tree proofs to graph proofs. Develops applications to many-valued logic as illustration. Illuminates conventional logic and its limitations while suggesting that "classical logicians...have been speaking multiple conclusions all their lives without knowing it." GHM

FOUNDATIONS, P. *Mathematical Logic: Proceedings of the First Brazilian Conference.* Ed: Ayda I. Arruda, Newton C.A. da Costa, Rolando Chuaqui. Lect. Notes in Pure and Appl. Math., V. 39. Dekker, 1978, xii + 303 pp, $29.50 (P). [ISBN: 0-8247-6772-1] 17 papers from a July 1977 conference held at the University of Campinas under the sponsorship of the newly formed Center of Logic, Epistemology, and History of Science. Most contributors are from Latin America; all papers are in English. LAS

FOUNDATIONS, P, L. *Mathematics, Science and Epistemology, Philosophical Papers, V. 2.* Imre Lakatos. Cambridge U Pr, 1978, x + 285 pp, $22.95. [ISBN: 0-521-21769-5] Selected papers, most previously unpublished, by this extraordinarily influential philosopher of mathematics. Includes analysis of infinite regress, empiricism (as a foundation for mathematics), proofs, and nonstandard analysis, as well as several papers on epistemology and some short position papers. LAS

COMBINATORICS, T(15-17: 1), S*, L*. *Matters Mathematical, Second Edition.* I.N. Herstein, I. Kaplansky. Chelsea, 1978, viii + 246 pp, $10.95. [ISBN: 0-8284-0300-7] Reprint (with minor additions and corrections) of the 1974 original edition (TR, March 1975). Although designed for a "math appreciation" course, its sophisticated reasoning makes it more suitable for mathematics majors, especially for prospective teachers. Topics: sets, number theory, permutations, group theory, finite geometry, game theory. LAS

COMBINATORICS, T(13-14: 1), S*, L*. *Dots and Lines.* Richard J. Trudeau. Kent St U Pr, 1976, x + 203 pp, $6.50 (P); $11. [ISBN: 0-87338] New paperback edition of a popularization of graph theory (TR, March 1977), together with a brief Afterword on the Appel-Haken proof of the four-color conjecture. LAS

NUMBER THEORY, T*(14-16: 1), S, L. *Elementary Number Theory, Second Edition.* Underwood Dudley. Freeman, 1978, ix + 249 pp, $15. [ISBN: 0-7167-0076-X] Corrected edition of the charming 1969 original edition (TR, January 1970; ER, November 1970), with many new (or revised) problems. This brief text contains well over 1000 problems, with answers and hints to many, and a special appendix on computer projects. LAS

LINEAR ALGEBRA, T(15-17), S, L. *Multilinear Algebra, Second Edition.* Werner Greub. Springer-Verlag, 1978, vii + 294 pp, $19.80 (P). [ISBN: 0-387-90284-8; 3-540-90284-8] A significant revision of the 1967 (TR, March 1968; ER, October 1969) edition. The material on exterior algebra has been expanded (particular additions--characteristic coefficients and the Pfaffian) and two chapters on Clifford algebras have been added. JAS

LINEAR ALGEBRA, T?(14-15: 1). *Algèbre Linéaire.* Jean Acher, Jean Gardelle. Dunod (US Distr: SMPF, 14 E. 60th St., NY 10022), 1978, x + 209 pp, 45F (P). [ISBN: 2-04-010331-7] Standard introductory topics in linear algebra are presented in an elementary way. There are numerous elementary exercises, but the claim that the approach contains "very little theory" combined with the Bourbakian sections on groups, rings, fields, characteristic functions of subsets and the like shows how much of a cultural variable "theory" is. JAS

LINEAR ALGEBRA, T(14: 1). *Vectors in Three-Dimensional Space.* J.S.R. Chisholm. Cambridge U Pr, 1978, xii + 293 pp, $33.50; $9.95 (P). [ISBN: 0-521-21832-2] The first three chapters deal with the algebra of vectors; the fourth, with transformations representing rotations and reflections; the fifth, with curves, surfaces and volumes; the sixth, with gradient, divergence and curl. This approach should appeal to those needing mathematics for physics and engineering. LLK

ALGEBRA, P. *Lecture Notes in Mathematics-641: Séminaire d'Algèbre Paul Dubreil.* Ed: M.P. Malliavin. Springer-Verlag, 1978, iii + 367 pp, $16 (P). [ISBN: 0-387-08665-X; 3-540-08665-X] Proceedings of the 1976-1977 seminar. JAS

ALGEBRA, T(15-16: 1, 2), P. *Grundzüge der Algebra.* Otto Schafmeister, Hartmut Wiebe. Teubner, Stuttgart, 1978, 247 pp, (P). [ISBN: 3-519-02754-2] A carefully written introduction to algebra intended for teachers and prospective teachers in German high schools. Should interest those who train mathematics teachers in the U.S. JD-B

ALGEBRA, T(17-18: 2, 3), P*, L. *General Lattice Theory.* George Grätzer. Pure and Appl. Math., V. 75. Acad Pr, 1978, xiii + 381 pp, $35. [ISBN: 0-12-295750-4]; Math. Reihe, B. 52. Birkhäuser, 1978, xiii + 381 pp, Fr. 78. [ISBN: 3-7643-0813-3] An encyclopaedic monograph, incorporating in the first two chapters the author's 1971 Freeman book *Lattice Theory: First Concepts and Distributive Lattices.* Remaining chapters cover congruences and ideals, modular and semimodular lattices, equational classes and free products. Includes hundreds of exercises, a 44-page bibliography, index and tables of notation. Published simultaneously in two distinguished monograph series, this volume will stand for years as the definitive treatment of lattice theory. LAS

ALGEBRA, P. *Lecture Notes in Mathematics-647: Chain Conjectures in Ring Theory.* Louis J. Ratliff, Jr. Springer-Verlag, 1978, 133 pp, $8 (P). [ISBN: 0-387-08758-3; 3-540-08758-3] An exposition of the catenary chain conjectures, their history, and their relations to other open problems in ring theory. TRS

FINITE MATHEMATICS, T(13: 1). *Finite Mathematics, Second Edition.* James W. Thomas, Ann M. Thomas. Allyn, 1978, xii + 459 pp, $14.95. [ISBN: 0-205-05996-1] Minor changes from the authors' successful first edition (TR, October 1973). TRS

FINITE MATHEMATICS, T(13-14). *Applied Finite Mathematics with Calculus.* Howard Anton, Bernard Kolman. Acad Pr, 1978, xvii + 760 pp, $15. [ISBN: 0-12-059560-55] The authors' *Applied Finite Mathematics, Second Edition* (TR, June-July 1978) constitutes the first ten chapters of this text. The last five chapters provide an intuitive introduction to calculus through the fundamental theorem. Presentation includes both e^x and log x, but neither sin x nor cos x. TRS

CALCULUS, T(13: 2-3). *Calculus with Analytic Geometry*. Marvin J. Forray. Macmillan, 1978, xiv + 1185 pp, $19.95. [ISBN: 0-02-338800-5] This text provides abundant examples and problems, with applications to the sciences, engineering, business, and social sciences. There is an extensive *Student Study Guide*. LLK

REAL ANALYSIS, P, L. *Lecture Notes in Mathematics-659: Differentiation of Real Functions*. Andrew M. Bruckner. Springer-Verlag, 1978, x + 246 pp, $12 (P). [ISBN: 0-387-08910-1; 3-540-08910-1] A comprehensive survey of properties of the class of derivatives of real-valued functions defined on connected subsets of the real line. Serves both as a guide to the literature and as a source for open problems. LAS

COMPLEX ANALYSIS, S(15-16), L. *Problems in the Theory of Functions of a Complex Variable*. L. Volkovysky, G. Lunts, I. Aramanovich. Trans: Victor Shiffer. MIR (US Distr: Imported Pub, 320 W. Ohio St., Chicago, IL 60610), 1977, 332 pp. An important collection of problems, with solutions, on pure and applied topics. Every teacher of complex variables should have access to this book. RBK

COMPLEX ANALYSIS, T(16-17: 1), S. *Complex Analysis*. Serge Lang. A-W, 1977, xi + 321 pp, $14.95. [ISBN: 0-201-04137-5] For advanced undergradutes or a first-year graduate course. Local theory developed by means of formal power series; homotopy and homology versions of Cauchy's Theorem. Second part develops classical topics presented with a modern flavor. Problems are interesting, challenging. RBK

COMPLEX ANALYSIS, T*(15-17: 1, 2), L. *Complex Analysis: An Introduction to the Theory of Analytic Functions of One Complex Variable, Third Edition*. Lars V. Ahlfors. McGraw, 1979, xiv + 331 pp, $21. [ISBN: 0-07-000657-1] Major changes from the second edition: a new, simpler proof of the general form of Cauchy's theorem, an introduction to the Riemann zeta function, and a rewritten, modern treatment of global analytic functions in the terminology of germs and sheaves. This remains a paradigm for introductory texts on the theory of complex functions. LAS

COMPLEX ANALYSIS, T*(16-17: 1, 2), S, L. *Functions of One Complex Variable, Second Edition*. John B. Conway. Grad. Texts in Math., V. 11. Springer-Verlag, 1978, xiii + 317 pp, $16.80. [ISBN: 0-387-90328-3; 3-540-90328-3] Additions to first edition (TR, March 1974) include: J.D. Dixon's proof of Cauchy's theorem; S. Grabiner's proof of Runge's theorem; a bibliographical appendix; and several new exercises. TRS

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-648: Nonlinear Partial Differential Equations and Applications*. Ed: J.M. Chadam. Springer-Verlag, 1978, 206 pp, $8.90 (P). [ISBN: 0-387-08759-1; 3-540-08759-1] The majority of the invited "pedagogical" lectures given during the special 1976-77 special seminar at Indiana University, Bloomington, JAS

DIFFERENTIAL EQUATIONS, P. *Elements d'Analyse*. J. Dieudonné. Gauthier-Villars (US Distr: SMPF, 14 E. 60th St., NY 10022), 1978. *Tome VII*, xvi + 296 pp, 210F (P) [ISBN: 2-04-010082-2]; *Tome VIII*, xiii + 330 pp, 240F (P). [ISBN: 2-04-10273-6] Chapter 23 (linear functional equations) in 75 sections filling two volumes. The main theme is the theory of partial differential equations concentrating on elliptic, hyperbolic and parabolic equations, drawing on results in all preceding chapters. LAS

DIFFERENTIAL EQUATIONS, P. *Nonlinear Equations in Abstract Spaces*. Ed: V. Lakshmikantham. Acad Pr, 1978, ix + 483 pp, $24. [ISBN: 0-12-434160-8] Invited and contributed papers from a symposium held at the University of Texas at Arlington in June 1977 on the general theme of the solvability of nonlinear differential and integral equations in Banach space. LAS

DIFFERENTIAL EQUATIONS, S(17-18), P. *Lecture Notes in Mathematics-637: Meromorphe Differentialgleichungen*. W.B. Jurkat. Springer-Verlag, 1978, vii + 194 pp, $10 (P). [ISBN: 0-387-08659-5; 3-540-08659-5] Notes from a 1977 summer course given at Universität Ulm. Includes historical notes and a modest bibliography. LAS

NUMERICAL ANALYSIS, T(15-16), L. *An Introduction to Numerical Analysis*. Kendall E. Atkinson. Wiley, 1978, xiii + 587 pp, $19.95. [ISBN: 0-471-02985-8] Root finding, interpolation and approximation, integration, numerical solution to differential equations, numerical linear algebra. Error analysis and convergence theory. Presumes linear algebra, differential equations and programming. Problems include applications. RWN

NUMERICAL ANALYSIS, T(15-16: 1), L. *Numerical Methods: Analysis, Algebra, Ordinary Differential Equations*. N.S. Bakhvalov. Trans: George Yankovsky. MIR (US Rep: Four Continent Book Corp., 156 Fifth Ave., NY 10010), 1977, 663 pp. Contains a large number and variety of methods collected according to their application to analysis, linear algebra and differential equations. Includes some of the relevant theory, at times presuming some introductory functional analysis. Problems are integrated into the text. RWN

NUMERICAL ANALYSIS, P. *Proceedings of the 1978 Army Numerical Analysis and Computers Conference*. US Army Research Office (PO Box 12211, Research Triangle Park, NC 27709), 1978, xi + 322 pp, (P).

NUMERICAL ANALYSIS, T(18: 1, 2), S, P*. *The Finite Element Method for Elliptic Problems*. Philippe G. Ciarlet. Stud. in Math. and its Appl., V. 4. North-Holland, 1978, xvii + 630 pp, $56.95. [ISBN: 0-444-85028-7] Presumes functional analysis, especially Hilbert and Sobolev spaces. Clearly written. Foundations of elliptic problems and the finite element method. Conforming and other methods, including use of isoparametric elements, for second order problems. A mixed method for the biharmonic problem. Nonlinear problems. Applications to plates and shells. Includes most of the convergence theory. RWN

FUNCTIONAL ANALYSIS, P. *Continuous Crossed Products and Type III von Neumann Algebras*. A. van Daele. London Math. Soc. Lect. Notes, No. 31. Cambridge U Pr, 1978, vii + 68 pp, $7.50 (P). [ISBN:

0-521-21975-2] Notes from the author's lectures given at the University of Newcastle-upon-Tyne. First part provides an introduction to crossed products of von Neumann algebras, and the second centers on crossed products with modular actions obtained by the Tomita-Takesaki theory. TRS

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-651: Functors and Categories of Banach Spaces.* Peter W. Michor. Springer-Verlag, 1978, 99 pp, $8 (P). [ISBN: 0-387-08764-8; 3-540-08764-8] Develops and investigates the duality between Banach operator ideals and tensor products using categorical and universal constructions. TRS

FUNCTIONAL ANALYSIS, T, S, P, L. *Functional Analysis, Fifth Edition.* Kôsaku Yosida. Grund. math. Wissenschaften, B. 123. Springer-Verlag, 1978, xii + 500 pp, $32.50. [ISBN: 0-387-08627-7; 3-540-08627-7] Corrected reprint of the 1974 *Fourth Edition* (TR, August 1975) enlarged by three pages of supplementary notes and additional references. LAS

FUNCTIONAL ANALYSIS, P. *Elements of Non-linear Functional Analysis.* Richard A. Graff. Memoirs No. 206. AMS, 1978, xii + 196 pp, $9.20 (P). [ISBN: 0-8218-2206-3] A new notion of Fréchet differentiability for dual Banach spaces is introduced and used to develop a theory of Banach manifold differential topology. Includes applications to some nonlinear problems in the calculus of variations. TRS

FUNCTIONAL ANALYSIS, P. *Some Recent Developments in Operator Theory.* Carl M. Pearcy. CBMS Reg. Conf. in Math., No. 36. AMS, 1978, v + 73 pp, $7.20 (P). [ISBN: 0-8218-1686-1] An expanded version of the author's lecture notes from the CBMS regional conference held at Bucknell University, August 11-15, 1975. TRS

ANALYSIS, P. *American Mathematical Society Translations, Series 2, Volume 111.* AMS, 1978, iii + 219 pp, $26. [ISBN: 0-8218-3061-9] Nine papers on analysis dating from 1954 to 1972, mostly from the sixties. JAS

ANALYSIS, P. *Lecture Notes in Mathematics-644 & 645: Vector Space Measures and Applications I & II.* Ed: R.M. Aron, S. Dineen. Springer-Verlag, 1978. *I*, viii + 451 pp, $16.50 (P) [ISBN: 0-387-08668-4; 3-540-08668-4]; *II*, viii + 218 pp, $8.90 (P). [ISBN: 0-387-08669-2; 3-540-08669-2] Proceedings of a conference held at Trinity College, University of Dublin, June 26-July 2, 1977. These volumes contain the written and expanded texts of most of the talks, with the exception of twenty papers on real and complex analysis in finite dimensions which will be published separately. JAS

ANALYSIS, T(15-17: 1, 2), S, P, L*. *Special Functions of Applied Mathematics.* B.C. Carlson. Acad Pr, 1977, xiv + 335 pp, $25. [ISBN: 0-12-160150-1] A book about four functions--denoted by Γ, R, S, ${}_2F_0$--unified by the method of Dirichlet averages of x^t and e^x. Each chapter concludes with a "formulary" and lots of exercises (with solutions at the back of the book); the volume concludes with an extensive bibliography and a detailed, very useful index. LAS

ANALYSIS, S(15-16). *Mathematical Methods in the Physical Sciences.* Merle C. Potter. P-H, 1978, xii + 466 pp, $18.95. [ISBN: 0-13-561134-2] Introduces many techniques needed in applications to physics and engineering, such as differential equations, series solutions, numerical methods, linear algebra and complex variables. The style is brief and to the point, though several examples are worked in each section. A little too much of a "how to" book for an effective textbook. Many problems. TLS

ANALYSIS, T(19: 1), S, P. *Treatise on Analysis, V. VI.* Jean Dieudonné. Trans: I.G. Macdonald. Pure and Appl. Math., V. 10-VI. Acad Pr, 1978, x + 239 pp, $25. [ISBN: 0-12-215506-8] Chapter 22 (Harmonic Analysis) in Dieudonné's *magnum opus*, translated from the 1975 French original edition. A concise outline of finite dimensional linear representations of locally compact groups, interspersed with problem sets nearly as extensive as the text. LAS

ANALYSIS, T(16-17: 1-3). *Foundations of Applied Mathematics.* Michael D. Greenberg. P-H, 1978, xvii + 636 pp, $18.95. [ISBN: 0-13-329623-7] An encyclopedic, introductory graduate text on classical applied mathematics, covering real and complex variables, linear analysis, ordinary and partial differential equations. Self-contained treatments of fluid mechanics, heat conduction, and Newtonian mechanics weave through the text and serve to unify material. Organization permits flexible use. Exercises. Bibliography. TRS

ANALYSIS, T(17-18: 1, 2), S, L. *An Introduction to Harmonic Analysis, Second Corrected Edition.* Yitzhak Katznelson. Dover, 1976, xiv + 264 pp, $4 (P). [ISBN: 0-486-63331-4] Corrected paperback republication of the 1968 Wiley first edition (TR, January 1969; ER, April 1970). Proceeds from Fourier series and Fourier analysis to commutative Banach algebras. LAS

DIFFERENTIAL GEOMETRY, P. *Lecture Notes in Mathematics-662: The Metric Theory of Banach Manifolds.* Ethan Akin. Springer-Verlag, 1978, xix + 306 pp, $13.50 (P). [ISBN: 0-387-08915-2; 3-540-08915-2] The author considers the category of metric manifolds (those manifolds which carry an atlas) all of whose transition maps are bounded by a fixed constant, using some standard Banach space norm. The main goal is to determine properties of this category. Since compact (or paracompact) manifolds are metric, many of their properties extend to this more general category. Many applications to dynamical systems and foliations. TLS

DIFFERENTIAL GEOMETRY, P. *Geodesics and Ends in Certain Surfaces Without Conjugate Points.* Patrick Eberlein. AMS, 1978, iv + 111 pp, $8.40 (P). [ISBN: 0-8218-2199-7] This monograph describes the geometric structure of certain collared neighborhoods of ends under certain bounds ($K \leq -c^2 < 0$) on the curvature of the surface. JAS

GEOMETRY, S(13-16), L. *Japanese Optical and Geometrical Art.* Hajime Ōuchi. Dover, 1977, 170 pp, $4.50 (P). [ISBN: 0-486-23553-X] One of the Dover Pictorial Archive Series; 746 motifs representing applications of geometrical and optical designs; no text; republication of the 1974 Japanese edition *Leading Part.* JNC

GEOMETRY, P. *Manifolds all of whose Geodesics are Closed.* Arthur L. Besse. Ergebnisse der Math., B. 93. Springer-Verlag, 1978, ix + 262 pp, $38. [ISBN: 0-387-08158-5; 3-540-08158-5] Much like Wolf's *Spaces of Constant Curvature* in that it applies a wide range of geometric techniques to answer a specific question, in this case a classification problem. Presentation is highly advanced, well into the research stage. The style varies from condensed and very formal to very relaxed and informative. Has a nice account of the history of the problem. Good bibliography. TLS

TOPOLOGY, P. *Lecture Notes in Mathematics-657 & 658: Geometric Applications of Homotopy Theory.* Ed: M.G. Barratt, M.E. Mahowald. Springer-Verlag, 1978. *I,* vii + 459 pp [ISBN: 0-387-08858-X; 3-540-08858-X]; *II,* viii + 487 pp, $18.50 (P) each. [ISBN: 0-387-08859-8; 3-540-08859-8] Papers presented at a March 21-26, 1977 conference held at Northwestern University, Evanston, Illinois. JAS

TOPOLOGY, P. *Lecture Notes in Mathematics-652: Differential Topology, Foliations and Gelfand-Fuks Cohomology.* Ed: Paul A. Schweitzer. Springer-Verlag, 1978, xiii + 252 pp, $12 (P). [ISBN: 0-387-07868-1; 3-540-07868-1] The Proceedings of the Symposium on Differential and Algebraic Topology held at the Pontifíca Universidade Católica do Rio de Janeiro from January 5-24, 1976. Approximately half the papers and courses are presented here; nearly all the rest are presented by way of precise bibliographic references. JAS

TOPOLOGY, S, L*. *An Introduction to Point Set Topology via Linearly Ordered Spaces.* J.R. Boyd, G.R. Gordh, Jr. J. of Undergrad. Math. (Dept. of Math., Guilford Coll., Greensboro, NC 27410), 1977, ii + 34 pp, $4.50 (P). A self-contained "Moore method" introduction to topology via a study of the real line. This monograph leads the student to a topological characterization of the real line among all linearly ordered spaces. Probably not enough material for a whole course but inviting as a supplement or special project. JAS

PROBABILITY, P. *Lecture Notes in Mathematics-649: Séminaire de Probabilités XII, Université de Strasbourg, 1976-77.* Ed: C. Dellacherie, P.A. Meyer, M. Weil. Springer-Verlag, 1978, viii + 805 pp, $32 (P). [ISBN: 0-387-08761-3; 3-540-08761-3] Presentation from the 1976-77 sessions of the seminar. JAS

STATISTICS, T(13-14: 1), S*, L*. *How to Use (and misuse) Statistics.* Gregory A. Kimble. P-H, 1978, xi + 290 pp, $14.95; $7.95 (P). [ISBN: 0-13-436204-7] An irreverent non-mathematical treatment of basic statistics designed to inculcate an appreciation of a style of thought and a respectable level of statistical literacy. "For those who find security in arithmetic, a final section...provides some of the technical tools." The informal style, mixing apt popular examples with tidbits of formulas and graphs is hard to put down: a student would have to work pretty hard to avoid learning something from this inviting book. LAS

STATISTICS, T*(15-16: 1), S. *Statistical Methods for Engineers and Scientists, A Students' Course Book.* A.C. Bajpai, I.M. Calus, J.A. Fairley. Wiley, 1978, xii + 444 pp, $14.50 (P). [ISBN: 0-471-99644-0] A programmed text where students work problems along the way and are supplied with answers. Quizzes and many supplementary exercises supplied, mostly from engineering. Includes chapters on nonparametric methods, quality control, and analysis of variance. FLW

STATISTICS, T(14-15: 2), S, L. *Statistics: Probability, Inference, and Decision, Second Edition.* Robert L. Winkler, William L. Hays. HR&W, 1975, xviii + 965 pp, $16.95. [ISBN: 0-03-014011-0] An introductory text which presumes some calculus but maintains a low mathematical level. Ideas are carefully explained, with valuable discussions of their meaning and relationship. Summary discussions. Excellent problems. Contains chapters on Bayesian inference, decision theory, regression correlation, sampling theory and analysis of variance, nonparametric methods. RBK

STATISTICS, T(15-16: 1, 2), S. *Probability and Statistics for Engineers.* Robert E. Little. Matrix Pub, 1978, xii + 540 pp, $13.95. [ISBN: 0-916460-04-5] Part one, intended for self study or supplementary reading while using part two as a text, relies heavily on computer simulation studies. Part two considers the theoretical aspects of probability, estimation, and hypothesis testing. FLW

STATISTICS, P. *Investigations in the General Theory of Statistical Decisions.* A.S. Holevo. Proc. of Steklov Inst. of Math., No. 124. AMS, 1978, vi + 140 pp, $40 (P). [ISBN: 0-8218-3024-4] A translation of a 1976 monograph which "is devoted to the 'noncommutative' theory of statistical decisions, which permits one to include, along with Wald's classical scheme, an essentially new class of statistical problems arising in the theory of quantum measurement and quantum communication channels." JAS

STATISTICS, P. *Selected Translations in Mathematical Statistics and Probability, V. 14.* AMS, 1978, v + 296 pp, $38. [ISBN: 0-8218-1464-8] 24 papers from various Russian and Hungarian sources, from 1952-1972. LAS

STATISTICS, S(16-18), P. *Applied Time Series Analysis.* Ed: David F. Findley. Acad Pr, 1978, x + 345 pp, $18. [ISBN: 0-12-257250-5] Paper based on talks given at a symposium on applied time series held at the University of Tulsa in 1976. Several papers include programs. FLW

STATISTICS, T(17: 1), S, P, L. *Nonparametric Probability Density Estimation.* Richard A. Tapia, James R. Thompson. Johns Hopkins U Pr, 1978, xi + 176 pp, $17.95. [ISBN: 0-8018-2031-6] Considers density estimation techniques in which very little is assumed. "Most concerned with maximum penalized estimation methods." Good historical introduction. Appendix on optimization theory in Hilbert space. FLW

STATISTICS, P. *Lecture Notes in Mathematics-636: Journées de Statistique des Processus Stochastiques.* Ed: Didier Dacunha-Castelle, Bernard van Cutsem. Springer-Verlag, 1978, 202 pp, $12.40 (P). [ISBN: 0-387-08658-7; 3-540-08658-7] Seven lectures from a meeting at Grenoble, June 2-3, 1977. JAS

STATISTICS, S(15-18), P, L*. *Outliers in Statistical Data.* Vic Barnett, Toby Lewis. Wiley, 1978, xi + 365 pp, $39.95. [ISBN: 0-471-99599-1] "Aims to bring together in a logical framework the vast amount of work on outliers" to be found in the literature. Extensive tables and bibliography. FLW

STATISTICS, S(17-18), P. *Lineare statistische Modelle und optimale Versuchspläne.* Olaf Krafft. Vandenhoeck & Ruprecht, 1978, xvi + 499 pp, DM 45 (P). [ISBN: 3-525-40144-2] Uses generalized inverses and other advanced ideas of linear algebra for an exposition of some moderately sophisticated statistical models. JAS

STATISTICS, S(16-18), P. *An Introduction to Bilinear Time Series Models.* Clive William John Granger, Allan Paul Andersen. Vandenhoeck & Ruprecht, 1978, 94 pp, DM 28 (P). [ISBN: 3-525-11239-4] Investigation of certain nonlinear variations on the Box-Jenkins time series models that produce stationary, invertible series. LAS

STATISTICS, T(13). *Fundamental Research Statistics for the Behavioral Sciences, Second Edition.* John T. Roscoe. HR&W, 1975, xvii + 483 pp, $12.95. [ISBN: 0-03-091934-7] Non-mathematical survey of descriptive, inferential statistical methods. Examples from education and psychology. Includes non-parametric tests, analysis of variance. Description of how and when to use various tests. Minimal number of problems. (First edition, TR, June-July 1974.) RBK

STATISTICS, T(13-14: 1), S, L. *Introduction to Statistics, Third Edition.* Sol Weintraub. U Statistical Tracts (75-19 171st St., Flushing, NY 11366), 1978, xi + 428 pp, (P). [ISBN: 0-931316-01-4] The usual non-calculus statistics topics plus appendices containing some proofs and information about other distributions. A large number of worked out examples. FLW

COMPUTER PROGRAMMING, T*(14-15: 1), S, L. *Programming in PASCAL.* Peter Grogono. A-W, 1978, vii + 359 pp, $9.95 (P). [ISBN: 0-201-02473-X] Pascal is an Algol-like language which should be of interest to mathematicians especially because of its data structuring capabilities. This rather thorough text is one of the better books to appear on the language so far. Includes recursion, procedures, types, dynamic structures, files and program design. RWN

COMPUTER PROGRAMMING, S. *Fortran with Style, Programming Proverbs.* Henry F. Ledgard, Louis J. Chmura. Hayden, 1978, 164 pp, $6.95 (P). [ISBN: 0-8104-5682-6] Proverbs (short rules) and guidelines for writing more accurate, error-free programs. Each rule is accompanied by explanations and sample programs demonstrating the rule's use or lack of use. LLK

COMPUTER PROGRAMMING, S(13). *BASIC from the Ground Up.* David E. Simon. Hayden, 1978, 222 pp, $6.69 (P); $8.95. [ISBN: 0-8104-5760-1] Simple introduction to computers, requiring a minimal knowledge of algebra. Covers one version of Basic statements with some of their variations. LLK

COMPUTER PROGRAMMING, L. *A Guide for Programmers.* Marilyn Bohl. P-H, 1978, viii + 216 pp, $10.95; $7.50 (P). [ISBN: 0-13-370551-X; 0-13-370544-7] Advice to programmers on program design, flowcharting, use of decision tables, program writing and checkout. IBM oriented. RWN

COMPUTER SCIENCE, T(15-17: 1), S, L*. *Introduction to Formal Language Theory.* Michael A. Harrison. A-W, 1978, xiv + 594 pp, $20.95. [ISBN: 0-201-02955-3] A clearly written introduction. Presumes a bit of automata theory. Contains motivation and proofs for most of basic theory especially that on context free languages. Includes examples, problems of varying difficulty and historical surveys. RWN

COMPUTER SCIENCE, T(15-17: 1), S, L. *Computer Algorithms: Introduction to Design and Analysis.* Sara Baase. A-W, 1978, xvii + 286 pp, $17.95. [ISBN: 0-201-00327-9] A survey of common algorithms (for sorting, for traversing graphs, for string matching, for polynomial and matrix algebra and for hard (NP-complete) problems) together with related analyses for correctness, efficiency, simplicity and optimality. Algorithms are described in informal statements that can be translated by students into any programming language. LAS

COMPUTER SCIENCE, P. *Feasible Computations and Provable Complexity Properties.* Juris Hartmanis. CBMS Reg. Conf. Appl. Math., No. 31. SIAM, 1978, vii + 62 pp, $7.75 (P). Investigates the structure of feasible computations; the provable properties of the complexity of computations; relationships among complete sets under log n-tape reductions of natural families of languages (NL, CSL, P, NP, PTAPE); and changes that occur if complexity properties must be formally verifiable. Discusses whether the N = NP problem may be independent of the axioms of set theory. References. RJA

COMPUTER SCIENCE, P, L. *Foundations of Secure Computation.* Ed: Richard A. DeMillo, *et al.* Acad Pr, 1978, x + 404 pp, $19.50. [ISBN: 0-12-210350-5] Proceedings, including transcripts of discussion, of a 1977 workshop held in Atlanta shortly after the national press revealed to the public the existence of (and controversy surrounding) "trapdoor" codes. Lack of index limits usefulness of this timely volume. LAS

COMPUTER SCIENCE, T(13: 1). *Information Processing Systems: An Introduction to Modern, Computer-Based Information Systems.* William S. Davis. A-W, 1978, xviii + 458 pp, $13.95. [ISBN: 0-201-00999-4] A broad, introductory perspective on computer systems and their usage especially in a business setting. Basic concepts, file processing and systems. Glossary. RWN

COMPUTER SCIENCE, T(13-18: 1, 2), S. *Introduction to Computers, Structured Programming, and Applications.* C. William Gear. SRA, 1978. *Module A: Applications and Algorithms in Science and Engineering,* vi + 179 pp, (P) [ISBN: 0-574-21188-8]; *Module A: Applications and Algorithms in Computer Science,* viii + 179 pp, (P) [ISBN: 0-574-21189-6]; *Module C: Computers and Systems,* ix + 125 pp, (P) [ISBN: 0-574-21191-8]; *Module P: Programming and Languages,* x + 146 pp, (P) [ISBN: 0-574-21187-X]; *FORTRAN and WATFIV Language Manual,* 108 pp, (P) [ISBN: 0-574-21192-6]; *Instructor's Guide,* 44 pp, (P). This text comprises a set of individual modularized volumes. A General Introduction discusses

computers and their uses, problem solving by computer, algorithms and program descriptions. *Module P* presents language structures that are fundamental for expressing algorithms in most programming languages. It is expected that the reader is concurrently learning a language from one of the manuals (Fortran, Pascal, PL/1, Basic, Algol) that is keyed to chapters in *P*. *Module C* contains the organization of computers and machine and systems-level programming. The two versions of *Module A* treat techniques and methods of solving numerical and nonnumerical problems that are of interest in science and engineering and in computer science. The *Instructor's Guide* proves valuable when attempting to use two or more of these modules together. Each module has its own index and answers to selected chapter problems.RJA

SYSTEMS THEORY, P. *Lecture Notes in Control and Information Sciences-5: Singular Optimal Control: The Linear-Quadratic Problem.* David J. Clements, Brian D.O. Anderson. Springer-Verlag, 1978, v + 93 pp, $9 (P). [ISBN: 0-387-08694-3; 3-540-08694-3] A monograph unifying recent advances in singular, linear-quadratic control. RWN

APPLICATIONS (BIOLOGY), P, L*. *Food Webs and Niche Space.* Joel E. Cohen. Monographs in Pop. Bio., V. 11. Princeton U Pr, 1978, xv + 189 pp, $14; $6.95 (P). "The aim of this work is to demonstrate that a niche space of dimension one suffices, unexpectedly often and perhaps always, to describe the trophic niche overlaps implied by real food webs in single habitats. Consequently, real food webs fall in a small subset of the set of mathematically possible food webs. That real food webs are compatible with one-dimensional niche spaces more often than can be explained by chance alone has not been noticed previously." This work, ten years in the making, was motivated by the use of interval graphs (which, Cohen says, "appeared to me in a dream") to model ecological niche space. An exciting, innovative contribution to this population series, this monograph concludes with a brief discussion of open problems to guide further work. LAS

APPLICATIONS (BIOLOGY), L. *Lecture Notes in Biomathematics-21: Theoretical Approaches to Complex Systems.* Ed: R. Heim, G. Palm. Springer-Verlag, 1978, 244 pp, $8.90 (P). [ISBN: 0-387-08757-5; 3-540-08757-5] Papers from a June 1977 symposium held in Tübingen dealing primarily with mathematical models of biological phenomena usually considered to be inaccessible to rigorous theory. LAS

APPLICATIONS (BIOLOGY), P. *Lecture Notes in Mathematics-653: Locally Interacting Systems and Their Application in Biology.* Ed: R.L. Dobrushin, V.I. Kryukov, A.L. Toom. Springer-Verlag, 1978, xi + 202 pp, $11 (P). [ISBN: 0-387-08450-9; 3-540-08450-9] A collection of papers developed from the papers presented at the School-Seminar on Markov Interaction Processes in Biology, held at Pushchino, Moscow Region, USSR in March 1976. The material is not biology but rather biologically-oriented mathematical research. The first third is devoted to systems "whose specific characteristics are found in their deterministic part;" automata theory flavors some of this work. The second part is heavily probabilistic. JAS

APPLICATIONS (BIOLOGY), S(16-18), P*, L*. *Studies in Mathematical Biology.* Ed: S.A. Levin. Stud. in Math., V. 15 & 16. MAA, 1978. *Part I: Cellular Behavior and the Development of Pattern,* xiv + 335 pp; *Part II: Populations and Communities,* xx + 327 pp, $27 set. [ISBN: 0-88385-100-8 set] 17 papers whose interfaces and common ground demonstrate the "maturity of mathematical biology as a discipline." A rich resource for senior seminars, a good primer for mathematicians preparing for more "applicable" careers, and a worthy contribution to the rapidly growing *Studies* series. LAS

APPLICATIONS (BIOLOGY), P, L. *The Dynamics of Arthropod Predator-Prey Systems.* Michael P. Hassell. Mono. in Pop. Bio., V. 13. Princeton U Pr, 1978, vii + 237 pp, $16; $6.95 (P). Difference equation models tested in diverse situations of insect ecology, emphasizing stability analysis in multigeneration studies. LAS

APPLICATIONS (CONTROL THEORY), P. *Control and Dynamic Systems: Advances in Theory and Applications, Volume 14.* Ed: C.T. Leondes. Acad Pr, 1978, xix + 384 pp, $18.50. [ISBN: 0-12-012714-8] Five papers devoted to models for large scale engineering systems: linear regulators, pressurized water reactors, jet engines, power systems. LAS

APPLICATIONS (ECONOMICS), S, P, L. *An Econometric Simulation Model of the U.S. Farm Sector and its Policies and Food Exports.* Earl O. Heady, Thomas M. Reynolds, Donald O. Mitchell. Vandenhoeck & Ruprecht, 1978, 61 pp, DM 22 (P). [ISBN: 3-525-11238-6]

APPLICATIONS (ECONOMICS), P. *New Results in the Variate Difference Method.* Gerhard Tintner, Heinrich Strecker. Vandenhoeck & Ruprecht, 1978, 102 pp, DM 28 (P). [ISBN: 3-525-11242-4] The variate difference method is a method for analysis of economic time series. This brief research paper discusses a variation of this method (the quotient method), suitable for small samples, as well as relations to weighted regression, control theory and diffusion processes. LAS

APPLICATIONS (ENGINEERING), T(15: 1), P. *Diagnosis & Reliable Design of Digital Systems.* Melvin A. Breuer, Arthur D. Friedman. Computer Sci Pr, 1976, ix + 308 pp, $18.95. [ISBN: 0-914894-57-9] A "how to" book for practicing engineers or for a university course in electrical engineering. Includes methods for finding failures in digital systems and for designing ultra-reliable systems. TH

APPLICATIONS (ENGINEERING), P. *Logical Systems for Industrial Applications.* Jerzy Jaczewski. Elsevier Sci Pub, 1978, xv + 452 pp, $59.75. [ISBN: 0-444-99804-7] Extensive treatment of binary logic, finite automata and logical control systems, with emphasis on realizations in industrial contexts. For engineers and students in technical schools. Revised and enlarged translation from 1970 Polish edition. Extensive bibliography of Central and Eastern European primary sources. GHM

APPLICATIONS (PHYSICS), S(17-18), P. *Lecture Notes in Physics-78: The Rigged Hilbert Space and Quantum Mechanics.* A. Böhm. Springer-Verlag, 1978, viii + 69 pp, $8 (P). [ISBN: 0-387-08843-1; 3-540-08843-1] "Rigged" Hilbert Space is a linear space equipped with a strong topology that insures representation by continuous operators and existence of eigenvectors of the momentum and position

operators. Introduced a decade ago by Gelfand and Maurin, it uses the rigor of distribution theory to synthesize the von Neumann and Dirac foundations for quantum mechanics. It is, according to Böhm, a "most beautiful piece of art." LAS

APPLICATIONS (PHYSICS), S(17-18), P. *Theoretical Principles in Astrophysics and Relativity.* Ed: Norman R. Lebovitz, William H. Reid, Peter O. Vandervoort. U of Chicago Pr, 1978, vii + 258 pp, \$23. [ISBN: 0-226-46989-1] The proceedings of the symposium held at the University of Chicago in honor of S. Chandrasekhar on May 27-29, 1975. Includes a common index to the ten papers included. JAS

APPLICATIONS (PHYSICS), S(18), P. *Path Integrals and Their Applications in Quantum, Statistical, and Solid State Physics.* Ed: George J. Papadopoulos, J.T. Devreese. Plenum Pr, 1978, x + 515 pp, \$49.50. [ISBN: 0-306-40017-0] Proceedings of a NATO Advanced Study Institute held at the University of Antwerp in July 1977, including 13 expositions of methods and applications of the Feynman integral. LAS

APPLICATIONS (PHYSICS), P. *Methods of Modern Mathematical Physics, IV: Analysis of Operators.* Michael Reed, Barry Simon. Acad Pr, 1978, xv + 396 pp, \$34. [ISBN: 0-12-585004-2] Continuing their presentation of the mathematics of nonrelativistic quantum mechanics from *Volume III,* the authors discuss perturbation theory and spectral analysis of self-adjoint operators on Hilbert space. TRS

APPLICATIONS (PHYSICS), S(15-17), L. *Introduction to the Theory of Relativity.* Peter Gabriel Bergmann. Dover, 1976, xii + 307 pp, \$4 (P). [ISBN: 0-486-63282-2] Corrected republication of the 1942 classic, supplemented by appendices on the laws of motion of ponderable bodies and on supplementary notes relating the original text to more current literature. LAS

APPLICATIONS (PHYSICS), P. *Lecture Notes in Mathematics-650: C*-Algebras and Applications to Physics.* Ed: R.V. Kadison, Huzihiro Araki. Springer-Verlag, 1978, 192 pp, \$8.90 (P). [ISBN: 0-387-08762-1; 3-540-08762-1] Proceedings of the Second Japan-U.S. Seminar on C*-Algebras and Applications to Physics. Contains extended versions of the five main talks and about half of the short seminar talks. Talks that appear elsewhere are only listed. JAS

APPLICATIONS (PHYSICS), P. *Lecture Notes in Physics-77: Topics in Quantum Field Theory and Gauge Theories.* Ed: J.A. de Azcárraga. Springer-Verlag, 1978, x + 378 pp, \$14.30 (P). [ISBN: 0-387-08841-5; 3-540-08841-5] The proceedings of the Seminar on Theoretical Physics which took place in Salamanca, Spain during the third week of June 1977. R. Hermann's lectures are included even though he was unable to attend. JAS

APPLICATIONS (PHYSICS), T(16-18: 2, 3), P, L. *Mathematical Methods of Classical Mechanics.* V.I. Arnold. Trans: K. Vogtmann, A. Weinstein. Grad. Texts in Math., V. 60. Springer-Verlag, 1978, x + 462 pp, \$24.80. [ISBN: 0-387-90314-3; 3-540-90314-3] A concise, well-outlined introduction to Newtonian, Lagrangian and Hamiltonian mechanics in the modern language of differential geometry. Text definitions plus 150 pages of appendices form a transition from the presumed advanced calculus background to the language of tangent bundles, Lie groups and Riemannian curvature. LAS

APPLICATIONS (PHYSICS), P. *Thermodynamic Formalism: The Mathematical Structures of Classical Equilibrium Statistical Mechanics.* David Ruelle. Ency. Math. and Its Appl., V. 5. A-W, 1978, xix + 183 pp, \$21.50. [ISBN: 0-201-13504-3] A complex, austere, formal treatment ("the first entirely rigorous account of the foundations of thermodynamics") masquerading as the sixth volume of a series emphasizing "clarity of exposition" and "accessibility to the non-specialist." Would that there were truth in packaging laws for mathematics books. LAS

APPLICATIONS (PHYSICS), P. *Lecture Notes in Physics-79: Group Theoretical Methods in Physics.* Ed: P. Kramer, A. Rieckers. Springer-Verlag, 1978, xviii + 546 pp, \$23.50 (P). [ISBN: 0-387-08848-2; 3-540-08848-2] Proceedings of the Sixth International Colloquium on Group Theoretical Methods in Physics held at Tübingen in July 1977. The variety of topics covered justifies the generality of the title. JAS

APPLICATIONS (PHYSICS), P. *International Conference on Mathematical Problems of Quantum Field Theory and Quantum Statistics, Part II: Fields and Particles. Mathematical Questions of Quantum Statistics.* Ed: V.S. Vladimirov. Proc. of Steklov Inst. of Math., No. 136. AMS, 1975, v + 450 pp, \$60 (P). [ISBN: 0-8218-3036-8] Second half of proceedings of the 1972 conference named in the title. First half is No. 135 in this series (TR, December 1978). LAS

APPLICATIONS (POLITICAL SCIENCE), S*(14-16), P, L*. *The Presidential Election Game.* Steven J. Brams. Yale U Pr, 1978, xix + 242 pp, \$3.95 (P); \$15. [ISBN: 0-300-02254-9] Decision theory and game theory applied to presidential elections: primaries, conventions, election, electoral college. Includes a game-theoretic analysis of the Supreme Court's decision in *United States vs. Nixon,* and a proposal for "approval voting" which, Brams argues, better supports basic democratic principles. LAS

APPLICATIONS (SOCIAL SCIENCE), P. *Quantitative Sociology: International Perspectives on Mathematical and Statistical Modeling.* Ed: H.M. Blalock, *et al.* Acad Pr, 1975, xii + 643 pp, \$21.75. [ISBN: 0-12-103950-1] An international selection of papers, many previously published, designed to facilitate cross-national communication in quantitative sociology. This is the English language edition; the collection will also appear in Russian, and possibly in other languages as well. LAS

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; Judith N. Cederberg, St. Olaf; John Dyer-Bennet, Carleton; Timothy Hoel, St. Olaf; Lorraine L. Keller, St. Olaf; Roger B. Kirchner, Carleton; George H. Mills, St. Olaf; R.W. Nau, Carleton; Thomas R. Savage, St. Olaf; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Timothy L. Strotman, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

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Members of the MAA are encouraged to send in suggestions, question, etc., about the operations of the Association. Communications will be referred to the appropriate officer of the Association for answering; from time to time, those of general interest may also be answered in one or both of the official journals. Communications should be addressed to: Suggestion Box, Mathematical Association of America, 1225 Connecticut Avenue., N.W., Washington, D.C. 20036.

CONGRESS ON EDUCATION

The Second Congress on Education will take place in Vancouver, British Columbia, June 17 to 20, 1979. It will feature hundreds of educational sessions, exhibits, workshops, luncheon roundtable discussions, and several keynote addresses. Last year's keynoters included Sam Lavenson, Dr. Lawrence Peter, Senator Daniel Patrick Moynihan, Senator Eugene McCarthy, Senator Eugene Forsey and the Premier and Minister of Education of Ontario. Art Linkletter has agreed to deliver a keynote address in Vancouver and several other personalities will be announced soon.

Educational topics at Congress II will be of interest to academics and lay persons alike — researchers, administrators, teachers, school board members, social and community agencies and educational organizations. Persons who wish to make a presentation can apply to the Congress Secretariat for an official abstract application form.

Congress II comes as a result of the success of the First Congress held in Toronto last June. The First Congress on Education, sponsored by the Canadian School Trustees' Association featured more than 300 presentations covering nearly 100 educational topics.

In addition to hundreds of contributed papers, workshops and poster sessions, Congress II will feature free-wheeling luncheon discussion roundtables on two days, where small groups of delegates can zero in on common educational concerns.

Further information about the Congress may be obtained from the Congress Secretariat, 191 College Street, Toronto, Ontario M5T,1P7.

PITT ESTABLISHES INSTITUTE

In July, 1978 the University of Pittsburgh established an Institute for Computational Mathematics and Applications (ICMA) in the Department of Mathematics and Statistics. The Institute will coordinate departmental research activities in the area of scientific computing, including current projects in multidimensional two-phase fluid flow, contour dynamics, numerical solution of nonlinear algebraic and differential systems, data analysis of flight control and applications of computer graphics.

Charter members of ICMA are George D. Byrne, Associate Professor of Mathematics and Chemical and Petroleum Engineering, Charles G. Cullen, Associate Professor of Mathematics, Charles A. Hall, Professor of Mathematics, P. R. Krishnaiah, Professor of Statistics, Thomas A. Porsching, Associate Professor of Mathematics, Werner C. Rheinboldt, Andrew Mellon Professor of Mathematics, and Norman J. Zabusky, Professor of Mathematics and Electrical Engineering. ICMA has a limited number of Graduate Research Assistantships and visiting faculty positions. The University appointed Charles Hall as the Executive Director of ICMA.

NINTH CONFERENCE ON STOCHASTIC PROCESSES AND THEIR APPLICATIONS

The conference will be held on August 6-10, 1979, at Northwestern University. This will be the ninth in a series of conferences organized by the Committee on Conferences on Stochastic Processes of the Bernoulli Society for Mathematical Statistics and Probability, a section of the International Statistical Institute. There will be about 18 invited papers, and several sessions for short contributed papers. The areas of emphasis for this year are Markov processes; random fields, measures, point processes; stochastic integrals; stochastic modelling, control, estimation, optimization; reliability, and search. The aim is to create a setting for discussions and reviews of recent results on applied and theoretical topics of general interest.

For further information please contact E. Çinlar, Northwestern University, Tech. 1744; Evanston, Illinois 60201; U.S.A. Telephone: (312) 492-3588.

BUCKNELL TEACHERS PLAN INTEGRATED PHYSICS-MATH COURSE

The interdependence of mathematics and physics is emphasized in a course being developed by two faculty members at Bucknell University through a $17,800 grant from the National Science Foundation.

"Integrated Physics-Mathematics Course," a two-year project directed by Dr. Stephen F. Becker, assistant professor of physics, and Dr. Eugene N. Luks, professor of mathematics, emphasizes the interdependence of the two disciplines and involves development of an eight-credit course (with laboratory) which will be team-taught by the grantees.

The teachers note that this new course "should establish a common language, now generally lacking, for advanced courses in both fields and encourage the use in courses in each field of principles and techniques from the other."

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

MAY MEETING OF THE MICHIGAN SECTION

The 1978 annual meeting of the Michigan section was held on May 5 and 6 at the Kellogg Center of Michigan State University, East Lansing. A total of 134 persons attended the sessions which were arranged locally by Professors J.E. Adney and E.A. Nordhaus.

Four hour addresses were given: Gary Chartrand of Western Michigan University on *An Historical Encounter with the Four Color Problem*, Maynard Thompson of Indiana University on *Some Models for Information Diffusion*, Brian Winkel of Albion College on *Contact: Cryptology and Mathematics*, and Peter Hilton of Batelle Institute and Case Western Reserve University on *The Catastrophe Theory Controversy*.

The Friday evening banquet speaker was George Feeman of Oakland University who spoke on *A Mathematical Connection to Nepal, or How to Deal with an Isolated Point*. Professor Feeman enhanced his address with a showing of slides made during his recent stay in that country.

At the business meeting, members heard reports on the Michigan Mathematics Prize Competition for secondary students, on summer conferences sponsored by the section at Northern Michigan University and at Hope College, and by section governor, Y. Alavi on several items including the new headquarters building being purchased by the MAA. Elliot Tanis reported for the nominating committee which presented Donal Malm, Oakland U., for chairperson, Delia Koo, Eastern Michigan U. for vice-chairperson, and Katherine Price, Highland Park C.C., for vice-chairperson. This slate of officers was elected by the membership.

On Saturday, a special community college "rap session" was held under the leadership of David Randall, Oakland Community College.

Several sessions of papers were presented as follows:

First session of papers —

A Computer Contest for Secondary Schools, Robert Kane and Michael Skaff, University of Detroit.

Project Coach: Preparing High School Students for a Mathematics Field Day. Lawrence Kugler, University of Michigan, Flint.

A New New Math: A Proposal for Restructuring, William Hoffman, Oakland University.

Applications of Mathematics in Automotive Design, Edward Moylan, Ford Motor Company.

Second session of papers —

How Many Ways Can you Color a Chessboard? (Polya's Enumeration Theorem), Michael Gilpin, Michigan Technological University.

The Inverse of a Sum Can be the Sum of the Inverse, Thomas Elsner, General Motors Institute.

Prospects for a Globally Convergent Algorithm for Computing Zeros and Fixed Points of Nonlinear Systems, Layne Watson, Michigan State University.

Finite Infinities, Victor Meyer, University of Detroit.

Modeling a Linear Elastic Collision, Karl Hudnut, Lake Superior State College.

R. CHAFFER, *Secretary-Treasurer*

OCTOBER MEETING OF THE NORTH CENTRAL SECTION

The 1978 Fall meeting of the North Central Section of the MAA was held October 20-21,1978 at the University of Saskatchewan, Saskatoon, Canada. There were 56 persons in attendance including 24 members.

Principal speakers were Professor Peter Hilton, 1st Vice President of the MAA, with an invited address entitled *The Applied Catastrophy Theory Controversy* and Professor N.S. Mendelsohn, University of Manitoba, with an invited address entitled *Combinatorial Designs in a Modern Algebraic Setting*. Professor Joe Konhauser, Chairman of the Section, presided over the business session.

Contributed papers were:

Beam Stability in Storage Rings and Fixed Point Properties, Roger Servranckx, University of Saskatchewan.

J-Diagrams for Abelian Groups, Ernest Ackermann, Gustavus Adolphus College.

A Partial Converse to Euler's Theorem: V-E+F=2, J.C. Fisher, University of Regina.

Rings of Witt Type, Murray Marshall, University of Saskatchewan.
Precision Specifications in Linear Experiemtnal Designs, David Maier, University of Saskatchewan.
Does the Equation $x^x y^y = z^z$ *Imply* $4xy = z^2$ *for Integers* $x, y, z, > 1$*?*, Daihachiro Sato, University of Regina.
An Overview of Forcing, J.R. Shellito, University of Regina.
Absolutely Essential Mappings, E. D. Tymchatyn, University of Saskatchewan.

CHARLES V. HEUER, *Secretary-Treasurer*

FALL MEETING OF THE NEW JERSEY SECTION

The New Jersey section of the Mathematical Association of America held its annual Fall Meeting on Saturday, November 4, 1978 at St. Peter's College, Jersey City, New Jersey.

At the business meeting the following new officers were elected: Chair elect, Jean Lane, Union College; Vice Chair for Two Year Colleges, Stephen Fellner, Bergen Community College; Public Information Officer, Judith Seery, Bell Labs.; Secretary James Magliano, Union College. In addition, the following officers were re-elected: Vice Chair for Innovations, S. Ashby Foote, Rutgers University; Vice Chair for Speakers, Charles J. Lewis, Monmouth College.

Samuel Greitzer, Vice Chairman for High School Contests, reported on the progress of the U.S. Team in the International Mathematical Olympiad. The U.S. Team finished second in the contest. Mark Kleiman, who gave a short sutdent talk at our last section meeting, had the only perfect score on the exam.

There was a short discussion of whether or not credit should be granted for developmental math courses. The subject was accepted as a panel topic for the Spring Meeting of the Section.

The morning session of the meeting was completed with an interesting talk by Thomas Banchoff of Brown University. The subject of his paper was *The Fourth Demension and Computer Animated Geometry*. After the luncheon there was another talk, *Some Questions which Arise in Calculus*, by Gerard P. Protomastro of Saint Peter's College.

The next meeting of the section will be at Monmouth College in West Long Branch, N.J. on Saturday, April 28, 1979 in association with MATYCNY and AMTNJ.

JAMES MAGLIANO, *Secretary*

NOVEMBER MEETING OF THE SEAWAY SECTION

The Fall Meeting of the Seaway Section of the M.A.A. was held at the University of Rochester on November 10, 11, 1978 with a total attendance of 118 people, including 86 members of the Association. Professor Violet Larney, Chairman of the Section, presided.

The Friday Evening Session, Professor Sanford Segal spoke on *Mathematical Anxiety: Real or Imaginary? A Proposal for Treatment.*

The Saturday Session featured an invited address on *Combinatorial Sequence Problems* by the Secretary of the M.A.A., Professor David P. Roselle, Virginia Polytechnic Institute.

Other papers presented were:
The Unimodular Row Problem, by Michael Barr, McGill University
Alternative Models in Rail Track Scheduling, by Eric Muller, Brock University.
The Orthogonal Least Squares Line: 1878-1978, by David Farnsworth, Eisenhower College.
The Two-Year College Mathematics Journal-The Past, The Present, and the Future, by Peter A. Lindstrom, Genesee Community College.
The Classical, Bayesian, and Structural Approaches to Statistical Inference, by Peter Tan, Carleton University.
Another Approach to Teaching Remedial Algebra to College Students, by Larry Copes, Ithaca College.
On Distinguished Subspaces of Summability Domain, by S-C Chang, Brock University.
A Teaching Model for Systems of Differential Equations, by Larry E. Knop, Hamilton College.

At the business meeting, on the recommendation of the Gehman Lecture Committee, chaired by Kenneth Magill of S.U.N.Y. at Buffalo, the Section unanimously approved the concept of publishing the Gehman Lectures. Through advertisement in the Monthly, the mathematical community will be invited to participate in this project. The members also recognized Professor Emmet Stopher of S.U.C. at Oswego, past Secretary-Treasurer of the Section, for his years of dedicated service to the Section.

EMMET STOPHER, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Fifty-ninth Summer Meeting, University of Minnesota, Duluth, August 21–23, 1979.

Sixty-third Annual Meeting, San Antonio, Texas, January 5–7, 1980.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, last weekend in April or first weekend in May. Deadline for papers 6 weeks before meeting.

FLORIDA, Hillsborough Community College, Tampa, March 2–3, 1979.

ILLINOIS, first Friday/Saturday in May.

INDIANA

INTERMOUNTAIN

IOWA, third weekend in April. Deadline for papers February 1.

KANSAS, Johnson County Community College, Overland Park, April 7, 1979.

KENTUCKY, early April. Deadline for papers 6 weeks before meeting.

LOUISIANA–MISSISSIPPI, Friday–Saturday before February 20. Deadline for papers 3 months before meeting.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK, Adelphi University, May 5, 1979.

MICHIGAN, first Friday and Saturday in May. Deadline for papers 6 weeks before meeting.

MISSOURI, University of Missouri, Kansas City, March 30–31, 1979.

NEBRASKA, April.

NEW JERSEY, early November and early May.

NORTH CENTRAL, end of October and April. Deadline for papers October 1 and April 1.

NORTHEASTERN, Saturday after Thanksgiving, and third week in June in odd-numbered years.

NORTHERN CALIFORNIA, Sonoma State University, Rohnert Park, February 24, 1979.

OHIO, Miami University, Middletown, April 20–21, 1979.

OKLAHOMA–ARKANSAS, Oklahoma State University, Stillwater, March 30–31, 1979.

PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 15–16, 1979.

PHILADELPHIA, Saturday before Thanksgiving.

ROCKY MOUNTAIN, University of Denver, April 27–28, 1979.

SEAWAY, first Saturday in November and Saturday in late April. Deadline for papers 6 weeks before meeting.

SOUTHEASTERN, University of Tennessee, Chattanooga, April 6–7, 1979.

SOUTHERN CALIFORNIA, University of Southern California, Los Angeles, March 10, 1979.

SOUTHWESTERN, usually in April. Deadline for papers 2 weeks before meeting.

TEXAS, Friday and Saturday in early April. Deadline for papers March 1.

WISCONSIN, Friday and Saturday between mid-April and first week in May. Deadline for papers 6 weeks before meeting.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, San Francisco, January 3–8, 1980.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES

AMERICAN MATHEMATICAL SOCIETY, University of Minnesota, Duluth, August 22–25, 1979.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION

ASSOCIATION FOR COMPUTING MACHINERY, Plaza Hotel, Detroit, Michigan, October 26–31, 1979.

ASSOCIATION FOR SYMBOLIC LOGIC, New York City, December 28–29, 1979.

ASSOCIATION FOR WOMEN IN MATHEMATICS

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS, Washington, D.C., August 13–16, 1979.

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Boston, Massachusetts, April 18–21, 1979.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Hyatt Regency Hotel, New Orleans, Louisiana, April 29–May 1, 1979.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Radisson Muehlbach, Kansas City, Missouri, November 8–10, 1979.

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"The problem sets are very important since students, if they can, will use the text only as a source of problems. Basically I thought the problem sets were good. There was sufficient length and sufficient number of straightforward and more challenging exercises."

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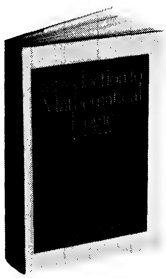

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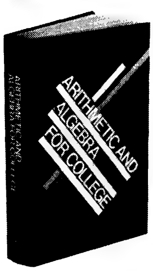

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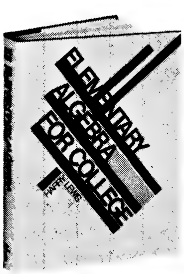

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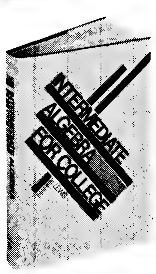

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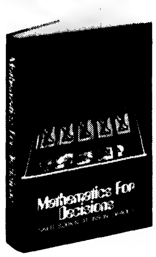

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MARCH

THE AMERICAN MATHEMATICAL MONTHLY

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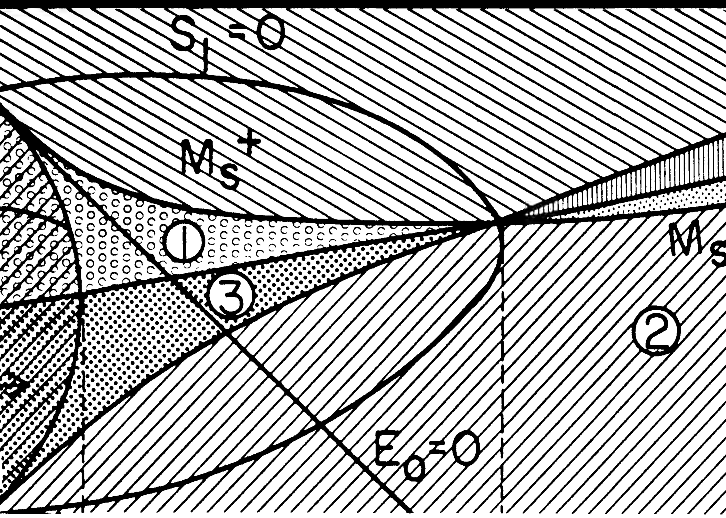

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1979

THE AMERICAN MATHEMATICAL MONTHLY

(FOUNDED IN 1894 BY BENJAMIN F. FINKEL)

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THE MATHEMATICAL ASSOCIATION OF AMERICA

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Research papers per se are unsuitable; see statement of policy (Vol. 85, p. 1).

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THE BANACH–TARSKI PARADOX

KARL STROMBERG

In this exposition we clarify the meaning of and prove the following "paradoxical" theorem which was set forth by Stefan Banach and Alfred Tarski in 1924 [1]. We were inspired to do this by a recent paper of A. M. Bruckner and Jack Ceder [2], where this theorem, among others, is brought into their interesting discussion of the phenomenon of nonmeasurable sets. We are grateful to Professor R. B. Burckel for calling this paper to our attention. We warmly recommend it to the reader. It is our intention here to present a strictly elementary account of this remarkable fact that will be accessible to readers with very little mathematical background. We do presume a little matrix theory and the elements of real analysis. We first state the main theorem and then give precise definitions before launching into its proof. We may as well admit in advance that its proof depends on Zermelo's Axiom of Choice, which is used in a very obvious way in the proof of Theorem C below (the set C selected there is *not* specified in a finitely constructable way).

BANACH–TARSKI THEOREM. *If X and Y are bounded subsets of $\mathbf{R}^3$ having nonempty interiors, then there exist a natural number n and partitions $\{X_j : 1 \leqslant j \leqslant n\}$ and $\{Y_j : 1 \leqslant j \leqslant n\}$ of X and Y, respectively (into n pieces each), such that X_j is congruent to Y_j for all j.*

Loosely speaking, the theorem says that if X and Y are any two objects in space that are each small enough to be contained in some (perhaps very large) ball and each large enough to contain some (perhaps very small) ball, then one can divide X into some finite number of pieces and then reassemble them (using only rigid motions) to form Y. This seems to be patently false if we submit to the foolish practice of confusing the "ideal" objects of geometry with the "real" objects of the world around us. It certainly does seem to be folly to claim that a billiard ball can be chopped into pieces which can then be put back together to form a life-size statue of Banach. We, of course, make no such claim. Even in the world of mathematics, the theorem is astonishing, but true.

DEFINITIONS. For $x=(x_1,x_2,x_3)$ in $\mathbf{R}^3$ we define the *norm* of x to be the number $|x|=(x_1^2+x_2^2+x_3^2)^{1/2}$. The *closed ball* of *radius* $r>0$ *centered* at $a\in\mathbf{R}^3$ is the set $\{x\in\mathbf{R}^3 : |x-a|\leqslant r\}$. A subset X of $\mathbf{R}^3$ is *bounded* if it is contained in some such ball, and X has *nonvoid interior*, if it contains some such ball. An *orthogonal matrix* is a square matrix with real entries whose transpose is also its inverse (its product with its transpose is the identity matrix). By a *rotation* we shall mean a 3×3 orthogonal matrix ρ whose determinant is equal to 1. We also regard such a ρ as a mapping of $\mathbf{R}^3$ onto $\mathbf{R}^3$ by writing $\rho(x)$ for the vector obtained by multiplying ρ by the column vector $x:\rho(x)=y=(y_1,y_2,y_3)$ where $x=(x_1,x_2,x_3)$,

$$\rho=\begin{pmatrix}\rho_{11} & \rho_{12} & \rho_{13}\\ \rho_{21} & \rho_{22} & \rho_{23}\\ \rho_{31} & \rho_{32} & \rho_{33}\end{pmatrix},\qquad y_i=\sum_{j=1}^{3}\rho_{ij}x_j$$

for $i=1,2,3$. A *rigid motion* (or *Euclidean transformation*) is a mapping r of $\mathbf{R}^3$ onto $\mathbf{R}^3$ having the form $r(x)=\rho(x)+a$ for $x\in\mathbf{R}^3$ where ρ is a fixed rotation and $a\in\mathbf{R}^3$ is fixed. We denote the 3×3

Karl Stromberg received his Ph.D. at the University of Washington in 1958 under the direction of Edwin Hewitt, with whom he is the coauthor of *Real and Abstract Analysis* (Springer-Verlag, 1965). He served on the faculty of the University of Oregon 1960–68 and has been Professor of Mathematics at Kansas State University since 1968. He has also spent single years on the visiting faculty at four other universities: Yale (1958–59), Chicago (1959–60), Uppsala (Sweden) (1966–67), York (England) (1974–75). His main research interests are in Analysis, both classical and abstract. He is the author of the forthcoming book *Classical Real Analysis* (Prindle, Weber, and Schmidt, 1979).—*Editors*

identity matrix by ι. Two subsets X and Y of $\mathbf{R}^3$ are said to be congruent and we write $X \cong Y$ if there exists some rigid motion r for which $r(X)=Y$. (Here, as usual, $r(X)$ denotes the set $\{r(x): x \in X\}$.) By a *partition* of a set X we mean a family of sets whose union is X and any two members of which are either identical or disjoint. Thus, to say that $\{X_j : 1 \leqslant j \leqslant n\}$ is a partition of X into n subsets means that

$$X = X_1 \cup X_2 \cup \cdots \cup X_n \quad \text{and} \quad X_i \cap X_j = \phi \text{ if } i \neq j.$$

It is allowed that some or all X_j be void.

The geometrical significance of our purely algebraic definition of a rotation is perhaps clarified by the next proposition.

PROPOSITION. *Let ρ be a rotation. Then we have the following.*

(i) *The image ρ of any line is a line*: $\rho(b+tc)=\rho(b)+t\rho(c)$ *for all* $b,c \in \mathbf{R}^3$ *and* $t \in \mathbf{R}$.

(ii) *Inner products are preserved by ρ; if $x,x' \in \mathbf{R}^3$, $\rho(x)=y$ and $\rho(x')=y'$, then*

$$\sum_{i=1}^{3} y_i y_i' = \sum_{j=1}^{3} x_j x_j'.$$

(iii) *Distances are preserved by ρ: if $x \in \mathbf{R}^3$, then $|\rho(x)|=|x|$.*

(iv) *If $\rho \neq \iota$, then the set $A=\{x \in \mathbf{R}^3 : \rho(x)=x\}$ is a line through the origin: there is a p in $\mathbf{R}^3$ such that $A=\{tp : t \in \mathbf{R}\}$ and $|p|=1$. We call A the axis of ρ.*

(v) *If q is any point of $\mathbf{R}^3$ having the two properties of p in* (iv), *then $q=p$* or *$q=-p$. We call p and $-p$ the poles of ρ.*

Proof. Assertion (i) is obvious and (iii) follows from (ii) by taking $x'=x$. To prove (v) notice that if $\{tp : t \in \mathbf{R}\} = \{tq : t \in \mathbf{R}\}$ and $|q|=|p|=1$, then $q=tp$ for some t and $t^2=t^2|p|^2=|tp|^2=|q|^2=1$ so t is 1 or -1.

To prove (ii), use the fact that ρ is orthogonal$\left[\sum_i \rho_{ij}\rho_{ik}=\iota_{jk}=1 \text{ or } 0 \text{ according as } j=k \text{ or } j\neq k\right]$ to write

$$\begin{aligned}\sum_i y_i y_i' &= \sum_i \Big(\sum_j \rho_{ij}x_j\Big)\Big(\sum_k \rho_{ik}x_k'\Big)\\ &= \sum_i \Big(\sum_j \sum_k \rho_{ij}\rho_{ik}x_j x_k'\Big)\\ &= \sum_j \sum_k \iota_{jk} x_j x_k' = \sum_j x_j x_j'.\end{aligned}$$

To prove (iv) we need a modest amount of matrix theory and real analysis. The characteristic polynomial $f(\lambda)=\det(\rho-\lambda\iota)$ of ρ is a cubic polynomial having real coefficients so the Intermediate Value Theorem assures that it has at least one real root. Let $\lambda_1,\lambda_2,\lambda_3$ be its three (complex) roots (counting multiplicity) where λ_1 is the largest real root. Then $f(\lambda)=(\lambda_1-\lambda)(\lambda_2-\lambda)(\lambda_3-\lambda)$ so

$$\lambda_1\lambda_2\lambda_3 = f(0) = \det\rho = 1. \tag{*}$$

If λ_k is a real root, then the system of equations

$$\sum_{j=1}^{3} (\rho_{ij}-\lambda_k\iota_{ij})x_j = 0 \qquad (i=1,2,3)$$

has a *real* solution x_1,x_2,x_3 (not all 0) so there is an $x \in \mathbf{R}^3$, $|x|\neq 0$, such that $\rho(x)=\lambda_k x$ from which (iii) yields $|\lambda_k|=1$, and so $\lambda_k=1$ or -1. If λ_2 is not real, then λ_3 is its complex conjugate and (*) becomes $\lambda_1|\lambda_2|^2=1$ so $\lambda_1=1$. If λ_2 is real, then so is λ_3, all three roots are 1 or -1, and (*) shows that $\lambda_1=1$ and $\lambda_2=\lambda_3$. Since $\lambda_1=1$, we can take $k=1$ in the above system to find a

vector $p \in \mathbf{R}^3$ with $|p|=1$ such that $\rho(p)=p$. Then $tp \in A$ for all $t \in \mathbf{R}$. Our job is to see that there are no other vectors in A. Assume that there is a $u \in A$ with $u \neq tp$ for all $t \in \mathbf{R}$. Choose a nonzero vector v that is perpendicular to the plane containing p, u, and 0; that is, $\Sigma v_j p_j = \Sigma v_j u_j = 0$. Since $\rho(p)=p$ and $\rho(u)=u$, it follows from (ii) that $\rho(v)$ is also perpendicular to this plane and thence from (iii) that $\rho(v)=v$ or $-v$. Any vector $x \in \mathbf{R}^3$ can be written as $x=\alpha p+\beta u+\gamma v$ for appropriate $\alpha,\beta,\gamma \in \mathbf{R}$ and, by (i), $\rho(x)=\alpha p+\beta u+\gamma\rho(v)$. Since $\rho \neq \iota$, we cannot have $\rho(v)=v$. Therefore $\rho(v)=-v$. The matrix

$$\sigma=\begin{pmatrix} p_1 & u_1 & v_1 \\ p_2 & u_2 & v_2 \\ p_3 & u_3 & v_3 \end{pmatrix}$$

has *nonzero* determinant (because p, u, and v are linearly independent) and the matrix product

$$\rho\sigma=\begin{pmatrix} p_1 & u_1 & -v_1 \\ p_2 & u_2 & -v_2 \\ p_3 & u_3 & -v_3 \end{pmatrix}$$

satisfies

$$-\det\sigma=\det(\rho\sigma)=(\det\rho)(\det\sigma)=\det\sigma$$

so $\det\sigma=0$. This contradiction completes the proof of (iv). ■

We now prove several theorems and lemmas which are of considerable interest in themselves as well as being vital stepping stones toward our main goal. The first three of these, of which Theorem C is the real key to our story, were set forth by Felix Hausdorff in 1914 [**4**, pp. 469–472]. We consider the two rotations

$$\psi=\begin{bmatrix} -1/2 & -\sqrt{3}/2 & 0 \\ \sqrt{3}/2 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\phi=\begin{pmatrix} -\cos\theta & 0 & \sin\theta \\ 0 & -1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix}$$

where θ is a fixed real number, to be chosen later. (Geometrically, ψ rotates $\mathbf{R}^3$ by 120° about the z-axis and ϕ rotates $\mathbf{R}^3$ by 180° about the line in the xz-plane whose equation is $x\cos\frac{1}{2}\theta=z\sin\frac{1}{2}\theta$.) One checks that the matrix ψ^2 is the same as the matrix ψ except that $\sqrt{3}$ is replaced by $-\sqrt{3}$ and that

$$\psi^3=\phi^2=\iota \tag{1}$$

where ι is the identity matrix. Now let G denote the set of all matrices that can be obtained as a product of a finite number of (matrix) factors, each of which is ϕ or ψ. Because of (1), it is clear that G is a group under matrix multiplication (if $\rho,\sigma \in G$, then ρ^{-1}, $\rho\sigma \in G$) and that each $\rho \neq \iota$ in G can be expressed in *at least* one way as a product

$$\rho=\sigma_1\sigma_2\cdots\sigma_n \tag{2}$$

where $n \geqslant 1$, each σ_j is ϕ or ψ or ψ^2, and if $1 \leqslant j<n$, then exactly one of σ_j and σ_{j+1} is ϕ. We call such expressions *reduced words* in the *letters* ϕ, ψ, and ψ^2. For example, the expression $\phi\psi^2\phi\phi\psi^2\phi$ is *not* a reduced word because of the two adjacent ϕ's, but it is equal to the reduced word $\phi\psi\phi$ ("equal" means that these products are the same matrix). Thus each element of G other than ι, ϕ, ψ, and ψ^2 can be expressed in at least one of the four forms

$$\alpha=\psi^{p_1}\phi\psi^{p_2}\phi\cdots\psi^{p_m}\phi, \qquad \beta=\phi\psi^{p_1}\phi\psi^{p_2}\cdots\phi\psi^{p_m},$$
$$\gamma=\phi\psi^{p_1}\phi\psi^{p_2}\cdots\phi\psi^{p_m}\phi, \qquad \delta=\psi^{p_1}\phi\psi^{p_2}\phi\cdots\phi\psi^{p_m} \tag{3}$$

where $m \geqslant 1$ and each exponent p_j is 1 or 2 (for $\delta, m>1$). These are the reduced words having more than one letter. Depending on our choice of θ, it may happen that two reduced words that appear to be different are actually equal; i.e., when multiplied out, they equal the same matrix. For example, if we choose $\theta=\pi$, one checks that $\psi\phi=\phi\psi^2$ and $\psi\phi\psi\phi=\iota$. However, we do have the following remarkable theorem which is the key to our later results.

THEOREM A. [4]. *If $\cos\theta$ is a transcendental number, then each element of G other than ι has exactly one expression as a reduced word in the letters ϕ, ψ, and ψ^2. That is, if*

$$\text{(i)} \quad \sigma_1\sigma_2\cdots\sigma_n=\rho_1\rho_2\cdots\rho_m$$

where each side of this equation is a reduced word, then $m=n$ and $\sigma_j=\rho_j$ for $1 \leqslant j \leqslant n$.

Proof. We need only show that no reduced word is equal to ι, for then if (i) held true with n as small as possible it would follow that $n=1$ and $\rho_1=\sigma_1$.

We first show that if α is as in (3), then $\alpha \neq \iota$. We have $\alpha=\sigma_m\sigma_{m-1}\cdots\sigma_2\sigma_1$ where each σ is either $\psi\phi$ or $\psi^2\phi$. That is, each σ is one of the two matrices

$$\sigma=\begin{bmatrix} \frac{1}{2}\cos\theta & \pm\frac{\sqrt{3}}{2} & -\frac{1}{2}\sin\theta \\ \mp\frac{\sqrt{3}}{2}\cos\theta & \frac{1}{2} & \pm\frac{\sqrt{3}}{2}\sin\theta \\ \sin\theta & 0 & \cos\theta \end{bmatrix}.$$

One checks by induction on m that if $K=(0,0,1)$, then $\sigma_m\sigma_{m-1}\cdots\sigma_1(K)=(\sin\theta P_{m-1}(\cos\theta), \sqrt{3}\sin\theta Q_{m-1}(\cos\theta), R_m(\cos\theta))$ where the P, Q, and R are certain polynomials with rational coefficients, their subscripts are their degrees, and their leading coefficients are

$$-\frac{1}{2}\left(\frac{3}{2}\right)^{m-1}, \pm\frac{1}{2}\left(\frac{3}{2}\right)^{m-1}, \left(\frac{3}{2}\right)^{m-1},$$

respectively. In fact, simple computations show that

$$P_0(x)=-\frac{1}{2}, Q_0(x)=\pm\frac{1}{2}, R_1(x)=x,$$

$$P_m(x)=\frac{1}{2}xP_{m-1}(x)\pm\frac{3}{2}Q_{m-1}(x)-\frac{1}{2}R_m(x)$$

$$Q_m(x)=\mp\frac{1}{2}xP_{m-1}(x)+\frac{1}{2}Q_{m-1}(x)\pm\frac{1}{2}R_m(x)$$

$$R_{m+1}(x)=(1-x^2)P_{m-1}(x)+xR_m(x).$$

This done, we see that since $\cos\theta$ is a root of no polynomial with rational coefficients, it is impossible that $\alpha(K)=K$ (else $R_m(\cos\theta)-1=0$) and so $\alpha\neq\iota$.

Now we see that no β as in (3) can equal ι, for otherwise $\alpha=\phi\beta\phi=\phi\iota\phi=\phi^2=\iota$. Similarly, if $\gamma=\iota$, then $\delta=\phi\gamma\phi=\iota$, so it remains only to rule out the possibility that $\delta=\iota$.

Assume that $\delta=\iota$ where δ is as in (3) and m is the smallest natural number for which this is true. Of course $m>1$. If $p_1=p_m$, then $\psi^{p_1+p_m}$ is either ψ^2 or $\psi^4=\psi$ so

$$\iota=\psi^{-p_1}\delta\psi^{p_1}=\phi\psi^{p_2}\cdots\phi\psi^{p_1+p_m}$$

is a reduced word of the form β which is impossible. Thus $p_1+p_m=3$. In case $m>3$, we have

$$\iota=\phi\psi^{p_m}\delta\psi^{p_1}\phi=\psi^{p_2}\phi\cdots\phi\psi^{p_{m-1}}$$

which is again of the form δ, contrary to the minimality of m. Therefore $m=2$ or 3. But $m=2$ yields $\iota=\psi^{p_2}\delta\psi^{p_1}=\phi$ while $m=3$ yields $\iota=\phi\psi^{p_3}\delta\psi^{p_1}\phi=\psi^{p_2}$ and these results are ridiculous. We conclude that $\delta=\iota$ is impossible. ■

We hereby choose and fix any θ such that $\cos\theta$ is transcendental. Of course all but countably many real numbers θ have this property. (Incidentally, it follows from the Generalized Linde-

mann Theorem that any nonzero algebraic θ will do; e.g., $\theta=1$.)

If an element $\rho\in G$ is expressed in its unique way as a reduced word as in (2), we call n the *length* of ρ and we say that σ_1 is the *first letter* of ρ or that ρ *begins with* σ_1. We write $l(\rho)=n$ and $l(\iota)=0$.

As usual, by a *partition* of a set X, we mean a pairwise disjoint family of subsets of X whose union is X.

THEOREM B. *There exists a partition* $\{G_1, G_2, G_3\}$ *of* G *into three nonvoid subsets such that for each* ρ *in* G *we have*

(i) $\rho\in G_1\Leftrightarrow\phi\rho\in G_2\cup G_3$,
(ii) $\rho\in G_1\Leftrightarrow\psi\rho\in G_2$,
(iii) $\rho\in G_1\Leftrightarrow\psi^2\rho\in G_3$.

(Note, for example, that $\phi\rho$ need not begin with ϕ. If $\rho=\phi\psi\phi$, then $\phi\rho=\psi\phi$ begins with ψ.)

Proof. Assign the elements of G inductively according to their lengths as follows. Put

$$\iota\in G_1, \phi\in G_2, \psi\in G_2, \psi^2\in G_3. \tag{4}$$

Suppose that $n\geqslant 1$ is some integer such that each $\sigma\in G$ with $l(\sigma)\leqslant n$ has been assigned to exactly one of G_1, G_2, and G_3. We now assign all elements of length $n+1$. If $l(\sigma)=n$ and σ begins with ψ or ψ^2, put

$$\begin{aligned}&\phi\sigma\in G_2 \text{ if } \sigma\in G_1,\\ &\phi\sigma\in G_1 \text{ if } \sigma\in G_2\cup G_3.\end{aligned} \tag{5}$$

If $l(\sigma)=n$ and σ begins with ϕ, put

$$\psi\sigma\in G_{j+1} \text{ if } \sigma\in G_j, \tag{6}$$

$$\psi^2\sigma\in G_{j+2} \text{ if } \sigma\in G_j \tag{7}$$

for $j=1$, 2, 3 where $G_4=G_1$ and $G_5=G_2$. By induction our partition is now formed. The assignment of any element of length n can be easily determined in n steps. For example, if $\rho=\psi\phi\psi\phi\psi^2\phi\psi^2$, then $l(\rho)=7$ and we note successively, beginning with the last letter, that

$$\psi^2\in G_3, \phi\psi^2\in G_1, \psi^2\phi\psi^2\in G_3, \phi\psi^2\phi\psi^2\in G_1$$
$$\psi\phi\psi^2\phi\psi^2\in G_2, \phi\psi\phi\psi^2\phi\psi^2\in G_1, \rho\in G_2.$$

One easily checks that the elements of length two satisfy

$$\{\phi\psi, \phi\psi^2, \psi^2\phi\}\subset G_1, \psi\phi\in G_3,$$

and therefore that (i)–(iii) hold if $l(\rho)\leqslant 1$ (for example, both sides of equivalence (i) are false unless $\rho=\iota$). For an inductive proof of (i)–(iii), suppose that $n>1$ is some integer and that these three equivalences are known to hold for all $\rho\in G$ having $l(\rho)<n$. Now let $\rho\in G$ with $l(\rho)=n$ be given.

CASE 1. Suppose that ρ begins with ϕ. Then (6) and (7), with $\sigma=\rho$, imply (ii) and (iii), respectively. Since $\phi\rho$ has length $n-1$, our induction hypothesis yields

$$\begin{aligned}\rho\notin G_1&\Leftrightarrow\phi(\phi\rho)=\rho\in G_2\cup G_3\\ &\Leftrightarrow\phi\rho\in G_1\Leftrightarrow\phi\rho\notin G_2\cup G_3\end{aligned}$$

and so (i) also holds for ρ.

CASE 2. Suppose that ρ begins with ψ. Then (i) follows from (5) with $\sigma=\rho$. We have $\psi\rho=\psi^2\sigma$ where $l(\sigma)=n-1$ and σ begins with ϕ, so (7) and (6) yield

$$\psi\rho=\psi^2\sigma\in G_2\Leftrightarrow\sigma\in G_3\Leftrightarrow\rho=\psi\sigma\in G_1\Leftrightarrow\psi^2\rho=\sigma\in G_3$$

which proves (ii) and (iii) for ρ.

CASE 3. Suppose that ρ begins with ψ^2. As in Case 2, (i) follows from (5). Here we have $\psi\rho = \sigma$ has length $n-1$ and begins with ϕ. So again (6) and (7) yield

$$\psi\rho = \sigma \in G_2 \Leftrightarrow \rho = \psi^2\sigma \in G_1 \Leftrightarrow \sigma \in G_2 \Leftrightarrow \psi^2\rho = \psi\sigma \in G_3$$

proving (ii) and (iii) in this final case. ■

THEOREM C. *There exists a partition $\{P, S_1, S_2, S_3\}$ of the unit sphere $S = \{x \in \mathbf{R}^3 : |x|^2 = x_1^2 + x_2^2 + x_3^2 = 1\}$ into four subsets such that*

$$\text{(i) } P \text{ is countable,} \qquad \text{(ii) } \phi(S_1) = S_2 \cup S_3,$$

$$\text{(iii) } \psi(S_1) = S_2, \qquad \text{(iv) } \psi^2(S_1) = S_3.$$

Proof. Let $P = \{p \in S : \rho(p) = p$ for some $\rho \in G$ with $\rho \neq \iota\}$. Since G is countable and each $\rho \neq \iota$ leaves just two points of S fixed (the poles of its axis of rotation) we see that (i) obtains. For each $x \in S \setminus P$, let $G(x) = \{\rho(x) : \rho \in G\}$. Each such $G(x)$ is a subset of $S \setminus P$ (if $\rho(x) \in P$ for some ρ, then $\sigma\rho(x) = \rho(x)$ for some $\sigma \neq \iota$ so $\rho^{-1}\sigma\rho(x) = x$, $\rho^{-1}\sigma\rho \neq \iota$, and $x \in P$), $x \in G(x)[x = \iota(x)]$, and any two such sets $G(x)$ and $G(y)$ are either disjoint or identical (if $t \in G(x) \cap G(y)$, say $\rho(x) = t = \sigma(y)$, and $z \in G(x)$, say $z = \tau(x)$, then $z = \tau(x) = \tau\rho^{-1}(t) = \tau\rho^{-1}\sigma(y) \in G(y)$; whence, $G(x) \cap G(y) \neq \phi \Rightarrow G(x) = G(y)$). Therefore, the family of sets $\mathfrak{F} = \{G(x) : x \in S \setminus P\}$ is a partition of $S \setminus P$. Next, choose exactly one point from each member of $\mathfrak{F}$ and denote the set of points so chosen by C. The set C has the properties:

$$C \subset S \setminus P, \tag{a}$$

$$c_1 \neq c_2 \text{ in } C \Rightarrow G(c_1) \cap G(c_2) = \phi, \tag{b}$$

$$x \in S \setminus P \Rightarrow x \in G(c) \text{ for some } c \in C \tag{c}$$

because $x \in G(c) \Leftrightarrow c \in G(x)$ for all $x, c \in S \setminus P$. Now define

$$S_j = G_j(C) = \{\rho(c) : \rho \in G_j, c \in C\}$$

for $j = 1,2,3$ where G_1, G_2, G_3 are as in Theorem B. Using (a) and the fact that $G(x) \subset S \setminus P$ if $x \in S \setminus P$, we see that $S_j \subset S \setminus P$ for each j. The fact that $G = G_1 \cup G_2 \cup G_3$ and (c) imply that $S \setminus P = S_1 \cup S_2 \cup S_3$. If $j \neq i$ in $\{1,2,3\}$, then $S_j \cap S_i = \phi$ (otherwise, for $x \in S_j \cap S_i$, we have $x = \rho(c_1) = \sigma(c_2)$ for some $c_1, c_2 \in C, \rho \in G_j, \sigma \in G_i$ so (b) yields $c_1 = c_2 = c$, say, and hence $\sigma^{-1}\rho(c) = c$ while $c \notin P$ from which $\sigma^{-1}\rho = \iota$ and $\rho = \sigma$ contrary to $G_j \cap G_i = \phi$). Therefore $\{P, S_1, S_2, S_3\}$ is a partition of S.

Finally, we apply (i)–(iii) of Theorem B to write

$$\phi(S_1) = \{\phi\rho(c) : \rho \in G_1, c \in C\} = \{\tau(c) : \tau \in G_2 \cup G_3, c \in C\} = S_2 \cup S_3,$$

$$\psi(S_1) = \{\psi\rho(c) : \rho \in G_1, c \in C\} = \{\tau(c) : \tau \in G_2, c \in C\} = S_2,$$

$$\psi^2(S_1) = \{\psi^2\rho(c) : \rho \in G_1, c \in C\} = \{\tau(c) : \tau \in G_3, c \in C\} = S_3$$

which proves (ii)–(iv). ■

The following lemma and its use in deducing Theorems D and E from Theorem C are contributions of W. Sierpiński (see [6]).

LEMMA. *If P is any countable subset of S, then there exists a countable set Q and a rotation ω such that $P \subset Q \subset S$ and $\omega(Q) = Q \setminus P$.*

Proof. The idea of the proof is very simple. We first select an axis of rotation for ω that contains no point of P, then we use the countability of $P \times P \times \mathbf{N}$ to select one of the uncountable supply of angles of rotation for ω that make ω satisfy $P \cap \omega^n(P) = \phi$ for all $n \geqslant 1$, and finally we put

$$Q = P \cup \bigcup_{n=1}^{\infty} \omega^n(P) \tag{8}$$

We now give details.

Among all vectors $v=(v_1,v_2,v_3)$ in S having $v_3=0$, there are only countably many for which v or $-v$ is in P. Select any $v=(v_1,v_2,0)\in S$ such that neither v nor $-v$ is in P. Writing $u=(1,0,0)$ and

$$\sigma=\begin{bmatrix} v_1 & v_2 & 0 \\ -v_2 & v_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we see that σ is a rotation, $\sigma(v)=u$, and the set $\sigma(P)$ contains neither u nor $-u$. For real numbers t, consider the rotations

$$\tau_t=\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos t & -\sin t \\ 0 & \sin t & \cos t \end{pmatrix}$$

that leave u fixed. For each triple (x,y,n) with $x,y\in\sigma(P)$ and $n\in\mathbf{N}$, it follows easily from the fact that $x_2^2+x_3^2>0$ that there exist either exactly n or exactly 0 values of t in $[0,2\pi[$ for which $\tau_t^n(x)=y$ according as $x_1=y_1$ or $x_1\neq y_1$. Since there are only countably many such triples in all, there are only countably many t for which the equality

$$\sigma(P)\cap\bigcup_{n=1}^{\infty}\tau_t^n\sigma(P)=\phi \tag{9}$$

fails. Fix any $t\in\mathbf{R}$ for which (9) obtains and write $\tau=\tau_t$. Now define $\omega=\sigma^{-1}\tau\sigma$ and define Q as in (8). Since $\tau^n\sigma=\sigma\omega^n$ for all n, (9) yields $\sigma(P\cap\omega(Q))=\sigma\left(P\cap\bigcup_{n=1}^{\infty}\omega^n(P)\right)=\phi$ from which we have $P\cap\omega(Q)=\phi$. But $Q=P\cup\omega(Q)$ so the proof is finished. ■

THEOREM D. *There exists a partition $\{T_j:1\leqslant j\leqslant 10\}$ of the unit sphere S into ten (disjoint) subsets and a corresponding set $\{\rho_j:1\leqslant j\leqslant 10\}$ of rotations such that $\{\rho_j(T_j):1\leqslant j\leqslant 6\}$ is a partition of S into six subsets and $\{\rho_j(T_j):7\leqslant j\leqslant 10\}$ is a partition of S into four subsets. Moreover, we can take T_7, T_8, and T_9 to all be rotates of S_1 and take T_1, T_2, T_3, and T_{10} to all be countable.*

Proof. We continue our previous notation and define

$$U_1=\phi(S_2),\ U_2=\psi\phi(S_2),\ U_3=\psi^2\phi(S_2),$$
$$V_1=\phi(S_3),\ V_2=\psi\phi(S_3),\ V_3=\psi^2\phi(S_3).$$

By Theorem C it is clear that $\{U_j,V_j\}$ is a partition of S_j for $j=1,2,3$ and that these six sets along with P form a partition of S into seven subsets. Now let

$$T_7=U_1,\ T_8=U_2,\ T_9=U_3,\ T_{10}=P,$$
$$\rho_7=\psi^2\phi,\ \rho_8=\phi\psi^2,\ \rho_9=\psi\phi\psi,\ \rho_{10}=\iota$$

and check that $\rho_{10}(T_{10})=P$ and $\rho_j(T_j)=S_{j-6}$ for $j=7,8,9$ so that $\{\rho_j(T_j):7\leqslant j\leqslant 10\}$ is indeed a partition of S. We shall now divide $S\setminus(T_7\cup T_8\cup T_9\cup T_{10})=V_1\cup V_2\cup V_3$ into six pieces. Let Q and ω be as in the preceding Lemma and define

$$T_1=\rho_8(S_1\cap Q),\ T_2=\rho_9(S_2\cap Q),\ T_3=\rho_7(S_3\cap Q)$$
$$T_4=\rho_8(S_1\setminus Q),\ T_5=\rho_9(S_2\setminus Q),\ T_6=\rho_7(S_3\setminus Q).$$

Plainly,

$$\{T_1,T_4\}\text{ partitions }\rho_8(S_1)=V_1,$$
$$\{T_2,T_5\}\text{ partitions }\rho_9(S_2)=V_2,$$
$$\{T_3,T_6\}\text{ partitions }\rho_7(S_3)=V_3,$$

and thus we see that $\{T_j:1\leqslant j\leqslant 10\}$ partitions S. Next define

$$\rho_4=\rho_8^{-1},\rho_5=\rho_9^{-1},\rho_6=\rho_7^{-1} \quad \text{and} \quad \rho_j=\omega^{-1}\rho_{j+3}$$

for $j=1,2,3$. Evidently,

$$\rho_{j+3}(T_{j+3})=S_j\backslash Q \qquad (j=1,2,3)$$

and, since $P\subset Q$, the union of these three sets is $S\backslash Q$. Finally, we have

$$\rho_j(T_j)=\omega^{-1}\rho_{j+3}(T_j)=\omega^{-1}(S_j\cap Q) \qquad (j=1,2,3)$$

so these three sets are disjoint and their union is $\omega^{-1}(Q\backslash P)=Q$. ∎

If, for a subset T of S, we write $T'=\{tx:x\in T,0<t\leqslant 1\}$, then $S'=\{y\in\mathbf{R}^3:0<|y|\leqslant 1\}$ is the punctured ball obtained from the solid unit ball $B=\{y\in\mathbf{R}^3:|y|\leqslant 1\}$ by removing the origin $O=(0,0,0)$, and it is clear that the first sentence of Theorem D remains true if we replace S by S' and T_j by T_j' throughout. We use this observation in the next proof.

THEOREM E. *There exists a partition* $\{B_k:1\leqslant k\leqslant 40\}$ *of the closed unit ball B into forty subsets and a corresponding set* $\{r_k:1\leqslant k\leqslant 40\}$ *of rigid motions such that* $\{r_k(B_k):1\leqslant k\leqslant 24\}$ *partitions B into twenty-four subsets and* $\{r_k(B_k):25\leqslant k\leqslant 40\}$ *partitions B into sixteen subsets.*

Proof. Apply the above Lemma to the case that P is the singleton set $\{u\}$ where $u=(1,0,0)\in S$ to obtain a countable set Q with $u\in Q\subset S$ and a rotation ρ_0 such that $\rho_0(Q)=Q\backslash\{u\}$. Next let $N_1=\{\frac{1}{2}(q-u):q\in Q\}$ and define the rigid motion r_0 by

$$r_0(x)=\rho_0\left(x+\tfrac{1}{2}u\right)-\tfrac{1}{2}u.$$

Plainly the vector 0 is in N_1 and $r_0(N_1)=N_1\backslash\{0\}$. Writing $N_2=B\backslash N_1$, $s_1=r_0$, $s_2=\iota$, and $M_h=s_h(N_h)$ for $h=1$ and 2, we see that $\{N_1,N_2\}$ partitions B and $\{M_1,M_2\}$ partitions $S'=B\backslash\{0\}$. We complete the proof by combining these partitions and rigid motions with the partition $\{T_j':1\leqslant j\leqslant 10\}$ of S' and the rotations $\{\rho_j:1\leqslant j\leqslant 10\}$ as in the remark following Theorem D.

Notice that, for each $j(1\leqslant j\leqslant 10)$, the family $\{T_j'\cap\rho_j^{-1}(M_i):1\leqslant i\leqslant 2\}$ partitions T_j' and that in turn $\{M_h\cap T_j'\cap\rho_j^{-1}(M_i):1\leqslant h\leqslant 2\}$ partitions $T_j'\cap\rho_j^{-1}(M_i)$ for $i=1$ and $i=2$. Thus $\{M_h\cap T_j'\cap\rho_j^{-1}(M_i):1\leqslant h\leqslant 2,1\leqslant i\leqslant 2,1\leqslant j\leqslant 10\}$ is a partition of S' into forty subsets and the forty sets

$$B_{hij}=s_h^{-1}\left[M_h\cap T_j'\cap\rho_j^{-1}(M_i)\right]$$

form a partition of B while for each fixed j the four sets

$$\rho_j s_h(B_{hij})=M_i\cap\rho_j(M_h\cap T_j') \tag{10}$$

$(1\leqslant h\leqslant 2,1\leqslant i\leqslant 2)$ form a partition of $\rho_j(T_j')$. We now invoke Theorem D to see that the families

$$\{\rho_j s_h(B_{hij}):1\leqslant h\leqslant 2,1\leqslant i\leqslant 2,1\leqslant j\leqslant 6\}$$

$$\{\rho_j s_h(B_{hij}):1\leqslant h\leqslant 2,1\leqslant i\leqslant 2,7\leqslant j\leqslant 10\}$$

are each a partition of S' while, for fixed i, (10) shows that the respective families of twelve and eight sets are each a partition of M_i which we can in turn map to partitions of N_i via s_i^{-1}. Therefore, writing $r_{hij}=s_i^{-1}\rho_j s_h$, we infer that

$$\{r_{hij}(B_{hij}):1\leqslant h\leqslant 2,1\leqslant i\leqslant 2,1\leqslant j\leqslant 6\}$$

and

$$\{r_{hij}(B_{hij}):1\leqslant h\leqslant 2,1\leqslant i\leqslant 2,7\leqslant j\leqslant 10\}$$

are partitions of B into twenty-four sets and sixteen sets, respectively. Finally, relabel the forty sets B_{hij} and the forty rigid motions r_{hij} with single subscripts $k=1,2,\ldots,40$. ∎

DEFINITION. We shall say that two subsets X and Y of $\mathbf{R}^3$ are *piecewise congruent* and we write $X\sim Y$ if, for some natural number n, there exist a partition $\{X_j:1\leqslant j\leqslant n\}$ of X into n

subsets and a corresponding set $\{f_j: 1 \leqslant j \leqslant n\}$ of rigid motions such that $\{f_j(X_j): 1 \leqslant j \leqslant n\}$ is a partition of Y. In case X is piecewise congruent to a subset of Y, we shall write $X \lesssim Y$.

Our next theorem gives some simple properties of the relations just defined.

THEOREM F. *For subsets X, Y, and Z of $\mathbf{R}^3$ we have*

(i) $X \sim X$,
(ii) $X \sim Y \Rightarrow Y \sim X$,
(iii) $X \sim Y$ *and* $Y \sim Z \Rightarrow X \sim Z$,
(iv) $X \sim Y \Rightarrow X \lesssim Y$,
(v) $X \lesssim Y$ *and* $Y \lesssim Z \Rightarrow X \lesssim Z$,
(vi) $X \subset Y \Rightarrow X \lesssim Y$,
(vii) $X \lesssim Y$ *and* $Y \lesssim X \Rightarrow X \sim Y$.

Proof. Since $Y \subset Y$, (iv) is banal. Since ι is a rigid motion, (i) and (vi) are obvious (with $n = 1$). Assertion (ii) follows from the fact that inverses of rigid motions are rigid motions.

To prove (v), suppose that $\{X_j: 1 \leqslant j \leqslant n\}$ and $\{Y_i: 1 \leqslant i \leqslant m\}$ are partitions of X and Y, respectively, and that $\{f_j: 1 \leqslant j \leqslant n\}$ and $\{g_i: 1 \leqslant i \leqslant m\}$ are sets of rigid motions such that $\{f_j(X_j): 1 \leqslant j \leqslant n\}$ is a partition of some $Y_0 \subset Y$ and $\{g_i(Y_i): 1 \leqslant i \leqslant m\}$ is a partition of some $Z_0 \subset Z$. Then one readily checks that the mn sets $A_{ij} = X_j \cap f_j^{-1}(Y_i)$ form a partition of X (for fixed j, the m sets $A_{1j}, A_{2j}, \ldots, A_{mj}$ are pairwise disjoint and their union is X_j) and, for fixed i, the n sets $f_j(A_{ij}) = Y_i \cap f_j(X_j)(1 \leqslant j \leqslant n)$ form a partition of $Y_i \cap Y_0$ so $\{g_i f_j(A_{ij}): 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n\}$ is a pairwise disjoint family whose union is some subset Z_1 of Z. Each composite mapping $g_i f_j$ is a rigid motion, so we have $X \sim Z_1$ and hence $X \lesssim Z$. This proves (v). The same argument proves (iii) by taking $Y_0 = Y$ and $Z_0 = Z$.

To prove (vii), suppose that $X \sim Y_0$ and $Y \sim X_0$ where $Y_0 \subset Y$ and $X_0 \subset X$. Let the notation be as in the preceding paragraph with $X = Z$ and $X_0 = Z_0$. We prove that $X \sim Y$ by copying a well-known proof of the Schröder–Bernstein Theorem. First define f on X and g on Y by $f(x) = f_j(x)$ if $x \in X_j$ and $g(y) = g_i(y)$ if $y \in Y_i$. For $E \subset X$, define $E' \subset X$ by

$$E' = X \setminus g[Y \setminus f(E)]. \tag{11}$$

Plainly,

$$E \subset F \subset X \Rightarrow E' \subset F' \tag{12}$$

Let $\mathfrak{D} = \{E: E \subset X, E \subset E'\}$. Notice that $\phi \in \mathfrak{D}$. Let $D = \bigcup \mathfrak{D}$ be the union of all the sets that belong to $\mathfrak{D}$. For each $E \in \mathfrak{D}$ we have $E' \subset D'$ by (12) so $E \subset D'$. Thus $D \subset D'$ and so (12) yields $D' \subset (D')'$; hence, $D' \in \mathfrak{D}$, $D' \subset D$, and $D' = D$. Put $E = D$ in (11) to obtain

$$D = X \setminus g[Y \setminus f(D)], X \setminus D = g[Y \setminus f(D)].$$

Clearly, $X \setminus D \subset X_0$. Now define, for $1 \leqslant j \leqslant n$ and $1 \leqslant i \leqslant m$,

$$A_j = D \cap X_j, A_{n+i} = g_i[Y_i \setminus f(D)], h_j = f_j, \quad \text{and} \quad h_{n+i} = g_i^{-1}.$$

It follows that $\{A_1, \ldots, A_n\}$ partition D, $\{A_{n+1}, \ldots, A_{n+m}\}$ partitions $X \setminus D$, $\{h_1(A_1), \ldots, h_n(A_n)\}$ partitions $f(D)$, and $\{h_{n+1}(A_{n+1}), \ldots, h_{n+m}(A_{n+m})\}$ partitions $Y \setminus f(D)$. Therefore $X \sim Y$. ■

Recall that a *closed ball* in $\mathbf{R}^3$ is any set of the form $A = \{x \in \mathbf{R}^3: |x - a| \leqslant \varepsilon\}$ where $a \in \mathbf{R}^3$ and $\varepsilon > 0$ are given. Recall also that a *translate* of a set $A \subset \mathbf{R}^3$ is any set of the form $A + b = \{x + b: x \in A\}$ where $b \in \mathbf{R}^3$ is given.

THEOREM G. *If $A \subset \mathbf{R}^3$ is a closed ball and if $A_1, A_2, \ldots, A_n$ are a finite number of translates of A, then*

$$A \sim \bigcup_{j=1}^{n} A_j.$$

Proof. We may suppose that $A=\{x\in\mathbf{R}^3:|x|\leqslant\varepsilon\}$ for some $\varepsilon>0$. Choose any $a\in\mathbf{R}^3$ for which $|a|>2\varepsilon$ and let $A'=A+a=\{y\in\mathbf{R}^3:|y-a|\leqslant\varepsilon\}$. We use Theorem E to show that $A\sim(A\cup A')$. So let the B_k and r_k be as in that theorem. For any set $D\subset\mathbf{R}^3$ and any $\delta>0$, let $\delta D=\{\delta x:x\in D\}$. We consider the partition $\{\varepsilon B_k:1\leqslant k\leqslant 40\}$ of A. Define rigid motions s_k by

$$s_k(x)=\varepsilon r_k\left(\frac{1}{\varepsilon}x\right) \quad \text{if } 1\leqslant k\leqslant 24,$$

$$s_k(x)=\varepsilon r_k\left(\frac{1}{\varepsilon}x\right)+a \quad \text{if } 25\leqslant k\leqslant 40.$$

(Note that if r is a rigid motion ($r(x)=\rho(x)+b$ where ρ is a rotation) and $s(x)=\varepsilon r\left(\frac{1}{\varepsilon}x\right)$, then s is a rigid motion because $s(x)=\rho(x)+\varepsilon b$.) From Theorem E we see that $\{s_k(\varepsilon B_k):1\leqslant k\leqslant 24\}$ partitions A, $\{s_k(\varepsilon B_k):25\leqslant k\leqslant 40\}$ partitions A', and so, since $A\cap A'=\phi$. $\{s_k(\varepsilon B_k):1\leqslant k\leqslant 40\}$ partitions $A\cup A'$. This proves that

$$A\sim(A\cup A').$$

We now prove the theorem by induction on n. The theorem is obvious if $n=1$. Suppose that $n>1$ is such that A is piecewise congruent to the union of any $n-1$ of its translates and let $A_1,\ldots,A_n$ by any n of its translates. By hypothesis $A\sim[A_1\cup\cdots\cup A_{n-1}]$ and it is obvious that $A_n\backslash[A_1\cup\cdots\cup A_{n-1}]$ is congruent (by translation) to a subset of A' so we have

$$A_1\cup\cdots\cup A_n\lesssim A\cup A'\sim A.$$

But clearly $A\lesssim A_1\cup\cdots\cup A_n$ so Theorem F yields $A\sim A_1\cup\cdots\cup A_n$. ■

We now state the Banach–Tarski Theorem again and then prove it.

THEOREM H. *If X and Y are bounded subsets of $\mathbf{R}^3$ having nonvoid interiors, then $X\sim Y$.*

Proof. Choose interior points a and b of X and Y, respectively, and then choose $\varepsilon>0$ such that $A=\{x\in\mathbf{R}^3:|x|\leqslant\varepsilon\}$ satisfies $A+a\subset X$ and $A+b\subset Y$. Since X is bounded, there exist a finite number $A_1,\ldots,A_n$ of translates of A whose union contains X. We therefore have, using Theorem G,

$$A\lesssim X\subset(A_1\cup\cdots\cup A_n)\sim A$$

so it follows from Theorem F that $X\sim A$. Similarly $Y\sim A$. Another application of Theorem F gives $X\sim Y$. ■

REMARKS. 1. The number 40 that appears in Theorem E is not the smallest possible. In fact, R. M. Robinson showed in 1947 [5] that there is a partition of B into five sets (one of them a singleton) which can be reassembled by rigid motions to form two disjoint closed balls of unit radius. Moreover, T. J. Dekker and J. deGroot proved [3] that these five sets can be chosen so that each is both connected and locally connected.

2. It follows from Theorem C that, since Lebesgue measure λ^3 on $\mathbf{R}^3$ is rotation invariant, none of the three sets $S_k'=\{tx:x\in S_k,0<t\leqslant 1\}, 1\leqslant k\leqslant 3$, can be Lebesgue measurable.

3. A poor analogue of Theorem C can be explicitly constructed (no Axiom of Choice) in the plane as follows. Fix any transcendental complex number c with $|c|=1$ (plenty of these exist, since there are only countably many z with $|z|=1$ that fail to be transcendental; we can take $c=e^i$). Now let X be the set of all complex numbers of the form

$$z=\sum_{j=0}^{n}a_jc^j$$

where n and $a_0,a_1,\ldots,a_n$ are nonnegative integers. Each $z\in X$ has a unique such expression. Let X_0 be the set of those z for which $a_0=0$ and let $X_1=X\backslash X_0$. Then $\{X_0,X_1\}$ partitions X. Define the rotation ρ of the plane by $\rho(z)=cz$ and define the translation τ by $\tau(z)=z+1$. Then $\rho(X)=X_0$ and $\tau(X)=X_1$ so the sets X_0 and X_1 are each congruent to X. The reason that this analogue is "poor" is twofold: X is both countable and unbounded.

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SEX DIFFERENCES IN MATHEMATICS: HOW *NOT* TO DEAL WITH THEM

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Even casual observation of this distinguished assemblage reveals sex differences among mathematicians. There are both male and female mathematicians! This may seem to be a vehement way of expressing the obvious. But it seems to be not at all obvious to those who portray the history of mathematics. A case in point is an important collection of portraits and biographies of mathematicians throughout the ages on a wall map entitled "Men of Modern Mathematics" [7]. There is a woman among them, Emmy Noether. But absent are other women who, despite enormous obstacles, contributed significantly to mathematics, e.g., Sophia Germain and Sonya Kovalevsky. In a similar vein, a well-known and otherwise excellent textbook on the history of mathematics has no women listed in the name index—and seemingly not mentioned in the text—not even Emmy Noether, although her father, Max Noether, is listed [3]. Still another well-known text on the history of mathematics referred to Hypatia of the fourth century as the first woman mathematician to be mentioned in the history of mathematics—but it referred to no other women, at least not in its first three editions, even as recently as 1969; however, there is a brief reference to Emmy Noether in the most recent edition of the text [5].

These are illustrations of ways in which *not* to deal with sex differences in mathematics. Do not ignore or overlook or hide the achievements of one sex. Let us find out more about these achievements and make them known to our colleagues, our students and the general public.

True, famous women mathematicians throughout history can be counted on one's fingers. But when mathematics students were asked to name such women, they usually did not reach even the first finger. For example, when the request to name famous women mathematicians was made of 26 mathematics majors in a junior-senior level algebra class, 24 did not list any names. In contrast, when they were then asked to name three to five famous mathematicians, 22 students answered, listing an average of four (male) mathematicians. It is important to increase the awareness of the contributions of women mathematicians in the past (cf. [**4**], [**20**]).

Nor should we *belittle* the women's contributions. At a recent conference on women in the history of mathematics, one of the participants remarked that on the whole she was disappointed

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in the achievements of the women whose names were recalled. She need not have been, if she took into account the conditions under which they worked. Nor should their accomplishments be attributed to others. I am thinking of the allegation that Sonya Kovalevsky's mathematics was really due to her mentor, Karl Weierstrass. It was a male mathematician who made the quip that there is some doubt about whether Kovalevsky was a mathematician and whether Emmy Noether was a woman. It is not worth repeating, except to illustrate the depths to which we can sink in belittling the accomplishments of women in the history of mathematics. Let us give them their due.

This should be done not only for women in the past, but also for contemporary women mathematicians. Yet even women are not doing so. How many of us have portraits in our offices of women mathematicians to motivate the interest of our students and colleagues? How many of us use our initials in publications so that the author is not readily identified as a woman? How many of us do not bother to publish what some of our male colleagues might readily rush into print? This point was revealed to me while interviewing male and female recent Ph.D.'s in mathematics. More women than men thought that their doctoral theses were *not* worthy of publication, even when other mathematicians, including their male advisors, thought that they were. How many of us know outstanding contemporary women mathematicians? Informal surveys which I have conducted at mathematics meetings suggest that women are less informed in this respect than men. When asked to name five outstanding *contemporary* women mathematicians, fewer women than men named them, and many women admitted that they did not know any. (These studies and the data which follow are described in **[15]**, **[16]**, **[17]**, **[18]**.)

In a questionnaire survey which asked for three outstanding women mathematicians, only one-third as many female mathematicians as males named three, 23% compared to 69%; over twice as many females as males did not name any, 50% compared to 21%; and over five times as many females admitted that they did not know any, 22% compared to 4%. They usually mentioned their own lack of acquaintance with the research literature and the insufficient visibility of women mathematicians. Both are situations that can be remedied. As Violet Larney pleaded **[10]**, "Female mathematicians, where are you?"

Another way not to deal with sex differences is to *discourage* women or to treat them *differently* for sexist reasons. That such tends to be the case was revealed in my NSF study in which questionnaires were sent to all members of the Association for Women in Mathematics (AWM) **[15]**. There were replies from about 40% of the membership, 350 females and 52 males.

Striking results were obtained in response to the question: What people or factors discouraged or hampered your decision to be a mathematician?

Discouraged by:	% Females	% Males
Family, friends	17	13
Teachers	21	8
Pre-college level	4	2
Undergraduate level	8	4
Graduate level	11	2
Advisors	11	4
Some person or persons	46	27

More women than men recalled being discouraged by family and friends. For example: "My mother worried that boys wouldn't like me or date me"; "My father thought I wasn't serious enough to be a mathematician." Three times as many women as men were discouraged by teachers or advisors, with the difference in each case most marked at the graduate level. For example: "The teachers expected less of the girls"; "My teacher paid attention only to the boys"; "My counselor said girls didn't do well in math"; "My advisor asked why I wasn't home having babies." Sexist reasons for the discouragement were given by a fifth of the women but none of the men. Almost half of the women recalled being discouraged by one or more persons,

which statistically is significantly more than for the men. But it is surely not insignificant that over one-quarter of the men—most of whom are now college professors—recalled being discouraged somewhere along the line.

Even more striking results were obtained when they were asked: Were you treated differently because of being female (or male) as a mathematics student or as a mathematics professional? That they encountered such treatment was reported by 80% of the women and 9% of the men! Moreover, such reports increased for the women as their training progressed but remained low for the man.

Different treatment at:	% Females	% Males
Pre-college level	23	6
Undergraduate level	26	8
Graduate level	43	8
Professional level	54	4
Any level	80	9

Furthermore, the frequency of complaints was about as great or greater for the younger women as for the older ones, e.g., complaints were made by 72% of those under age 30, by 71% of those over age 45.

Age	#	%
20–30	100	72
31–35	100	86
36–45	75	60
46–70	70	71

Among the complaints were that on the pre-college level some were treated as "strange" by their peers; they were told that boys do not like or are afraid of smart girls, especially math whizzes; their teachers did not encourage them to take mathematics courses; and they were advised to consider more traditional careers. At the undergraduate level, and even more on the graduate level, some of their teachers and advisors questioned their competence or did not take seriously their interest in mathematics. More than the men, they had to prove themselves. Some of them were assured that a woman would get married and have children and either not finish the degree or not pursue a career in mathematics. At the professional level, some reported denial of employment, receiving lower salary, or having less advancement potential than equally or less qualified males. According to our respondents' conceptions, affirmative action has not yet been effective. The younger women mentioned that they were frequently interviewed for positions but seldom offered them. In some cases there seemed to be a kind of backlash, e.g., the fear that if in the future a woman's contract was not renewed or she was not granted tenure then the university would be accused of having discriminated against women. On the other hand, a few women who were appointed as the token woman in the department had the feeling that this was done mainly to comply with affirmative-action requirements. Some said that they were given temporary positions, and it was expected that they would teach only the lower-level courses and would take care of such traditionally female responsibilities as school social functions. Another complaint was that they felt isolated from their male colleagues, socially and mathematically, or that they had the impression their colleagues felt restrained or uncomfortable when they were around. Interviews with male mathematicians suggest that there may be some basis for this feeling, because a few of them admit that they would feel uncomfortable with a woman colleague or even women graduate students. One mathematician claimed that because of the close relationship between doctoral advisor and advisee he would not want to have a woman doctoral student (and even if he did, his wife would not want him to). Moreover, because of deep-rooted social mores, he could feel protective or patronizing toward a woman mathematician but not at ease with her. He pointed out that it would take time to change such attitudes.

Returning to the AWM respondents, we find that over 90% agreed that changes should be made if women are to be encouraged to consider mathematics as a career. The female and male respondents tended to agree on the nature of the changes. For instance, the recommendations for the pre-college level were as follows:

Pre-college	% Females	% Males
Change attitude that math is unfeminine	36	42
Make teachers and advisors more aware of math career opportunities	21	21
More female role models	17	17
Equal treatment of males and females	14	19
Encourage, or don't discourage, women	12	10
Greater emphasis on math, advise four years of high school math	11	6
No changes needed	2	0

There is a close tie-in between the two most frequently recommended changes: weakening of the notion of mathematics as a masculine domain may be brought about by making teachers and advisors more aware of career opportunities in mathematics for both men and women. This was strikingly evident on visits to high schools where counselors admitted discouraging girls from pursuing mathematics because they did not think it offered opportunities for them. Moreover, students and parents also need to be made more aware of these opportunities. Of the students we questioned who had changed their minds about being math majors, a common reason for the switch was that they did not know what one did with mathematics except to teach it. Clearly, more information has to be spread about the varied career options which are opened up by training in mathematics. Women in particular need to know that mathematics is not exclusively a male discipline.

Another way *not* to deal with sex differences is much in vogue. It centers on math anxiety, which has been referred to as mathemaphobia [**6**], defined as "an irrational and impeditive dread of mathematics" ([**11**, p. 16], or mathophobia, a term attributed to Jerrold Zacharias of the Massachusetts Institute of Technology). There has been much anxiety about math anxiety. An article on it in *Ms.* magazine in 1976 resulted in more requests for reprints than any other article that had previously appeared in it [**23**]. The author, Sheila Tobias, referred to math anxiety as "a condition that disproportionately affects females" [**23**, p. 56]. She recognized: "There is some risk that in focusing on math anxiety in women, feminist educators may unintentionally support the prejudice and discrimination against women mathematicians.... But there is no question that math anxiety is a significant handicap for most women.... If we could develop a cure for math anxiety and 'bottle' it for women, [they] would show increased self-reliance and with it increased self-esteem" [**23**, p. 92].

The picture of women suffering from math anxiety is being widely circulated. Newsmagazines have written about it, as have women's magazines and even the *Wall Street Journal.* We run the very real risk of adding to the stereotypes concerning women and mathematics. The danger was dramatically revealed to me at a recent conference. A demonstration was held there in which volunteers discussed their anxiety about math. There they were, a dozen volunteers, all women, and two mathematician-counselors, both males!

Theories about sex differences in math ability and math anxiety also give us room for thought. Variants of psychoanalytic theories are offered as explanations by mathematicians,

educators, and psychologists. Plank and Plank (a review of this and other studies from which we have drawn is contained in [**9**]) investigated emotional components of arithmetic through autobiography and hypothesized that "successful early learning [is] dependent on the resolution of the pre-oedipal and oedipal conflicts" [**21**, p. 277]. They noted that an inability to learn mathematics might be due to a lack of detachment from the mother, since "to move on into a world of symbols from a world of objects, one has to detach oneself from the most loved object first" [**21**, p. 281]. They pointed to the importance of aggressive conflicts in the inhibition of mathematical activity, since in such activity "the child has to manipulate, dissect, destroy figures; this he can do only if he can allow himself to express aggression" [**21**, p. 287]. Since boys at all ages express aggression more readily than girls, does this explain why males seem to be better at mathematics? Moreover Rosen [**22**] found, in an analysis of a man who was mathematically gifted, that mathematical thinking could be used as a substitute for aggression. Keiser [**8**] maintained that the prototype for the disturbance in aspects of thinking that require the formulation of logical conclusions and the use of abstract thinking was in the failure to accept such bodily parts as the female genital organs, which can be known only by deduction. Aiken [**1**] concluded that girls are significantly more likely to suffer from a debilitating mathematics anxiety than are boys. Surveying such findings, Kogelman [**9**] wrote:

> In our society it is not unusual for mathematics to be identified as a masculine pursuit. Mathematics are often considered to be cold, withdrawn, and unemotional. This is in contrast to the idea that a woman is supposed to be warmer and more emotional. Hence, if there are conflicts around feminine identification, doing mathematics can be seen as incompatible with being a woman. Further ... it is culturally expected and acceptable for a woman to do poorly in mathematics. [**9**, p. 24]

He concluded that "debilitating mathematics anxiety was related to an inhibition of ego-function due to the displacement of both sexual and aggressive conflicts. This was supported by the finding that the onset of problems with mathematics tends to coincide with the beginning of adolescence.... The focus on detail which mathematics requires was seen to be threatening when there was a prohibition of sexual curiosity, fear of castration, or guilt over aggressive or oedipal wishes" [**9**, pp. 26–37]. What is not made clear, among other things, is why these problems should show up in mathematics in adolescence when, according to classical psychoanalytic theory, oedipal conflicts prevail at age six.

Another way *not* to deal with sex differences in mathematics is to assume that differences in mathematical *achievement* are due to inherent differences in mathematical *ability*. This assumption is implicit or explicit in many mathematical and psychological writings. It was also found among mathematicians and psychologists we interviewed. One pure mathematician characterized mathematical thinking as the purest, the highest type of thinking, devoid of the noise of reality. He implied that women do not attain to these heights or are more easily distracted. Care was usually taken to point out that women are not less intelligent than men but that they have different kinds of talents, skills, or intelligence (or that they are too smart to go into mathematics).

Let us pause to survey what *is* known about sex differences in mathematics. Very little that is conclusive! In their book on the psychology of sex differences [**19**], Eleanor Maccoby and Carol Jacklin write that there appear to be no sex differences in performances on quantitative tasks during the preschool years, or on mastery of numerical operations and concepts during the early school years, except that in disadvantaged populations the girls are ahead. Most studies show no sex differences in mathematics up to adolescence; but when differences are found in the age range of 9 to 13, they tend to favor boys. After this, the boys move ahead. They conclude that the findings suggest that it is not merely the amount of training the two sexes receive—e.g., that boys take more math courses—that is responsible for the difference in performance. It is also

interesting that verbal skills show very little sex difference from about ages 3 to 11. But from this age on, through the high school and college years, girls outscore boys in a variety of verbal skills. One hypothesis to account for these differences in aptitudes is that there are sex differences in the lateralization or specialization of the brain. It is known that the left hemisphere pertains to verbal functions and the right to spatial visualization, among other functions. One theory is that there is earlier and stronger development of lateralization in females which facilitates their verbal development, but that *spatial* skills call for a more *bilateral* cerebral representation and hence is facilitated in men, in whom laterality presumably is not so strong or developed so early. But there is a contradictory hypothesis that the localization of verbal functions in the left hemisphere and of spatial functions in the right tends to be weaker in women and stronger in men and that it is this difference which contributes to men's better spatial abilities. It is believed that cerebral lateralization is weak in left-handed men. Couple this with my own observation that left-handedness is high in male mathematicians (perhaps almost twice as high as in the general population, where it is about 14%) and conclude that perhaps it is *weak* lateralization which accounts for mathematical ability.

As mathematicians and educators, it seems fruitless to get involved in such controversies over the brain's role. It has even been suggested rather grimly that it is brain damage in boy babies which helps to make more of them mathematicians. We are not going to inflict brain damage on girl babies in the hope of making them mathematicians. It is equally fruitless to argue over whether or not spatial ability is a recessive trait carried in the X chromosome (cf. [**2**]). Let me go out on a limb and say that there is also relatively little value in quibbling experimentally over whether or not there are statistically significant sex differences in, say, spatial visualization. I am thinking of a study, funded for $100,000, which ended with inconclusive findings; some sex differences were statistically significant and others were not. This could have been predicted from the literature without expenditure of precious time and funds.

How should we deal with sex differences in mathematics? We should not ignore them or pretend they don't exist. It seems to be more fruitful to accept that, for a variety of reasons, there are some sex differences pertinent to mathematics, e.g., in a large population of high school and college age, the females in general may be better in verbal skills and the males in spatial skills, although this of course may not hold for a given male and female. Verbal and spatial skills are both involved in mathematics. How can we utilize sex differences in them to enhance mathematics teaching and learning for both sexes? This is an exciting and potentially productive way to deal with sex differences. I throw this open as a challenge to the members of the Mathematical Association. An added challenge is to take into account possible sex differences in attitudes and interests and in what has been called styles of mathematical thinking.

The path of prudence would dictate that I should offer no suggestions since the title of this talk is how *not* to deal with sex differences. However, let me describe some preliminary attempts by my husband, Abraham S. Luchins, and myself, which the reader can classify any way he or she chooses.

The Luchins' water-jar problems [**12**] have been said to yield sex differences, e.g., the female subjects were more susceptible to becoming set or less likely to recover from the set or Einstellung than the male subjects [cf. **19**, p. 102]. What we also found, but what is not so widely reported in the literature, is that girls tended to interpret the instructions differently, e.g., the instructions "Don't be blind" tended to be interpreted by the girls as "Don't be blind to the method the teacher illustrated or that worked in the earlier problems," while the boys more often interpreted it to mean "Watch out; don't get caught." Changes in instructions, changes in the social atmosphere, affected performances on the problems as well as did sex differences.

Many years ago my husband used Gestalt psychological principles [**13**], [**24**] to teach geometry to two classes of girls who had failed the New York State geometry regents examination. The focus was on how the problem or the diagram could lead to good errors or bad errors and how they could be restructured to lead to solution. The girls became adept at such analysis and restructurization. All of them passed the regents make-up examination, with an average grade in the 80's.

Recent work used geometric problems devised by Max Wertheimer, founder of Gestalt psychology [**14**], [**24**]. The problems require only elementary geometric facts presumably known by all the college students who served as subjects. However, spatial visualization and restructurization may be required before it is realized that this elementary knowledge suffices.

Analysis of the data suggests that there are some sex differences in responses but that they are less pronounced than differences in interests and attitudes. In group administration to calculus classes, the males did somewhat better than the females in one experiment; but the reverse was the case in another experiment where most of the females were mathematics majors. The matter was studied further in individual administration, with 80 subjects in one experiment and 120 in another. On the whole, the males had more solutions than the females. While the male mathematics and male non-mathematics majors had similar frequencies of solutions, the female mathematics majors had more solutions than the other females, e.g., 51% more on the average in one experiment in contrast to 26% difference between males and females. Were the female mathematics majors better at spatial visualization and restructurization than the other females, and why were the differences more pronounced than sex differences?

In individual administration, if a solution was not given in about 2.5 minutes, the subject was given an opportunity to receive a hint, if he wished, or to continue unaided. In this manner, a series of four increasingly stronger verbal hints were available to help the subject restructure each problem. More females than males asked for hints. However, the male non-mathematics majors required more frequent hints than the other males or than the females, whether or not the latter were mathematics majors. Thus, after three hints, fewer of the male non-mathematics majors had solved the problems, so that more of them needed all four hints. A possible clue may lie in the male students' apparently greater embarrassment in needing hints for such "easy problems." Another may be the females' greater acceptance of hints and better understanding of verbal hints.

The findings suggest that differences in attitudes toward mathematics and in task- and ego-orientation may be more influential than sex differences in spatial visualization and restructurization. They may also help to account for what has been called the elusive factor in these sex differences [**19**, p. 105]. The findings also point to the possible fruitfulness of further research on the roles of attitudinal factors in relation to sex-differences in spatial visualization and restructurization.

I conclude by asking that the Association consider concrete changes that can be made in methods of teaching and in the content and organization of the mathematics curriculum to take into account sex differences in mathematical achievements, attitudes, and interests.

This paper was an invited address delivered at the MAA meeting of Seaway Section, at Brock University, St. Catharines, Ontario, on May 5, 1978.

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THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

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The following results of the thirty-eighth William Lowell Putnam Mathematical Competition, held on December 3, 1977, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, five hundred dollars, was awarded to the Department of Mathematics of **Washington University**, St. Louis, Missouri. The members of its winning team were George T. Gilbert, Philip I. Harrington, and Tim J. Steger; each was awarded a prize of one hundred dollars.

The second prize, four hundred dollars, was awarded to the Department of Mathematics of the **University of California–Davis**, Davis, California. The members of its team were Daniel G. Knierim, David L. Knierim, and Stephen R. Peck; each was awarded a prize of seventy-five dollars.

The third prize, three hundred dollars, was awarded to the Department of Mathematics of the **California Institute of Technology**, Pasadena, California. The members of its team were Karl W.

Heuer, Peter W. Shor, and Albert L. Wells, Jr.; each was awarded a prize of fifty dollars.

The fourth prize, two hundred dollars, was awarded to the Department of Mathematics of **Princeton University**, Princeton, New Jersey. The members of its team were Eric S. Lander, Adam N. Rosenberg, and David J. Rusin; each was awarded a prize of fifty dollars.

The fifth prize, one hundred dollars, was awarded to the Department of Mathematics of the **Massachusetts Institute of Technology**, Cambridge, Massachusetts. The members of its team were Mark D. Haiman, Miller S. Puckette, and Thomas H. Spencer; each was awarded a prize of fifty dollars.

The five highest ranking individual contestants, in alphabetical order, were **Russell D. Lyons**, Case Western Reserve University; **Stephen W. Modzelewski**, Harvard University; **Michael Roberts**, Massachusetts Institute of Technology; **Adam L. Stephanides**, University of Chicago; and **Paul A. Vojta**, University of Minnesota–Minneapolis. Each of these students was designated as a Putnam Fellow by the Mathematical Association of America and awarded a prize of two hundred and fifty dollars by the Putnam Prize Fund.

The next five highest ranking individuals, in alphabetical order, were *Christopher S. Bretherton*, University of Colorado; *Daniel G. Knierim*, University of California–Davis; *Miller S. Puckette*, Massachusetts Institute of Technology; *Peter W. Shor*, California Institute of Technology; and *Tim J. Steger*, Washington University. Each of these five students was awarded a prize of one hundred dollars.

The following teams, named in alphabetical order, received honorable mention: *Case Western Reserve University* with team members Paul M. Herdeg, Russell D. Lyons, and Daniel L. Stock; *Harvard University* with team members David J. Groisser, Nathaniel S. Kuhn, and Stephen W. Modzelewski; *Pomona College* with team members Kenneth L. Brand, Bruce J. Levy, and Daniel Pritikin; *University of Waterloo* with team members Rajiv Gupta, Randolph R. Morrison, and Douglas R. Stinson; and *Yale University* with team members James E. Boyce, Jonathan H. Dorfman, and Samuel D. Zurier.

Honorable mention was achieved by the following twenty-nine individuals, named in alphabetical order: *Steven Alexander*, Princeton University; *Nicholas E. Baxter*, Stanford University; *Brian D. Boe*, Queen's University; *Edward J. Branagan*, Case Western Reserve University; *Michael P. Chandler*, California Institute of Technology; *Raymond A. Coley*, Princeton University; *Andrew Z. Fire*, University of California–Berkeley; *George T. Gilbert*, Washington University; *Theodore S. Goodman*, Princeton University; *Mark D. Haiman*, Massachusetts Institute of Technology; *Philip I. Harrington*, Washington University; *Karl W. Heuer*, California Institute of Technology; *David L. Knierim*, University of California–Davis; *Bruce D. Ladendorf*, Princeton University; *Pierre Lalonde*, Université Laval; *Andrew R. Latto*, Brown University; *Frank J. Lhota*, Wayne State University; *Neal N. Madras*, McGill University; *Dale A. Miller*, Lebanon Valley College; *Stephen R. Peck*, University of California–Davis; *N. Christopher Phillips*, University of California–Berkeley; *Adam N. Rosenberg*, Princeton University; *Kendall S. Stanley*, Purdue University–W. Lafayette; *Alan S. Stern*, Harvard University; *Douglas R. Stinson*, University of Waterloo; *Steven T. Tschantz*, University of California–Berkeley; *Anthony J. Vizzini*, Massachusetts Institute of Technology; *David E. Wallace*, Brown University; and *John L. Wojtowicz*, Case Western Reserve University.

The other individuals who achieved ranks among the top 100, in alphabetical order of their schools, were: Amherst College, *Peter V. N. Church*; Boise State University, *Gary A. Ray*; California Institute of Technology, *Stephen C. Jackson, Charles W. Schlindwein, Albert L. Wells, Jr.*; University of California–Berkeley, *Gregory S. Lee*; University of California–Davis, *Larry J. Romans*; University of California–San Diego, *Kevin C. Nunan*; Carleton University, *Jim D. Williams*; Carnegie–Mellon University, *Kevin A. Cline*; Case Western Reserve University, *James W. Davis, Paul M. Herdeg*; University of Chicago, *John R. Conlon, Jonathan S. Jones, Theodore G. Katzman*; Dartmouth College, *Geoffrey B. Crew, Barry Hayes*; Georgia Institute of Technology, *Hon W. Tam*; Harvard University, *David S. Anderson, Daniel S. Freed, David J. Groisser,*

Mark S. Manasse, Vlad G. Rutenburg; University of Manitoba, *T. Patrick Legg*; University of Maryland, *Eric Kuritzky*; Massachusetts Institute of Technology, *Jonathan J. Dagresta, Daniel V. D'Eramo, Robert E. Gompf*; University of Massachusetts–Amherst, *David A. Huse*; University of Massachusetts–Boston, *Paul S. Mason*; McGill University, *Patrick A. N. Smith*; Michigan State University, *Christopher H. Hall, Michael C. Slattery*; State University of New York–Binghamton, *Richard D. Bury*; University of Oklahoma, *Gary D. Kohler*; Pomona College, *Daniel Pritikin*; Princeton University, *Lawrence M. Ausubel, Mark E. Bassett, David D. Chambliss, Alan S. Geller, Eric S. Lander, Frank T. Leighton, David J. Rusin, Jeffrey O. Shallit, Joseph S. Weening, Matthew P. Wiener, Russell W. Young*; University of Santa Clara, *John P. Albert*; University of Southern California, *Niles D. Ritter*; Swarthmore College, *Dana W. Nance*; University of Toronto, *Paul W. Beame, Guy E. Moorhouse, Robert J. Rothwell*; Vanderbilt University, *Ronald W. Ausbrooks*; University of Waterloo, *Richard B. Cameron, Guy W. Hulbert, Geoffrey Mess, Randolph R. Morrison*; University of Wisconsin–Madison, *David S. Witte*; Yale University, *James E. Boyce, Samuel D. Zurier.*

There were 2,138 individual contestants from 332 colleges and universities in Canada and the United States in the competition of December 3, 1977. Teams were entered by 266 institutions.

The winner of the annual William Lowell Putnam Fellowship for the competition of the previous year (December 4, 1976) was Nathaniel S. Kuhn of Harvard University.

The Questions Committee for the thirty-eighth competition consisted of R. T. Bumby, J. D. E. Konhauser (Chairman), and L. A. Zalcman; they prepared the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS, PART A

Problem A-1

Consider all lines which meet the graph of

$$y=2x^4+7x^3+3x-5$$

in four distinct points, say (x_i,y_i), $i=1,2,3,4$. Show that

$$\frac{x_1+x_2+x_3+x_4}{4}$$

is independent of the line and find its value.

Problem A-2

Determine all solutions in real numbers x,y,z,w of the system

$$x+y+z=w,$$

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{1}{w}.$$

Problem A-3

Let $u,f,$ and g be functions, defined for all real numbers x, such that

$$\frac{u(x+1)+u(x-1)}{2}=f(x) \quad \text{and} \quad \frac{u(x+4)+u(x-4)}{2}=g(x).$$

Determine $u(x)$ in terms of f and g.

Problem A-4

For $0<x<1$, express

$$\sum_{n=0}^{\infty}\frac{x^{2^n}}{1-x^{2^{n+1}}}$$

as a rational function of x.

Problem A-5

Prove that

$$\binom{pa}{pb} \equiv \binom{a}{b} \pmod{p}$$

for all integers p, a, and b with p a prime, $p > 0$, and $a \geqslant b \geqslant 0$.

Notation: $\binom{m}{n}$ denotes the binomial coefficient $\dfrac{m!}{n!(m-n)!}$.

Problem A-6

Let $f(x,y)$ be a continuous function on the square

$$S = \{(x,y) : 0 \leqslant x \leqslant 1, 0 \leqslant y \leqslant 1\}.$$

For each point (a,b) in the interior of S, let $S_{(a,b)}$ be the largest square that is contained in S, is centered at (a,b), and has sides parallel to those of S. If the double integral $\iint f(x,y)\,dx\,dy$ is zero when taken over each square $S_{(a,b)}$, must $f(x,y)$ be identically zero on S?

PROBLEMS, PART B

Problem B-1

Evaluate the infinite product

$$\prod_{n=2}^{\infty} \frac{n^3-1}{n^3+1}.$$

Problem B-2

Given a convex quadrilateral $ABCD$ and a point O not in the plane of $ABCD$, locate point A' on line OA, point B' on line OB, point C' on line OC, and point D' on line OD so that $A'B'C'D'$ is a parallelogram.

Problem B-3

An (ordered) triple (x_1, x_2, x_3) of positive *irrational* numbers with $x_1 + x_2 + x_3 = 1$ is called "balanced" if each $x_i < 1/2$. If a triple is not balanced, say if $x_j > 1/2$, one performs the following "balancing act"

$$B(x_1, x_2, x_3) = (x_1', x_2', x_3'),$$

where $x_i' = 2x_i$ if $i \neq j$ and $x_j' = 2x_j - 1$. If the new triple is not balanced, one performs the balancing act on it. Does continuation of this process always lead to a balanced triple after a finite number of performances of the balancing act?

Problem B-4

Let C be a continuous closed curve in the plane which does not cross itself and let Q be a point inside C. Show that there exist points P_1 and P_2 on C such that Q is the midpoint of the line segment P_1P_2.

Problem B-5

Suppose that $a_1, a_2, \ldots, a_n$ are real $(n > 1)$ and

$$A + \sum_{i=1}^{n} a_i^2 < \frac{1}{n-1}\left(\sum_{i=1}^{n} a_i\right)^2.$$

Prove that $A < 2a_ia_j$ for $1 \leqslant i < j \leqslant n$.

Problem B-6

Let H be a subgroup with h elements in a group G. Suppose that G has an element a such that for all x in H, $(xa)^3 = 1$, the identity. In G, let P be the subset of all products $x_1ax_2a \cdots x_na$, with n a positive integer and the x_i in H.

(a) Show that P is a finite set.

(b) Show that, in fact, P has no more than $3h^2$ elements.

SOLUTIONS

In the 12-tuples $(n_{10}, n_9, \ldots, n_0, n_{-1})$ following each problem number below, n_i for $10 \geqslant n \geqslant 0$ is the number of students among the top 186 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solutions.

A-1. (137, 8, 2, 17, 1, 0, 1, 0, 3, 1, 2, 14)

A line meeting the graph in four points has an equation $y = mx + b$. Then the x_i are the roots of

$$2x^4 + 7x^3 + (3 - m)x - (5 + b) = 0,$$

their sum is $-7/2$, and their arithmetic mean $(\Sigma x_i)/4$ is $-7/8$, which is independent of the line.

A-2. (75, 39, 7, 4, 9, 9, 5, 2, 4, 3, 14, 15)

We show that w must equal one of x, y, z and that the remaining two unknowns must be negatives of each other. Let $s = x + y$ and $p = xy$. Then the given equations imply that $w - z = s$ and that

$$\frac{s}{p} = \frac{x+y}{xy} = \frac{1}{y} + \frac{1}{x} = \frac{1}{w} - \frac{1}{z} = \frac{z-w}{zw} = -\frac{s}{zw}.$$

Then $s/p = s/(-zw)$ implies that either $s = 0$ or $-zw = p$. If $s = 0$, then $y = -x$ and $w = z$. If $-zw = p = xy$, then $-z$ and w are the roots of the quadratic equation $T^2 - sT + p = 0$, which has x and y as its roots; this case thus leads to either $w = x$ and $-z = y$ or $w = y$ and $-z = x$.

A−3. (176, 3, 3, 0, 0, 0, 0, 0, 0, 1, 0, 3)

We show that there are an infinite number of expressions for $u(x)$ in terms of f and g; some of the simpler ones are:

$$\begin{aligned} u(x) &= g(x) - f(x+3) + f(x+1) + f(x-1) - f(x-3) \\ &= -g(x+2) + f(x+5) - f(x+3) + f(x+1) + f(x-1) \\ &= g(x+4) - f(x+7) + f(x+5) - f(x+3) + f(x+1). \end{aligned}$$

Let E be the shift operator on functions A defined by $EA(x) = A(x+1)$. Then $(E + E^{-1})u(x) = 2f(x)$ and $(E^4 + E^{-4})u(x) = 2g(x)$ are given. Thus $(E^2 + 1)u(x) = 2Ef(x)$ and $(E^8 + 1)u(x) = 2E^4 g(x)$. Motivated by the fact that $E^2 + 1$ and $E^8 + 1$ are relatively prime polynomials in E, one finds that

$$1 = \frac{1}{2}(E^8 + 1) - \frac{1}{2}(E^6 - E^4 + E^2 - 1)(E^2 + 1),$$

$$u(x) = \frac{1}{2}(E^8 + 1)u(x) - \frac{1}{2}(E^6 - E^4 + E^2 - 1)(E^2 + 1)u(x),$$

$$u(x) = E^4 g(x) - (E^6 - E^4 + E^2 - 1)Ef(x),$$

$$u(x) = E^4 g(x) + (-E^7 + E^5 - E^3 + E)f(x),$$

$$u(x) = g(x+4) - f(x+7) + f(x+5) - f(x+3) + f(x+1).$$

Other expressions are obtained using

$$\begin{aligned} g(y) &= -g(y-2) + f(y+3) + f(y-5) \\ &= -g(y+2) + f(y+5) + f(y-3). \end{aligned}$$

A-4. (73, 2, 26, 14, 7, 6, 3, 2, 6, 0, 9, 38)

$$\sum_{n=0}^{N}\frac{x^{2^n}}{1-x^{2^{n+1}}}=\sum_{n=0}^{N}\left(\frac{1}{1-x^{2^n}}-\frac{1}{1-x^{2^{n+1}}}\right)$$
$$=\frac{1}{1-x}-\frac{1}{1-x^{2^{N+1}}}\to\frac{1}{1-x}-1=\frac{x}{1-x}\text{ as } N\to\infty,$$

since $|x|<1$.

A-5. (66, 24, 9, 7, 1, 1, 1, 0, 5, 1, 22, 49)

It is well known that $\binom{p}{i}\equiv 0 \bmod p)$ for $i=1,2,\cdots,p-1$ or equivalently that in $Z_p[x]$ one has $(1+x)^p=1+x^p$, where Z_p is the field of the integers modulo p. Thus in $Z_p[x]$,

$$\sum_{k=0}^{pa}\binom{pa}{k}x^k=(1+x)^{pa}=\left[(1+x)^p\right]^a=[1+x^p]^a=\sum_{j=0}^{a}\binom{a}{j}x^{jp}.$$

Since coefficients of like powers must be congruent modulo p in the equality

$$\sum_{k=0}^{pa}\binom{pa}{k}x^k=\sum_{j=0}^{a}\binom{a}{j}x^{jp}$$

in $Z_p[x]$, one sees that

$$\binom{pa}{pb}\equiv\binom{a}{b}(\operatorname{mod} p)$$

for $b=0,1,\dots,a$.

A-6. (31, 10, 8, 4, 2, 4, 1, 2, 4, 18, 38, 64)

For (a,b) in S, let $I(a,b)$ be $\iint f(x,y)\,dx\,dy$ over the rectangle $0\leqslant x\leqslant a$, $0\leqslant y\leqslant b$. Also let (a,b) define inductively a sequence (a_n,b_n) using $a_1=a, b_1=b$, $a_{n+1}=a_n-b_n$ and $b_{n+1}=b_n$ when $0\leqslant b_n\leqslant a_n$, and $a_{n+1}=a_n$ and $b_{n+1}=b_n-a_n$ when $0\leqslant a_n<b_n$. Then the hypothesis implies that $I(a,b)=I(a_n,b_n)$ for all n. Since f is bounded on S and $\lim_{n\to\infty}a_n=0=\lim_{n\to\infty}b_n$, it follows that $I(a,b)=0$ for all (a,b) in S.

If $f(x,y)$ is not zero for all (x,y) in S, then f must be positive (or negative) in some rectangle $R=\{(x,y):c\leqslant x\leqslant d, h\leqslant y\leqslant k\}$ and hence $I=\int_R\int f(x,y)dxdy$ must be positive (or negative). But this contradicts

$$I=I(h,k)-I(h,d)-I(c,k)+I(c,d)=0.$$

Thus f is identically zero on S.

B-1. (32, 0, 42, 7, 2, 4, 1, 29, 16, 8, 7, 38)

$$\prod_{n=2}^{\infty}\frac{n^3-1}{n^3+1}=\lim_{k\to\infty}\left[\frac{2^3-1}{2^3+1}\cdot\frac{3^3-1}{3^3+1}\cdots\frac{k^3-1}{k^3+1}\right]$$

$$=\lim_{k\to\infty}\left[\frac{1\cdot 7}{3\cdot 3}\cdot\frac{2\cdot 13}{4\cdot 7}\cdot\frac{3\cdot 21}{5\cdot 13}\cdots\frac{(k-1)(k^2+k+1)}{(k+1)(k^2-k+1)}\right]$$

$$=\lim_{k\to\infty}\left[\frac{2}{3}\cdot\frac{k^2+k+1}{k(k+1)}\right]=\frac{2}{3}.$$

B-2. (5, 5, 22, 7, 16, 2, 2, 0, 7, 6, 17, 97)

Let O' be any point different from O on the line of intersection of planes AOC and BOD, e.g., O' may be the intersection of lines AC and BD. Let A' be the intersection of line OA with the line through O' and parallel to OC. Let C' be the intersection of line OC with the line through O' which is parallel to OA. Then $OA'O'C'$ is a parallelogram and its diagonals OO' and $A'C'$ bisect each other at a point M. Choosing B' and D' in the same way, one obtains a parallelogram $OB'O'D'$ whose diagonals OO' and $B'D'$ also bisect each other at the midpoint M of segment OO'. Hence segments $A'C'$ and $B'D'$ bisect each other (at M) and $A'B'C'D'$ is a parallelogram. (The parallelogram is not unique.)

B-3. (30, 6, 5, 2, 2, 2, 2, 0, 1, 2, 53, 81)

Let $x_i = \sum_{j=1}^{\infty} a_{ij} 2^{-j}$, with $a_{ij} \in \{0,1\}$, be the binary expansion of x_i. The triple is balanced if $a_{11} = a_{21} = a_{31} = 0$. Otherwise, $a_{i1} = 1$ for exactly one i and the balancing act produces $x_i' = \sum_{j=1}^{\infty} a_{i,j+1} 2^{-j}$. An unbalanced triple that remains unbalanced after any finite number of balancing acts is constructed by choosing the a_{ij} so that exactly one of a_{1j}, a_{2j}, a_{3j} equals 1 for each j while taking care that no one of sequences $a_{i1}, a_{i2}, \ldots$ repeats in blocks, i.e., that each x_i is irrational. One such solution has

$$\begin{aligned} a_{1j} &= 1 \quad \text{if and only if } j \in \{1, 9, 25, 49, \ldots\}, \\ a_{2j} &= 1 \quad \text{if and only if } j \in \{4, 16, 36, 64, \ldots\}, \\ a_{3j} &= 1 \quad \text{if and only if } j \in \{2, 3, 5, 6, \ldots\}. \end{aligned}$$

B-4. (38, 5, 1, 0, 0, 0, 0, 1, 0, 23, 87, 31)

We can assume that $Q = O$, the origin. Let $-C$ be the image of C under the reflection $P \to -P$. $-C$ is again a continuous closed curve surrounding O and $C \cap -C \neq \varnothing$ since they have the same diameter and both surround O (hence neither can be exterior to the other). Let $P_1 \in C \cap -C$. Then there exists $P_2 \in C$ such that $P_1 = -P_2$. These are the two desired points.

B-5. (6, 4, 4, 0, 0, 0, 0, 0, 9, 1, 47, 115)

From the Cauchy–Schwarz Inequality, one has

$$[(a_1 + a_2) + a_3 + a_4 + \cdots + a_n]^2 \leqslant [1^2 + 1^2 + \cdots + 1^2][(a_1 + a_2)^2 + a_3^2 + \cdots + a_n^2]$$

or

$$(\Sigma a_i)^2 \leqslant (n-1)[(\Sigma a_i^2) + 2a_1a_2] \quad \text{or} \quad [1/(n-1)](\Sigma a_i)^2 \leqslant (\Sigma a_i^2) + 2a_1a_2.$$

Using the hypothesis, one then has

$$A < -(\Sigma a_i^2) + \frac{1}{n-1}(\Sigma a_i)^2 \leqslant -(\Sigma a_i^2) + (\Sigma a_i^2) + 2a_1a_2 = 2a_1a_2.$$

Similarly, $A < 2a_ia_j$ for $1 \leqslant i < j \leqslant n$.

B-6. (6, 2, 11, 6, 3, 5, 6, 4, 9, 1, 26, 107)

Clearly $1 \in H$. Also $x \in H$ implies $x^{-1} \in H$. Then the hypothesis implies that $a^{-1} = a^2$ and that $xaxaxa = 1 = x^{-1}ax^{-1}ax^{-1}a$ when $x \in H$. Thus one easily shows that

$$\text{(i) } axa = x^{-1}a^2x^{-1}, \qquad \text{(ii) } a^2xa^2 = x^{-1}ax^{-1}.$$

Let

$$\begin{aligned} A &= \{xay : x, y \in H\}, \qquad B = \{xa^2y : x, y \in H\}, \\ C &= \{xa^2ya : x, y \in H\}, \quad \text{and} \quad Q = A \cup B \cup C. \end{aligned}$$

Each of A, B, C has at most h^2 elements; hence Q has at most $3h^2$ elements. Thus it suffices to prove that $x_1ax_2a\cdots x_na\in Q$ when each $x_i\in H$. We do this by induction on n.

For $n=1$, one sees that $x_1a=x_1a\cdot 1\in A\subseteq Q$. Now let $x_1,x_2,\ldots,x_{k+1}\in H$; $x_1ax_2a\cdots x_ka=q, x_{k+1}=z$, and $qza=p$. Inductively, we assume $q\in Q$ and seek to show $p\in Q$. The assumption implies that q is in A, B, or C. If $q=xay\in A$, then

$$p=(xay)za=xa(yz)a=x(yz)^{-1}a^2(yz)^{-1}\in B\subseteq Q,$$

using (i). If $q=xa^2y\in B$, then $p=xa^2yza\in C\subseteq Q$. If $q=xa^2ya\in C$, then

$$p=xa^2y(aza)=xa^2y(z^{-1}a^2z^{-1})=x\left[a^2(yz^{-1})a^2\right]z^{-1}$$
$$=x(yz^{-1})^{-1}a(yz^{-1})^{-1}z^{-1}\in A\subseteq Q,$$

using (i) and (ii).

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A STABILITY ANALYSIS OF A PROTOTYPE MOVING BOUNDARY PROBLEM IN HEAT FLOW AND DIFFUSION

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The reduction of a complex problem to a model whose analysis involves elementary mathematics, and which leads to results that may be verified by means of the data available, is exciting applied mathematics in spite of the fact that the mathematics used is not novel. This situation does not occur often.—A. H. Taub

1. A prototype problem. A stability analysis involving the controlled growth of a pure solid metal into a thermally undercooled liquid metallic bath **[12]** serves as an excellent example of the sort of problem which might have motivated Taub to make the above astute observation **[19]**. In this section we shall discuss that problem and then develop in Section 2 a linear stability analysis relevant to it.

The term *controlled* indicates that the mean position of the interface between the liquid and solid phases is advanced into the pure melt at a constant specified rate. Although this mean position could be of any desired shape, such as spherical or cylindrical, we shall assume in what follows that it is *planar* for ease of exposition. What is of interest here (and industrially) is that we determine solidification rates and bath temperatures for which the interface either maintains its planar shape during the whole solidification period or assumes instead a cellular (or dendritic) structure.

A two-dimensional mathematical model (**[1]**, **[24]**) can be employed which introduces a moving spatial coordinate system, denoted by (x,z), such that the x axis coincides with the mean

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interface position while the z coordinate measures the distance from it. The time variable for this model is, as usual, denoted by t. In an actual experiment the extent of the liquid and solid phases is naturally finite, but a simplifying assumption in models of this sort ([**18**]) extends z to positive and negative infinity. The model further postulates diffusion of heat in the absence of convection in such a way that thermal diffusivity is the same in both phases. In what follows, we shall consider the common value of this diffusivity as normalized to one.

The interface is defined at any time t to be the locus of points (x,z) which satisfy the relation

$$z=\varepsilon\zeta(x,t).$$

Here ζ must be determined and ε (a measure of the maximum deviation from a plane) is assumed for the purpose of linear theory to be much less than one in absolute value. Letting T and T' describe the temperatures in the liquid and solid, respectively, the governing diffusion equations can be written as follows ([**24**]). For $z>\varepsilon\zeta(x,t)$ (in the liquid):

$$\nabla^2 T+\frac{\partial T}{\partial z}-\frac{\partial T}{\partial t}=0. \tag{1.1}$$

For $z<\varepsilon\zeta(x,t)$ (in the solid):

$$\nabla^2 T'+\frac{\partial T'}{\partial z}-\frac{\partial T'}{\partial t}=0. \tag{1.2}$$

At the interface, $z=\varepsilon\zeta(x,t)$, we have the following conditions:

$$T=T'+1, \tag{1.3}$$

$$T=Q+\varepsilon\frac{\partial^2\zeta}{\partial x^2}\left[1+\varepsilon^2\left(\frac{\partial\zeta}{\partial x}\right)^2\right]^{-3/2} \tag{1.4}$$

and

$$\frac{\partial T'}{\partial \nu}-\frac{\partial T}{\partial \nu}=\left(1+\varepsilon\frac{\partial\zeta}{\partial t}\right)\left[1+\frac{1}{2}\varepsilon\frac{\partial^2\zeta}{\partial x^2}\left\{1+\varepsilon^2\left(\frac{\partial\zeta}{\partial x}\right)^2\right\}^{-3/2}\right] \tag{1.5}$$

where $\partial/\partial\nu=\partial/\partial z-\varepsilon(\partial\zeta/\partial x)\partial/\partial x$. The quantity Q in (1.4), a measure of the amount of undercooling, represents the difference between the melting temperature of the metal at a flat interface and the bath temperature for the liquid phase. Equation (1.3) represents continuity of temperature at the interface as measured from the same zero level (say absolute zero) in each phase, while equation (1.4) describes the alteration of that temperature from the normal melting temperature of the metal at a flat interface due to the curvature of the interface itself. When we use a general balance equation from continuum mechanics ([**5**], [**23**]), equation (1.5) follows from conservation of energy at the interface. The left-hand side of that equation represents net heat flux through the interface, while the right-hand side takes into account the heat released by solidification at the interface and the entropy necessary to stretch that surface from a plane into a larger nonplanar area.

Note that the model equations (1.1)–(1.5) have only one *controllable* parameter Q (i.e., the bath temperature can be changed), since the solidification rate has been held constant and the material parameters are determined by the particular metal selected. A more general model can be found in [**24**]. Also note that by assuming the liquid to be *undercooled*, we implicitly assume $Q>0$.

There exists an equilibrium solution to the governing system of equations which represents the uniform growth of a *planar* interface of solid metal into the molten phase. This planar interface solution is given by

$$\begin{aligned} &T=T_0(z)=Qe^{-z}, \qquad z>0,\\ &T'=T_0'(z)=(Q-1)e^{-z}, \qquad z<0,\\ &\zeta=0. \end{aligned}$$

We expect that far from the interface the influence of the shape of that interface on the temperature fields will become negligible. This means that

$$T \to T_0(z) \quad \text{as } z \to \infty; \qquad T' \to T_0'(z) \quad \text{as } z \to -\infty. \tag{1.6}$$

Equation (1.6) is the proper boundary condition to be applied as $|z| \to \infty$ and along with equations (1.1) and (1.2) and the boundary conditions (1.3)–(1.5) constitutes the mathematical formulation of our problem. In addition, x extends to positive and negative infinity, and we adopt the implicit requirement that the dependent variables remain bounded as $|x| \to \infty$.

2. The linear stability analysis. We have developed a system of nonlinear partial differential equations and boundary conditions which model our prototype problem. There are now two methods by which we might proceed. One is to obtain an explicit solution to (1.1) and (1.2) which satisfies (1.3)–(1.6). Advances in the theory and technique of solving nonlinear differential systems within the last twenty years have made it possible to obtain numerical or even analytical results which can be compared directly with experimental data. However, it may be difficult to determine from such a solution the relationship between the size of Q and the resulting interface shape: planar or nonplanar. We prefer, instead, the following approach.

All physical systems are subject to fluctuations in "controlled" quantities. Thus, the bath temperature and the solidification speed cannot be maintained exactly constant, and material properties may change slightly with time and variations in temperature. These tend to affect the shape of the advancing interface by producing infinitesimal perturbations on the basically planar shape. We shall show that, for some choices of the controllable parameter Q, all possible perturbations must die out with increasing time, but for other choices some perturbations must grow. In the latter case, the interface eventually changes from a planar to a cellular morphology. In order to distinguish mathematically between these cases, we can employ stability analysis (see Section 5). Although not explicitly producing a solution to a differential system, such analysis can provide valuable information about a phenomenon modeled by that differential system. Indeed, as pointed out by Gelder and Guy in [**11**], the *solution* is rarely the main priority. *It is more important to determine those factors which control a phenomenon and study how changes in these affect the behavior of that phenomenon.* Linear and nonlinear stability theory seem almost ideally suited for this task. In this section we shall introduce the concept of linear analysis (while leaving a discussion of nonlinear theory until Section 5) as applied to our prototype problem.

We can symbolically represent this analysis by considering a solution $\mathbf{q}(x,z,t;\varepsilon)$ of the basic equations of the form

$$\mathbf{q}(x,z,t;\varepsilon) = \mathbf{q}_0(z) + \varepsilon \mathbf{q}_1(x,z,t) + O(\varepsilon^2), \tag{2.1}$$

where $\mathbf{q}_0(z)$ is the planar interface solution, $|\varepsilon| \ll 1$ and

$$\mathbf{q}_1(x,z,t) = \mathbf{q}_{11}(z)\cos(\omega x)e^{\sigma t} \tag{2.2}$$

is an arbitrary component of a Fourier decomposable perturbation of wavelength $\lambda = 2\pi/\omega$. Substituting (2.1) into the basic equations, neglecting terms of order $O(\varepsilon^2)$, using (2.2) and canceling the common factor $\varepsilon\cos(\omega x)e^{\sigma t}$ yield an eigenvalue problem for σ with corresponding eigenvector $\mathbf{q}_{11}(z)$. To ensure the existence of nontrivial solutions, σ must satisfy the secular equation

$$\sigma + \left\{\tfrac{1}{2}(1-\omega^2)\right\} = 2\left\{Q - \omega^2 - \tfrac{1}{2}\right\}\sqrt{\sigma + \left\{\omega^2 + \tfrac{1}{4}\right\}} \tag{2.3}$$

where σ and ω are the growth rate and the wave number of the disturbance, respectively. The branch of the square root in (2.3) is defined so that

$$\mathrm{Re}\sqrt{\sigma + \omega^2 + \tfrac{1}{4}} > \tfrac{1}{2} \tag{2.4}$$

and has been selected to ensure that the perturbation quantities in temperature vanish far from the interface (i.e., as $|z| \to \infty$) in accordance with (1.6). It is possible that for some values of Q and ω^2 no σ will satisfy both (2.3) and (2.4) simultaneously. In that case, we have only the trivial solution—i.e., $\mathbf{q}_{11}(z) = \mathbf{0}$ identically.

It is the stability of the planar interface solution with which we are concerned. Since the time dependence of our infinitesimal perturbations is of the form $e^{\sigma t}$, we say that the planar interface solution is *asymptotically* stable, unstable, or neutrally stable to this type of disturbance for σ satisfying (2.3) and (2.4) according to whether

$$|e^{\sigma t}|\to 0, \infty, \text{ or } 1 \text{ as } t\to\infty.$$

This in turn depends on whether the sign of $\operatorname{Re}\sigma$ is less than, greater than, or equal to zero for such σ. If, for a certain range of parameter values, there exist no solutions σ to (2.3) and (2.4)—i.e., no eigenvalues—then $\mathbf{q}_{11}(z)=\mathbf{0}$ and we say that in this parameter range the planar interface solution is *identically stable* to infinitesimal disturbances. We have used a normal mode or separation of variables approach. The normal mode method analyzes the discrete or point spectrum of the eigenvalue σ while transform methods analyze both the continuous and the point spectrum. From the work of Sekerka [**16**] and Delves [**2**], it can be concluded that for this problem one can predict the onset of instability by determining where the point spectrum eigenvalues σ satisfy $\operatorname{Re}\sigma>0$ for a disturbance proportional to $e^{\sigma t}$.

In the next section, we shall develop a general criterion by which the stability of the system governed by (2.3) and (2.4) can be determined. We can then return to this problem and complete the stability analysis.

3. The stability criterion for a semi-quadratic. The secular equation (2.3) arising in Section 2 is a special case of the following secular equation, called a *semi-quadratic*:

$$\sigma+a=2b\sqrt{\sigma+c}\,, \tag{3.1a}$$

where a, b, and c are real numbers. The branch of the square root in (3.1a) must satisfy the *extraneous root condition* (corresponding to (2.4)):

$$\operatorname{Re}\sqrt{\sigma+c}\geqslant\alpha\geqslant 0. \tag{3.1b}$$

We now define the *most dangerous mode* of any secular equation in σ [**3, 14**] to be that value of σ satisfying the equation which has the *largest* real part, and hence governs the stability behavior. If we designate this most dangerous mode by σ_0, then we must obtain conditions on a, b, c, and α for which σ_0, and hence the sign of $\operatorname{Re}\sigma_0$, can be determined.

Note that each root of (3.1a) must also be a root of the *associated quadratic*:

$$\sigma^2+2S_1\sigma+S_0=0 \tag{3.2}$$

with

$$S_0=a^2-4b^2c \text{ and } S_1=a-2b^2, \tag{3.3}$$

so there can be at most two roots of (3.1a). The stability behavior of a quadratic is already well established ([**20**]), so if both values of σ which satisfy the quadratic secular equation (3.2) are also solutions of the semi-quadratic (3.1a) then there is nothing left to do. However, this is not always the case, since it is now necessary for such values of σ to satisfy, in addition, relation (3.1b). It is possible that certain roots of (3.2) may not fulfill this second condition for particular values of a, b, and c and hence not satisfy the semi-quadratic. We shall say such roots are *extraneous*. We note that this whole procedure depends on solving what is often called a radical equation and avoiding the "extraneous roots" by paying attention to the sign of the radical. Although this involves rather elementary mathematics, it is most interesting and encouraging in the spirit of our introductory quotation to see such equations arising in a genuine physical context.

Because they are needed for our subsequent analysis, we first summarize, without proof, the following standard results ([**20**]) for a general quadratic:

LEMMA. *Let x^+ and x^- be the roots of the equation*

$$x^2+2Ax+B=0$$

where A and B are real numbers and $\mathrm{Re}\,x^+ \geqslant \mathrm{Re}\,x^-$. *Then,*

(i) $\mathrm{Re}\,x^+ < 0$ *iff* $[A>0]\wedge[B>0]$,
(ii) $\mathrm{Re}\,x^+ > 0$ *iff* $[A<0]\vee[B<0]$,
(iii) $\mathrm{Re}\,x^+ = 0$ *iff* $\{[A=0]\wedge[B>0]\}\vee\{[A\geqslant 0]\wedge[B=0]\}$,
(iv) $\mathrm{Re}\,x^- < 0$ *iff* $[A>0]\vee[B<0]$,
(v) $\mathrm{Re}\,x^- > 0$ *iff* $[A<0]\wedge[B>0]$,
(vi) $\mathrm{Re}\,x^- = 0$ *iff* $\{[A=0]\wedge[B>0]\}\vee\{[A\leqslant 0]\wedge[B=0]\}$,
(vii) $\mathrm{Re}\,x^{\pm}=0$, $\mathrm{Im}\,x^{\pm}\neq 0$ *iff* $[A=0]\wedge[B>0]$.

We can now determine which of the roots of (3.2) is the most dangerous mode, σ_0, of (3.1) for given a, b, c, and α.

THEOREM 1 [Extraneous Roots]. *Let σ^+ and σ^- be the roots of* (3.2)–(3.3), *where* $\mathrm{Re}\,\sigma^+ \geqslant \mathrm{Re}\,\sigma^-$ *and define*:

$$E_1=\alpha-b \text{ and } E_0=\alpha^2-2\alpha b+a-c. \tag{3.4}$$

(i) *If $E_1\leqslant 0$, σ^+ is always a root of* (3.1) *and if in addition $E_0\geqslant 0$, σ^- is also a root.*
(ii) *If $0\leqslant b$, $E_1>0$ and $E_0\leqslant 0$, σ^+ is the only root of* (3.1).
(iii) *If $b<0$ and $E_0\leqslant 0$, σ^- is the only root of* (3.1).
(iv) *If $E_1>0$ and $E_0>0$,* (3.1) *has no roots.*

Proof. For each root σ of (3.2) define the quantity τ by

$$\tau=\sqrt{\sigma+c}\;-\alpha, \tag{3.5}$$

where the branch of the square root in the above is left unspecified. If we assume the branch of the square root in (3.1a) is also unspecified and use (3.5) to replace σ in that equation, then τ is seen to satisfy the quadratic

$$\tau^2+2E_1\tau+E_0=0. \tag{3.6}$$

In view of (3.1b), σ will be a root of our semi-quadratic if τ also satisfies

$$\mathrm{Re}\,\tau\geqslant 0. \tag{3.7}$$

Let us designate the roots of (3.6) by

$$\tau_1=\sqrt{\sigma_1+c}\;-\alpha \text{ and } \tau_2=\sqrt{\sigma_2+c}\;-\alpha \tag{3.8}$$

where $\mathrm{Re}\,\tau_1\geqslant\mathrm{Re}\,\tau_2$ and $\sigma_{1,2}$ satisfy (3.1a) in the manner described above. Then

$$\sigma_1-\sigma_2=2b\left(\sqrt{\sigma_1+c}\;-\sqrt{\sigma_2+c}\;\right)=2b(\tau_1-\tau_2)$$

and hence

$$\mathrm{Re}\,\sigma_1-\mathrm{Re}\,\sigma_2=2b(\mathrm{Re}\,\tau_1-\mathrm{Re}\,\tau_2). \tag{3.9}$$

Recalling that $\mathrm{Re}\,\sigma^+\geqslant\mathrm{Re}\,\sigma^-$ and making use of (3.8), we deduce from (3.9) that

$$\sigma_1=\begin{cases}\sigma^+ \text{ if } b\geqslant 0\\ \sigma^- \text{ if } b<0\end{cases}. \tag{3.10}$$

Now, identifying x, A, and B of the Lemma with τ, E_1, and E_0 of (3.6), using (3.7) and (3.10), we have the following.

If $E_1>0$ and $E_0>0$, then $\mathrm{Re}\,\tau_1<0$, so (3.1) has no roots, and (iv) is established.

If $E_1\leqslant 0$ and $E_0\geqslant 0$, then $\mathrm{Re}\,\tau_2\geqslant 0$ so both σ^+ and σ^- are roots of (3.1). Also, if $E_0<0$ and $E_1\leqslant 0$, then $\mathrm{Re}\,\tau_1\geqslant 0>\mathrm{Re}\,\tau_2$ and since $b>0$, we see that $\sigma_1=\sigma^+$ is the only valid root of (3.1). This establishes (i).

Finally, if $E_1>0$ and $E_0 \leqslant 0$, then $\operatorname{Re}\tau_1 \geqslant 0 > \operatorname{Re}\tau_2$ again, so σ_1 is the only valid root of (3.1). Now (ii) and (iii) follow directly from (3.10).

This completes the proof.

Motivated by our earlier discussion of the selection of the most dangerous mode of any secular equation in conjunction with Theorem 1, it follows that the semi-quadratic (3.1) must be identically stable whenever $E_1>0$ and $E_0>0$. Further, the most dangerous mode is given by σ^- whenever $b<0$ and $E_0 \leqslant 0$ and by σ^+ otherwise.

It remains only to determine asymptotic stability conditions for the semi-quadratic of (3.1). We shall do this by means of the following theorem.

THEOREM 2 [Stability]. *Let* σ_0 *be the most dangerous mode for the semi-quadratic* (3.1) *and let* S_0 *and* S_1 *be defined by* (3.3). *Then whenever* σ_0 *is nonextraneous, we have*:

(i) $\operatorname{Re}\sigma_0=\sigma_0=0$ *when* $[b \geqslant 0]\wedge[S_1 \geqslant 0]\wedge[S_0=0]$ *or* $[b<0]\wedge[S_1 \leqslant 0]\wedge[S_0=0]$.

(ii) $\operatorname{Re}\sigma_0=0, \operatorname{Im}\sigma_0 \neq 0$ *when* $[S_1=0]\wedge[S_0>0]$.

(iii) $\operatorname{Re}\sigma_0<0$ *when* $[b \geqslant 0]\wedge[S_1>0]\wedge[S_0>0], [b<0]\wedge[S_0<0]$ *or* $[b<0]\wedge[S_1>0]\wedge[S_0 \geqslant 0]$.

(iv) $\operatorname{Re}\sigma_0>0$ *when* $[b \geqslant 0]\wedge[S_1<0]\wedge[S_0 \geqslant 0], [b \geqslant 0]\wedge[S_0<0]$ *or* $[b<0]\wedge[S_1<0]\wedge[S_0>0]$.

Proof. Since

$$\sigma_0=\begin{cases}\sigma^+ \text{ if } b \geqslant 0\\ \sigma^- \text{ if } b<0\end{cases},$$

(except when σ^+ or σ^- is extraneous) this theorem follows virtually directly from the Lemma when the identification of σ, S_1, and S_0 for x, A, and B is employed, because (for example) a condition such as $[b<0]\wedge\{[S_1>0]\vee[S_0<0]\}$ is equivalent to the *two* conditions $[b<0]\wedge[S_0<0]$ and $[b<0]\wedge[S_1>0]\wedge[S_0 \geqslant 0]$.

This completes the proof.

In most applications involving the semi-quadratic, the quantities a, b, c, and α are functions of some set of parameters. It can then be convenient to determine the sign of $\operatorname{Re}\sigma_0$ in a graphical manner, by plotting the implicitly defined curves $E_0=0$, $E_1=0$, $b=0$, $S_0=0$, and $S_1=0$. Next the region of identical stability (where $E_0>0$ and $E_1>0$) and the region where $\sigma_0=\sigma^-$ ($b<0$ and $E_0 \leqslant 0$) can be determined. Of course $\sigma_0=\sigma^+$ everywhere else. Finally, the stability of the system can be determined in the nonextraneous region by using Table 1, which is a summary of the results of Theorem 2.

TABLE 1. Summary of the results of Theorem 2 indicating the sign of $\operatorname{Re}\sigma_0$ for combinations of possible choices of the signs for S_0, S_1, and b.

Sign of:	S_0	S_1	b	$\operatorname{Re}\sigma_0$
	+	+	+, 0, −	−
	+	0	+, 0, −	0*
	+	−	+, 0, −	+
	0	+	+, 0	0
	0	+	−	−
	0	0	+, 0, −	0
	0	−	+, 0	+
	0	−	−	0
	−	+, 0, −	+, 0	+
	−	+, 0, −	−	−

*$\operatorname{Im}\sigma_0 \neq 0$.

For the purposes of this graphical analysis, let us define the curves M_o, M_s^+ and M_s^- in terms of S_0 and S_1 of (3.3) by

$$M_o = [S_o > 0] \cap [S_1 = 0],$$
$$M_s^+ = [S_0 = 0] \cap [S_1 \geqslant 0],$$

and

$$M_s^- = [S_0 = 0] \cap [S_1 \leqslant 0].$$

It then follows from Theorem 2 that $\mathrm{Re}\,\sigma_0 = 0$ on the composite curve M defined by

$$M = \begin{cases} M_o \cup M_s^+ & \text{if } b \geqslant 0 \\ M_o \cup M_s^- & \text{if } b < 0 \end{cases},$$

and this curve separates the regions of stability (where $\mathrm{Re}\,\sigma_0 < 0$) from the regions of instability (where $\mathrm{Re}\,\sigma_0 > 0$). Thus, M is often referred to as the *marginal stability curve*. Note that on M_o, $\mathrm{Im}\,\sigma_0 \neq 0$, hence for this segment of the marginal curve the onset of instability occurs from an *oscillatory* state while on M_s, $\mathrm{Im}\,\sigma_0 = 0$, hence for this segment the onset of instability occurs from a *stationary* one. If M_o is the empty set, then the marginal state can be characterized by $\sigma_0 = 0$ and the *principle of exchange of stabilities* is said to be valid ([**14**]).

4. Application to the prototype problem. We shall now return to the prototype problem introduced in Sections 1 and 2, where we wished to determine the values of Q for which a planar or a nonplanar interface could be anticipated during the solidification process. As we have seen, this depends upon when the most dangerous mode, σ_0, for the secular equation (2.3) satisfies $\mathrm{Re}\,\sigma_0 < 0$ or $\mathrm{Re}\,\sigma_0 \geqslant 0$, respectively.

From (2.3), (2.4), and (3.1), we make the identification:

$$a = \tfrac{1}{2}(1-\omega^2), \quad b = Q - \omega^2 - \tfrac{1}{2}, \quad c = \omega^2 + \tfrac{1}{4}, \qquad \text{and } \alpha = \tfrac{1}{2}.$$

Since a, b, and c are functions of ω^2 and Q, we can plot the regions of asymptotic stability, neutral stability, and instability, as well as identical stability, in the $\omega^2 - Q$ plane using the results of Section 3. Toward that end, we list the curves relevant to that analysis in Table 2, where each is represented in the form $Q = Q(\omega^2)$. Note that the curve $S_0 = a^2 - 4b^2c = 0$ has been decomposed into the two branches: $S_0^+ = a - 2b\sqrt{c} = 0$ and $S_0^- = a + 2b\sqrt{c} = 0$. Also included in Table 2 is the curve $\Delta = 0$, where $\Delta = b^2 + c - a$ is the discriminant for the associated quadratic.

TABLE 2. The relevant extraneous root curves and stability curves for the prototype problem.

Curve	$Q = Q(\omega^2)$
$E_0 = \alpha^2 - 2\alpha b + a - c = 0$	$1 - \frac{1}{2}\omega^2$
$E_1 = b - \alpha = 0$	$1 + \omega^2$
$b = 0$	$\frac{1}{2} + \omega^2$
$S_0^+ = a - 2b\sqrt{c} = 0$	$\frac{1}{2} + \omega^2 + \frac{1}{2}(1-\omega^2)/(1+4\omega^2)^{1/2}$
$S_0^- = a + 2b\sqrt{c} = 0$	$\frac{1}{2} + \omega^2 - \frac{1}{2}(1-\omega^2)/(1+4\omega^2)^{1/2}$
$S_1 = a - 2b^2 = 0$	$\frac{1}{2} + \omega^2 \pm \frac{1}{2}(1-\omega^2)^{1/2}, 0 \leqslant \omega^2 \leqslant 1$
$\Delta = b^2 + c - a = 0$	$\frac{1}{2} + \omega^2 \pm \frac{1}{2}(1-6\omega^2)^{1/2}, 0 \leqslant \omega^2 \leqslant \frac{1}{6}$

In Figure 1 we plot the curves $b = 0$, $E_1 = 0$, and $E_0 = 0$ for this problem and indicate the regions of the $\omega^2 - Q$ plane which correspond to the various parts of Theorem 1. Note that in the physically realizable first quadrant of that plane there are no roots for $E_0 > 0$, while for $E_0 \leqslant 0$, σ^+ is the only root for $b \geqslant 0$ and σ^- the only root for $b < 0$.

In Figure 2 we plot the curves $S_1 = 0$, $S_0^- = 0$, $S_0^+ = 0$, $\Delta = 0$, $b = 0$, and $E_0 = 0$ in the first

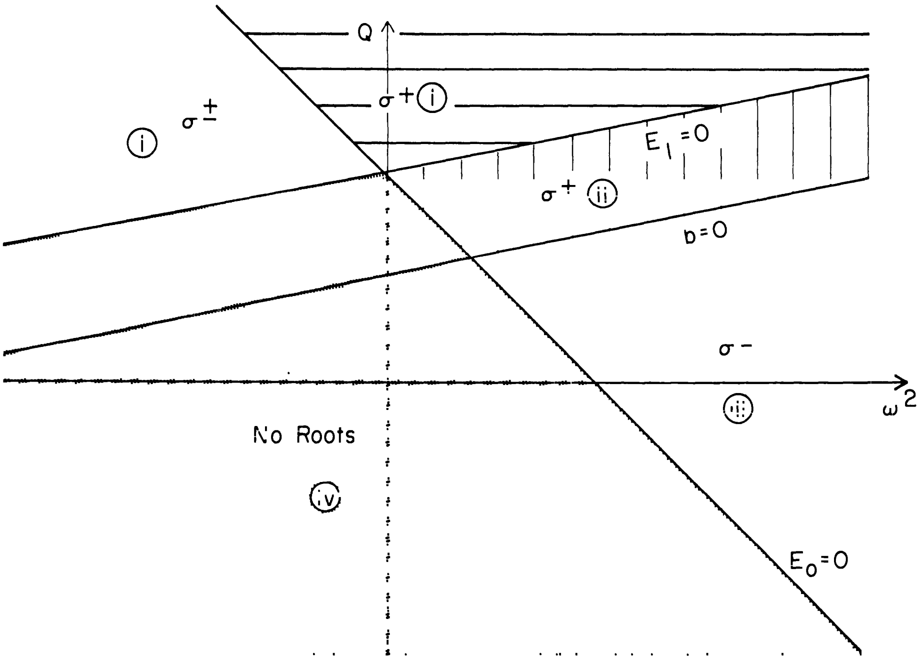

FIG. 1. A plot in the ω^2-Q plane identifying the various regions of extraneousness which correspond to those mentioned in Theorem 1 for our prototype problem. The circled numerals refer to the various parts of that theorem.

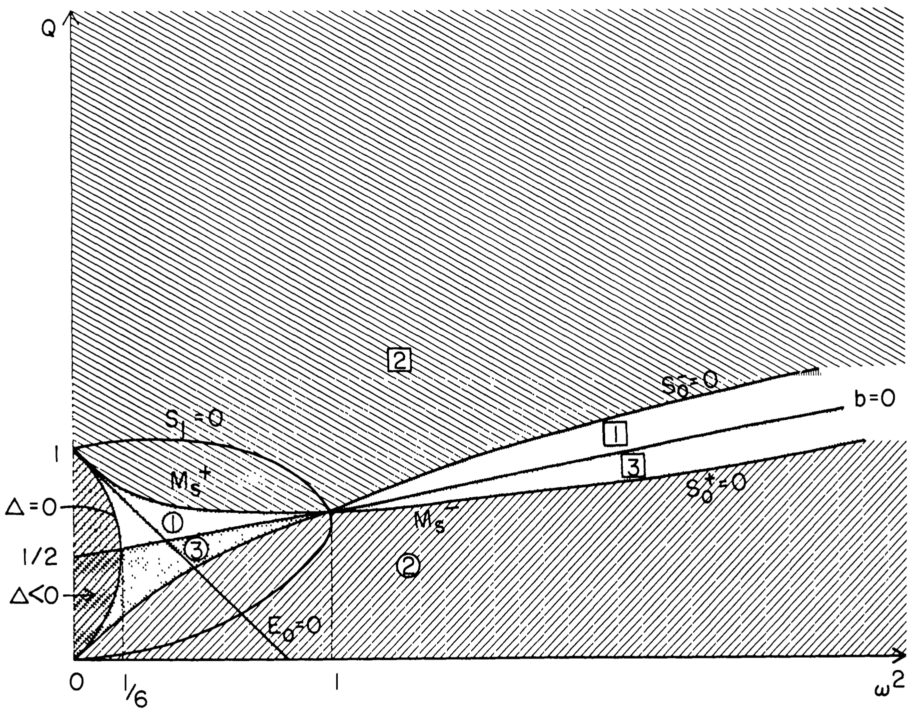

FIG. 2. A schematic plot in the physically realizable first quadrant of the ω^2-Q plane depicting the asymptotic stability behavior for our prototype problem as predicted by Theorem 2. The circled numbers correspond to the three stability subregions of part (iii) while the boxed numbers refer to the three instability subregions of part (iv).

quadrant of the $\omega^2 - Q$ plane and indicate the regions of asymptotic stability, instability, and neutral stability determined from Theorem 2. Note that M_o is empty for this problem and the marginal curve is given by

$$M_s = M_s^+ \cup M_s^- = [S_0^+ = 0] \tag{4.1a}$$

where

$$M_s^+ = [S_0^+ = 0] \cap [b \geqslant 0], \qquad M_s^- = [S_0^+ = 0] \cap [b < 0]. \tag{4.1b}$$

We now incorporate all the relevant stability information from Figures 1 and 2 into Figure 3. Since the locus $\Delta < 0$ lies in the region of identical stability which corresponds to part (iv) of Theorem 1, there exist only real solutions to the semi-quadratic and hence the most dangerous mode σ_0 is real. The marginal stability curve (4.1), which can be characterized by $\sigma_0 = 0$ and separates the region of instability where $\sigma_0 > 0$ from that of stability where $\sigma_0 < 0$, is given by

$$Q = Q_c(\omega^2) = \tfrac{1}{2} + \omega^2 + \tfrac{1}{2}(1-\omega^2)/(1+4\omega^2)^{1/2}. \tag{4.2}$$

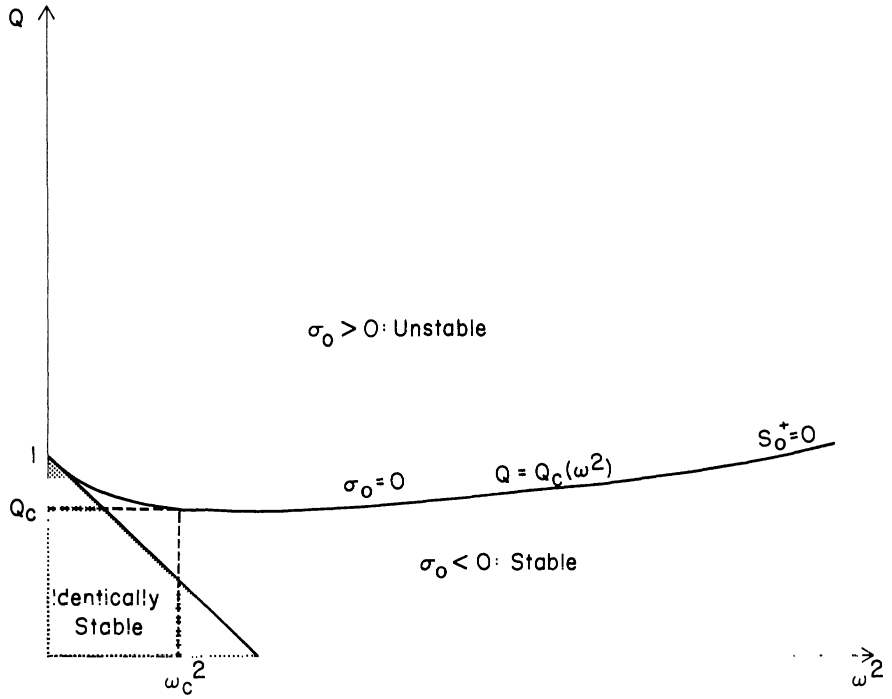

FIG. 3. Plot of Q, a dimensionless parameter measuring undercooling, versus ω^2, the square of the wave number of the disturbance. The curve $Q = Q_c(\omega^2)$ on which $\sigma_0 = 0$ is the marginal stability curve separating the unstable region where $\sigma_0 > 0$ from the stable one where $\sigma_0 < 0$. The region of the $\omega^2 - Q$ plane corresponding to identical stability is designated by shading.

As can be seen from Figure 3, the marginal curve (4.2) has an absolute minimum at $\omega = \omega_c$ where ω_c, the *critical wave number*, is such that

$$\frac{dQ_c}{d\omega^2}(\omega_c^2) = 0,$$

which implies

$$(4\omega_c^2 + 1)^{3/2} - \left(\tfrac{1}{2} + \omega_c^2\right) = 1. \tag{4.3}$$

Corresponding to ω_c is an associated value of Q (denoted by Q_c in Fig. 3) such that

$$Q_c = Q_c(\omega_c^2) = \tfrac{1}{2} + \omega_c^2 + \tfrac{1}{2}(1-\omega_c^2)/(1+4\omega_c^2)^{1/2}. \tag{4.4}$$

From Figure 4 we see that for $Q<Q_c$ there exist no wave-numbers, ω, such that $\sigma_0>0$ but for $Q>Q_c$ there exists a band of such wave-numbers corresponding to growing disturbances. This means that for $Q<Q_c$ one has stability and for $Q>Q_c$ one has instability.

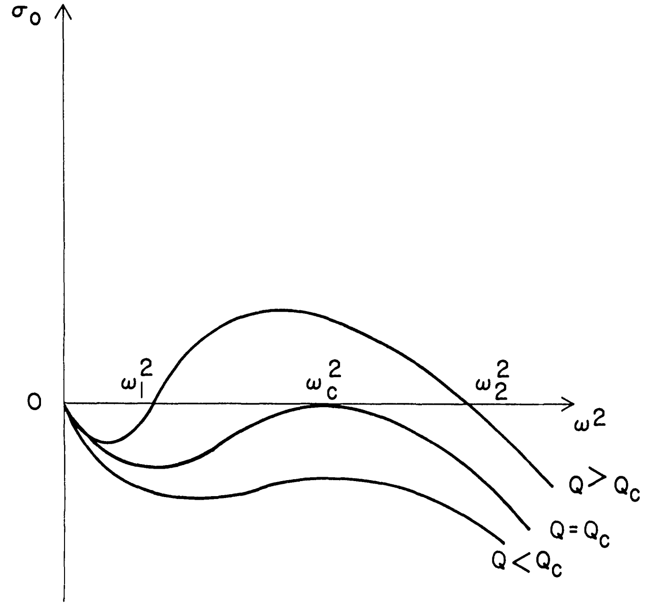

FIG. 4. A plot of σ_0 versus ω^2 corresponding to Figure 3, which shows that for $Q<Q_c$ there exist no ω^2 such that $\sigma_0>0$ while for $Q>Q_c$ there exists a band of such ω^2 centered about ω_c^2—i.e., $\omega^2 \varepsilon(\omega_1^2, \omega_2^2)$.

In an actual problem, one could solve (4.3) for ω_c^2 and insert this value into (4.4) to obtain the critical amount of undercooling Q_c. We could then physically interpret the results of the stability analysis as follows. For small degrees of undercooling ($Q<Q_c$), a planar interface would be expected since all perturbations decay. However, if the bath is undercooled sufficiently ($Q>Q_c$), the formation of cells can be anticipated (see Section 5 which follows for a discussion of this expectation).

5. Discussion. The main result of the linear stability analysis presented in this paper in connection with the planar solid-liquid interface is the determination of the *critical conditions* for the onset of instability to initially infinitesimal perturbations. Interpreting this, we have stated that one could expect unstable perturbations to give rise to a cellular morphology. Perhaps this latter result should be explained, since we have not explicitly shown it to be true.

The existence of planar, cellular, and dendritic growth patterns are all experimentally well documented ([**18**]). The interpretation of our mathematical results could be made with this fact in mind, but that is not necessary. It is possible to describe the time-dependent shape of a morphologically unstable solid-liquid interface for times sufficiently small that linear theory is still applicable ([**17**]) and the resulting interface shape *resembles* a cosine curve (i.e., a "corrugated" surface). This should be compared to our result,

$$z = \varepsilon \zeta_{11} \cos(\omega_c x) e^{\sigma_0 t}$$

(where ζ_{11} is a constant), which describes the interface shape due to the "most dangerous mode"

perturbation. Here, the cosine shape is clear, but the term $e^{\sigma_0 t}$ makes it appear that the amplitude of this disturbance might grow without bound (since $\sigma_0 > 0$ for an unstable perturbation). This "unrealistic" result arises because non-linear terms in the original model equations (1.1)–(1.6) have been neglected in our linear stability analysis.

Determining the long-time behavior of growing disturbances, and thus the eventual *visible* interface shape, as well as the effect of perturbations which are not initially infinitesimal, lies in the realm of nonlinear stability theory ([**21**], [**22**]). Although a nonlinear stability analysis has not been done for this pure metal problem, in a similar investigation involving dilute binary alloys [**21**] the non-linear terms are found to give rise to *stable* bands of width $\lambda = 2\pi/\omega_c$ when relatively large rates of solidification are employed. This is in qualitative agreement with the results of [**17**], and mathematical similarity *suggests* that analogous results would hold for our problem too, since our fixed rate of solidification is rather large. These problems have been done in two dimensions rather than a more realistic three dimensions. However, a three-dimensional nonlinear stability analysis for certain convective phenomena, which also sometimes display a cellular morphology [**14**], has shown that stable hexagons (i.e., cells) exist where a two dimensional analysis predicts bands. That *mathematical* prediction also suggests that a similar result might hold for our problem. Of course, to actually predict a cellular morphology, it would be necessary to do a three-dimensional nonlinear stability analysis for this pure metal problem. With this in mind, it should be noted that many commonly employed nonlinear stability techniques, such as the Stuart–Watson method, and the approaches of Eckhaus and Busse (all of which are surveyed in depth by Segel [**14**] and DiPrima [**3**]), as well as the more recent work of Kogelman and Keller [**6**] require complete knowledge of linear stability behavior. Hence, the linear stability results in this paper can be put to good use as the necessary first step in one of these nonlinear processes.

Our solidification problem is a prototype of a much more general class of physical phenomena that are included among those which can be modeled by the various techniques of continuum mechanics ([**7**], [**15**]). When modeled in this manner, this class of phenomena give rise to inherent surfaces at which one or more of the dependent variables suffer discontinuities. These are usually referred to as *surfaces of discontinuity*, and careful application of the conservation laws, which are postulated in a continuum mechanical treatment ([**7**], [**15**]), yields jump-type boundary conditions satisfied by the dependent variables across those surfaces ([**5**]). Physically, such surfaces of discontinuity are, in general, moving boundaries which can either be true interfaces separating phases at or near equilibrium, such as solid-liquid and liquid-vapor transformations ([**10**]), or represent convenient approximations for thin regions across which there are significant changes of properties, such as shocks and detonations in fluids ([**4**], [**25**]), fog boundaries ([**9**]), ionic transport ([**13**]), and flame fronts ([**8**]). Accordingly, the mathematical models for many of these phenomena can be classified as *Stefan Problems*, because there are parabolic diffusion equations involved which must be satisfied in a region or regions whose boundaries are to be determined ([**11**]). Historically, Stefan problems have been important mainly in the fields of fluid mechanics ([**7**]) and metallurgy ([**11**]), but their study has recently taken on a new significance. Now we see examples in the textile, glass, and chemical industries, as well as in the fields of geophysics, meteorology, biology, and astrophysics ([**9**], [**10**], [**11**]).

When analyzed by the stability methods with which we have been concerned, all of the related phenomena mentioned above as possessing surfaces of discontinuity give rise to our semi-quadratic secular equation or have a secular equation of the form

$$\sigma + a = 2\sum_{j=1}^{n} (b_j + d_j\sigma)\sqrt{\sigma + c_j}$$

where

$$\operatorname{Re}\sqrt{\sigma + c_j} \geqslant \alpha_j \geqslant 0 \tag{5.1}$$

for $j = 1, 2, \ldots, n$, or more generally of the form

$$F_m(\sigma; z_1, z_2, \dots, z_n) = 0,$$

where F_m is an mth order polynomial in the $n+1$ variables σ and $z_j, j=1,2,\dots,n$ for $z_j = \sqrt{\sigma + c_j}$ restricted as in (5.1). Here, then, is a series of complex phenomena from a variety of fields which can be modeled and in principle analyzed by the methods presented to yield results which may be experimentally verified.

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THE UMBRAL METHOD: A SURVEY OF ELEMENTARY MNEMONIC AND MANIPULATIVE USES

ANDREW P. GUINAND

1. Introduction. The basic idea of the umbral method consists in the use of a notation where certain exponents can be interchanged with suffixes. Its primary use is as an aid in dealing with sequences whose properties somehow resemble the properties of integral powers of an algebraic symbol.

The case most often encountered in the literature is that of the Bernoulli numbers, $\{B_n\}$, when they are expressed in the even suffix notation. That is, for $|x|<2\pi$,

$$\frac{x}{e^x-1}=\sum_{n=0}^{\infty} B_n\frac{x^n}{n!}=1-\tfrac{1}{2}x+\frac{1}{6}\frac{x^2}{2!}-\frac{1}{30}\frac{x^4}{4!}+\frac{1}{42}\frac{x^6}{6!}-\cdots.$$

These numbers satisfy the recurrence formula

$$\sum_{r=0}^{n-1}\binom{n}{r}B_r=\begin{cases}1, & (n=1),\\ 0, & (n=0,2,3,4,\ldots).\end{cases} \tag{1}$$

This formula can be written symbolically as

$$(B+1)^n-B^n\equiv\begin{cases}1, & (n=1),\\ 0, & (n=0,2,3,4,\ldots),\end{cases} \tag{2}$$

with the understanding that the expression on the left is to be expanded in powers of B, and then each term B^m is to be replaced by B_m. The symbol B is referred to as an "umbra," and the symbol $\equiv$ is used to denote symbolic or umbral equivalences, in which we have put $B^m\equiv B_m$ (**[8]**, **[9]**, **[13]**).

In (2) this umbral method is only being used as a mnemonic for the recurrence (1), but the method can also be a great aid in simplifying manipulations.

The notation is sometimes called the Blissard notation **[15]**, but other writers have attributed it to Lucas **[11]** or to Sylvester **[16]**. The method has been used informally by many writers, sometimes with virtually no explanation **[14]**. At the other extreme, rigorous presentations as an algebraic system have been given by Bell **[3]** and Temple **[18]**, and it has been expressed in terms of linear operators by Rota and Mullin **[16]**.

The aim of the present survey is to show that there is an intermediate level, almost completely neglected in the literature, at which the method can be used to great advantage. Instead of regarding umbral symbols as new algebraic or operational entities, we treat the method as a matter of notation, subject to rules of interpretation and manipulation, and give examples to show how the method simplifies both proofs and expressions of results.

2. The rules of the umbral method.

I. RULE OF INTERPRETATION. *Expressions involving one or several umbrae are to be interpreted by expanding as power series in the umbrae and replacing exponents by suffixes.*

II. RULE OF MANIPULATION. *Additions or linear combinations of equations involving umbrae are permissible, but multiplication is only valid when the factors have no umbra in common. In general,*

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any step in manipulation is valid if and only if it remains valid when interpreted in non-umbral form.

Rule I implies that we only give meanings to functions of umbrae which are analytic at the origin. The expressions B^{-1} and $\log B$ are meaningless, whereas e^{Bx} has the interpretation

$$e^{Bx} \equiv \sum_{n=0}^{\infty} \frac{(Bx)^n}{n!} \equiv \sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = \frac{x}{e^x - 1}, \qquad (0<|x|<2\pi). \tag{3}$$

In Rule II the ban on some multiplications is needed to avoid such fallacious arguments as: $B^2 \times B^4 \equiv B^6$, therefore $B_2 B_4 = B_6$: or $e^{2Bx} \equiv \{e^{Bx}\}^2$, therefore

$$\frac{2x}{e^{2x}-1} = \left\{ \frac{x}{e^x-1} \right\}^2.$$

The general principle contained in the latter part of Rule II ensures a check on such manipulations as

$$e^{Bx} \times e^x \equiv e^{(B+1)x}, \tag{4}$$

and

$$\frac{d}{dx}\{e^{Bx}\} \equiv B e^{Bx}. \tag{5}$$

Of these, (4) follows by Cauchy multiplication of exponential series. (Cf. [4], [19].) That is

$$e^{Bx} \times e^x \equiv \sum_{m=0}^{\infty} \frac{B^m x^m}{m!} \sum_{n=0}^{\infty} \frac{x^n}{n!} \equiv \sum_{p=0}^{\infty} \frac{x^p}{p!} \sum_{m+n=p} \frac{p!}{m!n!} B^m$$

$$\equiv \sum_{n=0}^{\infty} (B+1)^p \frac{x^p}{p!} \equiv e^{(B+1)x}.$$

Similarly (5) is validated by

$$\frac{d}{dx}\{e^{Bx}\} \equiv \frac{d}{dx}\left\{ \sum_{n=0}^{\infty} \left\{ B_n \frac{x^n}{n!} \right\} = \sum_{n=1}^{\infty} B_n \frac{x^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} B_{n+1} \frac{x^n}{n!} \right.$$

$$\equiv B \sum_{n=0}^{\infty} \frac{B^n x^n}{n!} \equiv B e^{Bx}$$

3. Applications of umbral methods to the Bernoulli numbers. (i) The recurrence formula. From (3) we have

$$e^{Bx} \equiv \frac{x}{e^x-1}. \tag{6}$$

Hence

$$e^{(B+1)x} - e^{Bx} \equiv x.$$

Expanding and equating coefficients of $x^n/n!$, we obtain the recurrence (2), as quoted earlier.

Changing the sign of x in (6), we have

$$e^{-Bx} \equiv \frac{-x}{e^{-x}-1} = \frac{xe^x}{e^x-1} \equiv e^{(B+1)x}. \tag{7}$$

Hence for all integers $n \geqslant 0$,

$$(B+1)^n \equiv (-B)^n.$$

With (2) this gives $B^n \equiv (-B)^n$ for $n \neq 1$ and $B_1 = -B_1 - 1$. Hence $B_1 = -\frac{1}{2}$, $B_3 = B_5 = B_7 = \cdots = 0$.

(ii) Power series for certain trigonometric functions. From (6) and (7) it follows that

$$\cosh Bx \equiv \tfrac{1}{2}(e^{Bx}+e^{-Bx}) \equiv \tfrac{1}{2}\frac{x}{e^x-1}(1+e^x) = \tfrac{1}{2}x\coth\tfrac{1}{2}x.$$

Replacing x by $2ix$ we deduce that $\cot x \equiv (\cos 2Bx)/x$. This can be regarded as a mnemonic for the non-umbral formula

$$\cot x = \sum_{n=0}^{\infty} (-)^n 2^{2n} B_{2n} \frac{x^{2n-1}}{2n!}, \qquad (|x|<\pi).$$

From the identities

$$\operatorname{cosec} x = \cot\tfrac{1}{2}x - \cot x, \qquad \tan x = \cot x - 2\cot 2x,$$

we get

$$\operatorname{cosec} x \equiv \frac{2\cos Bx - \cos 2Bx}{x}, \qquad \tan x \equiv \frac{\cos 2Bx - \cos 4Bx}{x}.$$

That is,

$$\operatorname{cosec} x = \sum_{n=0}^{\infty} (-)^{n-1}(2^{2n}-2)B_{2n}\frac{x^{2n-1}}{2n!}, \qquad (|x|<\pi),$$

and

$$\tan x = \sum_{n=1}^{\infty} (-)^{n-1}(2^{4n}-2^{2n})B_{2n}(x^{2n-1}/2n!), \qquad \left(|x|<\tfrac{1}{2}\pi\right).$$

(iii) The Bernoulli polynomials. These are polynomials $B_n(x)$, defined as the coefficients in the expansion ([2])

$$\frac{ze^{xz}}{e^z-1} = \sum_{n=1}^{\infty} B_n(x)\frac{z^n}{n!}.$$

In umbral form the left-hand side is $e^{(B+x)z}$. Hence

$$B_n(x) \equiv (B+x)^n \equiv \sum_{r=0}^{n} \binom{n}{r} B_{n-r}x^r.$$

The basic properties of the Bernoulli polynomials follow readily from this umbral form. We have immediately

$$B_n(0) \equiv B^n \equiv B_n, \quad \text{and} \quad B_n'(x) \equiv n(B+x)^{n-1} \equiv nB_{n-1}(x).$$

Further

$$e^{-(B+x)z} \equiv \frac{-ze^{-xz}}{e^{-z}-1} \equiv \frac{ze^{(1-x)z}}{e^z-1} \equiv e^{(B+1-x)z},$$

whence

$$(-)^n(B+x)^n \equiv (B+1-x)^n.$$

That is, the symmetry property ([2]) $(-)^n B_n(x) \equiv B_n(1-x)$. Also

$$\sum_{r=0}^{m-1} e^{(mB+mx+r)z} \equiv e^{m(B+x)z}\sum_{r=0}^{m-1} e^{rz} \equiv \frac{mze^{mxz}}{e^{mz}-1} \times \frac{e^{mz}-1}{e^z-1}$$

$$= \frac{z}{e^z-1} \times me^{mxz} \equiv me^{(B+mx)z}.$$

Hence, by equating coefficients of $z^n/n!$,

$$m(B+mx)^n \equiv \sum_{r=0}^{n-1} (mB+mx+r)^n \equiv m^n \sum_{r=0}^{n-1} \left(B+x+\frac{r}{m}\right)^n.$$

That is, the multiplication formula

$$B_n(mx) = m^{n-1} \sum_{r=0}^{m-1} B_n\left(x+\frac{r}{m}\right).$$

(iv) The Euler-Maclaurin summation formula for polynomials ([**8**], [**13**]). If

$$f(x) = \sum_{n=0}^{k} f^{(n)}(0)\frac{x^n}{n!}$$

is a polynomial of degree k in x, then

$$f(B+1)-f(B) \equiv \sum_{n=0}^{k} \frac{f^{(n)}(0)}{n!}\{(B+1)^n - B^n\} \equiv f'(0), \tag{8}$$

by the recurrence formula (2). Unit translations of this result give, successively,

$$\begin{aligned} &f(B+2)-f(B+1) \equiv f'(1),\\ &f(B+3)-f(B+2) \equiv f'(2),\\ &\cdots\cdots\cdots\cdots\cdots\cdots\\ &f(B+n)-f(B+n-1) \equiv f'(n-1). \end{aligned}$$

Adding these with (8) we get the umbral form of the Euler–Maclaurin summation formula

$$f'(0)+f'(1)+f'(2)+\cdots+f'(n-1) \equiv f(B+n)-f(B). \tag{9}$$

In particular, if p is a positive integer and $f(x)=x^{p+1}/(p+1)$,

$$1^p+2^p+3^p+\cdots+(n-1)^p \equiv \frac{(B+n)^{p+1}-B^p}{p+1}.$$

Also if we put $f(x)=(x+z)^p$ in (8) we get $B_p(z+1)-B_p(z)=pz^{p-1}$, which is the difference formula for the Bernoulli polynomials.

If we expand the right-hand side of (9) in powers of B by Taylor's theorem, it is to be interpreted as

$$\sum_{r=0}^{k} \frac{B^r}{r!}\{f^{(r)}(n)-f^{(r)}(0)\}.$$

If we put $\phi(x)=f'(x)$ and recall that $B_0=1$, $B_1=-\frac{1}{2}$, $B_3=B_5=\cdots=0$, then (9) gives

$$\tfrac{1}{2}\phi(0)+\phi(1)+\phi(2)+\cdots+\phi(n-1)+\tfrac{1}{2}\phi(n) = \int_0^n \phi(x)\,dx + \sum_{r=1}^{[\frac{1}{2}p+\frac{1}{2}]} \frac{B_{2r}}{2r!}\{\phi^{(2r-1)}(n)-\phi^{(2r-1)}(0)\}.$$

This is the usual non-umbral form of the Euler–Maclaurin summation formula.

4. The Euler numbers. The Euler numbers E_n in the expansion

$$\operatorname{sech} x = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = 1 - \frac{x^2}{2!} + 5\frac{x^4}{4!} - 61\frac{x^6}{6!} + \cdots, \qquad \left(|x| < \tfrac{1}{2}\pi\right)$$

can be associated with an umbra E. That is $E^n \equiv E_n$. Since $\operatorname{sech} x$ is an even function of x, it follows that $E_1=E_3=E_5=\cdots=0$. Then from

$$\frac{2}{e^x+e^{-x}} = \operatorname{sech} x \equiv e^{Ex}$$

we get

$$e^{(E+1)x}+e^{(E-1)x}\equiv 2.$$

and on equating coefficients, the recurrence formula

$$(E+1)^n+(E-1)^n\equiv\begin{cases}2, & (n=0),\\ 0, & (n=1,2,3,\dots).\end{cases}$$

If $f(x)$ is a polynomial in x of degree k, then the recurrence implies that

$$f(E-1)+f(E+1)\equiv 2f(0).$$

Hence

$$\begin{aligned}&f(1)-f(3)+f(5)-\cdots-f(4n-1)\\ &\qquad\equiv\tfrac{1}{2}\{f(E)+f(E+2)-f(E+2)-f(E+4)+\cdots-f(E+4n-2)-f(E+4n)\}\\ &\qquad\equiv\tfrac{1}{2}\{f(E)-f(E+4n)\}. \end{aligned}\tag{10}$$

This is the umbral form of an analogue for alternating series of the Euler–Maclaurin summation formula. In particular

$$1^p-3^p+5^p-7^p+\cdots-(4n-1)^p\equiv\tfrac{1}{2}\{E^p-(E+4n)^p\}$$

for positive integral p. In non-umbral form (10) becomes

$$f(1)-f(3)+f(5)-f(7)+\cdots-f(4n-1)=\tfrac{1}{2}\sum_{r=0}^{\left[\frac{1}{2}k+\frac{1}{2}\right]}\frac{E_{2r}}{2r!}\{f^{(2r)}(0)-f^{(2r)}(4n)\}. \tag{11}$$

5. **Asymptotic remainder formulas.** The Euler–Maclaurin summation formula and its analogue (11) are, of course, valid for functions other than polynomials, and many useful asymptotic formulas for remainders in summing series can be derived from them ([**8**]). For example, if we put $f(x)=1/x$ in (11), the result is the formula ([**5**])

$$\tfrac{1}{4}\pi=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots-\frac{1}{4n-1}+\tfrac{1}{2}\sum_{r=0}^{k}\frac{E_{2r}}{(4n)^{2r+1}}+O(n^{-2k-3}) \tag{12}$$

for fixed k and $n\to\infty$. This result cannot be deduced from (10) since $f(E)=1/E$ has no umbral interpretation, but another approach can give a short umbral proof.

For any algebraic quantity, by summing geometric series,

$$\sum_{s=0}^{m-1}\left\{\frac{(E+1)^s}{x^{s+1}}-\frac{E^s}{(x+1)^{s+1}}\right\}=\frac{\{E/(x+1)\}^m-\{(E+1)/x\}^m}{x-1-E}=O(x^{-m-1}) \tag{13}$$

for fixed m and $x\to\infty$. Similarly

$$\sum_{s=0}^{m-1}\left\{\frac{(E-1)^s}{x^{s+1}}-\frac{E^s}{(x-1)^{s+1}}\right\}=O(x^{-m-1}). \tag{14}$$

If we now regard E as the umbra for the Euler numbers, add (13) and (14), and use the recurrence formula, we get

$$\frac{2}{x}-\sum_{r=0}^{m-1}E^s\left\{\frac{1}{(x-1)^{s+1}}+\frac{1}{(x+1)^{s+1}}\right\}\equiv O(x^{-m-1}). \tag{15}$$

If we put $m=2k+2$, recall that odd order Euler numbers vanish, and divide by 2, then (15), in

non-umbral form, becomes

$$u(x)=\frac{1}{x}-\sum_{r=0}^{k}E_{2r}\left\{\frac{1}{(x-1)^{2r+1}}+\frac{1}{(x+1)^{2r+1}}\right\}=O(x^{-2k-3}). \tag{16}$$

Now $u(x)$ is a rational algebraic function of x, so for sufficiently large x it tends to zero monotonically as x increases. Setting $x=4n+1,4n+3,4n+5,\ldots$ it follows that

$$u(4n+1)-u(4n+3)+u(4n+5)-\cdots=O(n^{-2k-3})$$

as $n\to\infty$. That is

$$\frac{1}{4n+1}-\frac{1}{4n+3}+\frac{1}{4n+5}-\frac{1}{4n+7}+\cdots$$
$$-\sum_{r=0}^{k}E_{2r}\left\{\frac{1}{(4n)^{2r+1}}+\frac{1}{(4n+2)^{2r+1}}-\frac{1}{(4n+2)^{2r+1}}-\cdots\right\}$$
$$=O(n^{-2k-3}). \tag{17}$$

Since all terms after the first cancel out in the second part of (17), and by Leibniz's series

$$\tfrac{1}{4}\pi=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\cdots, \tag{18}$$

the required result (12) follows from (17).

The Leibniz series (18) is usually dismissed as useless for any practical calculation of π, though it is certainly the simplest series which could be so used. Using (12) with $n=2$, $k=1$ gives an estimate of π as 3.14133, an error of some 2.6×10^{-4}. Direct use of the Leibniz series (18) would require about 8000 terms to reach this order of accuracy.

If we set $x=4n+2,4n+4,4n+6,\ldots$ in (16), then similar arguments show that

$$\log 2=1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots-\frac{1}{2n}+\sum_{r=0}^{k}\frac{E_{2r}}{(4n+1)^{2r+1}}+O(n^{-2k-3}).$$

6. Reciprocal sequences and related identities. Two sequences $\{a_n\},\{b_n\},(n=0,1,2,\ldots)$ are said to be reciprocal if [1]

$$b_n=\sum_{r=0}^{n}(-)^r\binom{n}{r}a_r, \tag{19}$$

since it is then also true that

$$a_n=\sum_{r=0}^{n}(-)^r\binom{n}{r}b_r. \tag{20}$$

To prove this, designate a and b as umbrae. Then (19) becomes $b^n\equiv(1-a)^n$. Hence $e^{bx}\equiv e^{(1-a)x}$.

Replacing x by $-x$, we have

$$e^{ax}\equiv e^{(1-b)x}, \tag{21}$$

whence (20) follows on reversing the argument. Note that in this case a_0 and b_0 are not necessarily equal to one, unlike B_0 and E_0.

If $\{\alpha_n\},\{\beta_n\}$ is another pair of reciprocal sequences, then

$$e^{(a-\alpha)x}\equiv e^{ax}\times e^{-\alpha x}\equiv e^{(1-b)x}\times e^{-(1-\beta)x}\equiv e^{(\beta-b)x}.$$

Hence we get the identity

$$\sum_{r=0}^{n}(-)^r\binom{n}{r}a_r\alpha_{n-r}=\sum_{r=0}^{n}(-)^r\binom{n}{r}\beta_r b_{n-r}.$$

Identities involving Bernoulli numbers and reciprocal sequences occur in certain relations between angle-sums in n-dimensional simplexes ([7]). An extension of these results, in umbral form, is

$$(B+a)^n \equiv (-B-b)^n \equiv \tfrac{1}{2}\{(2B+a)^n+(-2B-b)^n\}.$$

That is, in non-umbral form,

$$\sum_{r=0}^{n}\binom{n}{r}B_r a_{n-r}=(-)^n\sum_{r=0}^{n}\binom{n}{r}B_r b_{n-r}=\sum_{r=0}^{n}\binom{n}{r}2^{r-1}B_r\{a_{n-r}+(-)^n b_{n-r}\}.$$

To prove this, it suffices to show that

$$e^{(B+a)x}\equiv e^{-(B+b)x}\equiv \tfrac{1}{2}\{e^{(2B+a)x}+e^{-(2B+b)x}\}.$$

From (3) and (21)

$$e^{(B+a)x}\equiv \frac{x}{e^x-1}e^{(1-b)x}\equiv \frac{-x}{e^{-x}-1}e^{-bx}\equiv e^{-Bx}\times e^{-bx}\equiv e^{-(B+b)x}.$$

Also

$$\tfrac{1}{2}\{e^{(2B+a)x}+e^{-(2B+b)x}\}\equiv \tfrac{1}{2}\left\{\frac{2xe^{ax}}{e^{2x}-1}+\frac{-2xe^{-(1-a)x}}{e^{-2x}-1}\right\}$$
$$\equiv \frac{xe^{ax}}{e^x-1}\equiv e^{(B+a)x},$$

as required.

7. Sums of products and dual umbral notation. Many identities involving sums of products of Bernoulli numbers have long been known. In particular, an identity of Euler, written in the even suffix notation, takes the form [12]

$$\sum_{r=1}^{n-1}\binom{2n}{2r}B_{2r}B_{2n-2r}=-(2n+1)B_{2n}. \tag{22}$$

Mordell [10] has commented that proofs of (22) in the literature are involved. Since $B_1=-\frac{1}{2}$ and other odd order Bernoulli numbers vanish, (22) can be written for even m as

$$\sum_{s=0}^{m}(-)^s\binom{m}{s}B_sB_{m-s}=(1-m)B_m. \tag{23}$$

For odd m, (23) is trivially true in that both sides vanish. The form of (23) suggests the use of a dual umbral notation, with two umbrae B and B', both for Bernoulli numbers, subject to the interpretation that terms $(B)^p(B')^q\equiv B_pB_q$. Then (23) becomes

$$(B-B')^m\equiv(1-m)B^m. \tag{24}$$

Such a notation has been used previously to simplify the expression of similar sums of products ([6], [17]). It can also suggest simple proofs; for Euler's identity in the form (24), we have

$$\sum_{m=0}^{\infty}(B-B')^m\frac{x^m}{m!}\equiv e^{(B-B')x}\equiv e^{Bx}\times e^{-B'x}$$
$$\equiv\frac{x}{e^x-1}\times\frac{-x}{e^{-x}-1}=\frac{x^2e^x}{(e^x-1)^2}$$
$$=-x^2\frac{d}{dx}\left\{\frac{e^{Bx}}{x}\right\}\equiv(1-Bx)e^{Bx}$$
$$\equiv\sum_{m=0}^{\infty}B^m\frac{x^m}{m!}-\sum_{m=0}^{\infty}B^{m+1}\frac{x^{m+1}}{m!}$$

$$\equiv \sum_{m=0}^{\infty} (1-m) B^m \frac{x^m}{m!}.$$

Equating coefficients of $x^m/m!$ we obtain (24).

8. Multiple umbral methods and other extensions. Endless similar results can be found by the methods of the previous sections. With a multiple umbral notation, $(B)^p(B')^q(B'')^r \cdots \equiv B_p B_q B_r \cdots$, we have

$$(B+B'+B'')^n \equiv 3\binom{n}{3}\left\{\frac{B^n}{n}+3\frac{B^{n-1}}{n-1}+2\frac{B^{n-2}}{n-2}\right\},$$

$$(B+B'-B'')^n \equiv 3\binom{n}{3}\left\{\frac{B^n}{n}+\frac{B^{n-1}}{n-1}\right\}, \tag{25}$$

$$(B+B'+B''+B'''+B'''')^n \equiv 5\binom{n}{5}\left\{\frac{B^n}{n}+10\frac{B^{n-1}}{n-1}+35\frac{B^{n-2}}{n-2}+50\frac{B^{n-3}}{n-3}+24\frac{B^{n-4}}{n-4}\right\}. \tag{26}$$

The pattern of coefficients in (25) and (26) follows that of

$$(x+1)(x+2)=x^2+3x+2 \quad \text{and} \quad (x+1)(x+2)(x+3)(x+4)=x^4+10x^3+35x^2+50x+24.$$

Another series of identities starts with the Euler identity (24), and continues with

$$(2B-B')^{2n} \equiv (1-2n)B^{2n},$$

$$(4B-B')^{2n} \equiv (1-2n)B^{2n}+n(2n-1)E^{2n-2},$$

$$(8B-B')^{2n} \equiv (1-2n)B^{2n}+n(2n-1)E^{2n-2}$$
$$+2n(2n-1)\left\{(2E-1)^{2n-2}+(2E)^{2n-2}+(2E+1)^{2n-2}\right\}.$$

Other identities with Euler numbers are

$$(2B+E)^n \equiv (4B+1)^n, \qquad (n+2)(E+E')^n \equiv 2^{n+2}(2^{n+2}-1)B^{n+2},$$
$$2(E+E'+E'')^n \equiv E^n - E^{n+2}.$$

The properties of the Euler polynomials can be derived in the same way as was used for the Bernoulli polynomials in Section 3 (iii). It is found that their umbral form is ([2]) $E_n(x) \equiv (\frac{1}{2}E+x-\frac{1}{2})^n$.

Proofs of the results of this section follow the same lines as preceding sections.

9. Remarks. The use of a special symbol, such as $\equiv$ or $\doteqdot$, is not essential; some writers use one ([9], [15]), others do not ([13]). I have used the former in this paper to lessen possible ambiguities.

The ban on multiplication of expressions with a common umbra, mentioned in Rule II, does not appear to have been stated explicitly in the literature. Presumably this is because it is usual for rules to tell what is permitted rather than what is forbidden. Riordan [15] does briefly imply that there are operations to be avoided; in so doing he also uses a form of dual umbral notation without any distinguishing marks for twin umbrae. For an expression such as $(a+a)^n$ he says, "Like sequences are treated exactly as unlike sequences." This becomes confusing if applied to the more involved expressions of Section 8, so I have preferred to make definite distinctions by using B, B', B'', and so on.

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THE SEVENTH U.S.A. MATHEMATICAL OLYMPIAD: A REPORT

SAMUEL L. GREITZER

The Seventh U.S.A. Mathematical Olympiad was held on May 2, 1978. From the Honor Roll of the Annual High School Mathematics Examination, 108 students, who had scored 118 points or better, were invited to take part, and 106 finally did participate. The papers were graded, first by Professors Michael Aissen and John Bender of Rutgers University, and then by Professor Murray Klamkin, of the University of Alberta, and this writer. A copy of the Olympiad problems appears below.

The table below compares the students' scores on the Olympiad with those on the Annual High School Mathematics Examination. As has been the case on all previous Olympiads, there is little or no correlation, which is not surprising, considering the differing content and goals of the contests.

Olympiad / H.S. Exam	0–10	11–20	21–30	31–40	41–50	51–60	61–70	71–80	81–90	91–100
148–150					1	1				2
145–147	1		1	1					1	1
142–144										
139–141	1			1	1	1		1		
136–138		1								
133–135	2	1				1		1		1
130–132	3	2	3	3	1		2			
127–129	2	4	3	4	1		1			
124–126	2	1	6	3	2	1	1			
121–123	2	3	5	1	3	1	1			
118–120	5	4	5	7	3					

The table merits inspection. For example, there were only 12 students who managed to score 61 points or better. Of course, the eight high scorers, who were honored on June 6 at special ceremonies in Washington, D.C., scored 70 or better. These high scorers were, in order of score:

Randall Dougherty	W. T. Woodson High School	Fairfax, Va.
Ehud Reiter	T. S. Wootton High School	Rockville, Md.
Mark Kleiman	Stuyvesant High School	New York, N.Y.
Michael Larsen	Lexington High School	Lexington, Mass.
Daniel Bloch	Bellport High School	Brookhaven, N.Y.
David Montana	Lawrenceville High School	Lawrenceville, N.J.
Victor Milenkovic	New Trier High School East	Winnetka, Ill.
Charles Walter	Centennial High School	Champaign, Ill.

Of the five problems, the one the students did best in was Number 5. This is a "chromatic graph" problem, but most of the students laboriously examined the many cases. The average score was 11.2 points out of 20.

Next in order was Number 1, where the average score was 9.4 out of 20. A good number of the contestants showed some knowledge of inequalities. The second problem was next in difficulty, with the average score 6.4 out of 20. It was obvious that the students had a very rudimentary knowledge of plane geometry.

Number 3 came next, with an average score of 4.5 points out of 20. Most students could not find their way through the terminology to solve what is otherwise a simple problem. The worst problem for the students was Number 4. It is evident that solid geometry has been successfully removed from the secondary-school curricula. Half the students had no concept of the meaning of the term "dihedral angle," and some were unacquainted with a tetrahedron. Whether this was good for the students is subject to argument.

As has been done previously, the team that represented the United States at the International Mathematical Olympiad was selected from among the high scorers. Also, as has been done previously, these students had a special "training session" to teach them the mathematics they needed to compete with the other nations on something like even terms.

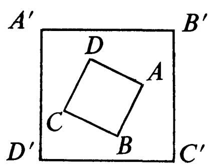
Seventh U.S.A. Mathematical Olympiad—May 2, 1978

1. Given that a, b, c, d, e are real numbers such that

$$a+b+c+d+e=8,$$
$$a^2+b^2+c^2+d^2+e^2=16.$$

Determine the maximum value of e.

2. $ABCD$ and $A'B'C'D'$ are square maps of the same region of a country but drawn to different uniform scales and are superimposed as shown in the figure. Prove that there is only one point O on the small map which lies directly over a point O' of the large map such that O and O' each represent the same place of the country. Also, give a Euclidean construction (straight-edge and compass) for O.



3. An integer n will be called *good* if we can write $n = a_1 + a_2 + \cdots + a_k$ where $a_1, a_2, \ldots, a_k$ are positive integers (not necessarily distinct) satisfying $1/a_1 + 1/a_2 + \cdots + 1/a_k = 1$. Given it is known that the integers 33 through 73 are good, prove that every integer $\geqslant 33$ is good.

4 (a) Prove that if the six dihedral angles (i.e., angles between pairs of faces) of a given tetrahedron are congruent, then the tetrahedron must be regular.
(b) Must the tetrahedron be regular if five dihedral angles are congruent?

5. Nine mathematicians meet at an international conference and discover that, among any three of them, at least two speak a common language. If each of the mathematicians can speak at most three languages, prove that there are at least three of the mathematicians who can speak the same language.

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FOURIER SERIES CAME FIRST

SALOMON BOCHNER

In an illuminating note on Carleson's theorem for Fourier series, Halmos (1978) starts out with the following sentence: "It is a historical misfortune ... that Fourier series were discovered before convergence" (this Monthly, volume 85, p. 33).

This sentence gave me a jolt. As I have always known—or believed that I knew—it was Fourier series that in effect "created" all modes of convergence of functions (and I mean of functions and not of mere numbers) and their principal consequences, so that a discovery of convergence before Fourier series would have been evolution running in reverse, much of the way.

After power series in Abel and others before him, the first major result about convergence of a series of functions was Dirichlet's theorem about convergence of Fourier series of functions that are piecewise monotone (1829) [**2**]. The most perduring key result about summability is Fejér's theorem on $(C,1)$-summability of Fourier series of continuous functions (1904)[**6**]. In between, Du Bois-Reymond (1876) [**5**] constructed a continuous function whose Fourier series does not converge everywhere, at a time when "counterexamples" were still at a premium; and C. Jordan (1893) [**11**] introduced the very important class of functions of bounded variation, apparently in order to express Dirichlet's convergent test more "elegantly" (1894) [**12**].

In 1907, Lebesgue integration and convergence in L^2-norm were put on the mathematical map by the Riesz–Fischer theorem [**7**], [**16**]; it was undoubtedly provoked by Fourier series, even if its application to general orthogonal expansion immediately captured the headlines. Fourier series were again of the essence in the 1927 result of Marcel Riesz, for $p > 1$, on L^p-convergence of L^p-Fourier series and the existence of conjugate functions [**17**]. The result in effect demonstrated that a Banach space can have a projection of norm greater than 1, even into a subspace spanned by a subset of a basis of the whole space.

Stochastic modes of convergence of probability distributions came to life when interpreted as modes of convergence of their Fourier transforms [**14**], [**1**]. When Norbert Wiener created his "Wiener process" (1923) [**14**], he modeled it on the body of Fourier series, which, as it happens, is not even the best way of modeling it. Also, Gelfand's creation of the theory of normed rings (1939) [**9**], was triggered by an attempt to disengage Wiener's Tauberian theorems (1932) [**20**] from their special setting amidst Fourier series and Fourier integrals.

Gauss had to work hard to derive his reciprocity law for Gaussian sums (1811) [**8**]. However,

Dirichlet did it smoothly after him with the Poisson summation formula for Fourier series (1835) [3]; and he established a new tier in the edifice of number theory, using as building stones his Dirichlet series, which are almost periodic objects of Fourier analysis [2]. Erich Hecke, Dirichlet's counterpart in the twentieth century, in a string of papers 1917–1920 [10], which created Riemann–Hecke functional equations for various zeta functions in algebraic number fields, bested eminent Fourier analysts of his generation by operating intricately with the Mellin transform on the body of the Poisson summation formula in several variables. The intricacy can be savored by reading the expository book of Landau (1918) [13], which does not even deal with Dirichlet characters, let alone Hecke's general quasi-characters. Decades later the role of the algebraically conceived idèles, which had been introduced for another purpose, was considerably enhanced when they were used by Tate (1950) [18] to relieve the intricacy in Hecke's work.

Riemann created his integral, the first of its kind, within his memoir on trigonometric series (1856, posthumously published 1867) [15]; and the underlying concept of a general function had been anticipated in Dirichlet's above-mentioned work on Fourier series (1829).

Georg Cantor, while still rather young, noticed that Riemann assumes without proof that if, for real A_n, α_n,

$$\lim_{n\to\infty} A_n \cos(nx+\alpha_n)=0 \tag{*}$$

for all x in $-\infty<x<\infty$, then $A_n \to 0$. He set out to fill in the gap and he succeeded; and he even proved that the hypothesis (*) can be dispensed with on certain "exceptional" sets. Actually the problem is a natural one for Lebesgue integration. It can then be easily seen that it suffices to assume (*) on some Lebesgue set of positive measure, even for

$$\lim_{n\to\infty} A_n \cos(\lambda_n x+\alpha_n)=0$$

if λ_n are any real exponents going to ∞; compare for instance [21]. But not having Lebesgue integration, Cantor had to make do with what he had. And he went about this business in such a clumsy way that he stumbled full-face into his glorious theory of sets. This is the honest truth (see Georg Cantor, *Gesammelte Abhandlungen*, edited by Ernst Zermelo, Berlin, 1932, p. 102, last four lines).

Yes, Fourier series did indeed come first.

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CORRECTIONS TO "IRREGULAR INTEGERS"

SCOTT GARTEN

An error exists in Lemma 5, pages 167–168 of [1]. When $l>1$ and $\mu(x)=1$, the condition $\nu(x)\geqslant 3$ is necessary and sufficient for the $n\mathbf{Z}$-irregularity of $x\in n\mathbf{Z}$. The second part of the proof reads when $l>1$, $\mu(x)=1$ and $\nu(x)\geqslant 3$, then $x=n^{\nu-1}nq$. Since $n=p_1p_2\cdots p_l$, rewrite nq as $(p_1q)(p_2\cdots p_l)$. Then a second distinct $n\mathbf{Z}$-prime factorization can be written. $X=(np_1q)(np_2\cdots p_l)n^{\nu-3}$. A counterexample is the number 128 in $4\mathbf{Z}$.

$128=4^2\cdot 8$. Notice $4=n=p_1p_2$ where $p_1=p_2=2$. $nq=8=4\cdot 2$. Hence $q=2$. Rewriting $x=(np_1q)(np_2)n^{\nu-3}$ where $x=128$ merely yields $128=(4\cdot 2\cdot 2)(4\cdot 2)4^{3-3}=4^2\cdot 8$, the same $4\mathbf{Z}$-prime factorization as the original.

This actually is the case for any p^7 in the system $p^2\mathbf{Z}$ where p is a $\mathbf{Z}$-prime. $p^7=(p^2)^2(p^2p)$. Here $n=p^2$, $p_1=p_2=p$ and $q=p$. The second $p^2\mathbf{Z}$-prime factorization is $p^7=(np_1q)(np_2)n^{\nu-3}=(p^2\cdot p\cdot p)(p^2p)(p^2)^{\nu-3}=(p^2)^2(p^2q)$. This is obviously identical to the first factorization.

With these results it is now possible to rewrite Theorem 1. When n is $\mathbf{Z}$-prime, the $n\mathbf{Z}$-irregular numbers in $n\mathbf{Z}$ are characterized by the condition $\mu(x)\geqslant 2$, $\nu(x)\geqslant 2$. When n is $\mathbf{Z}$-composite, $n\neq p^2$ for any p a $\mathbf{Z}$-prime, there are additional $n\mathbf{Z}$-irregular numbers characterized by $\mu(x)\leqslant 1$, $\nu(x)\geqslant 3$. In the case $n=p^2$ and $\mu(x)=1$, the condition $x\neq p^7$, $\nu(x)\geqslant 3$ guarantees x to be $n\mathbf{Z}$-irregular.

These corrections do not affect Theorem 2, the density theorem. In $p^2\mathbf{Z}$ there is exactly one element x with $\nu(x)=3$, $\mu(x)=1$, that is $p^2\mathbf{Z}$-regular; $x=p^7$. The $p^2\mathbf{Z}$-irregular integers with $\nu(x)\geqslant 3$, $\mu(x)=1$ comprise the set $C_3^1-\{p^7\}$. The set of $p^2\mathbf{Z}$-irregulars $I=D\cup C_3^0\cup[C_3^1-\{p^7\}]$, hence $\delta(I)=\delta(D)=\delta(C)=1/n$.

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PROGRESS REPORTS

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It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

Progress Reports is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses, what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal: usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

THE CLASS NUMBER PROBLEM

ROY W. RYDEN

It is always surprising when a long-standing problem in mathematics is cracked by a newcomer after many years of hard work by the best established people in the field. It is also remarkable when more than one independent solution for such a problem appears in print almost simultaneously. One famous example is the prime number theorem conjectured by Legendre and Gauss in the late eighteenth century and not proved until 1896 when Hadamard and de La Vallée Poussin both obtained the result. Another example is a conjecture of Gauss (again), on class number, which resisted all attempts at proof until 1966–67, when two young mathematicians, Baker and Stark, settled the problem using different methods. There is even more to the story, but a few details first.

The class number problem concerns the simplest extensions of the rational numbers, the quadratic fields. Let $\mathbf{Q}(\sqrt{d}\,)$ be the field obtained by adjoining $\sqrt{d}$ (where d is a square free integer) to the field $\mathbf{Q}$ of rational numbers. Just as $\mathbf{Q}$ contains $\mathbf{Z}$, the ordinary integers, $\mathbf{Q}(\sqrt{d}\,)$ contains a ring R of "integers", namely, those elements in $\mathbf{Q}(\sqrt{d}\,)$ which satisfy a monic polynomial equation with integer coefficients. In many respects the ring R is like $\mathbf{Z}$, but in one way R can be quite different: while integers in $\mathbf{Z}$ always factor *uniquely* into a product of primes, it is not necessarily true that integers in R do. (The integers in $\mathbf{Q}(\sqrt{-3}\,)$ have unique factorization; but those in $\mathbf{Q}(\sqrt{-5}\,)$ do not.)

For which d do the rings in $\mathbf{Q}(\sqrt{d}\,)$ have unique factorization? To answer the question, one studies the ideals of R—subsets of R which consist of all R-linear combinations of some set of elements of R. For example, for each element α in R there corresponds the principal ideal (α) consisting of all multiples of α. It is a famous theorem of Dedekind that the ideals of R *do* factor uniquely. Therefore, if each ideal of R is principal, then it follows that R has unique factorization. The converse is also true.

Unfortunately, R often has ideals that are not principal. We say two ideals A and B are equivalent if there exist principal ideals (α) and (β) such that $(\alpha)A=(\beta)B$. Of course the principal ideals all fall into one equivalence class, but there may be more classes (only finitely

many, however). The number of equivalence classes, $h(d)$, is called the class number of $\mathbf{Q}(\sqrt{d})$, and $h(d)=1$ precisely when R has unique factorization. Roughly speaking, the larger $h(d)$ is, the more complicated the ring R.

In his celebrated treatise *Disquisitiones Arithmeticae* (1801), Gauss introduced this concept of equivalence class and class number. Gauss was dealing with binary quadratic forms, a theory which has since evolved into the study of the fields $\mathbf{Q}(\sqrt{d})$ and their rings of integers. We are using the modern notation to describe the same ideas. In one of the articles in the *Disquisitiones*, Gauss conjectured that each class number has only finitely many *negative* d associated with it; and, in fact, he asserted that the tables of values d he had computed for some class numbers were complete. In particular, for $h(d)=1$, there are only nine fields listed, with $-d=1$, 2, 3, 7, 11, 19, 43, 67, and 163.

The first progress on the original conjecture of Gauss was made by Dirichlet (1839) when he related $h(d)$ to his L-functions, generalizations of Riemann's zeta-function. Later, Hecke (1918) found a connection between the conjecture and the Riemann Hypothesis for the Dedekind zeta-function. In 1933 Deuring showed that if, in particular, $h(d)=1$ for infinitely many negative d, then the (original) Riemann Hypothesis is true.

It was not until 1934 that Heilbronn, developing the earlier work of Hecke, Deuring, and Mordell, was able to verify Gauss's first conjecture by showing that $h(d)\to\infty$ for $d\to-\infty$. Heilbronn's method was much improved by Siegel (1935) and Brauer (1947), who established a general asymptotic formula for $h(d)$. Their formulas, however, were not effective and shed no light on Gauss's second conjecture (his list of d's).

Later in 1934 Heilbronn and Linfoot established that there were at most 10 numbers d such that $h(d)=1$. This was after some preliminary computational work by Dickson (1911) and D. H. Lehmer (1933), who showed that $h(d)$ could not be 1 for bounded but very large $-d$. In fact, Lehmer showed that $-d$ could be no smaller than 5 billion.

Gauss listed nine fields, and Heilbronn and Linfoot showed that ten is the most there could be. The existence (or nonexistence) of the tenth field eluded mathematicians for many years. In 1952 Heegner published a proof that no tenth field exists, but because of a gap in his proof it was received with much skepticism by specialists in the area. Heegner's method involved the solution of certain diophantine equations and the study of elliptic modular functions. Motivated by Heegner's use of diophantine equations, but with a desire to avoid modular functions (which had caused the doubt to be cast on Heegner's proof), Stark in 1967 was able at last to give a valid proof of the nonexistence of the tenth field. Later Stark (1968) and Deuring (1968) were each able to fill in the gap in Heegner's proof, verifying that Heegner was at least very close to the result.

Baker, on the other hand, was led to his solution of the class number 1 problem by his work in transcendental number theory. After extending the remarkable theorem of Gelfond–Schneider on transcendental numbers, which involves the linear forms of logarithms, Baker turned to some applications, and among these was a result on the class number problem developed by Gelfond and Linnik (1949). Baker, in 1967, was able to make the Gelfond–Linnik theorem effective and hence solved the class number 1 problem. (It was for Baker's work on transcendental numbers and the various applications of his theory that he received the Fields Medal in 1970.) Baker's approach is not entirely unrelated to Stark's method, but it is definitely independent and different.

The class number 2 problem for negative d was solved (once again) by both Baker and Stark independently in 1971, but this time they each managed to strengthen the Gelfond–Linnik–Baker inequalities to arrive at the result. Both of their theorems involved effective bounds, and Stark for a later paper used a programmable desk calculator to list the 18 negative numbers d for which $h(d)=2$.

The evaluation of negative d for which $h(d)>2$ remains an open question, and there is much lively work going on in this area. For positive d the whole class number problem seems to be beyond our reach.

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MATHEMATICAL NOTES

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A GENERALIZATION OF REGULATED FUNCTIONS

T. M. K. DAVISON

Following Dieudonné [1], we say that a real-valued function on an interval I of $\mathbf{R}$ is *regulated* if it has one-sided limits at every point of I. Regulated functions are shown [1, Chap. 8] to have many pleasant and useful properties vis-à-vis integration and differentiation. It seems reasonable then to ask what the regulated functions in $\mathbf{R}^n$ should be.

I take as the basic fact to generalize the proposition [1, 7.6.1] that a function on a compact interval is regulated if, and only if, it is a uniform limit of step functions. The generalization is contained in the Theorem below. It may be remarked that our proof of the sufficiency in the Theorem is basically identical to Dieudonné's (Lemma 5 is the key). However, the proof of the necessity differs from Dieudonné's since he uses the linear ordering of the reals to construct the desired step function.

We generalize step functions as follows. Let X be a topological space. A family $\mathfrak{F}$ of subsets of X is *appropriate* if it satisfies conditions A1, A2, A3 below:

A1. $\mathfrak{F}$ is a lattice with respect to union and intersection of sets,

A2. If A, $B\in\mathfrak{F}$, and $A\supset B$ then $A\setminus B(=\{x\in A : x\notin B\})$ is in $\mathfrak{F}$.

A3. The open sets of X which are in $\mathfrak{F}$ form a basis for the topology of X.

An example of an appropriate family on [0, 1] is the set of all finite unions of sub-intervals of [0, 1]. I can now introduce the basic concept:

DEFINITION. Let X be a topological space, and $\mathfrak{F}$ an appropriate family on X. Let $x\in X$. A function f: $X\to\mathbf{R}$ is *regulated at x (with respect to $\mathfrak{F}$)* if, given $\epsilon>0$ there is an $N\in\mathfrak{F}$, N a neighborhood of x (hence contains an open set containing x), such that $\exists\ A_1,\cdots,A_k\in\mathfrak{F}$ with $N=A_1\cup A_2\cup\cdots\cup A_k$ and $|f(s)-f(t)|\leqslant\epsilon$ for s, $t\in A_i$, $i=1,2,\ldots,k$.

Clearly if f has one-sided limits at x in [0, 1], then f is regulated at x (in the above sense, and with respect to the aforementioned appropriate family) for $(x-\delta, x+\delta)=(x-\delta,x)\cup\{x\}\cup(x,x+\delta)$ is a suitable decomposition (δ depending on ϵ of course).

If f: $X\to\mathbf{R}$ is regulated at all $x\in X$ we say *f is regulated* (with respect to $\mathfrak{F}$). Continuous functions on a locally compact (Hausdorff) space are regulated ($\mathfrak{F}$ does not matter). We let $R(=R(X,\mathfrak{F}))$ denote the set of all regulated functions on X.

LEMMA 1: *R is a linear subspace of the space of all functions from X to* $\mathbf{R}$.

Proof. Clearly, if f is regulated, so is αf, for $\alpha \in \mathbf{R}$. Now suppose $f, g \in R$. Let $x \in X$. Let $\varepsilon > 0$ be given. Choose nbds N, M of x, with $N, M \in \mathfrak{F}$ so that $N = \cup_{i=1}^{n} A_i$, $M = \cup_{j=1}^{m} B_j$ and $\forall$ s, $t \in A_i$, $\forall$ u, $v \in B_j$ $|f(s) - f(t)| \leqslant \varepsilon/2$, $|g(u) - g(v)| \leqslant \varepsilon/2$. Then $N \cap M$ is a nbd of x and $N \cap M \in \mathfrak{F}$ (A1). Moreover, $N \cap M = \cup_{i,j}(A_i \cap B_j)$ and $A_i \cap B_j \in \mathfrak{F}$. Let s, $t \in A_i \cap B_j$; then $|(f+g)(s) - (f+g(t)| \leqslant |f(s) - f(t)| + |g(s) - g(t)| \leqslant \varepsilon/2 + \varepsilon/2 = \varepsilon$. Hence, $f+g$ is regulated at x. Now x is arbitrary so $f+g$ is regulated, as claimed.

For $S \in \mathfrak{F}$ we let χ_S denote the characteristic function of S: that is, $\chi_S(x) = 1$ if $x \in S$, and $\chi_S(x) = 0$ if $x \notin S$.

LEMMA 2: *Let $S \in \mathfrak{F}$. Then χ_S is regulated.*

Proof. Let $x \in X$. Let $\epsilon > 0$ be given. Let $N \in \mathfrak{F}$ be a nbd of x. Then $N \cap S \in \mathfrak{F}$ (A1) and $N \setminus (N \cap S) \in \mathfrak{F}$ (A2). We can write $N = (N \cap S) \cup (N \setminus (N \cap S))$. Then χ_S is constant on $N \cap S$ (always 1 there), and constant on $N \setminus (N \cap S)$ (always 0 there). So $|\chi_S(s) - \chi_S(t)| = 0 \leqslant \epsilon$ for s, $t \in N \cap S$ or s, $t \in N \setminus N \cap S$.

We let E denote the linear subspace of R spanned by $\{\chi_S : S \in \mathfrak{F}\}$—these are our *appropriate step functions.* Then E is also a linear subspace of $B(=B(X, \mathbf{R}))$, the Branch space of bounded functions from X to $\mathbf{R}$ (with supremum norm). I will prove that if X is compact then R is in B, and R is the closure of E in B. The following result is useful in proving these statements.

LEMMA 3: *Suppose X is compact, and f is regulated with respect to $\mathfrak{F}$. Let $\epsilon > 0$ be given. Then $\exists$ $S_1, \cdots, S_m \in \mathfrak{F}$ such that*

(i) *$X = S_1 \cup S_2 \cup \cdots \cup S_m$, and*

(ii) *if x, $x' \in S_i$, then $|f(x) - f(x')| \leqslant \epsilon$.*

Proof. Since f is regulated, to each $x \in X$ there corresponds a nbd $N(x)$ of x, with $N(x) \in \mathfrak{F}$ such that $N(x) = \cup_{i=1}^{m(x)} A_i(x)$, where $A_i(x) \in \mathfrak{F}$, and if s, $t \in A_i(x)$, then $|f(s) - f(t)| \leqslant \epsilon$. Since $N(x)$ is a nbd, the interior of $N(x)$ ($=$int $N(x)$) contains x. Hence $\{\text{int } N(x) : x \in X\}$ is an open cover for X. Now X is compact so this open cover has a finite subcover: say $\{\text{int } N(x_1), \cdots, N(x_k)\}$. Then

$$X = \bigcup_{j=1}^{k} \bigcup_{i=1}^{m(x_j)} A_i(x_j), \quad \text{where} \quad A_i(x_j) \in \mathfrak{F} \quad \text{for all} \quad i,j.$$

After relabeling, this is the desired decomposition.

LEMMA 4: *Suppose X is compact. Then $R \subset B$. In words: if X is compact, then regulated functions are bounded.*

Proof. Let $f \in R$. Apply Lemma 3 with $\epsilon = 1$ to deduce that $X = \cup_{i=1}^{m} S_i$, $|f(s) - f(t)| \leqslant 1$ for s, $t \in S_i$. Let $y_i \in S_i$, for $i = 1, 2, \ldots, m$. Then $|f(x)| \leqslant \max_{1 \leqslant i \leqslant m}\{|f(y_i)| + 1\}$.

Next, we prove that R is closed by the familiar $\epsilon/3$ argument.

LEMMA 5: *If X is compact, then R is closed in B.*

Proof. Suppose $f_n \to f$, with $f_n \in R$. Let $x \in X$. We will show that f is regulated at x. Let $\epsilon > 0$ be given. Choose n such that $\|f - f_n\| \leqslant \epsilon/3$. Since $f_n \in R \exists N = A_1 \cup \cdots \cup A_k$, N a nbd of x, $A_i \in \mathfrak{F}$ such that $|f_n(s) - f_n(t)| \leqslant \epsilon/3$ for s, $t \in A_i$. Now suppose s, $t \in A_i$. Then

$$\begin{aligned} |f(s) - f(t)| &\leqslant |f(s) - f_n(s)| + |f_n(s) - f_n(t)| + |f_n(t) - f(t)| \\ &\leqslant \|f - f_n\| + \frac{\epsilon}{3} + \|f_n - f\| \\ &= \epsilon. \end{aligned}$$

We can now state and prove our result.

THEOREM. *Let X be a compact space, and $\mathfrak{F}$ an appropriate family on X. Then a function from X to $\mathbf{R}$ is regulated with respect to $\mathfrak{F}$ if, and only if, it is the uniform limit of step functions with respect to $\mathfrak{F}$.*

Proof. The sufficiency is clear by Lemmas 2 and 5. For proof of the necessity suppose $f \in R$, and let $\epsilon > 0$ be given. We will "construct" a function $f_\epsilon \in E$ such that $\|f - f_\epsilon\| \leqslant \epsilon$. We use Lemma 3 to decompose $X = \cup_{i=1}^{m} S_i$ (since $\mathfrak{F}$ is complemented we can suppose $S_i \cap S_j = \phi$, $(i \neq j)$) and $|f(s) - f(t)| \leqslant \epsilon$, $s, t \in S_i$. Let $y_i \in S_i$ be arbitrary, and define $f_\epsilon = \Sigma_{i=1}^{m} f(y_i)\chi_i$ where χ_i is the characteristic function of S_i. Then $|f(x) - f_\epsilon(x)| \leqslant \Sigma_{i=1}^{m} |f(x) - f(y_i)|\chi_i(x)$. $[1 = \Sigma_{i=1}^{m}\chi_i(x)$, since $S_i \cap S_j = \phi.]$ Now $\chi_i(x) = 0$ unless $x \in S_j$ (j unique); but then $x, y_j \in S_j$, so $|f(x) - f(y_j)| \leqslant \varepsilon$. Hence $|f(x) - f_\epsilon(x)| \leqslant \epsilon \forall x$. So $\|f - f_\epsilon\| \leqslant \epsilon$, as desired.

In $\mathbf{R}^n$ the family of finite unions of polytopes (a polytope is a finite intersection of half-planes) is appropriate. Consequently, when intersected with X, a compact subset of $\mathbf{R}^n$, it yields an appropriate family on X. One can prove that the regulated functions have "radial" limits everywhere, generalizing the situation when $n = 1$.

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REMARKS ON BILLIARDS

ROBERT SINE AND VLADISLAV KREĬNOVIČ

1. In this section we define billiards in the mathematical sense and give two elementary results. In the second section we will give a corollary of one of these results. While the corollary is immediate, its statement and significance require some deeper notions related to billiards, so we sketch these notions where they are needed.

Let K be a *convex body* (a closed bounded convex set with non-empty interior) in the plane or even in $\mathbf{R}^n$. The billiard ball is a point in K which will move in a straight line with unit velocity until it hits the edge of the table, ∂K. Here we want the billiard to bounce off the boundary and continue with unit velocity in a new direction until again it hits the wall. There may be ambiguities as to how the path should be continued—consider, for example, the point billiard ball running into a corner of a triangular table. We eliminate the ambiguities by assuming that K is a *smooth* convex set. That means for each point p in ∂K there is a unique supporting hyperplane so the bounce direction is determined by reflection off of the unique supporting hyperplane. The orbit is then the well-determined entire path that the moving point will generate in the course of time. In a note [4] the ball in $\mathbf{R}^3$ was characterized in terms of the geometry of billiard orbits as follows. Let K be a smooth convex body in $\mathbf{R}^3$. Then K is a ball if and only if each billiard orbit is contained in a two-dimensional affine (flat) subset of $\mathbf{R}^3$. We now use this result to generalize to higher dimensions.

THEOREM 1. *Let K be a smooth convex body in $\mathbf{R}^n$ for $n \geqslant 3$. Then K is a ball if and only if each billiard orbit is contained in a two-dimensional affine subset of $\mathbf{R}^n$.*

Proof. If K is a ball, it is clear the billiard property holds. So we assume $n \geqslant 3$ and each billiard orbit in K is contained in a two-dimensional affine subset of $\mathbf{R}^n$. For a point p in ∂K, let $N(p)$ denote the normal (to the unique supporting hyperplane) at p. We claim for each pair of points a and b in ∂K that $N(b)$ lies in the plane generated by $N(a)$ and b. For consider a billiard path from a to b and bouncing at b. The orbit lies in some two-plane, and clearly the normal at each point at which the orbit hits the boundary lies in this same two-plane. Thus $N(b)$ lies in the

two-plane generated by $N(a)$ and b. (If the segment joining a and b lies in ∂K, a slightly more complicated argument obtains upon selecting a third point c in ∂K but outside the segment.) Now let P be any three-space containing $N(a)$. By the argument above, the normals, at points of $\partial K \cap P$, belong to P (hence coincide with the normals to $\partial K \cap P$). The result for $n=3$ stated just before this theorem then applies, so $K \cap P$ is a three-ball. In particular, all the normals to $\partial K \cap P$ intersect at a single point. Now fix $a \neq b$ in ∂K. For any point c in ∂K, $N(c)$ goes through $p = N(a) \cap N(b)$, and the distance from c to p equals the distance from a to p. Thus all points of ∂K are equidistant from p, and K is then a ball with center p.

This raises the question of what happens in $\mathbf{R}^2$. The statement above is hopelessly false, but we do have a characterization of certain sets in $\mathbf{R}^2$ in terms of billiard orbits.

We need some definitions. Throughout, K will be a smooth convex body in $\mathbf{R}^2$. K is of *constant breadth* if the distance between distinct parallel supporting lines is a constant independent of the point of support. K has the *double normal property* if for each p in ∂K the normal segment N at p is also the normal segment for the antipodal point q (being the other point of $N \cap \partial K$). Finally, we will say that K has the *one-way orbit property* for billiard orbits if each orbit consists of only right-handed reflections or of only left-handed reflections.

THEOREM. *Let K be a smooth convex body in $\mathbf{R}^2$. The following are equivalent*:
(a) *constant breadth,*
(b) *double normal property,*
(c) *one way orbit property.*

REMARK. $a \Rightarrow b$ is trivial and well known. As for $b \Rightarrow a$ we quote Danzer [2]: "Though it has been well known probably for decades ... there seems to be no older reference than the paper by Besicovitch [1] ..."

Proof. In light of the Remark we will show only $b \Leftrightarrow c$. Suppose K has the double normal property and assume for the sake of contradiction that some billiard orbit in K contains a Z-shaped figure with reflection points a, b, c, and d on ∂K.

If $\angle abc$ and $\angle bcd$ are equal in magnitude, the normals $N(b)$ and $N(c)$ are parallel. It follows that the equivalent constant breadth property implies that the two line segments $\overline{ab}$ and $\overline{bd}$ lie in ∂K. But no set of constant breadth can contain a line segment in its boundary. Thus we can assume that $\angle abc$ is less than $\angle bcd$ in magnitude. Let b' be the antipodal point to b, and H the support line at c. It is seen at once that b' is forced to be on both sides of H. This contradiction gives the implication desired.

For the converse let b in ∂K have a normal N and assume $M \neq N$ where M is the reflection of N at the antipodal point b'. Consider the orbit which contains the path down M to b', then the reflection down N to b and a return along N after the reflection at b. It is clear that a small perturbation of the orbit gives rise to a billiard orbit which contains a Z-shaped figure. This gives the reverse implication and we are done.

2. Currently, as well as historically, a major facet of interest in billiard problems is their ergodic properties. The recent delightful set of notes of Sinai [3] can be consulted for a more complete exposition and references. For a given convex body we construct a phase space for billiards as follows. Let X be the space of pairs $x=(p,s)$ where p is a point of the "carrier space" K and s is a point on a $n-1$ dimensional sphere. If we are willing to reach a certain tolerant understanding at the boundary of K, there is a one-to-one correspondence between the points of billiard orbits in K and the points of X. The first coordinate p of (p,s) represents the instantaneous position of the billiard point and s represents the instantaneous velocity. The action of billiards in K can be transferred to motions in X so that a group of transformations of X is generated. There is a natural measurement of volume in X (a normalized positive measure) which is invariant under the group of transformations. One question of interest is how good a

job of stirring up the space is done by the group of transformations. The group of transformations is called *ergodic* if the only (measurable) sets which are left invariant by the group are either of volume zero or the complement of a volume zero set. If K is a ball, it is fairly easy to show the billiard transformations are not ergodic. Let A be the set of orbits whose closest approach to the center of the ball is over half the radius. It is easily seen that A corresponds to an invariant set in X. The invariant volume has the form $dv = da \cdot ds$ where da is the natural "area" measurement in K and ds is the natural surface "area" measurement of the $n-1$ sphere. This makes it apparent that the set to which A corresponds in X is not null or co-null with respect to the volume measurement of X. According to Sinai [3], ergodicity of billiards in a right triangle with incommensurable acute angles is an open problem. (The lack of smoothness is avoided by ignoring billiard orbits which ever hit a corner.) As an application of Theorem 2, we have:

COROLLARY. *Billiards in a smooth convex body of constant breadth in* $\mathbf{R}^2$ *is not ergodic.*

Proof. The decomposition of the set X into points belonging to left-hand orbits and those belonging to right-hand orbits gives a decomposition which establishes the result.

Added in proof. V. F. Lazutkin (Math USSR-Izv. 7, No. 1 (1973)) has a result which implies almost all problems are non-ergodic. All that is required is a sufficiently smooth boundary (553 continuous derivatives) but the machinery is deep.

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UNIFIED TREATMENT OF INEQUALITIES OF THE WEIERSTRASS PRODUCT TYPE

EMAD EL-NEWEIHI AND FRANK PROSCHAN

In a note by Klamkin and Newman [1], it was shown that

$$\prod_{i=1}^{n}(1+A_i) \geqslant (n+1)^n \prod_{i=1}^{n} A_i \tag{1}$$

$$\prod_{i=1}^{n}(1-A_i) \geqslant (n-1)^n \prod_{i=1}^{n} A_i, \tag{2}$$

where $A_i \geqslant 0$ $(i=1,2,\ldots,n)$ and $\sum_{i=1}^{n} A_i = 1$. Under the same conditions it was shown by Klamkin in [2] that

$$\frac{\prod_{i=1}^{n}(1+A_i)}{(n+1)^n} \geqslant \frac{\prod_{i=1}^{n}(1-A_i)}{(n-1)^n} \tag{3}$$

with equality if $A_i = \frac{1}{n}$. In [2] the author refers to a similar inequality to (2) by Ky Fan [4, p.

363] under tighter conditions on A_i but more relaxed conditions on ΣA_i, namely,

$$\prod_{i=1}^{n}(1-A_i)\geqslant\left\{\frac{n-\Sigma A_i}{\Sigma A_i}\right\}^n\prod_{i=1}^{n}A_i\quad\text{for } 0<A_i\leqslant\tfrac{1}{2}. \tag{4}$$

It should be noted that (3) can be derived from (4) by letting $A_i'=\dfrac{1-A_i}{2}$. (Reported in this MONTHLY, 83 (1976) 800.) Inequalities (1), (2), and (3) are proved by the authors as extensions of the Weierstrass product inequalities (see [1]).

The main purpose of this note is to show that the inequalities (1), (2), (3), and (4) can be obtained by a uniform approach using the powerful tools of majorization and Schur-functions. Majorization (defined in Section 2) is a partial ordering in R_n, the n-dimensional Euclidean space. A Schur-function is a function that is monotone with respect to this partial ordering. In Section 2 we show that inequalities (1), (2), (3), and (4) follow immediately by observing that certain functions are Schur-functions, thus providing a unifying proof of these inequalities.

We now give the standard definitions and results of majorization and Schur-functions needed for proving inequalities (1), (2), (3) and (4).

Given a vector $\mathbf{x}=(x_1,\dots,x_n)$, let $x_{[1]}\geqslant x_{[2]}\geqslant\cdots\geqslant x_{[n]}$ denote a decreasing rearrangement of $x_1,\dots,x_n$.

DEFINITION 1. A vector $\mathbf{x}$ is said to majorize a vector $\mathbf{x}'$ if

$$\sum_{i=1}^{j}x_{[1]}\geqslant\sum_{i=1}^{j}x'_{[1]},\qquad j=1,\dots,n-1,$$

and

$$\sum_{i=1}^{n}x_{[i]}=\sum_{i=1}^{n}x'_{[i]};$$

in symbols $\mathbf{x}\succ\mathbf{x}'$.

DEFINITION 2. A function $f:R_n\to R$ is said to be a *Schur-convex* (*Schur-concave*) function if $\mathbf{x}\succ\mathbf{x}'$ implies that $f(\mathbf{x})\geqslant(\leqslant)f(\mathbf{x}')$. Functions which are either Schur-convex or Schur-concave are called *Schur-functions*. A function $\phi:R\to R$ is said to be log-convex (log-concave) if log $\phi:R\to R$ is convex (concave).

THEOREM 1. *Let $\phi:R\to R$ be a log-convex (log-concave) real-valued function. Let $g:R_n\to R$ defined by $g(\mathbf{x})=\prod_{i=1}^{n}\phi(x_i)$, where $\mathbf{x}=(x_1,\dots,x_n)$. Then g is Schur-convex (Schur-concave).*

Theorem 1 follows immediately from a theorem by Ostrowski (1952) [3]. It also follows from Hardy, Littlewood and Pólya [4, Theorem 1, p. 164].

We are now ready to prove inequalities (1), (2), (3), and (4). Let $\mathbf{A}=(A_1,\dots,A_n)$, where $A_i\geqslant 0,\Sigma_{i=1}^{n}A_i=1$. Clearly $\mathbf{A}\succ(\frac{1}{n},\dots,\frac{1}{n})$. Now let $\phi_1(x)=(1+x)/x$, where $x\geqslant 0$. Then $\phi_1(x)$ is log-convex. Therefore by Theorem 1, $\prod_{i=1}^{n}\phi_1(x_i)$ is Schur-convex. Hence

$$\prod_{i=1}^{n}\frac{(1+A_i)}{A_i}\geqslant(n+1)^n$$

which establishes (1).

Now let $\phi_2(x)=(1+x)/(1-x), 0\leqslant x<1$. Then $\phi_2(x)$ is log-convex and so by Theorem 1, $\prod_{i=1}^{n}\phi_2(x_i)$ is Schur-convex. It then follows that

$$\prod_{i=1}^{n}\frac{(1+A_i)}{(1-A_i)}\geqslant\frac{(n+1)^n}{(n-1)^n},$$

which establishes (3). Equality holds if $A_i=\frac{1}{n}$ since $\phi_2(x)$ is strictly log-convex.

Now let $\phi_3(x)=(1-x)/x$; then $\phi_3(x)$ is log-convex for $0<x\leqslant\frac{1}{2}$ and log-concave for $\frac{1}{2}\leqslant x<1$. Also let $\mathbf{A}=(A_1,\ldots,A_n)$, where $0\leqslant A_i\leqslant\frac{1}{2}$. Clearly

$$\mathbf{A}\succ\left(\frac{\Sigma A_i}{n},\ldots,\frac{\Sigma A_i}{n}\right),$$

so that

$$\prod_{i=1}^{n}\left(\frac{1-A_i}{A_i}\right)\geqslant\left[\frac{1-\Sigma A_i/n}{\Sigma A_i/n}\right]^n=\left[\frac{n-\Sigma A_i}{\Sigma A_i}\right]^n,$$

establishing (4).

Finally, let $\mathbf{A}^1=((n-1)A_1,\ldots,(n-1)A_n)$ and let $A^2=(1-A_1,\ldots,1-A_n)$ where $A_i\geqslant 0,\Sigma A_i=1$. It can be easily verified that $\mathbf{A}^1\succ\mathbf{A}^2$. By Theorem 1, $g(\mathbf{x})=\Pi_{i=1}^n x_i$ is a Schur-concave function. It then follows that

$$(n-1)^n\prod_{i=1}^{n}A_i=\prod_{i=1}^{n}(n-1)A_i\leqslant\prod_{i=1}^{n}(1-A_i),$$

proving inequality (2).

It is apparent that many additional inequalities of the Weierstrass product type can be formulated and proved by choosing the appropriate log-concave function, forming products to obtain a Schur-convex function, and then using Definition 2 above.

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CLASSROOM NOTES

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REGULATED FUNCTIONS: BOURBAKI'S ALTERNATIVE TO THE RIEMANN INTEGRAL

S. K. BERBERIAN

1. Introduction. At the outset, I hasten to say that I remain a "Riemann loyalist": pound for pound, the Riemannian circle of ideas can't be beat for its instructional value to the beginning

THE NORMAL BASIS THEOREM

WILLIAM C. WATERHOUSE

Several books treating Galois theory [1], [2], [3] prove the normal basis theorem for infinite fields by starting with linear independence of characters and going *via* algebraic independence of characters. For teaching, it may be useful to have on record a variant argument where the intermediate step is replaced by a simpler result. We begin still with linear independence.

THEOREM. *Let L/K be a Galois field extension with group G. If $\Sigma_{\sigma\in G} r_\sigma\sigma(b)=0$ for all b in L, then all r_σ are zero.*

The proof is by contradiction, taking a relation with as few nonzero r as possible and then subtracting $0=\Sigma_\sigma r_\sigma\sigma(cb)=\Sigma_\sigma(\sigma(c)r_\sigma)\sigma(b)$ for appropriate c to shorten it; details are given in all the books cited. Our intermediate step now is really part of a proof that $L\otimes_K L\approx\Pi_G L$, but it makes sense without tensor products and is interesting in itself.

COROLLARY. *For each σ in G let d_σ in L be given. Then there is a finite sequence of pairs a_i, b_i in L such that $\Sigma_i a_i\sigma(b_i)$ equals d_σ for every σ.*

Proof. Consider the vector space $L\times\cdots\times L$ indexed by σ in G. No linear function $\Sigma r_\sigma x_\sigma$ vanishes on all n-tuples of the form $x_\sigma=\sigma(b)$, so these $\langle\sigma(b)\rangle$ span the L-space. Hence we can write any $\langle d_\sigma\rangle$ as $\Sigma a_i\langle\sigma(b_i)\rangle$ for some collection of a_i and b_i in L.

THEOREM (Normal Basis). *Let L be a Galois extension of an infinite field K. Then there is some c in L whose images $\sigma(c)$ for σ in* Gal(L/K) *are linearly independent over K.*

Proof. Choose a_i, b_i in L with $\Sigma a_ib_i=1$ and $\Sigma a_i\sigma(b_i)=0$ for $\sigma\neq\text{id}$. Consider the determinant $\det_{\sigma,\tau}[\Sigma_i X_i\tau^{-1}\sigma(b_i)]$, a polynomial in variables X_i. Setting $X_i=a_i$ we get the determinant of the identity matrix, so the polynomial is not identically zero. As K is infinite, there are values x_i in K with $0\neq\det[\Sigma x_i\tau^{-1}\sigma(b_i)]=\det[\tau^{-1}\sigma(\Sigma x_ib_i)]$. We take c to be any such Σx_ib_i, so that the matrix $[\tau^{-1}\sigma(c)]$ is nonsingular. Suppose then $0=\Sigma u_\sigma\sigma(c)$ with u_σ in K. Applying τ^{-1}, we find $0=\Sigma_\sigma u_\sigma\tau^{-1}\sigma(c)$ for all τ. By nonsingularity all u_σ are zero, and thus the $\sigma(c)$ are independent.

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THE KANTOROVICH THEOREM WITH OPTIMAL ERROR BOUNDS

G. J. MIEL

The Kantorovich theorem for Newton's method is of basic importance in numerical analysis. An instructive history of the theorem is given in [4, p. 428]. Using majorant functions, Ortega [3] gave an elegant proof of the convergence. To this proof, Tapia [5] added a derivation of certain non-optimal error bounds. Best possible lower and upper error bounds were presented by Gragg and Tapia [1]. In the derivation of one of the optimal upper bounds, Gragg and Tapia resorted to the original Kantorovich recurrence relations. The establishment of these relations is somewhat lengthy and involved. In this note, we incorporate in Ortega's proof a derivation, well

suited for the classroom, of the Gragg-Tapia optimal upper bounds.

In what follows, X and Y are Banach spaces, D is an open convex subset of $X, F: D \to Y$ is Fréchet differentiable on D and such that

$$\|F'(x)-F'(y)\| \leqslant K\|x-y\|, \qquad x,y \in D.$$

The open ball $\{x: \|x-x_0\| < r\}$ and its closure are denoted by $S(x_0,r)$ and $\bar{S}(x_0,r)$, respectively.

THEOREM. *For some $x_0 \in D$, assume that $F(x_0)^{-1}$ is defined on all of Y, and that*

$$\|F(x_0)^{-1}Fx_0\| \leqslant a,\ \|F(x_0)^{-1}\| \leqslant b,\ \bar{S}(x_0,t^*) \subset D,$$

where $t^* = (2a/h)(1-\sqrt{1-h})$, $h \equiv 2Kab \leqslant 1$. Let $t^{**} = (2a/h)(1+\sqrt{1-h})$. Then the iterates $x_{n+1} = x_n - F'(x_n)^{-1}Fx_n$ exist, remain in $S(x_0,t^*)$, and converge to the only root x^* of F in $D \cap S(x_0,t^{**})$. Moreover, the error bounds

$$\|x^*-x_n\| \leqslant \frac{4\sqrt{1-h}}{h}\,\frac{\theta^{2^n}}{1-\theta^{2^n}}\|x_1-x_0\| \quad \text{if } h<1, \tag{1}$$

$$\|x^*-x_n\| \leqslant 2^{-n+1}\|x_1-x_0\| \quad \text{if } h=1, \tag{2}$$

$$\|x^*-x_n\| \leqslant \theta^{2^{n}-1}\|x_n-x_{n-1}\|, \qquad \theta = t^*/t^{**}, \tag{3}$$

are valid and best possible.

Proof. First, we outline Ortega's proof [3]. Define the scalar function

$$f(t) = \frac{K}{2}t^2 - \frac{1}{b}t + \frac{a}{b} = \frac{K}{2}(t^*-t)(t^{**}-t)$$

and the iterates

$$t_0 = 0, \qquad t_{n+1} = t_n - f'(t_n)^{-1}f(t_n). \tag{4}$$

From the Banach lemma, if $x \in S(x_0,t^*)$ then $F'(x)^{-1}$ is defined on all of Y and

$$\|F'(x)^{-1}\| \leqslant b/(1-bK\|x-x_0\|). \tag{5}$$

If also $Gx = x - F'(x)^{-1}Fx \in S(x,t^*)$, then using $Fx + F'(x)(Gx-x) = 0$ and the mean-value theorem,

$$\|G(Gx)-Gx\| \leqslant \frac{\frac{bK}{2}\|Gx-x\|^2}{1-bK\|Gx-x_0\|}. \tag{6}$$

Simple calculations show that $\lim t_n = t^*, t_n < t_{n+1}$, and

$$t_{n+1}-t_n = \frac{\frac{bK}{2}(t_n-t_{n-1})^2}{1-bKt_n}. \tag{7}$$

If x_n exists and $\|x_i - x_{i-1}\| \leqslant t_i - t_{i-1}$, $i \leqslant n$, as is true by assumption for $n=1$, then $\|x_n - x_0\| \leqslant t_n - t_0 \leqslant t^*$. Evaluate (6) with $x = x_{n-1}$ and use (7) to get $\|x_n - x_{n-1}\| \leqslant t_n - t_{n-1}$. Consequently,

$$\|x_{n+p}-x_n\| \leqslant t_{n+p}-t_n, \tag{8}$$

for arbitrary positive integers n and p. Therefore, $\{x_n\}$ converges to some x^* and the error bounds

$$\|x^*-x_n\| \leqslant t^*-t_n \equiv e_n \tag{9}$$

are valid. That $Fx^* = 0$ follows from the continuity at x^* of F and F'. The uniqueness statement follows by consideration of the modified Newton method. The bounds (9) are standard and contained in [3].

Next, along the lines of Gragg and Tapia [1], we work with the quantities e_n of (9). Rewrite

(4) as

$$t_{n+1}=t_n+e_nE_n/(e_n+E_n), \qquad E_n=t^{**}-t_n. \tag{10}$$

Hence

$$e_{n+1}=\frac{e_n^2}{2e_n+\Delta}, \qquad \Delta=t^{**}-t^*, \qquad e_0=t^*. \tag{11}$$

If $h=1$, then $\Delta=0$ and

$$e_n=2^{-n+1}a.$$

If $h<1$, let $e_n=(\Delta/2)(s_n-1)$ to get $s_{n+1}=(s_n^2+1)/2s_n$. As in [**1**, p. 11],

$$\frac{s_{n+1}-1}{s_{n+1}+1}=\left(\frac{s_n-1}{s_n+1}\right)^2, \qquad \frac{s_n-1}{s_n+1}=\left(\frac{s_0-1}{s_0+1}\right)^{2^n}=\theta^{2^n},$$

and consequently,

$$e_n=\frac{4\sqrt{1-h}}{h}\,\frac{\theta^{2^n}}{1-\theta^{2^n}}\,a.$$

The bounds (1) and (2) follow from the fact that the quantities e_n/a are monotone increasing functions of a.

Now, for the proof of (3), let $\hat{a}=\|x_1-x_0\|$ and use (9) to get

$$\|x^*-x_1\|\leqslant\left(\frac{t^*-\hat{a}}{\hat{a}}\right)\hat{a}=\Omega(\hat{a},b)\hat{a},$$

where

$$\Omega(a,b)=2\omega(h)-1, \qquad \omega(h)=\frac{1}{h}(1-\sqrt{1-h}\,).$$

Since $\omega'(h)\geqslant 0$, for $h\in(0,1)$, the function Ω is monotone increasing in a. Thus

$$\|x^*-x_1\|\leqslant\Omega(a_0,b_0)\|x_1-x_0\|$$

where $a_0=a$, $b_0=b$.

Use (8) and (5) to show that

$$\|F'(x_n)^{-1}Fx_n\|\leqslant a_n=-f'(t_n)^{-1}f(t_n), \qquad \|F'(x_n)^{-1}\|\leqslant b_n=-f'(t_n)^{-1}. \tag{12}$$

Define

$$t_n^*=\frac{2a_n}{h_n}\left(1-\sqrt{1-h_n}\,\right), \qquad h_n=2Ka_nb_n.$$

For any t,

$$(f')^2-2f''f=\text{discriminant of } f. \tag{13}$$

Use (13) to get $\left(\frac{d}{dt}\right)f/(f')^2\leqslant 0$, $t\in(0,t^*)$. Therefore,

$$h_n=2Kf(t_n)/f'(t_n)^2\leqslant 2Kf(0)/f'(0)^2=h_0\leqslant 1$$

and t_n^* is a real number.

From (13) and (12),

$$f'(t_n)^2-2Kf(t_n)=f'(0)^2-2Kf(0), \qquad \frac{1}{b_n}\sqrt{1-h_n}=\frac{1}{b}\sqrt{1-h}\,, \qquad t_n^*+t_n=t^*. \tag{14}$$

Use (14) to show that $\bar{S}(x_n, t_n^*) \subset \bar{S}(x_0, t^*) \subset D$. It follows that

$$\|x^* - x_{n+1}\| \leqslant \Omega(a_n, b_n)\|x_{n+1} - x_n\|.$$

Again from (14),

$$\Omega(a_n, b_n) = \frac{t_n^* - a_n}{a_n} = \frac{t^* - t_{n+1}}{t_{n+1} - t_n}.$$

Thus

$$\|x^* - x_n\| \leqslant \alpha_n \|x_n - x_{n-1}\|, \qquad \alpha_n = (t^* - t_n)/(t_n - t_{n-1}). \tag{15}$$

Recall now the definitions of e_n, E_n, a_n given in (9), (10), (12), respectively. From (4),

$$e_{n+1} = \frac{e_n^2}{e_n + E_n}, \qquad E_{n+1} = \frac{E_n^2}{e_n + E_n}, \qquad a_n = \frac{e_n E_n}{e_n + E_n}. \tag{16}$$

Consequently, $\alpha_{n+1} = e_{n+1}/a_n = e_n/E_n$ and $e_{n+1}/E_{n+1} = (e_n/E_n)^2$. The solution of the difference equation

$$\alpha_{n+1} = \alpha_n^2, \qquad \alpha_1 = t^*/t^{**}, \tag{17}$$

yields (3). The bounds (9) and (15) are clearly attained with $F = f$, $x_0 = t_0$. This completes the proof.

The quadratic polynomial f shows that the uniqueness statement in the theorem cannot be improved. The error bounds for $h = 1$ are valid for all $h \leqslant 1$. The recurrence relations (11) and (17) are useful for on-line computation of the bounds (9) and (15). The explicit bounds (1) and (2) arise from the fact that the difference equation (4) for the majorizing sequence has been solved in closed form by Gragg and Tapia [**1**, p. 11]. Their derivation of (3) relies on the original recurrence relations of Kantorovich, whereas the argument given here works directly with the majorizing sequence.

As shown in [**2**], a natural modification of this argument yields optimal bounds

$$\|x^* - x_n\| \leqslant \hat{\alpha}_n \|x_n - x_{n-1}\|^2, \qquad \hat{\alpha}_n = (t^* - t_n)/(t_n - t_{n-1})^2.$$

An explicit form for $\hat{\alpha}_n$ is easily found. From (16), $\hat{\alpha}_{n+1} = e_{n+1}/a_n^2 = 1/(e_{n+1} + \Delta)$. Thus, since e_n is known in closed form,

$$\hat{\alpha}_n = \frac{h}{4a\sqrt{1-h}}(1 - \theta^{2^n}) \quad \text{if } h < 1, \qquad \hat{\alpha}_n = \frac{2^{n-1}}{a} \quad \text{if } h = 1.$$

Finally, we note that bounds of type (15) are valid for a class of Newton-type methods, see [**2**].

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MAINTAINING MATHEMATICAL MOMENTUM IN THE STATE OF JEFFERSON

RICHARD G. MONTGOMERY AND ROY RYDEN

Although the boundaries of the State of Jefferson have eluded political demarcation, it is generally agreed that this region encompasses sizable portions of northern California and southern Oregon and sometimes includes a thin strip of northwestern Nevada. As early as 1852 and as recently as 1941, bonds of common interests and mutual disenchantments prompted serious attempts to establish the State of Jefferson as a political reality. Traces of those endeavors remain; for example, a newspaper and an equestrian association have incorporated "State of Jefferson" into their names.

There is also the State of Jefferson Mathematics Congress. Organized jointly by the Mathematics Departments of Humboldt State University and Southern Oregon State College to help overcome the geographical isolation of these institutions from other centers of mathematical activity, the Congress encourages mathematics in the State of Jefferson by bringing people together to share and discuss current ideas on mathematics, applications, and curricular directions.

The inaugural session of the Congress, then known as the Weaverville Mathematics Congress, was held near Weaverville, California, during May 1972; each succeeding May the Congress has reconvened for two days. The most recent sessions, held on the shores of Whiskeytown Lake, have attracted camping mathematicians and their families from five institutions: Chico State University, Humboldt State University, Sonoma State College, Southern Oregon State College, and the University of Nevada at Reno.

Held in a (somewhat primitive) National Park Service group-campground, the Congress has a formal program consisting of two one-hour talks and an occasional shorter talk. Topics have included research mathematics recalled or in progress, applications of mathematics to areas such as biology and computers, and mathematical history and lore. In addition, the annual Round-Table-Under-the-Oaks Meeting identifies and discusses common concerns of our various mathematical programs. The Congress formally adjourns with the annual "business" meeting, held around an evening campfire.

This brief sketch of the State of Jefferson Mathematics Congress provides another example of mathematically stimulating activities which can endure through the energy of common interest and in spite of meager budgets. (See also references [1]–[6].) And, of course, should you just happen to be wandering by the Whiskeytown National Recreation Area (near Redding, California) on May 18–20, 1979, the Eighth Session of the *State of Jefferson Mathematics Congress* welcomes your attendance.

References

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STUDENT PACING OR INSTRUCTOR PACING?

RICHARD B. THOMPSON AND JAN S. MCCOY

1. Introduction. Many colleges and universities now use some type of Personalized System of Instruction (PSI) in their basic and remedial mathematics courses. There is strong evidence [1] suggesting that mastery-unit testing is the essential ingredient in the successful application of PSI to mathematics. Under this plan the material to be learned is divided into relatively short units which are studied in sequence. A student progresses to a new unit only after demonstrating mastery of the preceding unit by achieving a predetermined level of performance on a unit quiz.

The administrative procedures for PSI vary widely among institutions. (See [2], for example.) However, it is clear that provision must be made to allow students some flexibility in their quiz-taking schedules, while at the same time encouraging them to progress through the course. Such systems are usually classified as *student-paced* or *instructor-paced.*

Student-paced programs provide students with a schedule showing the normal or expected rate of progress and allow them to take quizzes whenever they feel they are ready to do so. This is a rational system that allows for individual differences among students. Unfortunately, it also invites unlimited procrastination and has often been associated with very low completion rates.

Instructor-paced programs force students to complete various parts of their course on a fixed schedule. Such a system is somewhat antithetical to the basic idea of PSI and must be designed with some flexibility if it is not to negate the benefits of mastery-unit testing. On the other hand, instructor pacing can provide the pressure that some students need to complete the course.

In a given setting for PSI it is often unclear how the relative advantages and disadvantages of student pacing and instructor pacing balance each other. We will show that there is no single answer to the question of whether student pacing or instructor pacing is better. Instead, the choice of method seems to depend heavily upon the mathematical level of the course and upon the nature of the students.

2. The experiment. The authors are directors of a large program at the University of Arizona called the Individual Learning System (ILS). This system, which averages 3000 students per semester in Introductory, Intermediate, and College Algebra, was run on a student-paced basis through the fall of 1976. A description and evaluation of the ILS is given in [3].

Of particular interest here is that the ILS has provision for flexible credit enrollment. A student who registers for a complete 3-credit course may, at any time during the semester, drop back to a 2-credit program that consists of two-thirds of the course, or to a 1-credit program that consists of one-third of the course. At the end of the semester, students receive final grades for the parts of the course that they have completed and then register for the remaining part or parts during subsequent semesters.

In the spring of 1977, prompted by the desire to raise completion rates, we converted the ILS to a type of instructor pacing that involves three tracks. Students in an entire 3-credit course are given a schedule showing the normal completion date and make-up (retesting) dates for each of the twelve units that are part of their program. No quizzes over a given unit are available to a student after the last make-up date for that unit.

Students who cannot maintain this pace change their registration to a 2-credit program and receive a schedule of the normal completion and make-up dates for the eight units that they will study. Finally, students who cannot keep up with the 2-credit pace change their registration to a 1-credit program and receive a new slower-paced schedule. Thus, each student in the ILS must maintain a 12-unit-per-semester (one week per unit) pace, an 8-unit-per-semester (one and one-half weeks per unit) pace, or a 4-unit-per-semester (three weeks per unit) pace.

Several factors enter into a comparison of the relative performance of students in the ILS with student pacing versus the ILS with instructor pacing. Over a period of several years, we have observed a uniform decline in some areas of student performance each spring semester

when compared to the preceding fall semester. Thus a direct comparison of fall 1976 and spring 1977 does not give the complete answer to our question. A comparison between spring 1976 and spring 1977 eliminates the fall–spring bias but introduces a new problem, since several minor changes were made in the ILS between the academic years of 1975–76 and 1976–77.

To minimize the problems outlined above, we have compared the performance of 100 students per semester in each of Introd ctory, Intermediate, and College Algebra for four semesters: fall 1975, spring 1976, fall 1976, and spring 1977. These students were selected from similar sections each semester and represent the first 100 students on the class rolls who began work in the 3-credit course. Since the placement procedures used during the four semesters were identical, these represent comparable groups of students.

To help assess the effects of the semester-to-semester variations that were unrelated to student or instructor pacing, we have also computed *expected performances* for instructor pacing in spring 1977. These are based on the following proportion involving the three preceding student-paced semesters.

$$\frac{\text{expected performance spring 1977}}{\text{actual performance fall 1976}} = \frac{\text{actual performance spring 1976}}{\text{actual performance fall 1975}}$$

3. **The results.** *College Algebra.* Both 1975–76 and 1976–77 appear to be typical years in College Algebra, with no program changes during either academic year except the change from student pacing to instructor pacing between fall 1976 and spring 1977. However, a significant upgrading of course standards, which was instituted between the two academic years, resulted in a general lowering of completion rates during 1976–77.

	Student-paced				Instructor-paced
	Fall 1975	Spring 1976	Fall 1976	Expected Performance Spring 1977	Spring 1977
% who passed entire 3-credit course	45%	32%	38%	27%	44%
% who passed 1-, 2-, or 3-credit program	78%	61%	61%	44%	62%
Mean number of credit hours earned per student	1.85	1.34	1.44	1.04	1.57
Mean score for all 30-point unit quizzes	22.8	22.4	23.6	23.2	23.6
Mean final exam score per student (100 points possible)	55.4	54.7	61.3	60.5	63.0

TABLE 1. College Algebra

The first three rows of Table 1 show that the introduction of instructor pacing caused a substantial improvement in completion rates and credits earned when compared with the predicted values for student pacing. In absolute terms the introduction of instructor pacing more than offset the normal fall to spring decline in course completion.

The last two rows of Table 1 indicate that the level of performance on unit quizzes has remained relatively constant throughout the four semesters and that the level of final examination performance improved slightly under instructor pacing.

Intermediate Algebra. No program changes were made in Intermediate Algebra during the academic years 1975–76 and 1976–77 other than the change from student to instructor pacing between fall 1976 and spring 1977. However, course standards were raised between the two academic years, causing a general drop in completion rates for both of the 1976–77 semesters.

It is our subjective feeling that the fall semester of 1975 may not have been quite typical. The completion rates were somewhat higher than normal and the quality of work on the final examination appeared to be lower than usual. These aberrations, if they are such, would tend to lower the predicted completion rates and raise the predicted final examination scores for spring 1977.

	Student-paced				Instructor-paced
	Fall 1975	Spring 1976	Fall 1976	Expected Performance Spring 1977	Spring 1977
% who passed entire 3-credit course	55%	46%	43%	36%	47%
% who passed 1-, 2-, or 3-credit program	74%	68%	62%	57%	68%
Mean number of credit hours earned per student	1.92	1.69	1.54	1.31	1.73
Mean score for all 30-point unit quizzes	24.8	24.4	24.4	24.0	24.4
Mean final exam score per student (100 points possible)	58.4	63.8	67.6	73.9	67.4

TABLE 2. Intermediate Algebra

The first three rows of Table 2 give different measures of the amount of work that students completed during each semester. Instructor pacing reversed the usual fall to spring decline as well as the general 1975–76 to 1976–77 decline caused by increased standards.

The fourth row of Table 2 indicates no change in quiz scores during the experimental period. The bottom line shows that final examination scores under instructor pacing were essentially as good as or better than those under student pacing; however, the low scores in fall 1975 resulted in a high expected performance in spring 1977 that was not matched by the actual scores.

Introductory Algebra. This course was not offered for credit in the fall of 1975. Thus our data are given for only three semesters and no expected performance figures can be computed. However, spring 1976, fall 1976, and spring 1977 were all conducted under the same course plan, with the single exception of the switch to instructor pacing in spring 1977. In this case the spring 1976 and spring 1977 results should be directly comparable.

As shown in the first three rows of Table 3, the various measures of course completion were all significantly lower under instructor pacing. The final two rows indicate that unit quiz scores have remained relatively constant but that final examination scores were definitely lower under instructor pacing.

	Student-paced		Instructor-paced
	Spring 1976	Fall 1976	Spring 1977
% who passed entire 3-credit course	57%	45%	43%
% who passed 1-, 2-, or 3-credit program	70%	59%	54%
Mean number of credit hours earned per student	1.87	1.54	1.45
Mean score for all 30-point unit quizzes	24.6	23.8	25.0
Mean final exam score per student (100 points possible)	64.6	65.2	60.5

TABLE 3. Introductory Algebra

4. Conclusions. Our results indicate that instructor pacing was definitely superior to student pacing in College Algebra. The overall completion rate was up substantially from the expected rate, and the mean number of credit-hours earned increased by .53. This was all accomplished with no loss in quality of work on unit quizzes and with the highest final examination scores that we have yet had. We note that the change to instructor pacing did not cause very much student complaint about loss of freedom. It was, in fact, welcomed by many students as needed pressure to help them complete the course.

In contrast to the success of instructor pacing in College Algebra, this method was highly unsatisfactory in Introductory Algebra. Despite the absence of the increased course standards that were present for Intermediate and College Algebra, we experienced the worst completion rates and the lowest final examination scores we have ever had in Introductory Algebra.

The effect of instructor pacing on Intermediate Algebra appears to rest somewhere between the unqualified success in College Algebra and the failure in Introductory Algebra. Comparison of results of instructor pacing with the expected performance levels under student pacing indicates a gain in completion rates at the expense of quality of work on the final examination. However, the previously discussed anomalies of fall 1975 in Intermediate Algebra may have exaggerated both this gain in quantity and loss in quality. On balance it still appears that instructor pacing is preferable in Intermediate Algebra.

In attempting to analyze the wide variation of the effectiveness of instructor pacing versus student pacing, we have arrived at the following conclusions. Introductory Algebra is our lowest-level course and is highly remedial. Students in this course have had difficulty with mathematics, are afraid of the subject, and often have very low motivation. In this setting any extra pressure to keep up with a fixed pace simply makes their problems worse and causes them to drop out. In contrast to this, the cautious approach of allowing these students to take quizzes when they feel they are ready seems to build confidence and to get them started toward understanding mathematics.

Unlike the students in Introductory Algebra, the students in College Algebra are better able to cope with learning mathematics and do not find the pressure of instructor pacing as destructive as does the former group. Since the material covered in College Algebra is more difficult than that in lower-level courses, it is more tempting to put off quizzes as long as possible. This procrastination is effectively reduced by instructor pacing.

Intermediate Algebra students are a mixed group, having some characteristics of the less advanced and some characteristics of the more advanced students. We suspect that the net advantage of instructor pacing with this course arises from our use of mandatory placement

based upon a pretest. This tends to move most of the students with really serious mathematical learning problems out of Intermediate Algebra and to concentrate them in Introductory Algebra.

Considerable care must be exercised in interpreting the results of educational experiments. Our results were obtained in a large program involving mandatory placement. It is possible that different outcomes would occur in smaller classes or with a different student population. However, we do feel safe in concluding that very low-level students profit more from the security of student pacing and that more advanced students benefit from the extra pressure of instructor pacing.

When considering these results, one must bear in mind that our instructor pacing is modified by the availability of flexible-credit enrollment. Completely rigid instructor pacing is probably incompatible with the basic goals of a PSI program.

The question of student pacing versus instructor pacing must not overshadow the fact that neither method is an end in itself. These are simply necessary devices that allow the use of mastery-unit testing.

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3. R. Thompson, The individual learning system for algebra service courses, in Innovative Teaching Methods in Introductory College Mathematics, ed. by M. Hassett and R. Thompson, Rocky Mountain Mathematics Consortium, 1977, pp. 1–11.

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PROBLEMS AND SOLUTIONS

Edited by A. P. Hillman

The editors receive more problems than they can use, but they are always on the lookout for interesting and novel problems. The task of selection will be much easier if proposers will keep the following guidelines in mind:

Send all problems (both elementary and advanced) to A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131, in duplicate if possible. The editors urge proposers to include any solutions or information that will help the editors, including reasons why the problem is interesting. Problems in well-known textbooks and results that appear in generally accessible sources are not acceptable.

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution. If you submit a problem without a solution, you should tell the editors whether you know (or somebody else knows) how to solve the problem. If you are uncertain, the problem might be more suitable for the Research Problems section; see the instructions given there.

Proposers are asked to aim for the same audience as for the rest of the MONTHLY*: a rule of thumb is to think of people who have had at least a year of graduate work in mathematics, very likely some years ago. Proposals that can be both understood and solved by high school students, or by those in the first two years of college, are usually more appropriate for the Mathematics Magazine or the Two-Year College Mathematics Journal. No precise line can be drawn between elementary and advanced problems, but in general an elementary problem will involve notions no more difficult than those encountered in the first year of graduate work and should be solvable by methods common at the same level. An advanced problem may deal with more advanced concepts or require more sophisticated techniques for its solution.*

A problem that takes half a page or more to state, or that involves several ad hoc definitions, is usually unsuitable. The statement of a problem should avoid specialized jargon and symbols whenever the English language is more readily understood: for example, "f is a continuous function" is preferable to "$f \in C$."

Before submitting a problem, be sure that it says exactly what you intend it to say. Solvers are not telepathic and will usually solve what was proposed, not what you had in mind.

PROBLEMS DEDICATED TO EMORY P. STARKE

Solutions of the problems in this issue dedicated to Professor Emory P. Starke should be mailed to Prof. A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, N.M. 87131 (USA) before July 31, 1979. To facilitate consideration, solutions should be typed (with double spacing).

S 6. *Proposed by M. S. Klamkin and A. Meir, University of Alberta*

Let $x_i > 0$ for $i = 1,2,\ldots,n$ with $n \geqslant 2$. Prove that

$$(x_1)^{x_2} + (x_2)^{x_3} + \cdots + (x_{n-1})^{x_n} + (x_n)^{x_1} \geqslant 1.$$

S 7. *Proposed by George E. Andrews, Pennsylvania State University, and Richard Askey, University of Wisconsin, Madison*

Let $p_n(x) = (x+1)(x+q)\cdots(x+q^{n-1})$, $n = 1,2,\ldots, p_0(x) = 1$. Find the coefficients $a(k,m,n)$ defined by

$$p_n(x)\cdot p_m(x) = \sum_{k=0}^{m+n} a(k,m,n)\cdot p_k(x). \tag{1}$$

S 8. *Proposed by R. Johnsonbaugh, Chicago State University*

Call a function f from a metric space (M,d) into itself *a weak contraction map* if whenever $x,y \in M$ with $x \neq y$, we have $d(f(x),f(y)) < d(x,y)$.

(i) Give an example of a weak contraction map on a complete metric space with no fixed point.

(ii) Show that even on a compact metric space a weak contraction map need not be a contraction map; i.e., it need not satisfy $d(f(x),f(y)) \leqslant cd(x,y)$ for $0 < c < 1$.

(iii) Prove that a weak contraction map on a compact metric space has a unique fixed point.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA (USA) 94303. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (in duplicate, with double spacing) and should be mailed before July 31, 1979. Please enclose a self-addressed label or card (for acknowledgment).

E 2761. *Proposed by Ron Adin, undergraduate, Technion, Haifa, Israel*

Let $P(z)$ be a polynomial of degree at least 2 with complex coefficients, not all of them real. Prove that the equation $P(z)P(-z) = P(z)$ has roots in both the upper and lower open half-planes, $\operatorname{Im} z > 0$ and $\operatorname{Im} z < 0$.

E 2762. *Proposed by Peter Hoffman, University of Waterloo, Canada*

Let $A_1,\ldots,A_n$ be $k\times k$ matrices over a field F, such that $A=A_1+\cdots+A_n$ is invertible. Show that the block-matrix

$$B=\begin{bmatrix} A_1 & A_2 & A_3 & \cdots & & A_n & & 0 & \cdots & 0 \\ 0 & A_1 & A_2 & \cdots & & A_{n-1} & & A_n & \cdots & 0 \\ \vdots & \vdots & \vdots & & & & & & \vdots & \vdots \\ 0 & 0 & \cdots & A_1 & A_2 & & \cdots & & \cdots & A_n \end{bmatrix}$$

has full rank, i.e., rank $(B)=mk$ where m is the number of block-rows.

E 2763. *Proposed by Lorraine L. Foster, California State University, Northridge.*

Let $f(n)=n^3+396n^2-111n+38$. Prove that the congruence $f(n)\equiv 0 \pmod{3^a}$ has precisely 9 solutions $(\operatorname{mod} 3^a)$ for all integers $a\geqslant 5$.

E 2764. *Proposed by Ioan Tomescu, University of Bucharest, Romania.*

Let X be a finite set. Prove that

$$\sum|A_1\cup A_2\cup\cdots\cup A_k|=(2^k-1)\sum|A_1\cap A_2\cap\cdots\cap A_k|,$$

where the sums are over all choices of $A_1,\ldots,A_k\subseteq X$. (Here $|S|$ is the number of elements in S.)

E 2765. *Proposed by Naoki Kimura, University of Arkansas, Fayetteville.*

Establish the two following equations:

$$\int_{-1/2}^{3/2} f(3x^2-2x^3)\,dx=2\int_0^1 f(3x^2-2x^3)\,dx,$$

$$\int_{-1/2}^{3/2} xf(3x^2-2x^3)\,dx=2\int_0^1 xf(3x^2-2x^3)\,dx,$$

for all functions f continuous on $-1/2\leqslant x\leqslant 3/2$.

Is there a quadratic polynomial $g(x)$ such that

$$\int_{-1/2}^{3/2} f(3x^2-2x^3)\,dx=\int_0^1 g(x)f(3x^2-2x^3)\,dx$$

for every continuous function f?

E 2766. *Proposed by I. Borosh and D. Hensley, Texas A & M University.*

Let r be a positive rational number but not an integer. Prove that there are infinitely many positive integers n such that $[nr]$ is prime. (Here $[x]$ is the greatest integer in x.)

SOLUTIONS OF ELEMENTARY PROBLEMS

Integer Points on an Ellipse

E 2682 [1977, 738]. *Proposed by Douglas Hensley, University of Illinois at Urbana–Champaign.*

Let E be an ellipse in the plane whose interior area $A\geqslant 1$. Prove that the number n of integer points on E satisfies $n<6A^{1/3}$.

Solution by the proposer. Take the center of E as origin of $\mathbf{R}^2$ and let C be the circle

$x^2+y^2=A/\pi=r^2$. Let u be a linear transformation of $\mathbf{R}^2$ such that $u(C)=E$ and $\det(u)=1$. (Such u exists because the interior areas of E and C are the same.) Let k be an integer, $k>2(\pi^2A)^{1/3}$. Since $A\geqslant 1$, we have $k\geqslant 5$. Let P be the regular k-gon inscribed in C whose vertices are $p_s=r\exp(2\pi is/k)$, $s\in\mathbf{Z}$, where we identify $\mathbf{C}$ with $\mathbf{R}^2$, as usual.

If $v_s=\overrightarrow{p_sp_{s+1}}$, then

$$|v_s|<2\pi r/k=\frac{2}{k}(\pi A)^{1/2}.$$

Since the angle between v_s and v_{s+1} is equal to $2\pi/k$, we have

$$|v_s\times v_{s+1}|=|v_s|^2\sin\frac{2\pi}{k}<\frac{8\pi^2}{k^3}A<1.$$

Since u preserves the area, we also have $|u(v_s)\times u(v_{s+1})|<1$. Assume that there exist three integer points a,b,c on E between $u(p_s)$ and $u(p_{s+2})$. Then $|(b-a)\times(c-b)|$ is a non-zero integer and hence it is $\geqslant 1$. Thus $|u^{-1}(b-a)\times u^{-1}(c-b)|\geqslant 1$. This is impossible since $u^{-1}(a),u^{-1}(b)$, $u^{-1}(c)$ are between p_s and p_{s+2}, and in that arc the absolute value of the cross product of successive chord vectors is maximal with v_s and v_{s+1}. Hence there are at most two integer points on each of the closed arcs $(u(p_s),u(p_{s+2}))$ of E. It follows that there are at most k integer points on E. If k is the least integer satisfying $k>2(\pi A)^{1/3}$ then $k<6A^{1/3}$.

Also solved by George Andrews.

Editor's Comment. Andrews proves the following stronger result. Let C be any strictly convex region in the plane whose interior area $A\geqslant 1$. Then the number of integer points on the boundary of C is less than $6A^{1/3}$. His proof is based on the method of Bohuslav Diviš: Lattice points on convex curves, Monatsh. Math., 77 (1973), 389–395. He also remarks that the main result of his paper "On Estimates in Number Theory" (this MONTHLY, 70 (1963) 1063–1065) can now be improved.

Egyptian Fractions

E 2689 [1978, 47]. *Proposed by L.-S. Hahn, University of New Mexico.*

Is there a non-empty finite set S of positive integers such that

(1) $n\in S\Rightarrow n-1\in S$ or $n+1\in S$,

(2) $\sum_{n\in S}\frac{1}{n}$ is an integer?

Solution by Peter L. Montgomery, System Development Corporation, Huntsville, Alabama. Yes. Two solutions are

$$S_1=\{1,2,7,8,13,14,39,40,76,77,285,286\}\quad\text{and}$$
$$S_2=\{2,3,4,5,6,7,9,10,17,18,34,35,84,85\}.$$

In each case, the Egyptian fractions sum to 2.

The second example was also found by Dean Hickerson.

Classical Inequalities

E 2691 [1978, 48]. *Proposed by Živojin M. Mijalković, Pirot (Yugoslavia) and J. B. Keller, Courant Institute.*

If $x_i>0$ $(1\leqslant i\leqslant n)$ show that

$$(\Pi x_i)^{\frac{1}{n}\Sigma x_i}\leqslant\Pi x_i^{x_i}\leqslant\left(\frac{\Sigma x_i^2}{\Sigma x_i}\right)^{\Sigma x_i}.$$

Solution by Russell Lyons, undergraduate, Case Western Reserve University. In fact we have

$$(\Pi x_i)^{\Sigma x_i/n} \leqslant \left(\frac{\Sigma x_i}{n}\right)^{\Sigma x_i} \leqslant \Pi x_i^{x_i} \leqslant \left(\frac{\Sigma x_i^2}{\Sigma x_i}\right)^{\Sigma x_i}$$

with equality in each case iff $x_1 = \cdots = x_n$. Each of these follows from the weighted arithmetic-geometric mean inequality: If a_i, w_i are positive then

$$\left(\frac{\Sigma w_i a_i}{\Sigma w_i}\right)^{\Sigma w_i} \geqslant \Pi a_i^{w_i}$$

with equality iff $a_1 = \cdots = a_n$. In the first case let $w_i = 1$, $a_i = x_i$; in the second, let $w_i = x_i$ and $a_i = 1/x_i$; and in the third, let $w_i = a_i = x_i$.

Also solved by David Anderson, M. T. Bird, Robert Breusch, Stavros Busenberg, David Carlson, Jesse Deutsch, Michael Dixon, C. T. Giel, Gustaf Gripenberg (Finland), Benjamin Klein, Kenneth Klinger, Olaf Krafft (Germany), L. Kuipers (Switzerland), O. P. Lossers (Netherlands), L. E. Mattics, Jean-Marie Monier, John Myers, Akihiro Nishi, Ingram Olkin, Richard Ostrander, Victor Pambuccian (Romania), "Nakonets Reshil" (USSR), Terence Shore, Constantino Unguriano, Gillian Valk, Benjamin Zipperer, and the proposer.

A Dense Subset of the Unit Circle

E 2697 [1978, 116]. *Proposed by William Anderson and William Simons, West Virginia University.*

Is there a dense subset S of the unit circle such that each point in S has rational coordinates and the (Euclidean) distance between any pair of points in S is also rational?

I. *Solution by J. G. Mauldon, Amherst College.* Let T be the set of all points $e^{i\alpha}$ with $\tan\alpha$ rational. The set $S = \{z^4 | z \in T\}$ is dense on the unit circle and consists of rational points. If $z = e^{i\alpha}$ and $u = e^{i\beta}$ are in T then

$$\begin{aligned}|z^4 - u^4| &= 2|\sin(2\alpha - 2\beta)|\\ &= 2|\sin 2\alpha \cos 2\beta - \cos 2\alpha \sin 2\beta|.\end{aligned}$$

Since sine and cosine of 2α and 2β are rational numbers, $|z^4 - u^4|$ is also rational. Thus any two points of S are at a rational distance.

II. *Solution by James C. Smith, undergraduate, University of South Alabama.* For real t let P_t be the point

$$\left(\frac{4t(t^2-1)}{(t^2+1)^2}, \frac{4t^2-(t^2-1)^2}{(t^2+1)^2}\right),$$

which lies on the unit circle. A routine (but lengthy) computation shows that the distance between P_s and P_t is the absolute value of

$$\frac{4(s-t)(st+1)}{(s^2+1)(t^2+1)}.$$

Hence we can take S to be the set of all points P_t with t rational.

Also solved by Heiko Harborth (Germany), N. Miku (Netherlands), S. C. Locke (Canada), Kenneth Bernstein, Matt Wyneken, L. E. Mattics, John McVoy, Aage Bondesen & Torkil Heiede (Denmark), and the proposers.

Gerald B. Huff notes that a solution of this problem appears in the paper of N. H. Anning and P. Erdős, Integral Distances, Bull. Amer. Math. Soc., 51 (1945) 598.

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to Professor Roger C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109. To facilitate their consideration, solutions of Advanced Problems in this issue should be typed (in duplicate with double spacing) and should be mailed before July 31, 1979.

6258. *Proposed by John S. Lew, IBM, Yorktown Heights, N.Y.*

Let $X=(x_{jk})$ be an $m\times n$ matrix, where $1<m<n$ and the x_{jk} are algebraically independent indeterminates over the field C of complex numbers. Let X' be the transpose of X. Prove that $\det(XX')$ is an irreducible polynomial over C.

6259. *Proposed by William D. Blair, Northern Illinois University, and James E. Kettner, Hanover Park, Ill.*

Let R be a commutative ring with unity and $R[x,x^{-1}]$ be the ring of Laurent polynomials $f(x)=\sum_{i=-m}^{n}a_ix^i$ over R. Find necessary and sufficient conditions on the coefficients a_i of $f(x)$ for $f(x)$ to be invertible.

6260. *Proposed by Eric Langford, California Polytechnic State University, San Luis Obispo*

If X is a subset of a topological space S, then it is known that there can be formed at most six new sets by repeated formations of closures and interiors iterated in any order. (This is related to the famous Kuratowski closure-and-complement problem; see E. Langford, Characterization of Kuratowski 14-sets, this MONTHLY 78 (1971) 362–367 for details.) It is also known that if we further allow the formation of unions then no more than six new sets can be generated for a maximum total of thirteen (Problem 5996 [1974, 1034], [1978, 283]). Given that we start with X and the additional six sets described in the first sentence, what is the *minimum* number of new sets that can occur when we further allow unions?

6261. *Proposed by Hugh Noland, California State University, Chico*

Let S be an uncountable set of real numbers and let A be a countable subset of S. Must there exist an open set U, containing A, such that $S-U$ is uncountable?

6262. *Proposed by A. Blass and F. Harary, University of Michigan, and W. T. Trotter, Jr., University of South Carolina*

What is the probability that a tree selected at random has a fixed point? More specifically, let t_n be the number of (nonisomorphic) trees with n points and let f_n be the number of such trees T with at least one point fixed under all automorphisms of T. Calculate $\lim_{n\to\infty}f_n/t_n$.

6263. *Proposed by David Pokrass, Emory University*

In a simple nonassociative ring R, let

$$(a,b,c)=(ab)c-a(bc),\qquad [a,b]=ab-ba,\qquad a\circ b=ab+ba.$$

If R satisfies the identity $w\circ(x,y,z)=0$ and has no elements of additive order 2, show that R is either associative or anticommutative, i.e., R satisfies either $(x,y,z)=0$ or $x\circ y=0$ identically.

SOLUTIONS OF ADVANCED PROBLEMS

The Sum of the Elements of the Inverse of a Matrix

6162 [1977, 575]. *Proposed by the late Ray Latham*

If $A=(a_{ij})$ is the $n\times n$ matrix defined by $a_{ij}=1/(1-4(i-j)^2)$, and $\mathbf{x}=(x_i)$ is the unique vector such that $A\mathbf{x}=\mathbf{e}$ (the all 1's vector), show that

$$\sum_{i=1}^{n} x_i=\binom{n+1}{2}.$$

Solution by F. Djorup, George W. Soules, and M. Sweet, Institute for Defense Analyses, Princeton, N.J. The following more general problem is solved.

Let $n>0$ be an integer, α be a real number not $0,1,2,\ldots,$ and $A=(a_{ij})$ $(0\leqslant i,j\leqslant n)$ be the $(n+1)$-square matrix defined by $a_{ij}=r_{|i-j|}$, where for $i=0,1,\ldots,$ $r_i=\binom{-\alpha-1}{i}\Big/\binom{\alpha}{i}$. Show that when A is invertible (e.g., $\alpha=-3/2$, the stated case) that the sum of the entries in A^{-1} is $\binom{n-2\alpha-1}{n}$.

The problem is solved by exhibiting $\mathbf{x}=(x_0,\ldots,x_n)$ such that $A\mathbf{x}=\mathbf{e}$ and $\Sigma x_i=\binom{n-2\alpha-1}{n}$. Let $A=A^n$ and $\mathbf{x}=\mathbf{x}^n$. By solving $A\mathbf{x}=\mathbf{e}$ for small n and writing x_i^n in a Pascal triangle, it can be seen that

$$x_i^n=(-1)^n\binom{\alpha}{i}\binom{\alpha}{n-i}.$$

The above x_i^n satisfy the recursion

$$nx_i^n=(n-\alpha-1)\left(x_i^{n-1}+x_{i-1}^{n-1}\right)-(n-2\alpha-2)x_{i-1}^{n-2}$$

for $n>0$ and $0\leqslant i\leqslant n$, where $x_i^n=0$ outside that range.

The proof that $A^n\mathbf{x}^n=\mathbf{e}$ is obtained by showing

(1) $(A^n\mathbf{x}^n)_0=1$, and

(2) if $A^k\mathbf{x}^k=\mathbf{e}$ for $k<n$, then $(A^n\mathbf{x}^n)_j=1$ if $j>0$.

Proof of (1). Let $\beta=-1-\alpha$, and $f(t)=(1-t)^\alpha=\Sigma_{i\geqslant 0}(-1)^i\binom{\alpha}{i}t^i$. Then

$$(A^n\mathbf{x}^n)_0=\sum_{i=0}^{n}a_{0i}x_i^n=\sum_{i=0}^{n}(-1)^n\binom{\beta}{i}\binom{\alpha}{n-i}$$

is the coefficient of t^n in

$$f(t)\sum_{i\geqslant 0}(-1)^i\binom{\beta}{t}t^i=(1-t)^{\alpha+\beta}=(1-t)^{-1}$$

which has all coefficients equal to 1.

Proof of (2). This follows directly from the recursion, since A^{n-1} is obtained from A^n by striking out the last row and column, and since the recursion says $\mathbf{x}^n$ is a linear combination of the $(n+1)$-vectors $(x^{n-1},0)$, $(0,x^{n-1})$, and $(0,x^{n-2},0)$.

Lastly, $\Sigma_{i=0}^n x_i^n$ is the coefficient of t^n in $f^2(t)$, since

$$f^2(t)=\sum_{n\geqslant 0}\left[\sum_{i=0}^{n}(-1)^i\binom{\alpha}{i}(-1)^{n-i}\binom{\alpha}{n-i}\right]t^n.$$

But $f^2(t)=(1-t)^{2\alpha}$ so

$$\sum_{i=0}^{n}x_i^n=(-1)^n\binom{2\alpha}{n}=\binom{n-2\alpha-1}{n}.$$

When $\alpha = -1/2$, A is constant; and it is suspected (but not known) that A is invertible, and indeed positive definite, for $\alpha \neq -1/2$.

Soules expresses interest in a proof using the fact that the matrix is a positive definite Toeplitz matrix, and such a proof is supplied by A. A. Jagers (Netherlands). Olga Taussky Todd remarks that there are other cases where the sum of the entries in A^{-1} can be found, and refers to Knuth, *Art of Computer Programming*, vol. 1, page 36.

Also solved by Paul S. Bruckman, Clark Givens & Otto G. Ruehr, and L. E. Mattics.

Escaping from an Infinite Maze

6163 [1977, 575]. *Proposed by John Myhill, State University of New York at Buffalo.*

Devise an algorithm for escaping from a connected, countably infinite, locally finite maze. "Countably infinite" means the number of edges and nodes is $\aleph_0$, "locally finite" means only a finite number of edges meet at each node. "Algorithm" means this: You are lost in the middle of the maze, having no idea of where the exit is. Your only possibility of escape therefore is to devise a tour which will take you through every node of the maze after a finite number of steps. In order to keep track of your route, you are given an everlasting pencil and an infallible eraser; at each node, and at the roadside of each road near the node, is a board on which you can write and erase. However, you have only a finite alphabet to write with, and there is a fixed bound on how many characters you can write on the boards. (In particular, then, you cannot keep on any board a record of how many times you have passed it.) Your field of vision is limited to being able to see, from any node, what is written on the board at that node and what is written on the nearby roadside boards.

(*) Can this procedure be altered to solve the locally infinite case?

Several readers pointed out that the locally infinite case is impossible unless there is provided at each node a method of successively selecting edges at that node in such a way that each edge will ultimately be selected. We assume then that, if the number of edges at a node is infinite, they are labeled by the natural numbers.

Solution by Robert McNaughton, Rensselaer Polytechnic Institute. At the beginning, label the starting vertex as "old," leaving all other vertices and branches unlabeled. Thereafter, repeat alternately the following two steps:

(1) Tour the entire portion of the graph labeled "old." At each vertex in this portion select the first unlabeled branch (if any) and label it and also the node at the opposite end, if it is not already labeled, as "new."

(2) Tour the entire portion of the graph labeled "new" and "old," relabeling all "new" branches and vertices as "old."

Steps (1) and (2) are possible, since at any time the set of branches and nodes to be toured form a finite connected subgraph, and algorithms for touring a finite connected graph within the constraints of the problem as posed are well known. (See, for example, Chapter 3 of B. A. Trakhtenbrot's Algorithms and Automatic Computing Machines, D. C. Heath, Boston, 1963.) The touring algorithm requires marking branches and nodes. For alternate steps we can use different markers, and at each step we can erase the marks used in the previous step.

Finally, it is clear that this algorithm provides a tour of the entire graph.

Also solved by Robin Ault, S. Thomas McCormick, Walter Taylor, and, in the locally finite case, by Bruce T. Smith and the proposer.

Cauchy Random Variables

6164 [1977, 575]. *Proposed by Ignacy I. Kotlarski, Oklahoma State University*

Let the random variable $Z_1 = X$ follow the Cauchy distribution with the probability density function $f(x) = [\pi(1+x^2)]^{-1}$, $x \in R$. Show that for $n = 2, 3, \ldots$, the random variables

$$Z_2 = \frac{2X}{1-X^2},\ Z_3 = \frac{3X - X^3}{1-3X^2},\ Z_4 = \frac{4X-4X^3}{1-6X^2+X^4}, \ldots,$$

$$Z_n = \frac{\binom{n}{1}X - \binom{n}{3}X^3 + \binom{n}{5}X^5 - \cdots}{1 - \binom{n}{2}X^2 + \binom{n}{4}X^4 - \binom{n}{6}X^6 + \cdots}, \ldots,$$

also follow the same Cauchy distribution.

Solution by Marcel F. Neuts, University of Delaware. The random variable Z_n may be written as

$$Z_n = \frac{1}{i}\frac{(1+iX)^n - (1-iX)^n}{(1+iX)^n + (1-iX)^n} = \tan[n \arctan X],$$

by using the polar forms of $1 \pm iX$. If X has a standard Cauchy distribution, $\arctan X$ has a uniform distribution on $(-\pi/2, \pi/2)$. Using the fact that $n \arctan X$ is then uniformly distributed on $(-n\pi/2, n\pi/2)$ and using the periodicity of the function $\tan y$, an easy consideration shows that Z_n has the same distribution as $\tan(\arctan X) = X$.

Also solved by Mangho Ahuja, Lane Bishop, Paul S. Bruckman, L. E. Clarke, Emeric Deutsch, Ellen Hertz, David C. Hoaglin, A. A. Jagers (Netherlands), L. F. Kemp, J. H. B. Kemperman, Kenneth Klinger, Gérard Letac (France), William D. Markel, L. E. Mattics, R. R. Miller, G. S. Rogers, David Shelupsky, Michael Skalsky, Wolfe Snow, Lajos Takács, and the proposer.

Functions Approximated by Their Mean Values

6165. [1977, 575–576]. *Proposed by A. G. O'Farrell, St. Patrick's College, Kildare, Ireland*

Suppose $f(x)$ is a real-valued function on $\mathbb{R}^n$ and define

$$M(x,r) = \frac{\int_{|x-y| \leq r} f(y)\,dy}{\int_{|x-y|\leq r} 1\,dy}, \text{ for } x \in \mathbb{R}^n, \quad r > 0.$$

Suppose

$$\frac{M(x,r) - f(x)}{r^2} \to 0 \text{ as } r \downarrow 0$$

for each $x \in \mathbb{R}^n$. Show that $f(x)$ is harmonic.

Solution by Roger L. Cooke, University of Vermont. The statement as given is false. If

$$f(x) = f(x_1, \ldots, x_n) = \operatorname{sgn} x_1 = \begin{cases} 1 & \text{if} \quad x_1 > 0 \\ 0 & \text{if} \quad x_1 = 0 \\ -1 & \text{if} \quad x_1 < 0, \end{cases}$$

then $M(x,r)=f(x)$ if $x_1=0$ or $r \leqslant |x_1|$.

It certainly follows, then, that

$$\lim_{r \downarrow 0} r^{-2}(M(x,r)-f(x))=0.$$

Nevertheless $f(x)$ is not even continuous, hence is not harmonic.

If we assume that $f(x)$ is continuous, the statement is true. It is in fact a well-known characterization of harmonic functions by means of a generalized Laplacian. See Radó, Subharmonic functions, *Ergebnisse der Mathematik*, 5 (1937) 14.

Ivan Netuka and Jiři Vesely, Praha, Czechoslovakia, also observe that continuity is required, and refer, for a more general result, to M. Brelot, *Eléments de la théorie classique du potentiel*, Paris, 1969, Chap. 2. They also make the following comment.

The same is true, if the volume average is replaced by the surface average. Then the corresponding theorem is usually attributed to W. Blaschke ([1], 1916). But it was proved two years earlier by M. Plancherel in [4]. What is more interesting is the fact that the theorem for both kinds of integral means follows from a result published by S. Zaremba ([7], 1905)(see also a remark in [2]). Note, finally, that related conditions were studied in connection with subharmonic functions by S. Saks [6] and I. Privaloff [5]. (Many results on harmonic functions and their mean value property are included in the survey [3].)

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Condition to Be Constant

6167 [1977, 576]. *Proposed by Charles R. Williams and Joseph C. Warndof, Midwestern University.*

Suppose $f: R^n \to R^{n-1}$ and for each point $a \in R^n$, the limit

$$\lim_{x \to a} \frac{|f(x)-f(a)|}{|x-a|}$$

exists. Is f necessarily a constant function? (The answer is "yes" if $n=2$.)

Solution by W. Emerson, Queens College, N.Y., J. Komlós, Hungarian Academy of Sciences, & R. Pollack, New York University.

(A) Let

$$\lim_{x \to a} \frac{|f(x)-f(a)|}{|x-a|} = \lambda_a \quad \text{and} \quad Z = \{a | \lambda_a = 0\}.$$

Letting Z' denote the complement of Z, we first show that $\mu(Z')=0$ and then conclude that f is a constant.

Suppose that $\mu(Z')>0$; then for any $\theta>1$ there is an n such that the set $Z'=\{a|\theta^n \leqslant \lambda_a < \theta^{n+1}\}$ has positive measure. Similarly, for any $\varepsilon>0$ there is a $\delta>0$ such that the set

$$A=\{a\in Z_n'|(1-\varepsilon)\lambda_a|x-a|<|f(x)-f(a)|<(1+\varepsilon)\lambda_a|x-a| \quad \text{if} \quad |x-a|<\delta\}$$

has positive measure. Now choose a sphere S of radius $r<\delta/2$ which is almost filled by A, i.e., $\mu(A\cap S)>(1-\varepsilon')\mu(S)$. Finally, on the sphere concentric with S and with radius $r/2$ inscribe a simplex with edge length r_1 and about each of the vertices of the simplex place a sphere of radius $(\varepsilon/2)r_1$. Since the volume of these spheres is $>\varepsilon'\mu(S)$, each of them contains a point of S. Call these points $a_1,a_2,\ldots,a_{n+1}$. Then

$$(1-\varepsilon)\theta^n|a_i-a_j|<|f(a_i)-f(a_j)|<(1+\varepsilon)\theta^{n+1}|a_i-a_j|$$

and

$$(1-\varepsilon)^2\theta^n r_1<|f(a_i)-f(a_j)|<(1+\varepsilon)^2\theta^{n+1}r_1.$$

Hence

$$\frac{(1-\varepsilon)^2}{(1+\varepsilon)^2}\cdot\frac{1}{\theta}<\frac{|f(a_i)-f(a_j)|}{|f(a_k)-f(a_l)|}<\frac{(1+\varepsilon)^2}{(1-\varepsilon)^2}\cdot\theta,$$

which is impossible in R^{n-1} if ε is small enough and θ close enough to 1. Hence $\mu(Z')=0$.

(B) Now take any two points a,b in R^n and consider all the minor arcs of circles joining a and b. Almost all of them intersect Z' in a set of linear measure 0. Let C be such an arc of length l. Let $B_k=\{a\in C|k<\lambda_a\leqslant k+1\}$, $k=0,1,\ldots$; $B=\cup_{k=0}^{\infty}B_k$. Each B_k has linear measure 0, so we can cover it by a union D_k of disjoint arcs $D_{k,i}$ of total length $\varepsilon/2^k$. Now if a is in C and not in B, cover a by a sphere of radius δ_a such that

$$|f(x)-f(a)|<\varepsilon|x-a| \quad \text{if} \quad |x-a|<2\delta_a.$$

If a is in B_k, cover a by a sphere s of radius δ_a such that $C\cap s$ lies in some $D_{k,i}$ and

$$|f(x)-f(a)|<(k+2)|x-a| \quad \text{if} \quad |x-a|<2\delta_a.$$

A finite number of these spheres cover C, say with centers $a_1,a_2,\ldots,a_L$ in order along C from a to b. Hence, with $a=a_0$, $b=a_{L+1}$,

$$\begin{aligned}|f(b)-f(a)|&=\left|\sum_{i=0}^{L}f(a_{i+1})-f(a_i)\right|\\&\leqslant\sum_{i=0}^{L}|f(a_{i+1})-f(a_i)|<\varepsilon l+2\Sigma(k+2)\mu(D_k)<\varepsilon(l+20),\end{aligned}$$

whence $f(b)=f(a)$.

Also solved by the proposer.

The Number of Terms in a Binomial Expansion Modulo a Prime

6170 [1977, 659]. *Proposed by Paul W. Haggard, East Carolina University*

Let D be an integral domain with characteristic the prime p and let x and y be indeterminates. In $D[x,y]$, consider expansions of $(x+y)^n$ for nonnegative integers n.

(a) If p_1 is an odd prime, prove that the expansion of $(x+y)^{p_1}$ has an even number, N, of terms.

(b) When and how can n be obtained such that the expansion of $(x+y)^n$ will have a given number, N, of terms?

Solutions were given by L. Carlitz, Jeffrey Mitchell Cohen, Marguerite Gerstell, Seymour Kass, John O. Kiltinen, L. E. Mattics, and the proposer. All used the observation that if $n=a_0+a_1p+\cdots+a_tp^t, 0\leqslant a_i<p$, then $(x+y)^n$ has $N=(a_0+1)(a_1+1)\cdots(a_t+1)$ terms. It follows that if N is odd, then n must be even, and that N is attainable for some n if and only if N is a product of

factors b_i, $1 \leqslant b_i \leqslant p$.

Cohen and Kiltinen observed that it is not necessary that D be an integral domain. Kass generalized the result, showing that, for p a prime and n as before, the number of coefficients of $(x_1 + \cdots + x_r)^n$ not divisible by p is $\prod_{i=0}^{t} \binom{a_i + r - 1}{a_i}$; see Abstract 614–17, A.M.S. Summer Meeting, 1964.

REVIEWS

Edited by J. Arthur Seebach, Jr., and Lynn A. Steen
with the assistance of the mathematics departments of St. Olaf, Carleton, and Macalester Colleges
Collaborating Editor for Films: Seymour Schuster, Carleton College

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Encyclopedic Dictionary of Mathematics. Edited by Shôkichi Iyanaga and Yukiyosi Kawada. MIT Press, 1977. *Vol. I*: xv + 833 pp; *Vol. II*, 864 pp. $125 set. (Telegraphic Review, February 1978.)

This is a remarkable undertaking, a monument to the industriousness and determination of the Japanese mathematical community, which one would not have deemed possible 20 years ago. Once before in history has such an attempt been made to give a complete account of contemporary mathematics: this was the famous *Enzyklopädie der mathematischen Wissenschaften*, begun under the guidance of Felix Klein in the last years of the nineteenth century; its publication was slowed down by the First World War and dragged on until after 1930, desperately trying to catch up with the progress of the various parts of mathematics during that period.

Although arranged in alphabetical order instead of by great divisions (such as Arithmetic, Algebra, Analysis, Geometry, etc.) as in the *Enzyklopädie*, the general presentation of the *Encyclopedic Dictionary* has remained the same: most definitions are given *in extenso*, as well as the most significant results of each theory, and of course there are almost no proofs. A tremendous gain of space has been achieved by eliminating much of the discursiveness of the old *Enzyklopädie*; the great majority of its historical information (which would have been a mere duplication); a large amount of results of secondary importance which needlessly cluttered many articles; and finally, all the parts devoted to astronomy, geodesy, mechanics, and physics which had no significant mathematical content. It has thus been possible to compress into about one-tenth of the bulk of the *Enzyklopädie* a more valuable amount of information on a science which certainly at present is at least ten times more extensive than it was in 1900.

One should particularly mention the general excellence of the cross-referencing system (a dagger † calls attention to notions not defined in an article, and their definitions can be located in other articles by consulting the general index), the abundant bibliography, and the marvelous tables and indexes which occupy almost half of the second volume, giving, for many topics, precise information quite difficult to obtain elsewhere.

One of the big problems in such a work is how to divide the material among the various headings: the difficulty is how to strike a reasonable balance between massive and forbidding chapters and a scattering of very numerous entries, among which the nonspecialist will feel completely lost. The total number of entries in the *Encyclopedic Dictionary* has been reduced

from 593 in the first Japanese edition to 436 in the 1968 edition. In my opinion, it is still too large, and the fragmentation has been carried too far; a better solution would have been the one adopted by the recent French *Encyclopaedia Universalis*, where the subjects have been regrouped under a smaller number of larger articles, but all key words are duly mentioned in their alphabetical order, with references to the general articles in which they belong. As it is, a nonspecialist will have some trouble getting acquainted with a given theory by reading the *Encyclopedic Dictionary*. When, for instance, "Algebraic topology" is divided into no less than 25 articles (all duly mentioned in a "Systematic list of articles" at the end of vol. 2), where is one to start, and in what order should one proceed? In that case, there is a short general and historical survey under the heading "Topology," but it falls far short of giving a useful guide to the reader (it refers only to 5 of the 25 articles on the subject!). It is hard to understand why there are separate articles on "Symbolic logic," "Propositional logic," and "Predicate logic," or why "Chain complexes" is disjoint from "Homological algebra." Sometimes it is quite difficult to discover where a given topic may be hidden: for instance, the very active recent theory of Dynamical Systems (Smale, Peixoto, Anosov, etc.) is not mentioned in the "Systematic list of articles." In the Index, one is referred to the article "Ordinary differential equations (qualitative theory)," where no mention whatsoever is made of these recent results. Finally, it turns out that they are mentioned in part under "Topology of differentiable manifolds," in part under "Ergodic theory," and the rest under "Three body problem"!

As it is to be expected in a work in which so many mathematicians have collaborated, the intellectual level of the articles is subject to large variations. Excluding the articles on applied mathematics (which I have no competence to judge), it is convenient to distinguish three parts, which for simplicity I call "living mathematics," "classical mathematics," and "historical articles." By "living mathematics," I mean those theories which at the present time are the focus of the most lively and successful research. The corresponding articles in the *Encyclopedic Dictionary* are generally excellent; they obviously have been written by very competent and broad-minded specialists, well aware of the most recent discoveries and exercising very good taste in evaluating their importance. Articles such as "Algebraic varieties," "Fiber bundles," "Harmonic analysis," "Lie algebras," "Topology of differentiable manifolds," "Transcendental numbers," "Unitary representations," and "Zeta functions" are real masterpieces of exposition, and the number of omissions of significant results in such articles is quite small.

Not so much praise can be given to the part concerned with "classical mathematics," where it seems that, on the whole, the editors have failed to exercise their function and have allowed many articles to be included that are quite unworthy of the high standard of the ones mentioned above. A large number of them (at least 40) seem to have been written by people who have no knowledge at all of present-day mathematics and cling to incredibly old-fashioned and obsolete presentations. For instance, in the article on "Contact transformations," the author writes as if he had never heard of a tangent bundle; most of the article on "Integral equations" might have been bodily lifted from a book written in 1910; the articles on "Coordinates" and "Curves" are fantastic hodge-podges of totally unrelated questions; and so on.

Finally, the historical articles are mediocre—at best, commonplace; at worst, totally lacking in historical perspective and containing amazing errors. For instance, nothing is said of Euler's work on number theory or of Poincaré's papers on potential theory; and one is told that "the concept of connection forms the basis" of E. Cartan's thesis, whereas the thesis was written in 1894, and the first traces of the general idea of connection only emerge in Cartan's papers of the 1920's!

On the whole, however, these only constitute minor blemishes on an extraordinary book, and it would certainly be easy to remedy these defects in a later edition. The *Encyclopedic Dictionary* will no doubt remain as the standard reference for anyone who wants to get acquainted with any part of the mathematics of our time.

J. Dieudonné, Nice, France

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S(13-18), L. *Mathematics Today: Twelve Informal Essays*. Ed: Lynn Arthur Steen. Springer-Verlag, 1978, viii + 367 pp, $12. [ISBN: 0-387-90305-4; 3-540-90305-4] The book's objective is to tell "the intelligent nonmathematician something of the nature, development and use of mathematical concepts." The essays, on very diverse topics, are at or somewhat above the *Scientific American* level. Only an intelligent nonmathematician can tell whether the book attained its objective; but most mathematicians should find at least some of the essays interesting and informative. RPB

GENERAL, T(13: 1). *Consumer Mathematics*. Linda L. Thompson. Glencoe Pub, 1978, viii + 470 pp, $10.96. [ISBN: 0-02-479280-2] Covers a wide variety of topics from both an informational and (elementary) mathematical point of view, e.g., mathematical skills review, map, table and graph reading, metric system, income, budgeting, installment buying, borrowing and investing money, health care, transportation and housing costs, taxes, etc. RSK

GENERAL, P. *Transactions of the Moscow Mathematical Society, 1978, Issue 2*. AMS, 1978, iii + 252 pp, $48 (P). Translation of *Volume 34 (1977)*. LAS

PRECALCULUS, T(13: 2, 3). *Essentials of Technical Mathematics with Calculus*. Richard S. Paul, M. Leonard Shaevel. P-H, 1978, xiv + 878 pp, $17.95. [ISBN: 0-13-289199-9] Contents are roughly 3/4 pre-calculus and 1/4 calculus. Concepts are not proved, but are clearly explained and illustrated by many examples. Includes topics from elementary and advanced algebra, trigonometry, functions, analytic geometry, differential and integral calculus, with applications. Treatment of calculus topics is somewhat terse. RSK

PRECALCULUS, T?(13: 1), S. *Mathematics for Health Professionals*. Ida M. Martinson, G.R. Kepner. Springer Pub, 1977, xii + 240 pp, $11.95 (P). [ISBN: 0-8261-1870-4] Primarily a collection of rules, illustrated with realistic examples from the field of health. Explanations are sparse and the presentation is not always mathematically accurate. Too concise for practical use as a text, but the examples are very instructive. RSK

EDUCATION, T(15-17: 2). *Basic Concepts of Elementary Mathematics, Third Edition*. John M. Peterson. Prindle, 1978, x + 501 pp, $15.95. [ISBN: 0-87150-247-X] Revision of the 1974 *Second Edition* (TR, February 1975), designed for current or prospective elementary school teachers. Major changes include a rewritten chapter on probability and statistics, and new chapters on metric measure and transformation geometry. Also includes more worked out examples, some new sections, and an appropriate article from *The Arithmetic Teacher* at the end of each chapter. RSK

FOUNDATIONS, P. *Selected Papers on Łukasiewicz Sentential Calculi*. Ed: Ryszard Wójcicki, Grzegorz Malinowski. Polish Acad of Sci, 1977, 199 pp, Zł 70. Notable in this collection is R. Giles' formalization of the theory of physical experimentation in terms of an infinite-valued logic, plus a short history and bibliography of Łukasiewicz logics; the other papers are technical. PJC

FOUNDATIONS, P. *Mordchaj Wajsberg: Logical Works*. Ed: Stanisław J. Surma. Polish Acad of Sci, 1977, 216 pp, Zł 65. Complete published papers in English translation, treating mainly the axiomatization of many-valued logics, together with a short biography. PJC

FOUNDATIONS, T(18: 1, 2), P. *Topos Theory*. P.T. Johnstone. Acad Pr, 1977, xxiii + 367 pp, $34.25. [ISBN: 0-12-387850-0] A well-written exposition with an historical overview given in the introduction. Some exercises with each chapter, a reasonable index of definitions and an extensive bibliography make this exposition suitable for a relatively broad audience, but the presentation is not for the mathematically naive. JAS

COMBINATORICS, P. *Studies in Foundations and Combinatorics*. Ed: Gian-Carlo Rota. Acad Pr, 1978, ix + 265 pp, $29. [ISBN: 0-12-599101-0] First volume, comprising 12 advanced survey papers, in *Advances in Mathematics Supplementary Studies*, a series of volumes sold separately from the journal "to facilitate publication" of already-accepted articles. LAS

COMBINATORICS, P*. *Lecture Notes in Mathematics-642: Theory and Applications of Graphs*. Ed: Y. Alavi, D.R. Lick. Springer-Verlag, 1978, xiv + 635 pp, $23 (P). [ISBN: 0-387-08666-8; 3-540-08666-8] Proceedings of the International Conference on the Theory and Applications of Graphs, held at Western Michigan University in May 1976. Includes a brief history of graph theory. CEC

NUMBER THEORY, S(15), P. *Variationen über ein zahlentheoretisches Thema von Carl Friedrich Gauss*. Herbert Pieper. Birkhäuser, 1978, 183 pp, sFr. 20. [ISBN: 3-7643-0959-8] A collection of mathematical results which are all related to the Law of Quadratic Reciprocity. It begins with Gauss's first proof, and includes various other approaches, material on Gaussian sums and ends by considering the factorization of certain polynomials in finite fields. The point of view is historical. A bibliography is included. CEC

NUMBER THEORY, T(13), S*, L, *Some "Prime" Comparisons*. Stephen I. Brown. NCTM, 1978, x + 106 pp, $6 (P). [ISBN: 0-87353-131-0] Through topics that deal with prime numbers the author illustrates much of what the mathematical method is all about. A well-written, entertaining and usable monograph. Includes many good exercises. CEC

NUMBER THEORY, S(16-17), P, L? *Exercices de théorie des nombres*. D.P. Parent. Gauthier-Villars ([illegible]: SMPF, 14 E. 60th St., NY 10022), 1978, 307 pp, 59F (P). [ISBN: 2-04-010200-0] This book consists of 166 number theory problems and their solutions. Topics include prime numbers, additive number theory, rational series, algebraic number theory, uniform distribution, transcendental numbers, modular forms, quadratic forms, continued fractions and p-adic analysis. Each section begins with a summary of the fundamental theorems with bibliographic references. A valuable resource. CEC

LINEAR ALGEBRA, T(14-15: 1, 2), *Introduction to Matrices and Linear Transformations, Third Edition*. Daniel T. Finkbeiner II. Freeman, 1978, xii + 462 pp, $18. [ISBN: 0-7167-0084-0] Rewritten exposition and beginning promote understanding before encounter with general ideas. Gaussian elimination used as unifying computational technique. Clean uncluttered appearance. Flexible, permitting a variety of courses in linear algebra. More illustrative examples, new exercises, and greatly enlarged solutions section. Linear mapping notation now in conformity with function notation. (First Edition, TR March 1967.) JK

ALGEBRA, T(18: 2), S, P, *General Theory of Lie Algebras*. Yutze Chow. Gordon, 1978. [ISBN: 0-677-03890-9] *V. 1*, xxii + 461 pp; *V. 2*, xx + 436 pp, $72 set. Standard treatment of classical theory for characteristic zero. *Volume 1* goes through Levi decomposition, *Volume 2* contains classification theory and introduction to representations and cohomology. No exercises; index and bibliography found only in *Volume 2*. Offset, overpriced at $72. JS

ALGEBRA, T(18), S, P, *Introduction to Lie Algebras and Representation Theory, Second Printing, Revised*. James E. Humphreys. Grad. Texts in Math., V. 9. Springer-Verlag, 1972, xii + 171 pp, $16.80. [ISBN: 0-387-90052-7; 3-540-90052-7] Essentially the same book as appeared in first edition (TR, May 1973), changes are "correcting minor errors and improving a few arguments" as well as adding appendix to section 24 in Weyl's formula. JS

ALGEBRA, S(18), P, *Lecture Notes in Mathematics-655: Quadratic Forms Over Semilocal Rings*. Ricardo Baeza. Springer-Verlag, 1978, vi + 199 pp, $11 (P). [ISBN: 0-387-08845-8; 3-540-08845-8] A treatment of the algebraic theory of quadratic forms over semi-local rings (as opposed to fields). Includes bibliographical references. CEC

ALGEBRA, S(18), P, *Polynomial Rings and Affine Spaces*. Masayoshi Nagata. CBMS Reg. Conf. in Math., No. 37. AMS, 1978, 33 pp, $5.20 (P). [ISBN: 0-8218-1687-X] A brief collection of lecture notes from the CBMS Regional Conference held at Northern Illinois University in July 1977. Includes bibliographical references. CEC

FINITE MATHEMATICS, T(13: 2), *Mathematics with Applications*. William C. Stewart. Business Pub, 1979, xi + 455 pp, $15.95. [ISBN: 0-256-02114-7] A text for students in business and economics which combines some precalculus and calculus topics with mathematics of finance, matrices, and linear programming. All applications are aimed at this audience. LLK

FINITE MATHEMATICS, T(13: 1), *Finite Mathematics*. Steven C. Althoen, Robert J. Bumcrot. Norton, 1978, xiv + 590 pp, $14.95. [ISBN: 0-393-09046-9] An approach to problem solving and decision making suitable for both business and liberal arts students. An honest attempt at finding real world problems with all topics introduced through the problems. LLK

CALCULUS, T(13: 2, 3), *Calculus and Analytic Geometry, Third Edition*. Douglas F. Riddle. Wadsworth, 1979, xiii + 959 pp, $23.95. [ISBN: 0-534-00626-4] This edition retains the three-level approach to limits and continuity. Changes are in the problem sets, a more algebraic approach to vectors, and the inclusion of line and surface integrals. (*First Edition*, TR April 1970; *Second Edition*, TR October 1974.) LLK

REAL ANALYSIS, T(15-16: 2), *Advanced Calculus, An Introduction to Analysis, Third Edition*. Watson Fulks. Wiley, 1978, xiv + 731 pp, $20.95. [ISBN: 0-471-02195-4] Major changes from previous editions (First Edition, TR January 1970) include separation of continuity and differentiation of functions of one real variable and modernization of the definition of the derivative of a vector function of a vector variable. JK

DIFFERENTIAL EQUATIONS, T(15-18: 2), S, P, L, *Advanced Mathematical Methods for Scientists and Engineers*. Carl M. Bender, Steven A. Orszag. McGraw, 1978, xiv + 539 pp, $29.50. [ISBN: 0-07-004452-X] A departure from the ordinary. Asymptotic and perturbative analysis methods for differential and difference equations. Computer-generated plots and tables compare approximate with exact solutions. Care stressed over rigor. Usable at several levels. Every section and problem labelled for difficulty. Self-contained but not for the meek. JK

DIFFERENTIAL EQUATIONS, P, *Equations of Mixed Type*. M.M. Smirnov. Transl. of Math. Mono., V. 51. AMS, 1978, iii + 232 pp, $27.20. [ISBN: 0-8218-4501-2] Existence and uniqueness results for the Tricomi and Frankel problems. RBK

DIFFERENTIAL EQUATIONS, P, *Lecture Notes in Mathematics-668: The Structure of Attractors in Dynamical Systems*. Ed: N.G. Markley, J.C. Martin, W. Perrizo. Springer-Verlag, 1978, vi + 264 pp, $14.25 (P). [ISBN: 0-387-08925-X; 3-540-08925-X] 23 research papers presented at the June 1977 CBMS Regional Conference at North Dakota State University. The ten principal lectures by the late Rufus Bowen are not included. LAS

DIFFERENTIAL EQUATIONS, T(15-16: 1, 2). *Differential Equations.* C. Ray Wylie. McGraw, 1979, xiii + 593 pp, [illegible] [ISBN: 0-07-072197-1] A comprehensive, classical introductory text treating, in addition to standard elementary topics, Bessel functions, heat and wave equations, numerical methods, linear difference equations, and qualitative theory of nonlinear equations. It lacks a modern view (no linear algebra, no calculators or computer problems, no help with mathematical modelling) but its many worked-out examples provide a thorough treatment of the basics. LAS

DIFFERENTIAL EQUATIONS, P. *Boundary-Interior Layer Interactions in Nonlinear Singular Perturbation Theory.* F.A. Howes. Memoirs No. 203. AMS, 1978, iv + 108 pp, $7.60 (P). [ISBN: 0-8218-2203-9]

NUMERICAL ANALYSIS, S(13-18), L. *Algorithms for RPN Calculators.* John A. Ball. Wiley, 1978, xii + 330 pp, $17. [ISBN: 0-471-03070-8] Contains an impressively organized and clearly presented discussion of the characteristics of reverse Polish notation (RPN) calculators: history, organization, operation, utilization in scientific work. Detailed chapter presentations on (1) standard numerical algorithms; (2) iterative solutions of elementary transcendental equations; (3) curve fitting; (4) numerical integration, differentiation, and interpolation. An extensive list of specific algorithms for RPN calculators is included in an appendix. Many exercises with answers. References. Index. RJA

FUNCTIONAL ANALYSIS, T*(17-18: 1), S*, P, L*. *Bounded Integral Operators on L^2 Spaces.* P.R. Halmos, V.S. Sunder. Ergebnisse der Math., V. 96. Springer-Verlag, 1978, xv + 134 pp, $18.20. [ISBN: 0-387-08894-6; 3-540-08894-6] A concise, direct exposition of the "natural generalization" of operators induced by matrices, replete with examples, open questions, exercises embedded in the text, and plenty of plain language. Uses "<2,1> compactness" as the chief tool to ask, in the final three sections, which operators can be integral, which must be, and which are? LAS

OPTIMIZATION, T(18: 1). *Principles of Optimal Control Theory.* R.V. Gamkrelidze. Transl: Karol Makowski. Math. Concepts and Methods in Sci. and Eng., V. 7. Plenum Pr, 1978, xii + 175 pp, $24.50. [ISBN: 0-306-30977-7] Contains "principles of general control theory and proofs of the maximum principle and basic existence theorems of optimal control theory." Restricted to the "time-optimal problem with fixed end points." RSK

OPTIMIZATION, S(15). *Programmation Linéaire.* J. Acher, J. Gardelle. Dunod (US Distr: SMPF, 14 E. 60th St., NY 10022), 1978, 87 pp, 32F (P). [ISBN: 2-04-010347-3] A short, typical treatment of the simplex method for linear programming, with discussion of duality, the decomposition principle, and the transportation method. A small number of standard exercises. TAV

ALGEBRAIC GEOMETRY, P. *Lie Groups: History, Frontiers and Applications, V. VIII: Hilbert's Invariant Theory Papers.* Trans: M. Ackerman; Comments: R. Hermann. Math Sci Pr, 1978, ix + 336 pp, $35 (P). [ISBN: 0-915692-26-0] First four chapters are translations of four of Hilbert's papers in Invariant Theory (1885-1893). Two additional chapters by Hermann attempt to put Hilbert's work in a contemporary context and notation. No index, limited bibliography. Offset paperback; price is steep at $35. JS

ALGEBRAIC GEOMETRY, P. *American Mathematical Society Translations, Series 2, V. 112: Nine Papers on Hilbert's 16th Problem.* D.A. Gudkov, G.A. Utkin. AMS, 1978, iv + 172 pp, $27.60. [ISBN: 0-8218-3062-7] A series of related papers, translated from Russian, that solve the classical problem of algebraic geometry concerning the location of ovals of nonsingular sixth order curves. LAS

DIFFERENTIAL GEOMETRY, P. *Selected Papers.* Shiing-shen Chern. Springer-Verlag, 1978, xxxi + 476 pp, $24.80. [ISBN: 0-387-90339-9; 3-540-90339-9] Approximately one-third of Chern's published papers, including, at Chern's choice, especially those that are shorter and less accessible. Includes a personal remembrance by André Weil, a mathematical analysis of Chern's work by Phillip Griffiths, a professional autobiography by Chern, a complete Chern bibliography, and a list of Ph.D. theses supervised by Chern. LAS

GEOMETRY, S(14), L. *Polyhedra Primer.* Peter Pearce, Susan Pearce. Van N-Rein, 1978, viii + 134 pp, $5.95 (P). [ISBN: 0-442-26496-8] Aimed at readers requiring a knowledge of spatial geometry. An illustrated glossary of over 200 terms, arranged in order of increasing complexity, covering polygons, tessellations, polyhedra, space filling and open packings. Great for browsing. Good index. Skimpy bibliography. Unsophisticated and non-mathematical but a steal at the price. JK

GEOMETRY, P. *C-types of n-dimensional Lattices and 5-dimensional Primitive Parallelohedra (with Application to the Theory of Coverings).* S.S. Ryškov, E.P. Baranovskiĭ. Proc. of Steklov Inst. of Math., No. 137. AMS, 1976, iv + 140 pp, $40 (P). [ISBN: 0-8218-3037-6] Research report on geometric investigations dealing with partitioning Euclidean space. LAS

TOPOLOGY, T(14-18: 1, 2), S, L. *General Topology.* Akos Császár. Adam Hilger, 1978, 488 pp, £20.50. [ISBN: 0-85274-275-4] Extraordinarily complete text, yet a special effort is maintained throughout to maintain accessibility to the beginner. General background is assumed to be that of a first undergraduate course in analysis. Many exercises. Hints and intermediate steps in solutions of exercises provided when appropriate. Attention given to discussion of motivating examples and interesting special cases. Excellent chapter exercises. Format of the text is a plus. General references. Subject index. List of notation. RJA

PROBABILITY, T(17-18: 2). *Probability Theory: Independence, Interchangeability, Martingales.* Yuan Shih Chow, Henry Teicher. Springer-Verlag, 1978, xv + 455 pp, $24.80. [ISBN: 0-387-90331-3; 3-540-90331-3] Theoretical presentation of the measure-theoretic foundations of probability theory and some of the resulting theorems, particularly in the areas mentioned in the title. Also emphasizes stopping times, both to prove theorems and in their own right. RSK

PROBABILITY, P. *Lecture Notes in Mathematics-672; Stochastic Convergence of Weighted Sums of Random Elements in Linear Spaces.* Robert L. Taylor. Springer-Verlag, 1978, vii + 216 pp, $12.50 (P). [ISBN: 0-387-08929-2; 3-540-08929-2] Unified presentation of known results, inspired by "recent interest in representing stochastic processes as random variables in function spaces." Good bibliography. RSK

PROBABILITY, P*. *Lecture Notes in Mathematics-675: Characterizations of Probability Distributions. A Unified Approach with an Emphasis on Exponential and Related Models.* Janos Galambos, Samuel Kotz. Springer-Verlag, 1978, viii + 169 pp, $9.80 (P). [ISBN: 0-387-08933-0; 3-540-08933-0] Brings together results on characterizations and unifies existing theory, restricting attention to the exponential distribution, both univariate and multivariate, and its monotonic transformations. Good bibliography. RSK

PROBABILITY, T(15-16), S, L. *Probability, Statistics, and Queueing Theory, With Computer Science Applications.* Arnold O. Allen. Comp. Sci. and Appl. Math. Acad Pr, 1978, xvi + 390 pp, $29.50. [ISBN: 0-12-051050-2] Somewhat formal presentation of those areas of probability, statistics, and queueing theory which are most applicable to computer science problems, amply illustrated with computer science examples. Emphasis is on queueing theory, with a minimal treatment of statistical inference. Graded exercises. RSK

PROBABILITY, T(17-18), P*. *Probability Theory II, Fourth Edition.* M. Loève. Grad. Texts in Math., V. 46. Springer-Verlag, 1978, xvi + 413 pp, $19.80. [ISBN: 0-387-90262-7; 3-540-90262-7] Second half of the revision of the 1963 Van Nostrand *Third Edition*. (First half TR, March 1978.) This volume covers the last two parts of the previous edition: Dependence and Elements of Random Analysis. New material includes a chapter on Brownian motion and limit distributions. RSK

PROBABILITY, T(17: 1), P. *Characterizations of the Normal Probability Law.* A.M. Mathai, G. Pederzoli. Halsted Pr, 1977, x + 149 pp, $8.50. [ISBN: 0-470-99322-7] "Mathematically semi-rigorous" investigation of those properties that are unique to the normal distribution. Nine chapters deal with the univariate normal law, one with the multivariate normal law, and one is on characterizations for information and statistical measures. Good sets of exercises; extensive bibliography. RSK

PROBABILITY, P. *Probability on Banach Spaces.* Ed: James Kuelbs. Dekker, 1978, ix + 521 pp, $49.50. [ISBN: 0-8247-6799-3] Four papers on Poisson and Gaussian measures, cluster points, subgaussian processes, and processes with finite value, followed by Part II (in 250 pp.) of W. Woyczyński's major paper "Geometry and Martingales in Banach Space." LAS

PROBABILITY, P. *Lecture Notes in Biomathematics-23: Geometrical Probability and Biological Structures: Buffon's 200th* [illegible]. Ed: R.E. Miles, J. Serra. Springer-Verlag, 1978, xii + 338 pp, $13.50 (P). [ISBN: 0-[illegible]; 3-540-08856-3] A diverse collection of papers on biology, mathematical morphology, stereology, integral geometry and geometric probability from a bicentenary symposium held in Paris at the *Jardin des Plantes* which were established by the naturalist Buffon who in 1777 published in *Essai d'Arithmétique morale* his celebrated probabilistic analysis of a needle dropped at random on a grid of lines. LAS

PROBABILITY, P. *Studies in Probability and Ergodic Theory.* Ed: Gian-Carlo Rota. Acad Pr, 1978, xiii + 293 pp, $32. [ISBN: 0-12-599102-9] 16 papers in the second *Supplementary Studies* volume of *Advances in Mathematics*. Like the journal in content, this volume is marketed (and catalogued in libraries) like a book. LAS

PROBABILITY, P. *Stochastic Approximation Methods for Constrained and Unconstrained Systems.* Harold J. Kushner, Dean S. Clark. Appl. Math. Sci., V. 26. Springer-Verlag, 1978, x + 261 pp, $9.80 (P). [ISBN: 0-387-90341-0; 3-540-90341-0] Stochastic approximation deals with recursive Monte-Carlo algorithms for optimizing, tracking or regulating stochastic systems. The treatment here makes extensive use of compactness methods, and gives results under considerably more general conditions than do the classical methods--the approach appears particularly useful when weak convergence methods are used. Definitely for those familiar with the field. TAV

PROBABILITY, P**. *Brownian Motion and Classical Potential Theory.* Sidney C. Port, Charles J. Stone. Prob. and Math. Stat. Acad Pr, 1978, xii + 236 pp, $22.50. [ISBN: 0-12-561850-6] A clearly written development of the relationship between Brownian motion and potential theory. Much has been written about each topic by various authors, but this is one of the first successful attempts to show their intimate relationship. Assumes a knowledge of probability theory at the graduate level (e.g., Chung's book). Contains an extensive bibliography. TAV

PROBABILITY, P. *Stochastic Analysis.* Ed: Avner Friedman, Mark Pinsky. Acad Pr, 1978, xii + 340 pp, $18. [ISBN: 0-12-268380-3] Proceedings of an international conference held at Northwestern University in April, 1978. LAS

STATISTICS, T(15). *Analyse de Données, Applications et Méthodes Partiques.* A. Henry-Labordère. Masson (US Distr: SMPF, 14 E. 60th St., NY 10022), 1977, vii + 94 pp, $20.30 (P). [ISBN: 2-225-46134-1] An elementary treatment of data analysis, for students and practitioners of applied statistics. Case-study approach to elements of linear regression and factor analysis. Assumes some linear algebra and probability. No exercises. TRS

STATISTICS, P*. *Developments in Statistics, V. 1.* Ed: Paruchuri R. Krishnaiah. Acad Pr, 1978, xii + 339 pp, $34.50. [ISBN: 0-12-426601-0] First in a series of volumes designed to provide a medium for the publication of papers, in both theory and applications of statistics, which are too long for journals but too short to be separate monographs. Contains six invited papers on various topics in the general areas of stochastic control theory, point processes, multivariate distribution theory, time series, nonparametric methods, and factorial designs. RSK

STATISTICS, P*. *Discrete Discriminant Analysis*. Matthew Goldstein, William R. Dillon. Wiley, 1978, x + 186 pp, $16.95. [ISBN: 0-471-04167-X] In the Wiley Series in Probability and Mathematical Statistics. Comprehensive treatment, including both theory and applications, of the problem of discrimination between groups having discrete valued multivariate observations. Includes Fortran programs to implement the general classification procedures discussed. RSK

STATISTICS, S**(13-16), L**. *Statistics: A Guide to the Unknown, Second Edition*. Ed: Judith M. Tanur, *et al*. Holden-Day, 1978, xxxi + 496 pp, $7.50 (P). [ISBN: 0-8162-8605-1] Revision of the well-known 1972 *First Edition* (TR, January 1973; ER, April 1974), containing two new essays and a set of problems for each of the essays. The new essays are those contained in two of the three "Mini-SAGTUS" which have appeared to date (TR, November 1976, October 1977, January 1979). A very readable collection of 46 essays describing a wide variety of applications of statistics. RSK

STATISTICS, P. *The Information in Contingency Tables*. D.V. Gokhale, Solomon Kullback. Statistics, V. 23. Dekker, 1978, x + 365 pp, $29.75. [ISBN: 0-8247-6698-9] Unified approach to the analysis of contingency tables and other count data, based on the principle of "minimum discrimination information," which leads naturally to log-linear models. Emphasis is on methodology with many illustrative examples. Good bibliography. RSK

STATISTICS, P. *Asymptotic Optimality of Likelihood Ratio Tests in Exponential Families*. W.C.M. Kallenberg. Math. Centre Tracts, No. 77. Math Centrum, 1978, viii + 126 pp, Dfl. 16 (P). Optimality is investigated with respect to the criteria of the "shortcoming" of a test and the concept of "Bahadur deficiency." Derives necessary results on probabilities of large deviations. RSK

STATISTICS, T(16-17: 1, 2), P. *The Scientific Use of Factor Analysis in Behavioral and Life Sciences*. Raymond B. Cattell. Plenum Pr, 1978, xxii + 618 pp, $32.50. [ISBN: 0-306-30939-4] Comprehensive two-part presentation, moving from concrete to abstract. Part I presents basic concepts and processes of factor analysis; Part II deals with more complex statistical procedures. Much explanatory material reflecting the author's considerable experience in this area. Good set of references. No exercises. RSK

STATISTICS, S(17-18). *Problems and Solutions in Theoretical Statistics*. D.R. Cox, D.V. Hinkley. Chapman and Hall, 1978, vii + 193 pp, $12.95 (P). [ISBN: 0-470-26299-0] Outline solutions and discussion of the "Further results and exercises" in the authors' 1974 text *Theoretical Statistics* (TR, November 1975). Includes a brief summary of the main ideas required. RSK

STATISTICS, P. *Optimal Stopping Rules*. A.N. Shiryayev. Transl: A.B. Aries. Appl. of Math., No. 8. Springer-Verlag, 1978, x + 217 pp, $24.80. [ISBN: 0-387-90256-2; 3-540-90256-2] General theory of optimal stopping rules for Markov processes, both discrete and continuous time, with applications to an optimal selection problem and to the "disruption" problem. RSK

STATISTICS, T(13: 1). *Understandable Statistics: Concepts and Methods*. Charles Henry Brase, Corrinne Pellillo Brase. Heath, 1978, xv + 392 pp, $11.95. [ISBN: 0-669-08342-9] Elementary introduction requiring only one year of high school algebra. Minimal selection of topics; artificial problems. RSK

STATISTICS, T(15-17: 1), S. *Correlational Procedures for Research*. Robert M. Thorndike. Gardner Pr, 1978, xi + 340 pp, $21.50. [ISBN: 0-470-15090-4] Primarily an intuitive overview of multivariate analysis, which may serve as a readable introduction to more mathematical treatments. Begins with an extensive treatment of bivariate relationships. Other topics include partial and multiple correlation, canonical analysis, discriminant analysis, cluster analysis and factor analysis. Includes some matrix algebra. No exercises. RSK

STATISTICS, T(14-17: 1), P, L. *Applied Nonparametric Statistics*. Wayne M. Daniel. HM, 1978, xiii + 510 pp, $16.95. [ISBN: 0-395-25795-6] Well-organized collection of common nonparametric techniques, utilizing much real data. Presentation tends to be matter-of-fact, but all pertinent information is given. Extensive references and good problem sets. Presumes course in elementary statistics. RSK

STATISTICS, T?(16: 1, 2). *Statistical Treatment of Experimental Data*. J.R. Green, D. Margerison. Elsevier, 1977, x + 382 pp, $34.75. [ISBN: 0-444-41615-3] Introduction for experimentalists, particularly in the physical sciences. Theoretical explanations are minimal and the treatment of probability is sparse. Extensive treatment of regression, including polynomial regression through a fixed point. Statistical methods are illustrated with real data. No exercises. Overpriced. RSK

STATISTICS, T(13: 1, 2). *Basic Statistics for the Behavioral Sciences*. Kenneth D. Hopkins, Gene V. Glass. P-H, 1978, xii + 436 pp, $13.95. [ISBN: 0-13-069377-4] Divided approximately evenly between descriptive and inferential statistics, but at a sophisticated level. No nonparametrics, but includes multiple-comparison techniques. Each chapter ends with a summary, a list of significant terms, concepts and symbols, a diagnostic "mastery" test, a collection of problems, and answers to the test and problems. Mathematical review notes are integrated into the chapters where needed. RSK

STATISTICS, P. *The H-Function with Applications in Statistics and Other* [illegible] .M. Mathai, R.K. Saxena. Wiley, 1978, xii + 192 pp, $9.95. [ISBN: 0-170-26380-6] "All [illegible] " concerning the functions known in the literature as generalized Mellin-Barnes, or generalized G-, or Fox's H-functions. LAS

COMPUTER PROGRAMMING, T*(1). *BASIC: An Introduction to Computer Programming*. Robert J. Bent, George C. Sethares. Brooks/Cole, 1978, xii + 239 pp, $10.95 (P). [ISBN: 0-8185-0250-9]; *Instructor's Manual*, 138 pp, (P). Clear presentation, using many examples, within a problem-solving framework. Good problem sets. Review true or false quizzes end each chapter, with answers in an appendix. *Instructor's Manual* contains sample tests, student projects, and detailed solutions to problems not solved in the text. RSK

COMPUTER PROGRAMMING, S*, P*. *Programming Programmable Calculators*. Harold S. Engelsohn. Hayden, 1978, 211 pp, $9.95 (P). [ISBN: 0-8104-5105-0] A book which is designed to teach programming on programmable calculators. It specifically covers the SR52, SR56, TI57, TI58 and TI59 by Texas Instruments and the Commodore PR100. It is more extensive and readible than the manuals which are supplied by the manufacturers. Includes many sample programs and programming exercises. CEC

COMPUTER SCIENCE, T(14-15: 1), L. *Data Structures and Programming Techniques*. Herman A. Maurer. Trans: Camille C. Price. P-H, 1977, vii + 228 pp, $13.50. [ISBN: 0-13-197038-0] Defines and gives procedures for implementing and manipulating various data structures using a subset of PL/I. Linked lists, stacks, queues, trees, graphs and multilinked structures. Searching, merging, and sorting. RWN

SYSTEMS THEORY, T(17-18: 1), P. *Dynamic System Identification: Experiment Design and Data Analysis*. Graham C. Goodwin, Robert L. Payne. Math. in Sci. and Eng., V. 136. Acad Pr, 1977, x + 291 pp, $26. [ISBN: 0-12-289750-1] Concerned with the problem of obtaining mathematical models of physical systems from "noisy" observations. Presumes background in probability and statistics, stochastic processes, and linear systems theory, key results from which are reviewed in the earlier chapters or summarized in the appendices. Four main chapters cover models for dynamic systems, estimation for dynamic systems, experiment design, and recursive algorithms. Includes outline solutions of all problems. Good bibliography. RSK

APPLICATIONS (BIOLOGY). *Some Mathematical Questions in Biology, IX*. Ed: Simon A. Levin. Lect. on Math. in Life Sci., V. 10. AMS, 1978, ix + 244 pp, $11.20 (P). [ISBN: 0-8218-1160-6] Catastrophe and chaos, chemical diffusion, representation of visual information, control of growth: five papers from the 1977 annual AAAS symposium. LAS

APPLICATIONS (BIOLOGY), P, L. *Caste and Ecology in the Social Insects*. George F. Oster, Edward O. Wilson. Mono. in Pop. Bio., V. 12. Princeton U Pr, 1978, xv + 352 pp, $20; $7.50 (P). The foundation of sociobiology: a mathematical theory of the ecological and evolutionary aspects of caste, based on an evolutionary strategy to enhance ergonomic efficiency. Most of the models involve optimization, dealing with efficient allocation of resources. LAS

APPLICATIONS (BIOLOGY), P. *Lecture Notes in Biomathematics-24: The Measurement of Biological Shape and Shape Change*. Fred L. Bookstein. Springer-Verlag, 1978, viii + 191 pp, $8.90 (P). [ISBN: 0-387-08912-8; 3-540-08912-8] Excellent summary of mathematical techniques, and their shortcomings, used in quantifying biomedical shape data and in measuring their variations and changes. Much of the essay is author's dissertation in statistics and zoology at the University of Michigan. Formulates some new statistical methods for shape. Extensive list of references from the literature of many scientific disciplines. JK

APPLICATIONS (ECONOMICS), T(18: 1), S, P. *Functional Equations in Economics*. Wolfgang Eichhorn. Appl. Math. and Comp., No. 11. A-W, 1978, xviii + 260 pp, $16.50 (P); $28.50. [ISBN: 0-201-01949-3] A sequel to Aczél's 1966 classic *Lectures on Functional Equations and Their Applications* providing applications to economics: prices, production, index numbers, growth processes, aggregation, etc. Extensive index of notation and definitions; author and subject indices; bibliography. LAS

APPLICATIONS (ENGINEERING), P. *Relativistic Theories of Materials*. Aldo Bressan. Tracts in Nat. Philo., V. 29. Springer-Verlag, 1978, xiv + 290 pp, $49. [ISBN: 0-387-08177-1; 3-540-08177-1] An exposition of the progress (primarily of the last fifteen years) made in understanding the interpretations of relativity theory in terms of more basic "materials science." An introductory chapter which reviews relativity theory leads to a theoretical treatment of thermodynamics "from an Eulerian point of view" followed by "Materials From a Lagrangian Point of View." JAS

APPLICATIONS (ENGINEERING), T(15-18: 1, 2), S, P, L. *Practical Approach to Pattern Classification*. Bruce G. Batchelor. Plenum Pr, 1974, xii + 243 pp, $23. [ISBN: 0-306-30796-0] Pattern classification is the study of decision-making procedures and of the devices which realize such procedures. Approach is heuristic and experimental with attention paid to the engineering problems involved in building such machines. Methods of classifying patterns and comparisons of such; parameter calculations; construction of pattern classifier in analogue or digital equipment; application areas; pattern classifier design needs; data used in testing. Many helpful diagrams and illustrations. Chapter summaries and references. Appendices. Subject index. RJA

APPLICATIONS (ENGINEERING), P. *Two-Dimensional Digital Signal Processing*. Ed: Sanjit K. Mitra, Michael P. Ekstrom. Benchmark Papers in Elec. Eng. and Comp. Sci., V. 20. Dowden, Hutchinson & Ross, 1978, xiii + 371 pp, $32. [ISBN: 0-87933-320-0] Papers in this volume are divided among three categories: deterministic or statistical two-dimensional digital signal processing, or implementation techniques. In turn, each category is further subdivided into two or more groupings, each of which is preceded by appropriate editorial introductions and comments. 43 papers. Author and subject indices. RJA

APPLICATIONS (PHYSICS), P. *Lecture Notes in Physics-80: Mathematical Problems in Theoretical Physics*. Ed: G. Dell'Antonio, S. Doplicher, G. Jona-Lasinio. Springer-Verlag, 1978, vi + 438 pp, $16.50 (P). [ISBN: 0-387-08853-9; 3-540-08853-9] 34 main lectures (plus a few short communications) from the International Conference on the Mathematical Problems in Theoretical Physics held in Rome in June 1977. An impressive survey of a diverse frontier. LAS

Reviewers Whose Initials Appear Above.

Richard J. Allen, St. Olaf; Ralph P. Boas, Northwestern University; Paul J. Campbell, Beloit; Clifton E. Corzatt, St. Olaf; Lorraine L. Keller, St. Olaf; Roger B. Kirchner, Carleton; Richard S. Kleber, St. Olaf; Joseph Konhauser, Macalester; R.W. Nau, Carleton; Thomas R. Savage, St. Olaf; John Schue, Macalester; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; T.A. Vessey, St. Olaf.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1529 18th Street, N.W., Washington, D.C. 20036

PERSONAL ITEMS

Associate Professor John R. Hubbard, Lycoming College, Williamsport, Pennsylvania, has been appointed Chairman of the Mathematics Department.

Dr. John R. Rasmussen, Bowdoin College, has joined the Consulting Services Department of Rainier National Bank, Seattle, as Senior Consultant and Assistant Vice President.

Professor Zeev Nehari, Carnegie-Mellon University, died on September 1, 1978. He was a member of the Association for twenty-seven years.

Associate Professor and Chairman Francis A. Varrichio, Saint Peter's College, Jersey City, New Jersey, died on December 4, 1978, at the age of 46. He was a member of the Association for twenty-three years.

SECOND INTERNATIONAL CONFERENCE OF SCIENTIFIC EDITORS

The second International Conference of Scientific Editors will be held in Amsterdam on October 13-17, 1980. It is aimed at furthering the discussions begun at the first meeting on the growth of scientific publications, their social and intellectual functions and diverse techniques. Further information may be obtained from Helena Tombal, Elsevier Scientific Publishing Company, P.O. Box 330, 1000 AH Amsterdam, The Netherlands.

SUMMER SEMINAR ON STATISTICAL CONCEPTS AND METHODS

The North Central Section, in cooperation with the School of Statistics at the University of Minnesota (Twin Cities), will hold a seminar on statistical concepts and methods on the campus of the University of Minnesota (Duluth) June 18-22, 1979.

The seminar will consist of a mini-course for mathematicians who are either statistical neophytes or who feel the need of some review of basic statistical concepts, followed by a series of lectures on modern methods and applications.

Details and application form are available from Professor Michael F. Miller, Department of Mathematical Sciences, 320 Mathematics-Geology Building, University of Minnesota-Duluth, Duluth, Minnesota 55812.

SEVENTH ANNUAL MATHEMATICS AND STATISTICS CONFERENCE

The Seventh Annual Mathematics and Statistics Conference at Miami University, Oxford, Ohio, will be held September 28-29, 1979. The theme for this year's conference will be "Geometry." Featured speakers will include Professors Branko Grunbaum of the University of Washington, and Ernest E. Schult of Kansas State University. There will be sessions of contributed papers, which should be suitable for a general audience of mathematicians and students who are not necessarily experts in geometry. Abstracts should be sent by June 1, 1979 to professor David Kullman, Department of Mathematics and Statistics, Miami University, Oxford, Ohio 45056. Information concerning preregistration, housing, etc. may also be obtained from the above address.

The Ohio Delta Chapter of Pi Mu Epsilon will also hold its annual student conference September 28-29, 1979. Undergraduate mathematics students are invited to contribute papers, and should send abstracts to Professor Milton Cox, Department of Mathematics and Statistics, Miami University, Oxford, Ohio 45056.

FIFTH ANNUAL RELIABILITY TESTING INSTITUTE

The fifth annual Reliability Testing Institute will be held at the University of Arizona, Tucson, on May 14-18, 1979. It is designed for engineers of all ranks including reliability engineers and managers, product assurance engineers and managers, quality control engineers and managers, manufacturing, sales and service engineers and managers, staff engineers, project engineers, statistical engineers, industrial engineers, R & D engineers, design engineers, all other engineers, statisticians, government representatives, industry representatives, and college and university teachers. Undergraduate college mathematics or its equivalent is

necessary to qualify for and benefit from this Institute. Knowledge of statistics and probability is desirable.

For further information contact Special Professional Education, College of Engineering, University of Arizona, Old Engineering Building, Tucson, Arizona 85721.

TWO JUNE WORKSHOPS IN APPLICABLE MATH

The MD-DC-VA section of the MAA will sponsor two five-day workshops at Salisbury State College, Maryland. *Catastrophe Theory and Its Applications* will be led by Dr. Alexander Woodcock of Amherst College. He will apply catastrophe theory to the physical, social, life and environmental sciences. This workshop will be held 4-8 June, 1979.

The second workshop, *Mathematics and the Microcomputer* will be led by Mr. A.F. Falcoff, Dupont Senior Engineer, and Dr. H. H. Suber of Salisbury State College. They will present the organization and architecture of a microcomputer and its role in applications. This workshop will be held 11-15 June.

These workshops are designed for teachers in two and four-year colleges. The total cost, including room & board, is $140 for each workshop. For more information, write Dr. B. A. Fusaro, Dept. Math Sciences, SSC, Salisbury, MD 21801.

CALL FOR COMMENTS ON CRYPTOLOGY COURSES

During the past five years there has been renewed interest in studying the mathematics of cryptology. There are a number of courses being taught in various areas around the country. Some courses are taught in mathematics or statistics departments, some in the area of computer science and electrical engineering and others in lighter modes such as interim short courses, night school avocational courses, and seminars with attention given to many areas of cryptology, not just mathematics.

CRYPTOLOGIA, in its third year of publication, is a quarterly journal devoted to *all* aspects of cryptology with special attention given to mathematics, statistics and computer science. The editors encourage those who have had experience teaching in this area to share these with your colleagues. In each issue a column is reserved for such exchanges. Anyone wishing to share these ideas or wanting to know more about the relationships between cryptology and mathematics or the journal itself may write to: Brian J. Winkel, Department of Mathematics, Albion College, Albion, MI 49224.

EARLY ADMISSION TO GRADUATE STUDY

The Department of Mathematics at the University of Rhode Island offers admission to graduate study for mathematically talented and promising students without a bachelor's degree.

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In case of doubt, an entrance examination administered by the department may also be required. The early admission program is designed to lead to the M.S. and Ph.D. degrees. It does not yield a bachelor's degree.

MAA CHRISTIE FUND

The Northeastern Section of the MAA is honoring one of its most distinguished colleagues, Dan E. Christie of Bowdoin College, through a fund drive. The proceeds of the drive will be applied as a principal-only payment to reduce the principal of the mortgage on the new MAA headquarters building in Washington, D.C.

Professor Christie, who died in 1975, was one of the earliest members of the Northeastern Section and served it in many capacities, including those of Chairman of the Section and twice Governor from the Section. He was a member of numerous MAA committees, including the Committee on the Undergraduate Program in Mathematics, the Committee on an Internship Program in Mathematics Education, the Committee on Assistance to Developing Colleges, and the Committee on Publications. In addition to his distinguished service to the Association, Professor Christie also found time to write three books and to direct several prestigious NSF institutes.

Dan Christie was greatly beloved by both his colleagues and his students. The officers of the Northeastern Section invite everyone to join them in honoring Dan Christie as a mathematician, a teacher, and a true friend by sending a donation to Professor Small, Department of Mathematics, Colby College, Waterville, Maine 04901. Checks are to be payable to the MAA Christie Fund.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

FALL MEETING OF MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA SECTION

The 1978 Fall Meeting (November Meeting) of the Maryland-District of Columbia-Virginia Section of the MAA was held at the United States Naval Academy on Saturday, November 18, 1978. There were 124 persons in attendance including 108 members. The principal speakers were Alfred B. Willcox, Executive Director, MAA and Professor Brindell Horelick of the University of Maryland, Baltimore County, who spoke respectively on *Mathematics: Where are We Going and What Educators Can Do About It* and *Some Mathematical Models for One-Variable Calculus Students*.

The Chairman of the Section, Professor Orville M. Thomas, presided over a brief business session. He announced that the Spring Meeting will be held at George Mason University on April 28, 1979.

The contributed papers included:

How to Cut a Cake Fairly, Walter Stromquist, Falls Church, Virginia

A Recursive Function for Data Compression and Security, John Hays, Naval Research Laboratory

Learning from Research, Clifford J. Maloney, Bethesda, Maryland

Functions of Exponential Type are Differences of Functions of Bounded Index, Shantilal N. Shah, Hampton Institute

The Traveling Salesman Problem and Some Variants, Bruce L. Golden, University of Maryland

Representation Theorems for Some Classes of Analytic Functions, Edward J. Moulis, U.S. Naval Academy

Linguistics and Design, Stefan Shrier, Booz, Allen and Hamilton, Inc., Bethesda, Maryland

Heat Equation in a Metal Bar, Howard Penn, U.S. Naval Academy

First A.I.D., Richard L. Eisenman, University of Maryland

The Propositional Calculus of Speech, John C. Baez, Great Falls, Virginia

Some Relationships Among Digital Filters, Numerical Integration, Digital Simulation and Data Reconstruction, Allen Darling, Booz, Allen & Hamilton, Inc., Bethesda, Maryland

Thrust Effects on the Orbit Determination Problem, R. N. Pal (presented paper), M. Fan, J.W. Kennedy, R. L. Smith, Computer Sciences Corporation, Silver Spring, Maryland

Extensions of Khatri's Matrix Product, Rana Singh, Virginia State College

The Section was shown two films, *Sampling and Estimation* and *Hypothesis Testing*, which were made available by the Media Guild.

REUBEN C. DRAKE, *Secretary*

NOVEMBER MEETING OF THE NORTHEASTERN SECTION

The twenty-fourth annual meeting of the Northeastern Section of the MAA was held at Bunker Hill Community College, Charlestown, Massachusetts on November 18, 1978; there were 104 people in attendance. Section Chairman Donald B. Small presided, assisted by Sectional Governor Donald L. Kreider.

The following talks were given at the morning session:

Vector Spaces of Magic Squares, James E. Ward, Bowdoin College

Curves, Surfaces and Knots, David Eisenbud, Brandeis University

At the afternoon business meeting it was announced that the first Section Short Course would be held at the University of Maine, June 18-22, 1979. Maynard Thompson of Indiana University is to be the principal lecturer and his topic will be *Applications of Mathematics in Medicine and Biology*. Chairman Small reported that to date contributions to the MAA building fund in the name of Dan Christie totaled $1201. Sectional Governor Donald Kreider announced that the Executive Committee had created a distinguished lectureship program. The lectureship is named the Christie Lecture in memory of former Sectional Governor Dan E. Christie.

The Christie Lecture will be given by a distinguished member of the mathematical community and is to be a part of each annual meeting. The topic of the lecture will be an issue of current importance in mathematics. And, although the Section is rich with people who have made contributions to research and teaching, the choice of the Christie Lecturer is not restricted to a member of the Section.

In order to perpetuate the Lectureship, Don Kreider moved that the Section Chairman be directed to appoint each year a committee of at least 3 members of the Section to select a Christie Lecturer for the next fall meeting. The motion was passed unanimously. The meeting closed with the announcement that the next Section meeting would be held at the University of Maine, Orono, Maine, June 22-23, 1979. The following talks completed the program:

Mathematics Placement in a Total Systems Approach, Joan M. McGowan, Bunker Hill Community College

Linear Programming: The New Calculus? Sue H. Whitesides, Dartmouth College

Somewhat Sensitive Languages, William E. Marsh, Hampshire College

GEORGE BEST, *Secretary-Treasurer*

ACADEMIC MEMBERS ELECTED INTO THE ASSOCIATION

In accordance with the amendment adopted at the Business Meeting of the Association at Stillwater on August 30, 1961, the Board of Governors at its meetings of August 13, 1977, in Seattle, Washington, January 5, 1978, in Atlanta, Georgia, and on August 7, 1978 in Providence, Rhode Island, elected the thirtieth, thirty-first, and thirty-second sets of applicants for academic membership (for the previous lists of applicants, see the April, 1977 issue, pages 322-3, and the references cited there). Approval of election was given to the following applicants for academic membership:

Alcorn State University
Appalachian State University
Auburn University at Montgomery
Bloomsfield State College
Centenary College
Christopher Newport College
County College of Morris
DeKalb Community College
Edmonds Community College
El Paso Community College
Fayetteville State College
Georgia Southwestern College
Golden West College
Hong Kong Polytechnic Institute
Indiana University of Pennsylvania
John Carroll University
Lyndon State College
Mansfield State College
Marymouth College of Kansas
Matematiska Institutionen, Lindköping, Sweden
Millsaps College
North Georgia College
North Hennepin Community College
Pan American University
Richland College
Rose Hulman Institute of Technology
Somerset County College
Sonoma State College
Southeastern Missouri State University
Rul Ross State University

MAA NEWS LETTERS

A number of MAA Sections publish monthly news letters. The information contained therein is of particular interest to Section members since the news letters tell of activities within the member institutions as well as personal endeavors of individual members. In addition to such items, there are often items that are of general interest to the mathematical community at large. Your News and Notices Editor finds some of the stories helpful in providing material as fill-in for completion of pages in that section of the MAA Monthly. The receipt of copies of your news letters is appreciated.

CALENDAR OF FUTURE MEETINGS

Fifty-ninth Summer Meeting, University of Minnesota, Duluth, August 21–23, 1979.

Sixty-third Annual Meeting, San Antonio, Texas, January 5–7, 1980.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Westminster College, New Wilmington, Pennsylvania, April 27–28, 1979.

FLORIDA, Hillsborough Community College, Tampa, March 2–3, 1979.

ILLINOIS, Northern Illinois University, De Kalb, April 27–28, 1979.

INDIANA, Butler University, Indianapolis, April 21, 1979.

INTERMOUNTAIN, Idaho State University, Pocatello, May 4–5, 1979.

IOWA, Cornell College, Mt. Vernon, April 20–21, 1979.

KANSAS, Johnson County Community College, Overland Park, April 7, 1979.

KENTUCKY, Morehead State University, Morehead, April 6–7, 1979.

LOUISIANA-MISSISSIPPI, Friday–Saturday before February 20. Deadline for papers 3 months before meeting.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK, Adelphi University, May 5, 1979.

MICHIGAN, University of Detroit, May 4–5, 1979.

MISSOURI, University of Missouri, Kansas City, March 30–31, 1979.

NEBRASKA, Nebraska Wesleyan University, Lincoln, April 20–21, 1979.

NEW JERSEY, Monmouth College, West Long Branch, April 28, 1979.

NORTH CENTRAL, College of St. Teresa, Winona, Minnesota, April 27–28, 1979.

NORTHEASTERN, University of Maine, Orono, June 22–23, 1979.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO, Miami University, Middletown, April 20–21, 1979.

OKLAHOMA–ARKANSAS, Oklahoma State University, Stillwater, March 30–31, 1979.

PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 15–16, 1979.

PHILADELPHIA, Saturday before Thanksgiving.

ROCKY MOUNTAIN, University of Denver, Denver, April 27–28, 1979.

SEAWAY, SUNY, College at Oneonta, May 4–5, 1979.

SOUTHEASTERN, University of Tennessee, Chattanooga, April 6–7, 1979.

SOUTHERN CALIFORNIA, University of Southern California, Los Angeles, March 10, 1979.

SOUTHWESTERN, University of Texas, El Paso, April 6–7, 1979.

TEXAS, Texas Tech University, Lubbock, April 6–7, 1979.

WISCONSIN, University of Wisconsin, Eau Claire, March 30–31, 1979.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, San Francisco, January 3–8, 1980.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES

AMERICAN MATHEMATICAL SOCIETY, University of Minnesota, Duluth, August 22–25, 1979.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION

ASSOCIATION FOR COMPUTING MACHINERY, Plaza Hotel, Detroit, Michigan. October 29–31, 1979.

ASSOCIATION FOR SYMBOLIC LOGIC, New York City, December 28–29, 1979.

ASSOCIATION FOR WOMEN IN MATHEMATICS

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS, Washington, D.C., August 13–16, 1979.

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Boston, Massachusetts, April 18–21, 1979.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Hyatt Regency Hotel, New Orleans, Louisiana, April 30–May 2, 1979.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Radisson Muehlbach, Kansas City, Missouri, November 8–10, 1979.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Royal York Hotel, Toronto, Canada, June 11–13, 1979.

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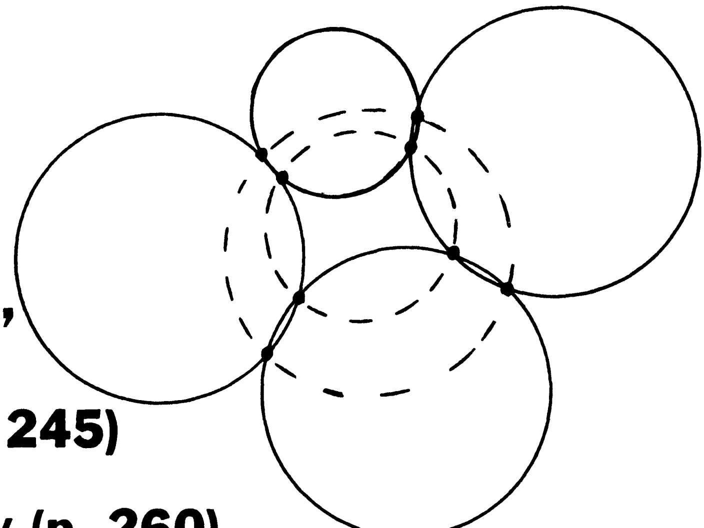

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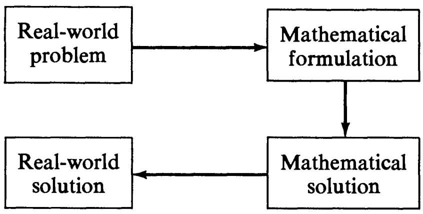
THE ROLE OF APPLICATIONS IN PURE MATHEMATICS

DOROTHY L. BERNSTEIN

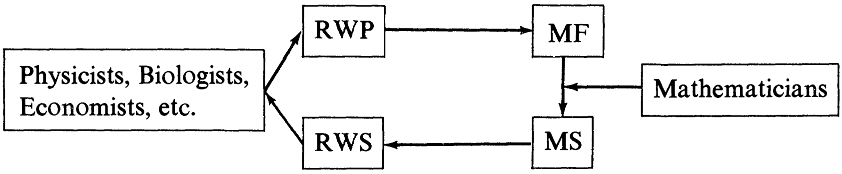
In recent years, the MAA and other mathematical groups have paid considerable attention to applied mathematics and the use of mathematics in natural and social sciences and in government and industry. A glance at the programs of the national and sectional meetings of the past few years will show a generous sprinkling of hour-long talks and panels dealing with applied topics. The usual diagram that is supposed to represent the role of mathematics in applications goes something like this:



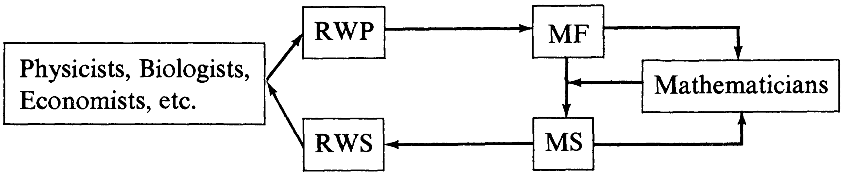
Of course, one might proceed by experiment or observation directly from the real-world problem to the real-world solution; but this may be impossible or prohibitively expensive, and so the fruitful detour through mathematical models is taken. I have no quarrel with this general description, although I point out that it says nothing about who formulates the mathematical problem, who solves it, and who interprets the solution in real-world terms. I am concerned with the impression it leaves, either expressed or implied, that this is all irrelevant to the development of pure mathematics. Mathematicians stand over here, inventing axiom systems and abstract spaces and proving theorems about them, which are available for use when needed in the above scheme. Physicists and biologists and economists and all the others are over there, proposing their problems and then gratefully receiving the solutions.*

Dorothy L. Bernstein was an undergraduate at the University of Wisconsin in Madison, where she began her graduate work; she continued her graduate study at Brown University, writing her dissertation on "The Double Laplace Integral" under Professor Jacob Tamarkin, and receiving her Ph.D. in 1939. Since September 1937, she has been teaching full-time in colleges and universities, except for sabbatical leaves for research. She has taught at Mount Holyoke College, the University of Wisconsin, the University of California (Berkeley), the University of Rochester, and Goucher College. Her sabbaticals have been spent at the Institute of Advanced Study, University of California (Los Angeles), Brown University, and the University of Tennessee. At present she is Chairman of the Mathematics Department at Goucher College. Her mathematical interests are in analysis, especially integration, measure theory, and partial differential equations. She also has a strong interest in probability and statistics, and in the use of computers in the mathematics curriculum. She has been active in the affairs of the MAA; she was First Vice-President in 1972–73, and has been President since January, 1978. This paper is based on talks given to the Southeastern Section and the Iowa Section of the MAA, in April 1978.—*Editors*

*This point of view is expressed in the following quotation, translated from a letter from Hermite to Stieltjes, 28 November 1882: "I am only an algebraist and have never gone outside the domain of pure mathematics. I am nevertheless completely convinced that the most abstract speculations of Analysis are evidence of realities that exist outside ourselves and will eventually come to our attention. I even think that the work of the pure geometers is directed, without their being aware of it, toward such an end, and the history of Science seems to me to prove that a mathematical discovery comes about at the precise time when it is needed for each new advance in the study of those phenomena of the real world that are amenable to calculation."



It is my thesis that there is something missing from this flow-chart. Real-world problems and their mathematical formulation often are the source of interesting problems in pure mathematics, and mathematical solutions of these models are sometimes the stimulation for generalizations which yield important concepts and theorems in mathematics. I propose that we examine some specific areas of mathematics in which applications have played an important part and then try to draw some general conclusions about the role of applications in mathematics (let's drop the "pure"—I'm never quite sure what it means in a given discussion).



Example 1. Our first example is geometry. As its name indicates, it began as an empirical science, that of land measurement by the Egyptians. Because of the flooding of the Nile, the size of a man's farm was never the same from one year to the next; and since the amount of taxes he paid was determined by how much land he owned, it was extremely important to have procedures for accurate measurement of land. It was the Greeks, culminating in Euclid, who made geometry into a postulational system with theorems and proofs that was a model of mathematical thinking for the next 2,000 years. However, it was still considered a description of the real world, and much of its authority during this long period came from this fact. In the eighteenth and nineteenth centuries, mathematicians failed in attempts to prove Euclid's fifth postulate by showing that its denial led to a contradiction. Instead, there came a gradual realization, probably first by Gauss, that the hyperbolic geometry of Saccheri, Bolyai, and Lobachevsky did not lead to a contradiction. In fact, as Klein, Beltrami, and Poincaré showed, both hyperbolic geometry and the elliptic geometry of Riemann and Schläfli were as logically consistent as Euclidean geometry. All were abstract mathematical systems in the modern sense, each with its own power and beauty, but none with a special claim as a description of the physical space we inhabit. Cayley was able to show that elliptic geometry was closely related to projective geometry, which had also been developed during the eighteenth and nineteenth centuries [**1**].

In the first part of the twentieth century, however, precisely at the time mathematicians were completing the abstraction of geometry, the flow was reversed. Einstein, looking for a basis for his genéral theory of relativity, found it in the geometry of Riemann. The idea that physical space is finite but unbounded is a cliché of modern-day physics; but, to quote the physicist Freeman Dyson, "Einstein took the revolutionary step of identifying our physical space-time with a curved non-Euclidean space . . . on the basis of very general arguments and aesthetic judgments. The observational tests of the theory were made only after it was essentially complete, and they did not play any part in the creative process. Einstein himself seems to have trusted his mathematical intuition so firmly that he had no feeling of nervousness about the outcome of the observations. The positive results of the observations were, of course, decisive in convincing other physicists that he was right" [**2**]. That it was possible for the same man to hold both points of view is illustrated by Hilbert, who made important contributions to both axiomatic geometry and to general relativity.

And now there is a final swinging of the pendulum. Says Roger Penrose [3]: "The debt to pure geometry that relativity had owed has now been amply repaid. For many of the ideas of the modern subject of differential geometry received their initial stimulus from concepts arising from Einstein's general relativity." These include manifolds, tangent spaces, and parts of complex geometry.

Example 2. My second example is the area of numerical calculations. John Napier invented logarithms for the very practical and utilitarian purpose of making multiplication easier by essentially replacing it by addition. Nevertheless, logarithms and the number e, which is the base of natural logarithms, became indispensable in the development of modern analysis. Infinite series were used as an aid to calculation long before anyone worried about convergence, and before they assumed their central role.

Indeed, the whole subject of numerical analysis has developed right alongside mathematics. According to Philip Davis, numerical analysis is a branch of both applied mathematics and computer science. It formulates algorithms, analyzes errors (not blunders), studies rates of convergence, and compares algorithms. Such work, in this day of computers, is very important. But its real influence on mathematics itself has been to change its flavor. Instead of relying almost exclusively on existential methods, as was the case 50 years ago, mathematicians are now also using constructive or algorithmic methods, under the stimulating necessity of getting numerical solutions.

Of course, computers have changed the way both scientists and mathematicians regard numerical computation. For the former, they have opened up previously inaccessible areas to mathematical and numerical treatment. For the latter, they have changed what we consider important. Some of you have heard Henry Pollak talk on this subject. I would like to illustrate it by a simple example which I first heard from the late George Forsythe: The usual solution of the quadratic equation $ax^2+bx+c=0$, where $ac\neq 0$, is, setting $\Delta=b^2-4ac$:

$$(1a)\ x_1=\frac{-b+\sqrt{\Delta}}{2a},\qquad (1b)\ x_2=\frac{-b-\sqrt{\Delta}}{2a}.$$

However, since neither root of the quadratic is 0, an equivalent equation can be found by dividing by x^2 and setting $t=1/x$: $ct^2+bt+a=0$. If we find the roots t_1 and t_2 of this equation and then take reciprocals, we get alternative formulas for x_1 and x_2:

$$(2a)\ x_1=\frac{2c}{-b-\sqrt{\Delta}},\qquad (2b)\ x_2=\frac{2c}{-b+\sqrt{\Delta}}$$

Mathematically, the two sets of formulas give identical results. But due to round-off errors and the inexact process of taking square roots on a computer, there are times when one choice is much more nearly exact than the other. Indeed, Forsythe proves that one should use (1a) and (2b) when the coefficient $b\geqslant 0$, and (2a) and (1b) when $b<0$.

Example 3. Calculus, and the whole group of ideas which followed it and which we lump under the heading of analysis, or theory of functions, is one of the cornerstones of modern mathematics. It began, quite specifically, with attempts to develop a theory to account for the observations and quantitative measurements of various phenomena made by Galileo, Copernicus, Kepler, and others in the sixteenth and seventeenth centuries. Galileo found that the distance traveled by falling bodies is proportional to the square of elapsed time of fall. Kepler made some early attempts at integration in order to measure the volume of kegs, or bodies bounded by curved surfaces. Finally, Isaac Newton, in his *Principia Mathematica*, stated in mathematical form certain physical principles which he said governed *all* matter. His law of universal gravitational attraction and his three laws of motion became the foundation of the science of mechanics. In the course of developing these laws he invented his calculus of fluxions as a tool. The concepts of *Principia Mathematica* were quickly accepted in the Western scientific

world, and are still, according to the meteorologist P. D. Thompson, "the keystones in the study of the behavior of physical systems in natural environments encountered on earth" [4].

But for 150 years after Newton there was only an inexact semi-physical formulation of the calculus, although during that period some very significant advances were made. Often the same men worked in both mathematics and mechanics. Thus Euler and D. Bernoulli studied the mechanical properties of gases and liquids, as well as making fundamental contributions to mathematics. Euler, Gauss, Jacobi, Cauchy, Riemann—all had varying standards of abstraction and rigor in the mathematics they did. It was not until Weierstrass that analysis became abstract and unempirical. (I wonder how much of this was because Weierstrass had been a lawyer before he became a mathematician.)

Example 4. My fourth example is the body of mathematics lumped under the term *Fourier series*, or more generally, harmonic analysis. When Fourier was studying the phenomenon of heat conduction in the early nineteenth century, he realized that trigonometric functions like $\sin x$, $\cos 2x$, $5\sin 3x$, were periodic and had different periods and amplitudes, so that linear combinations of them could represent fairly intricate periodic phenomena. But it turned out that what he needed was a representation of a function $f(x)$ by an infinite series of such terms:

$$f(x) \sim \sum (a_n \cos nx + b_n \sin nx),$$

which we have since come to call Fourier series. Although, by that time, power series expansions of the form

$$f(x) \sim \sum a_n x^n$$

were fairly well understood, such an expansion requires that f have derivatives of all orders in the neighborhood of the origin, since $a_n = f^{(n)}(0)/n!$. On the other hand, Fourier series could represent functions which were continuous, or even piecewise continuous (if you accepted the mean value at points of discontinuity as representing the function), since formally at least the coefficients are of the form

$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\sin nx\,dx, \qquad a_n = \frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx, \qquad a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(x)\,dx,$$

and this depends in turn on relations of the form

$$\int_0^{2\pi} \sin nx \sin mx\,dx = 0, \qquad \int_0^{2\pi} \sin nx \cos mx\,dx = 0, \qquad \int_0^{2\pi} \cos nx \cos mx\,dx = 0 \qquad (n \neq m).$$

The study of the convergence properties of Fourier series, and of which functions could be represented by them, interested many mathematicians during the nineteenth century and led them in surprising directions. For example, Georg Cantor was working on this when he found he had to deal with the infinite sets of numbers at which such a series could converge; from this he was led to the study of infinite sets in general and his far-reaching work on cardinal and ordinal numbers. Henri Lebesgue, on the other hand, was intrigued by the fact that the usual Riemann integral was not good enough to allow him to compute the Fourier coefficients of general functions $f(x)$, and so he extended the notion of the integral to take care of this—the indispensable Lebesgue integral.

While the study of Fourier series went on during the nineteenth century, many other series of special functions with analogous properties were discovered, often in connection with a particular physical problem and its resulting differential equation, associated with names like Legendre, Jacobi, Laguerre, Bessel, and Hermite. In each case, one had a denumerably infinite set of functions orthogonal to each other—the integral of the product of two different functions of the set over the basic interval was 0—allowing expansions of fairly arbitrary functions in infinite series of terms from this set. All of these became known as Fourier series; and they have played a fundamental role in the modern development of vector spaces, since they furnish nontrivial

examples of complete orthonormal bases in certain inner-product spaces. In addition, they have led to many important generalizations.

Example 5. The use of differential equations to describe physical processes has gone on since Newton first stated, in his second law of motion, that when a point-mass moves in a given direction under the action of a force its acceleration, the second derivative of distance, is proportional to the force. In my previous example, I referred to the solution of certain kinds of second-order differential equations, called Sturm-Liouville equations, which were of great interest to nineteenth-century mathematicians.

When we consider linear nth order equations, with constant coefficients, we are immediately struck by their analogy to nth order algebraic equations, obtained by replacing the kth order derivative by the kth power of a variable. It was an engineer, Oliver Heaviside, who first exploited this in a systematic way at the end of the nineteenth century, to get a set of arbitrary rules for solving such differential equations, which he called the operational calculus. Many mathematicians were horrified since there was absolutely no theoretical justification for what he was doing; but it worked, and more and more engineers, physicists, and chemists used it. Slowly, mathematicians, beginning with Bromwich in Australia and then Plancherel in France, Doetsch in Germany, Wiener and Widder in the United States, realized that if one applied a certain linear operator to a function $F(t)$, which consisted in multiplying by e^{-st} and integrating over the interval $[0, \infty)$ the resulting function $f(s)$ (called the Laplace transform of F) behaved in such a way that differentiation of F with respect to t corresponded essentially to multiplication of $f(s)$ by s. Consequently, a differential equation in F became an algebraic equation in $f(s)$, which was then solved and the inverse transform evaluated by the theory of residues. At about the same time other integral transforms, such as the Fourier transform and the Mellin transform, were developed. I should mention, by the way, that Laplace first used the transform which bears his name in his great work on celestial mechanics.

One can describe the motion of a point in the plane by a simple system of differential equations which represent the components of force applied to the x and y coordinates, and get solutions of the form $(x(t),y(t))$. Early in the twentieth century, Poincaré made a revolutionary study of the path of such a point considering the solutions as giving parametric equations of the path. He was particularly interested in its asymptotic behavior—what happens as t becomes large—and showed that in general it led to either singular points or limit cycles. The study of such singular points and limit cycles, called the Poincaré-Bendixson theory, has led to solution of many engineering problems in servo-mechanisms and automatic control. It has also had a profound effect on what has come to be called the qualitative theory of differential equations and the theory of differential equations in the large. In fact, this is one area where the distinction between pure and applied mathematics disappears completely, as anyone recognizes who has been at the Center for Dynamical Systems at Brown.

Example 6. My next example is graph theory, which started in a mathematical sense with Euler's Königsberg bridge problem in 1735, but where the earliest contributions were made by Kirchhoff in 1847 with his study of electrical networks, and by Cayley in 1857 with his work in organic chemistry. A graph, let me remind you, is a diagram consisting of a finite number of points, some of which are connected by edges, usually directed. The famous problem of whether you can color any plane map using at most 4 colors belongs in this area—it was proposed to de Morgan by Guthrie in 1850. Interest in the subject of graphs among mathematicians fluctuated over the next hundred years. Among those who contributed to graph theory during this period, let me mention Kuratowski in Russia and Whitney and MacLane in the United States. Since 1950, there has been a steady increase in interest in the subject, due to its many applications. It has been said that physics, chemistry, genetics, psychology, economics, engineering, operations research, all use the results of graph theory. The important question to computer and electronic engineers of when an electrical circuit can be realized as a printed circuit, for example, has led to an extensive analysis of the corresponding problem in graphs.

A look at *Studies in Graph Theory*, volumes 11 and 12 of the MAA series, edited by Fulkerson, will give you an idea of the present ramifications of the subject. Fulkerson himself has written a paper on flow network and combinatorial operations research. A. J. Hoffman has a paper on eigenvalues of graphs where he uses the notion of adjacency matrices. Whitney and Tutte have an interesting paper on Kempe chains and the four-color problem. By the way, Appel and Haken at the University of Illinois have apparently proved the four-color conjecture, using a computer. I say "apparently" because it brings up the very interesting question of just what constitutes a proof in this age of computers.

Example 7. Twentieth-century physics has become completely intertwined with mathematics. I have already referred to general relativity theory. Another basic development has been quantum mechanics. When Cayley invented matrices in the nineteenth century, he prided himself on their uselessness; but in 1925, Heisenberg found them to be precisely the tool he needed to describe his conception of atomic structure—so-called matrix mechanics. Schrödinger, at about the same time, used the Sturm-Liouville theory of differential equations to describe his concept of atomic structure—so-called wave mechanics. The two theories were soon recognized as equivalent; indeed, in 1927, von Neumann unified the theory in terms of linear operators in a complex Hilbert space.

Group theory, which originated in the study by Lagrange and Galois of symmetry and symmetric transformations on the roots of an equation, became a central idea in algebra, and, indeed, in all of mathematics, during the nineteenth century. In the twentieth century, the group became a fundamental concept in the mathematical description of the physical world. For example, crystallography begins with the description of all possible space groups—230 of them. More recently, group theory has been used in the description of the elementary particles. It provided a key to the classification of recently discovered particles, called the 8-fold way, which enabled Gell-Mann and Ne'eman to predict the existence of a certain baryon before it was discovered experimentally in 1964. In turn, the mathematical study of continuous Lie groups and their resulting Lie algebras have been stimulated by these physical applications. (I should mention in passing that recent interest in large finite simple groups has been stimulated by problems in algebraic coding theory.)

The theory of analytic functions of a complex variable has been called the greatest achievement of nineteenth century mathematics. And, of course, its applications to physics and engineering have been numerous. But the theory of functions of several complex variables seemed to belong to the domain of pure mathematics. There were some early results by Weierstrass, Poincaré, and Hartogs, but then interest languished until after World War II. Since then there has been an awakened interest and much fundamental work has been done, largely because of the requirements of quantum field theory. It turns out that the probability distributions of collisions of elementary particles are best described by piecewise analytic functions of several complex variables. The theory of residues has been extended to integrals over hypersurfaces in n-dimensions—the so-called Feynman integrals were invented by a physicist.

Example 8. Probability theory started with the absolutely practical request of a gambler to Vieta and Pascal to explain the odds in tossing two coins. Much work was done on Bernoulli trials—repeated experiments where the probability of success in a single trial is constant, and also with the celebrated normal curve which appeared over and over as an empirical law. However, according to Mark Kac, after Laplace probability almost disappeared as a mathematical discipline in the Western world until the 1920's, even though during the nineteenth century physicists like Maxwell and Boltzmann used it spectacularly in the study of Brownian motion. (Exception must be made for the work of Russian mathematicians like Chebysheff and Markov.) But then Poincaré and Hilbert revived interest in the subject. Poincaré's little book on the calculus of probability is still a model of clear exposition which I recommend to any graduate student preparing for a French exam. And with the work of Kolmogoroff in 1935, the theory of

probability finally became a precise and sophisticated discipline, based on Lebesgue measure; Paul Lévy and Norbert Wiener also played significant roles in its development.

The subject of statistical mechanics, which had been studied by Boltzmann and Gibbs, gave rise to the ergodic hypothesis—that a quantity depending on a mechanical system with a large number of components would have identical averages over *time* and over *space*. Mathematicians like von Neumann and G. D. Birkhoff gave a clear mathematical formulation of this hypothesis and proved the first theorems in 1931. Then one of the familiar stochastic variables arising from independent repeated trials (Bernoulli trials) was shown to be an example of an ergodic system **[5]**. During the past twenty years, students of Kolmogoroff have generalized another physical concept—that of entropy—which was useful in the study of such probabilistic systems and is now being used to tackle other problems in statistical mechanics. There is a clear exposition of this entire matter in a series of lectures given by Sinai at Moscow State University, and translated in 1977 by V. Scheffer **[6]**.

Example 9. Some economists, like Lawrence Klein, say flatly that economics is a mathematical discipline. They point to the use of optimization, game theory, linear programming, and mathematical models of price equilibrium (now being rephrased in probabilistic terms). However, I shall turn for my next example to the general theory of optimization which has had applications to, and draws from, operations research, managerial science, control theory, statistics, and mathematics. Optimization became viable with the computer; in fact, it has been estimated that one-fourth of all scientific computing involves optimization. According to Dantzig and Eaves, mathematical optimization does four things: (1) develops a mathematical structure called a program, which models a real world situation; (2) investigates existence and attributes of optimal, or near optimal, solutions, and how to characterize them; (3) designs algorithms for the computation of optimal solutions; and (4) implements mathematical solutions in a particular application, evaluates results and then modifies it. In volume 10 of the MAA series called *Studies in Optimization*, which appeared in 1974, Dantzig and Eaves say "Optimization theory is a fertile ground for new and promising problems, problems upon which to build new mathematical theories. It could be a source of new, exciting, and relevant problems which would motivate the mathematical student." Besides this volume, containing papers by Kuhn, Tucker, and the two editors, one can also look at "Towards Global Optimization" by L. Dixon and G. P. Szego.

Example 10. I have left my own field, partial differential equations, for the last, and I shall discuss it briefly. Following well-known physical prototypes (wave mechanics, heat conduction, potential theory) boundary value problems in partial differential equations with constant coefficients were classified into three general types: hyperbolic, parabolic, and elliptic. The same classification was easily generalized to linear p.d.e. with variable coefficients, and boundary value problems were proposed and solved, and in some cases, applied to solve real-world problems in many areas. Meanwhile, the theory was extended to quasilinear and to non-linear p.d.e.; in each case, the classification into the three types was preserved, sometimes without real justification. Using the notion of a generalized function called a *distribution* invented by Laurent Schwartz 25 years ago, the concept of a solution was extended and this in turn was helpful in solving many physical problems. The physical notions of conservation of energy and momentum were used as the basis for studying a whole class of equations known as conservation laws; solutions for boundary value problems for such equations were obtained in terms of distributions. The concept of boundary layers, taken from aerodynamics, led to a general theory of singular perturbations, but much remains to be done in the way of a general existence theory of partial differential equations.

I could cite other examples, such as the whole area of statistics and experimental design; but instead, let us try to formulate some general conclusions:

1. There has been a continuous and fruitful interplay between science and mathematics from the very beginning. Von Neumann, in his essay "The Mathematician" [7], classifies science into three groups: descriptive, experimental, theoretical. Or rather, these are three stages through which a science passes. The descriptive ones call most on mathematics and have the least to give. The experimental sciences use mathematics freely and begin to return the investment. In the theoretical sciences, there is a free interchange of ideas with mathematics. Notice, I said interchange, not identification. A mathematician's motives are internal: his intellectual curiosity, his sense of form and pattern, his taste. A scientist, even a theoretical one, finds his motives elsewhere, as von Neumann points out.

2. No one can predict what mathematics will become useful for applications or when this will happen.

3. Mathematical ideas may originate as abstract concepts and then have useful applications or they can originate in the context of applications and be generalized to abstract concepts.

Well, what about the future? The danger of overspecialization that many people have pointed out is not one that bothers me. At present, most of us are willing to sit and listen to what others have to say about advances in their fields. But that is not the same as becoming creatively interested. I was reading the book by F. D. Thompson in which he describes the remarkable breakthrough that occurred in numerical weather prediction during the period 1951–55 [4]. In 1949, von Neumann had organized a special group at the Institute of Advanced Study to go over the partial differential equations of weather prediction, which had been formulated by Richardson in 1922. Computers had been introduced, but they could only handle data. By the time the programmers had written the instructions necessary to handle the data collected from weather stations, it was too late to predict the weather for a given day. It was under the pressure of this problem that he conceived the revolutionary idea of storing programs as well as data—that is, essentially letting the computer decide how to handle data. With this established, the project was under way and, according to Thompson, "It was a small dedicated group of people who knew something about meteorology, physics, mathematics, numerical analysis, and computer science who were able, in the short period of four years, to completely solve the problem of mathematical weather prediction." Incidentally, now it is a new ballgame, since one considers relativistic hydrodynamics of probabilistic solutions.

I have said nothing about new problems facing our society in conserving energy and handling the environment, nor about the whole range of biomedical problems. All of these need new mathematical ideas, and I am convinced that it is only by a combined attack like that just described that they can be handled. How should this be accomplished? I am not a believer in giant national programs; I think interested individuals can accomplish more. To the young Ph.D.'s I say: Talk to your friends—the young men and women of your own age in physics, biology, economics, anthropology, psychology. Find out what they are interested in—at some point, you will find a common bond with someone, and from then on, with both of you talking furiously and each learning from the other, you may very well find something of real mathematical interest to you as well as something helpful to them. Remember, they probably know the cut-and-dried mathematics; it's the dreamy, kooky ideas I'm talking about. Next, to the graduate students—if you have the time, or even if you haven't, try to take a graduate course in some field outside of mathematics in which you have some interest and knowledge. I do not mean an introductory survey course—I mean an advanced graduate course where you really find out what the new thinking is in the subject. You may find yourself over your head sometimes, but it will open your eyes to new possibilities; and as you talk to other graduate students, you may give them an idea of what modern mathematics is all about. The undergraduates—well, arrange some joint meetings of your mathematics clubs with clubs in other fields. And, again, talk to your friends. You note there is one group I have left out, the one to which I belong, the senior faculty. We know something of what is going on elsewhere, and some of us, as my examples have shown, have been able to bridge the gap between mathematics and

applications. But our methods of thought are set, which makes it much harder for us to see new connections and new relations. I believe that young people, starting in the ways I have indicated, can make real progress in established areas by solving new problems, and in new fields by defining mathematical problems and beginning to solve them.

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THEORETICAL AND APPLIED COMPUTER SCIENCE: ANTAGONISM OR SYMBIOSIS?

KEITH HARROW

1. Introduction. Many computer science students complain bitterly about being forced to take a course in theoretical computer science (typically, either formal languages and automata theory or computability theory). The complaints center about two main points: the course is not relevant to computer science (i.e., programming) and the course is too mathematical (i.e., abstract).

Of course these comments can be answered directly that computer science is not just programming and needs a mathematical foundation. (See, for example, R. Schneider's paper [23].) However, we will discuss a different answer, one that will be both more surprising and more interesting to the student: the direct interaction between theoretical and applied computer science. By this we mean specific topics, ideas, proof techniques, etc., that are used in theoretical computer science but which the student has already seen or which the student will soon see in applied areas. In other words, when introducing a new topic in a theoretical course, one can say, "This is also useful in your --- course," or, "This is just like--- that you learned last term." Most students will be more impressed with (and will learn faster from) a series of immediate reinforcements of this type than with vague promises of eventual future use.

Of course, it is unreasonable to expect the majority of students to become theoreticians or even to enjoy these courses as much as those in which they actually create programs. At the very least though, much of the mystery surrounding theoretical computer science can be eliminated. In particular, by stressing the two-way relationships between theoretical and practical areas, we can show our students that theoretical computer science often provides useful tools for other studies. We can also show them a more surprising result—that topics discussed in applied courses are used in theoretical areas as well.

To a large degree, we will ignore the well-known relationship between automata and formal language theory and compiler theory. See the new book by P. Lewis, D. Rosenkrantz, and R.

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Stearns [16], or several of the articles in either [1] or [24], for a detailed study of this question. For the moment, we will also ignore the argument that theoretical computer science is "good" for a student and the fact that the ability to think in a clear and consistent way is needed in pragmatic as well as theoretical areas. Instead, we will concentrate on showing the student direct evidence of how theoretical computer science can be of use.

Section 2 will discuss three major areas of interaction: algorithms, recursion, and simulation. In section 3 we will mention several other topics in which knowledge of one part of computer science will complement knowledge of another. These topics include structured programming, trade-offs, a universal machine, unsolvability, and several others.

To answer the question raised in the title, there is definitely a symbiotic relationship between theoretical and applied computer science, and each needs the other for its continued growth.

2. Three Major Areas of Interaction. First, let us discuss a few ideas that are absolutely fundamental to computer science. We will see that they come up repeatedly in both theoretical and practical discussions. The most important of these topics include algorithms, recursion, and simulation.

The concept of an algorithm is central to applied computer science. D. Knuth begins the first volume [15, p. 1] of his series of books by stating: "The notion of an *algorithm* is basic to all of computer programming, so we should begin with a careful analysis of this concept." In fact, the title of this first volume is "Fundamental Algorithms."

The concept of an algorithm is also important in theoretical computer science. J. Hopcroft and J. Ullman [13] base their study of formal language theory on the notion of devising procedures and algorithms to answer certain questions, e.g., Given a finite automation, does it accept an infinite number of strings? An algorithm is usually taken to be a mechanical process (i.e., a procedure) that always gives an answer. By Church's Thesis, this can be interpreted to be a total program in some language (theoretical or real) or a function computed by a Turing machine which always halts, etc. Note that a given problem may have many possible algorithms in the formal sense.

These same ideas can be seen in the concrete study of algorithms. An algorithm for searching or sorting can be implemented in many possible ways. The student should recognize that an analysis of the sort is more concerned with the underlying technique (the algorithm) than with the program using this technique. This distinction is also important in more theoretical discussions. A student accustomed to thinking in terms of algorithms (instead of specific implementations of these algorithms) will be more easily persuaded of the difference between a function and any given representation of that function. In formal language theory, we must differentiate between a language and a particular grammar for that language. For example, a language may be unambiguous even though one particular grammar for it is ambiguous [13]. This level of abstraction is of course essential to being a good programmer.

Recursion (sometimes called primitive recursion) is a theoretical operation in which the value of a function at a given point is defined in terms of that same function, but at smaller input values. In the simplest case, $f(0)=k$ (a constant) and $f(x+1)=h(x,f(x))$ where $h(a,b)$ is some known function. That is, $f(3)$ is some function of $f(2)$, which is itself a function of $f(1)$, and so on. By putting limits on the number of recursions allowed or bounds on the growth rate of the function being defined, we obtain many well-known hierarchies of function classes within the parent class of primitive recursive functions. Each of these classes (e.g., the Grzegorczyk hierarchy [11] or loop program classes [17]) is important as a theoretical model of what types of functions real computers can compute.

In applied computer science, recursion has two main uses. First, the very nature of several programming languages (notably LISP and SNOBOL) encourages solution to a large class of problems using recursive techniques [19]. This usually means one or more recursive subprocedures, i.e., in the course of the computation a procedure will call itself, typically with simpler parameters. See [3] or [5] for an excellent discussion of recursive programming methods. Other

languages, including APL, PL/I, and ALGOL, allow recursion, but programmers usually do not take advantage of this feature because of ignorance or fears of inefficiency (we will comment on this below). Perhaps even more significant is the use of recursion in the description of algorithms (even if a particular implementation uses a stack or some other method of faking recursion). Good examples of this include algorithms for traversing a tree in preorder or postorder and lexical analysis in a compiler. Thus, the second main use of recursion is for algorithmic specification and analysis.

This illustrates the importance of recursion to both aspects of computer science. But the question remains: How do these relate to each other?

Students seeing a recursive algorithm for the first time are often baffled. Most will have a vague feeling of uneasiness, thinking that nothing is being done. In addition, the student is usually convinced that recursion is a very complicated topic. The algorithm works in some mysterious way, but the student is not sure how and certainly would never think of using recursion again in the future.

This fear of recursion is unfortunate, since, as pointed out above, recursion can be very useful in programming. Even worse, it is unnecessary, because recursion is really not that complicated and mysterious. In fact, a knowledge of theoretical computer science can help enormously in understanding recursion as it is used in applied areas. A student who is comfortable working with (theoretical) recursive functions will be much more confident when dealing with a description of a recursive algorithm, since the underlying concept of recursion is identical in both. The converse is also true: if a student is familiar with the idea of presenting an algorithm in a recursive format, or if the student has learned how to write recursive procedures, then the introduction of the formal concept of primitive recursion will be relatively painless. In other words, a student seeing a concept for the second time, even in a different context, will understand that concept much more clearly than he did the first time.

From an educational viewpoint, the very fact that the concept is being presented in different contexts will sometimes shed more light on the idea than any individual presentation. These two ideas—the relative ease of relearning and the extra reinforcement of a new approach—apply to almost all of our comments, not just to recursion.

There is another, potentially critical, reason for studying recursion. In the past few years, many real-world problems have been solved efficiently using recursion. A. Aho, J. Hopcroft, and J. Ullman [2] have a detailed discussion of a general technique which they call "divide and conquer." Basically, it consists of breaking up a large problem into a series of smaller ones, such that the sum of the complexities of the smaller problems is still less than the complexity of the original. Recursively, the smaller problems are then further broken down, and so on. The savings over more traditional methods are often dramatic. For example, a straightforward computation of the kth largest of a group of n elements takes roughly $n \log n$ steps if the numbers are first sorted and the selection is made from the sorted list. But [2] presents a recursive procedure which takes no more than $4n$ steps to find the kth largest of a group of n. For large values of n, this algorithm is significantly faster than the method of sorting and, in fact, can be shown to be within a constant factor of being optimal. Thus, instead of being inefficient, the recursive algorithm proves to be optimal! This is just one example of how recursion is now being used to solve practical problems. (Note: An algorithm of great theoretical importance, to recognize on a non-deterministic Turing machine within polynomial time if a given integer is prime, has also been solved recursively by V. Pratt [20], after resisting for years all previous efforts.) As more computer scientists become familiar with mathematical ideas, we can expect this phenomenon to be repeated, as yesterday's theoretical idea becomes today's state of the art.

Simulation is our last major concept common to both theoretical and pragmatic computer science. The simulation of one type of machine or computing device by another is the basis of several texts in computability theory (see M. Davis [8] or M. Minsky [18]). The fact that any

model of a computer can act like any other is also used to lend support to Church's Thesis (one version of which states that computing machines, under any of a series of equivalent definitions, can do precisely what is "mechanical"). Formal language theory, in its study of the Chomsky hierarchy, is to a large degree the study of how four types of machines and grammars can be made to simulate each other. For example, when we prove that the class of sets accepted by non-deterministic pushdown automata is precisely the class of context-free languages, we do this by showing how a context-free language can be made to simulate a pushdown automaton, and vice versa [13]. A student will grasp the essential ideas of the proofs rather easily if they are presented from this point of view.

Simulation also has many applications in the real world. These include simulation of one machine or computer language by another, e.g., an IBM 360 emulating an IBM 7090, a LISP program being interpreted line by line, or the idea of a virtual machine. Simulation methods are often used in studying mathematical, physical, or economic models, e.g., a model of the United States economy or a model of traffic flow in a city. Note once more that, regardless of whether the student is exposed first to the theoretical or first to the practical course, the work spent in understanding the concept the first time will make relearning the idea in a new format relatively easy.

In the next section, we will look at a few more specific places in which theoretical and practical areas interact.

3. Other Areas of Interaction. Let us begin with a look at the operations used to define the classes of partial recursive, recursive, and primitive recursive functions [**8**]. These operations are recursion, composition, and minimalization. Recursion has already been discussed. Composition is simply a way of using the results of one or more computations in another computation. (Given functions $f, g_1, \ldots, g_m$, we say h is defined from f and the g_i by composition if:

$$h(x_1,\ldots,x_n)=f(g_1(x_1,\ldots,x_n),\ldots,g_m(x_1,\ldots,x_n)).)$$

The natural programming analogue consists of feeding the results of one or more subprocedure calls into another subprocedure. Starting from a small set of initial functions, we can construct the class of primitive recursive functions by applying only compositions and recursions. The operation of minimalization searches successively through $0,1,2,\ldots$ until a number is found which satisfies a certain equality. The (unbounded) search continues until such a number is found and then outputs that number (the function is undefined if no integer satisfies the equality); if the search is a bounded minimalization, then we only try values up to the given bound, and output 0 if there is no success up to this point. Functions for which an unbounded minimalization always gives an answer are called regular. The class of recursive functions is obtained from the initial functions by applying recursion, composition, and minimalization of regular functions. Arbitrary minimalization leads to the class of partial recursive functions, which includes non-total functions. Observe that the essential difference between the three classes is the type (if any) of minimalization allowed. This should be stressed as a theoretical result that mirrors the real world. It is well known that a program with an unbounded search is to be avoided at all costs; a search that is guaranteed to halt is better, while a bounded search is, of course, the nicest situation. The output of 0 in case a bounded minimalization fails to find a value satisfying the conditions can be likened to a search which exits gracefully in case of error.

Anyone who has programmed is aware of a harsh fact: a program which runs very quickly will often use much more storage space than a somewhat slower program which uses space more efficiently, and vice versa. A fast sort may require large amounts of auxiliary storage, while a not-so-fast sort may require no extra storage. A program which handles one specific problem will usually be more efficient on it than a general-purpose problem-solver, which must worry about exceptions, special cases, and other time-consuming situations. We usually refer to this phenomenon as a trade-off between two or more competing factors, e.g., utilization of space versus running time. To a large degree, every programming or design decision is a trade-off and should be clearly stated as such when the decision is made.

The question of trade-offs also comes up in theoretical discussions. To take a trivial example, a one-head Turing machine to recognize the set of palindromes over the alphabet $\{0, 1\}$ must keep running back and forth between the two halves of the tape, while a two-head Turing machine can send one head to the end of the tape and then march the two heads toward each other, matching symbol by symbol. The one-head machine requires time proportional to the square of the length of the input string, but the two-head machine can recognize a string in a number of steps proportional to the length of the string (and the two-head machine does not have to write on the tape). Another well-known trade-off is the simulation of a k-tape Turing machine (operating in $T(n)$ steps) by a 2-tape machine (in $T(n) \cdot \log T(n)$ steps) **[13]**. R. Daley **[7]**, Hopcroft and Ullman **[14]**, W. Savitch **[22]**, and many others have studied trade-offs between program size and complexity, time and tape used, deterministic and non-deterministic Turing machines, etc. Thus there is a wide range of research in theoretical computer science concerned with trade-offs, reflecting the importance of this topic to pragmatic computer science.

Dovetailing or interleaving is a theoretical technique used when we wish to run several Turing machines, grammars, or more general procedures in parallel rather than in series. (For example, we might wish to find any Turing machine which halts on the number 7.) The process consists of spending a certain amount of time on the first subprocess, then a slightly larger amount on the first two, then still more time on each of the first three, etc. If we run the procedures in this way (rather than running the first to completion and so on), the possibility of an infinite loop in any one subprocess will not stop the entire process. This idea of shifting back and forth from one job to another after a specified number of steps, rather than waiting until the job has finished executing, might seem mysterious to many students. But it becomes much clearer when it is compared to a concept used in almost every large computer: multiprogramming or timesharing, where the CPU works on several programs "simultaneously," each for a specified time period. In fact, the method of dovetailing described above would be considered far too simple-minded for real-world situations. Recently, there have been proposals **[12]**, **[21]** to allow a dovetailing statement in theoretical or real programming languages. An actual application to program testing is discussed in **[12]**.

In the same vein as our previous comment, we note that students are often initially skeptical about the concept of a Gödel numbering—that is, a method to associate a unique number with a program or arbitrary string of symbols. The student will accept this idea much more readily if reminded of the many times more complicated (and efficient) encoding of a program or data into just 0's and 1's given by EBCDIC.

One of the ideas most widely discussed in computer science today is structured programming. Although it embraces many other concepts as well, the term *structured programming* usually implies to some degree a restriction on the use of the GO TO or branch instruction. Curiously, a theoretical class of programs (the loop programs of A. Meyer and D. Ritchie **[17]**) provides a justification for the recommendation to avoid the GO TO.

The loop program language consists of a few simple assignment statements ($X=0, X=X+1, X=Y$) which can be contained within LOOP X–END pairs. Note that there are no branch instructions. A LOOP X–END pair means execute all instructions within the loop, including possible further loops, X times. If X is zero, the loop is skipped over. As would be expected, more and more complex functions can be computed as the depth of nesting of the loops increases (e.g., $X+Y$ can be computed using one loop, $X \cdot Y$ using two, etc.). The entire class of loop programs computes precisely the class of primitive recursive functions. Thus, without any type of branching mechanism (aside from the most elementary form of iteration), we can compute a rather sophisticated class of functions.

On the other hand, if we include a conditional branch (in fact, if we consider just a conditional branch TO A IF $X \neq 0$, $X=X+1$, and $X=X-1$ as our basic instruction set), we can compute the entire class of partial recursive functions **[8]**. Conditional branching allows us to define wildly pathological functions, while the absence of branching restricts us to a nicely behaved class of functions which is still large enough to include all functions used in actual

programming practice. No wonder E. Dijkstra's famous letter **[10]** was entitled "GO TO Statement Considered Harmful"! We also note the article by C. Böhm and G. Jacopini **[4]** providing a theoretical justification for the use of DO WHILE, IF THEN ELSE, and assignment statements as the basic instruction set for actual programming languages. A student familiar with the GO TO controversy will appreciate these striking examples of the importance of restricted control structures. In addition, the idea of limiting a language to a few simple instructions (rather than a whole slew of programming language constructs which often lead to undesirable complexity) is justified by the power of these theoretical languages with just a handful of elementary instructions. Reference to the instruction set of a typical assembler will reinforce the idea that much can be done using only the most simple of building blocks. The excellent article **[6]** by R. Constable and A. Borodin contains a much more detailed discussion of some of the issues raised here.

A universal program in computability theory refers to a program which receives as input the encoding of another program P and an encoding of an input X to P. The universal program executes P on X, typically by decoding the program and doing a step-by-step simulation. A discussion of this can shed much light on more practical areas.

First, students are intrigued by the idea of a single universal program which can do anything. It is very useful to discuss exactly why this is not feasible in the real world. Hopcroft and Ullman **[13]** provide a particularly easy-to-understand universal Turing machine. One of its nicest features is that the student sees immediately that the universal program refers again and again to the original program P, and thus operates (in a real-world sense) very inefficiently. The student should also observe that the universal program will take hundreds of steps to do what P does in one or two steps. This reinforces many of the ideas brought up in a study of simulation.

Second, the idea of a program receiving another program as input should be emphasized. This is much more common in actual computing practice than most people suspect; consider, for example, a compiler or a pre-processor. In LISP, the equivalence between programs and data is exploited by the interpreter. A universal program is often used as part of a step-counting program to see if a particular program halts in a given number of steps or, more generally, after using a given amount of any "resource." The student can see in this the rudiments of an operating system in which one program can oversee or monitor another.

One last somewhat strained analogy: the universal program can be viewed as a microcosm of computer science. The universal program itself is the hardware (e.g., an IBM 370 or CDC 6600), the program P to be simulated is the software (e.g., a FORTRAN or PL/I compiler), the encoded input to program P is the data (e.g., a particular FORTRAN or PL/I program with data). Given the appropriate software and data (the encoded program P and the encoded input to P), the hardware is universal, i.e., it can do essentially anything.

Ideas about binding times, including the difference between a compile-time and an execution-time decision, can be brought up in a study of theoretical languages. Assume that our language contains an assignment statement $X = X + 1$ and a way to set $X = 0$ (possibly in a series of instructions). Then we can eliminate the instruction $X = k$ (where k is a constant) by the macro:

$$\begin{array}{l} X=0 \\ \left.\begin{array}{l} X=X+1 \\ \vdots \\ X=X+1 \end{array}\right\} k \text{ times} \end{array}$$

The natural next step is: there is a macro for $Z = W$ which uses a conditional branch to loop W times. Can this be replaced by:

$$\begin{array}{l} Z=0 \\ \left.\begin{array}{l} Z=Z+1 \\ \vdots \\ Z=Z+1 \end{array}\right\} \text{W times?} \end{array}$$

Explaining why this is not legal (whereas the expansion a constant number of times certainly is) should vividly illustrate the difference between actions that can be taken before a program runs and actions that depend upon explicit values for the variables, that is between compile-time and run-time.

A problem is said to be unsolvable if there is no algorithm (or by our remarks above, if there is no mechanical procedure) to answer the problem in all cases. Students (and colleagues) often scoff at unsolvability results, saying that no real-world problem could ever be unsolvable and that the typical "unsolvable" problem (e.g., Does a Turing machine ever print a 2?) is not relevant.

However, R. DiPaola **[9]** has given an example of a decision problem that arose in connection with an actual data-retrieval system (an automatic question–answer system). An attempt was made to isolate a class of questions to the system that would be considered "reasonable." The class of definite formulas from logic was selected as the prototype for the class of reasonable questions. DiPaola showed that it is unsolvable to determine if a given formula is definite and thus that it is unsolvable to determine if a given question is "reasonable."

This also helps to answer the objections of those who say that "unsolvable" is too broad a claim, since it only means unsolvable by computer and thus puts no limitations on what humans can do. For those who say "unsolvable" is too limited a claim, since problems that are solvable within an incredibly large amount of time are not really "solvable," we can point to the recent study of feasible (i.e., polynomially bounded) problems **[2]**.

Finally, the entire subject of solvable and unsolvable problems should give the student a healthy skepticism toward those who claim computers (or humans) can, or soon will be able to, do "everything."

Note that we have restricted ourselves to simple topics in which knowledge of one area (theoretical or pragmatic computer science) reinforces the other. The student should also realize the existence of much more widespread applications. Two very useful books which present some of these are Aho **[1]** and Yeh **[24]**. In addition, it is hoped that each teacher can find new examples of this interaction to further solidify the ties between these two subjects.

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GEOMETRY ACCORDING TO EUCLID

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1. Introduction. The geometry of a two-dimensional vector space over a commutative field—that is, affine geometry—has much in common with Euclidean geometry [5, Chap. 13]. It is the purpose of this note to point out the less well known but nevertheless important

1.1. PRINCIPLE. *Virtually* all of Euclid's geometry is valid in affine planes over *almost any* field.

The words in italics will be explained in section 2 (see Remark 2.9); roughly, the "virtually" excludes any claim by Euclid of existence that depends on order or continuity, while the "almost any" refers to those (commutative) fields that admit a separable quadratic extension.

Now 1.1 is hardly a novel observation; in more primitive forms it has probably been around since the invention of commutative fields. I have seen five recent treatments of this idea (described in §6), all independent of one another. This multiplicity of rediscoveries suggests both that the principle is widely applicable and that it is not as well known as it should be. Before we look at this previous work and how it was applied, we'll look in sections 2 and 4 at just what Euclid's geometry is and how Euclid himself laid the groundwork for the present study. In fact, my main point will be that Euclid's theorems and procedures remain valid in a general setting virtually as Euclid himself stated them. Section 3 is devoted to examples showing what one can expect from the general theory. In section 5 there is an example of how Principle 1.1 can be used, in order to prepare the reader for the applications that follow. A reader who is not interested in metamathematics, or whose intuition tells him that the principle is obvious, may turn immediately to section 6, where the applications are discussed. Throughout this paper I maintain the standard of rigor appropriate to the mathematics of the third century B.C.

2. A Model for Euclid's Postulates. By *Euclid's geometry* (which we shall find to be more general than what is usually thought of as Euclidean geometry), I mean the consequences of his five postulates:

1. A straight line may be drawn from any point to any other point.

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2. A finite straight line may be extended continuously in a straight line.
3. A circle may be described with any center and any radius.
4. All right angles are equal.
5. Through a given point, precisely one straight line can be drawn parallel to a given straight line.

REMARKS: As is commonly done today, I've replaced Euclid's Postulate 5 by "Playfair's Axiom." The two are easily shown to be equivalent in the presence of the other four postulates (see the comment by Heath on Proposition I.29 in [**14**]). On the other hand, I've retained the other postulates as Heath translated them, despite the appearance of words having awkward connotations—particularly *finite* and *continuously*. My justification for this soon will be apparent (see Remark 4.2). Although such words certainly suggest order and continuity, the full force of these notions were never required by Euclid. In regard to his requirements, let us define a finite straight line (or line segment) to be a pair of points. We may then render the first two postulates by, "A straight line (or, more simply, a line) is a set of points having at least three elements. Any two points are contained in one and only one line." For the technical words used by Euclid in his postulates and propositions (such as *circle* and *parallel*), his definitions can be used without modification. A more thorough discussion of the definitions and postulates is found in the commentary to Book I provided by Heath in [**14**, pp. 153–240].

Here now is our goal: Find a model that satisfies the five postulates; then all of Euclid's geometry will be valid in that model, even though it may be more general than what Euclid might have intended.

Postulates 1, 2, and 5 are the axioms for the affine plane [**5**, §12.1]. In the model that is relevant here (see, for example, [**10**, §3.6,7]) the point P is an ordered pair of elements (p_1,p_2) from some field $\mathbf{R}$, and a line is a point set

$$\{(x,y)|ax+by+c=0, \quad a,b,c\in\mathbf{R}, \quad a,b \text{ not both zero}\}.$$

In order to discuss a model that satisfies 3 and 4, let us restrict our attention to those fields in which x^2+1 has no zeros. This is a convenient restriction, since all resulting concepts and notations will have a familiar appearance; thus no special prerequisites are required of the reader beyond knowledge of the complex numbers. The natural extension of the theory to fields that are even more general will be discussed in Remark 2.9.

2.1. NOTATION. $\mathbf{C}=\mathbf{R}(i)$ will denote the extension of $\mathbf{R}$ by one of the roots i of $x^2+1=0$.

Recall that every element $p\in\mathbf{C}$ can be written uniquely as $p=p_1+p_2i$, for some $p_1,p_2\in\mathbf{R}$. Thus, as in the usual Argand diagram [**5**, §9.3], the point $P=(p_1,p_2)$ corresponds to the "complex number" p. I shall consistently use capital and small letters in this way, except when I require the variable point $Z=(x,y)$ and its corresponding $z=x+iy\in\mathbf{C}$.

2.2. DEFINITIONS. $\bar{z}\equiv x-iy$
$$d^2\equiv z\bar{z}\equiv|z|^2 \qquad (=x^2+y^2\in\mathbf{R})$$

An affine plane equipped with this distance function is called a *metric affine plane*. Note that the triangle inequality is not necessarily valid in this general setting, so that the plane need not be a metric space in the sense common to analysis and topology. Nevertheless, d^2 does enjoy many of the properties of the square of the usual Euclidean distance—namely, those that are not imposed by the order properties of the field. In [**32**, II.3] and [**19**] such properties are stated precisely. In particular, Postulate 3 is satisfied: The circle with center $A=(a_1,a_2)$ and radius squared $r\in\mathbf{R}$ is $(x-a_1)^2+(y-a_2)^2=r$, or, more briefly, $|z-a|^2=r$.

The concept of angle is our final requirement.

2.3. DEFINITION. A *directed angle*, $\angle PQR$, is an ordered triple of points.

For angle equality, let's follow Euclid and use a notion of "superposition" (see Heath's comments following Common Notion 4, p. 224 ff., and Proposition I.4, p. 249, in **[14]**). In particular,

2.4. DEFINITION. *An angle-preserving transformation* is an element of the group $\mathbf{S}^+$ of *direct similarities*,

$$\sigma: z \to az+b, \qquad a,b \in \mathrm{C}, \qquad a \neq 0.$$

I shall justify this definition in §3. What is important here is that, as in the usual Euclidean plane, there is one and only one direct similarity that takes the point pair P,Q to P',Q', namely,

$$\sigma(PQ;P'Q'): z \to \frac{p'-q'}{p-q}z + \frac{pq'-p'q}{p-q}.$$

We can then map $\angle PQR$ to a standard $\angle EOR^{\sigma}$ where E and O correspond to 1 and 0 respectively, and $\sigma = \sigma(PQ;EO)$. Then we naturally want $\angle PQR = \angle P'Q'R'$ only when $(R')^{\sigma'} \in OR$ (where $\sigma' = \sigma(P'Q';\ EO)$).

But this is not a complete definition: as in Figure 2A, let E' correspond to -1, and $R \neq E, E'$ be a point on the unit circle with center O. Then for the side-angle-side theorem to be valid, $\Delta EOR \simeq \Delta E'OR$ only if $\angle EOR = \angle E'OR$ are right angles (i.e., when $r = \pm i$). This requires a "positive half-line" in order to distinguish $\angle EOR$ from its supplement $\angle ROE'$.

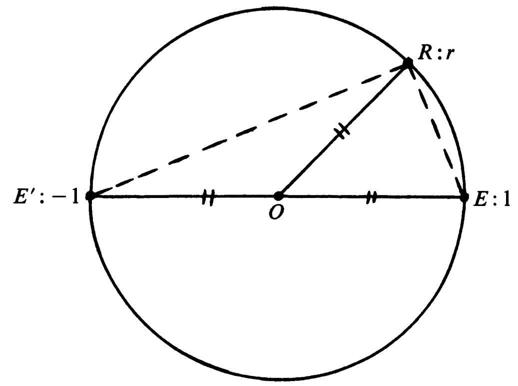

FIG. 2A. Supplementary angles $\angle EOR$ and $\angle ROE'$.

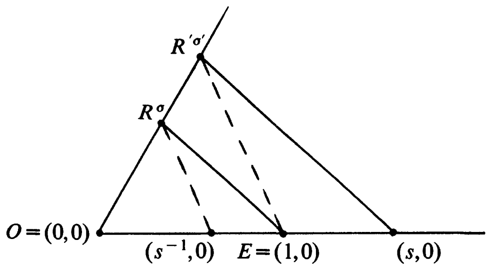

FIG. 2B. Definition 2.6 of angle equality.

2.5. DEFINITION. The *positive elements* $\mathbf{P}$ form a subset of $\mathbf{R}$ that satisfies

(a) For $r \neq 0$, $r \in \mathbf{P}$ if and only if $-r \notin \mathbf{P}$.

(b) $\mathbf{P}$ is a group under the multiplication of $\mathbf{R}$; in particular, $1 \in \mathbf{P}$, $0 \notin \mathbf{P}$.

REMARKS: The axioms for an *ordered field* require further that $\mathbf{P}$ be closed under addition. This certainly is not wanted here—it would imply that the field be infinite (of characteristic zero). Another familiar property that may be missing is that the sum of squares be itself a square. The consequences of these further conditions are discussed in **[30]**.

2.6. DEFINITIONS. *Two directed angles*, $\angle PQR$ and $\angle P'Q'R'$, *are equal* if and only if $(R')^{\sigma'} \in OR^{\sigma}$ (where $\sigma = \sigma(PQ;EO)$ and $\sigma' = \sigma(P'Q';EO)$) and the line through $(R')^{\sigma'}$ parallel to ER^{σ} meets the x-axis in a point of $\mathbf{P}$. When $(R')^{\sigma'} \in OR^{\sigma}$ and the line through $(R')^{\sigma'}$ parallel to ER^{σ} meets the x-axis in a point of $\mathbf{R} \setminus \mathbf{P}$, then $\angle PQR$ and $\angle R'Q'P'$ are *supplementary*. An angle is a *right angle* if and only if it equals its supplement. See **[5a]** for a brief discussion of directed angles and for further references.

Note that the conditions on **P** ensure that equality is an equivalence relation. For example, it is reflexive since $1 \in \mathbf{P}$, and it is symmetric (as shown in Figure 2B) since $s^{-1} \in \mathbf{P}$ if and only if $s \in \mathbf{P}$. Other geometric consequences of this modified ordering of the coordinatizing field have been studied by Sperner **[29]** (see also **[23**, Chap. 9] for further references), Kustaanheimo **[32]**, Segre **[27]**, and Ilkka **[17]**.

So far we have discussed only directed angles. Much of Euclid's geometry requires a notion of *undirected angles*, for which no distinction is made between $\angle PQR$ and its mirror image $\angle RQP$; in particular,

2.7. Definition. *Two undirected angles*, $\angle PQR$ and $\angle P'Q'R'$, *are equal* if and only if either they or the pair $\angle RQP$ and $\angle P'Q'R'$ are equal in the sense of Definition 2.6.

Since it is always clear from the context whether or not an angle is directed, I will not bother to distinguish between them by my notation. For example, Postulate 4 clearly involves an undirected right angle and is therefore satisfied using 2.7; similarly, Euclid never distinguishes between *directly* and *oppositely* congruent figures. On the other hand, we need directed angles in order to define addition in the usual way by juxtaposition:

2.8. Definition. $\angle PQR + \angle P'Q'R' = PQ(R')^{\sigma}$ where σ is the direct similarity $\sigma(P'Q'; RQ)$ that takes P' to R and Q' to Q.

One checks easily that the conditions on **P** ensure that equals added to equals are equal. Furthermore, although we shall have no need for it, we may now introduce an angle measure as a function from ordered triples of points to the factor group $\mathbf{C}^*/\mathbf{P}$. Compare this with the usual measure of $\theta = \angle PQR$ in which all positive multiples of $(r-q)/(p-q)$ are identified with $e^{i\theta}$ **[22**, §47.9]. As usual, addition of angles corresponds to multiplication in $\mathbf{C}^*/\mathbf{P}$.

2.9. Remark. Most of this theory is valid over any field **R** which has a separable quadratic extension. The details are, for the most part, straightforward (see, for example, **[19]** or **[25]**). Although the propositions are the same, the interpretation becomes a bit bizarre. For example, if the characteristic of **R** be 2, then, among other things, perpendicular lines never intersect, and no angle can have a supplement **[1]**, **[6]**. Ilkka in **[16]** gives a lucid description of the anomalies to be expected in finite planes when -1 is a square.

3. Examples and some more concepts. We can define the distance-preserving transformations just as it is done for the Euclidean plane **[22**, §43.1], but we must be careful with the similarities, which cannot in general be described in the usual way as products of isometries and dilatations.

3.1. Definitions. The group generated by the *isometries* $z \to az + b$ or $z \to a\bar{z} + b, |a| = 1$, and the *dilatations* $z \to rz + b$, $r \in \mathbf{R}$, is the *Euclidean group*, denoted by **E**. Two sets of points are called *comparable* **[13]** if they are in the same orbit of **E**.

These definitions are motivated by

Example 1: Let $\mathbf{R} = \mathbf{Q}$, the rational numbers. In the rational plane there is no isometry that takes $y = 0$ to $y = x$. In fact, there is no transformation of **E** that takes either line to the other: to see this note that $y = 0$ is a diameter of $x^2 + y^2 = 1$, so that its images under **E** would have to be diameters of $(x - b_1)^2 + (y - b_2)^2 = r^2, r \in \mathbf{Q}$; on the other hand, circles for which $y = x$ is the diameter must be of the form $(x - b_1)^2 + (y - b_1)^2 = 2r^2$ (see Fig. 3A). One similarity that does take $y = 0$ to $y = x$ is $z \to (1+i)z$. This is a dilative rotation of the plane over the reals (the product of the dilatation $z \to \sqrt{2}\, z$ and the rotation $z \to (1+i)z/\sqrt{2}$), but it cannot be so described over **Q**. It is for this reason that, in the definition of angle equality, the group of

"motions" for our version of geometry must include all the direct similarities, not just the isometries.

With this example in mind, it is easy to verify in the general case that

3.2. (a) *Two lines are comparable if and only if there is a direct isometry* $z \to az + b, |a| = 1$, *that takes one to the other.*

(b) *Parallel lines are always comparable.*

(c) *Intersecting lines are comparable if and only if a circle centered at the point of intersection intersects one whenever it intersects the other.*

(d) *Two circles* $|z-a|^2 = r$ *and* $|z-a'|^2 = r'$ *are comparable if and only if* $r = k^2 r'$ *for some* $k \in \mathbf{R}$.

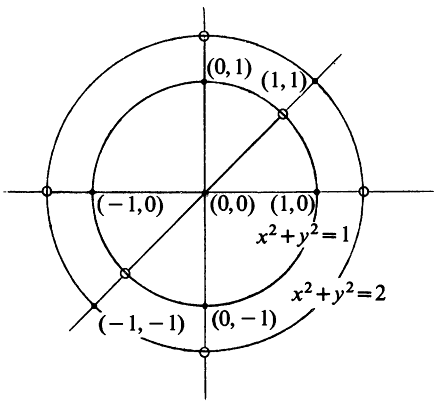

FIG. 3A. $x^2+y^2=1$ and $x^2+y^2=2$ in the rational plane.

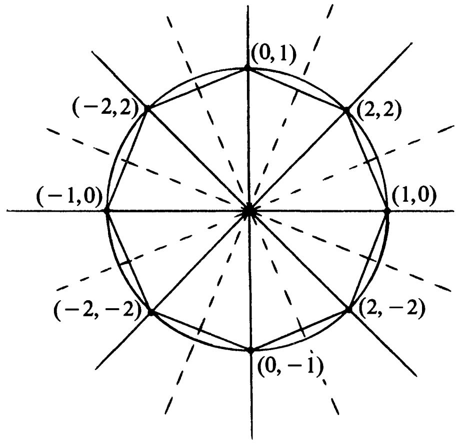

FIG. 3B. $x^2+y^2=1$ in the plane over GF(7).

Example 2: Let $\mathbf{R} = \mathrm{GF}(7)$. In the plane over GF(7) the unit circle $|z| = 1$ has 8 points on it, as shown in Figure 3B. Of the 8 lines through the origin, 4 meet the unit circle in a pair of points and 4 miss it. $\mathbf{P}$ is the set $\{1,2,4\}$ of nonzero squares of $\mathbf{R}$.

The stabilizer of the circle in $\mathbf{S}^+$ is C_8, the cyclic group of order 8. This is the group $z \to az$ where $|a| = 1$ (here a is one of the 8 solutions to $z^8 = 1$—recall that over $\mathbf{C} = \mathrm{GF}(q^2), \bar{z} = z^q$ so that $z\bar{z} = |z|^2 = z^{q+1}$ [4, §7]). In the Euclidean group the stabilizer of the unit circle is D_8, the complete symmetry group of the regular 8-gon. The unit circle can thus be regarded as a regular octagon, the 8 lines through O corresponding to the 8 lines of symmetry of the octagon. The 16 possible angles of the plane correspond in a natural way to the appropriate angles formed by diagonals or sides of the regular octagon.

The lines of the plane fall into 2 orbits under $\mathbf{E}$: The diameters of the unit circle (corresponding to diagonals of the regular 8-gon) are pairwise comparable, but are not comparable with the other 4 lines through O (which correspond to the joins of the midpoints of the opposite sides). This follows immediately from the familiar properties of D_8.

The above discussion actually applies to planes over any finite field $\mathrm{GF}(q), q$ a prime power. In particular, the following facts are easily verified [4, Theorem 7.5]:

3.3. (a) *The stabilizer of the unit circle in the Euclidean group* $\mathbf{E}$ *is isomorphic to* D_{q+1}, *the symmetries of the regular* $(q+1)$*-gon.*

(b) *If q is odd there are 2 orbits of lines under* $\mathbf{E}$. *One may be described as all lines parallel to the diameters of the unit circle. If q is even, all lines are in the same orbit.*

(c) *If q is odd there are 2 orbits of circles under* $\mathbf{E}$. *One may be described by* $\{|z-a|^2 = r \mid r$ *is a nonzero square in* $\mathrm{GF}(q)\}$. *If q is even, all circles are in the same orbit.*

Example 3: Let $\mathbf{R}$ be the smallest extension of the rationals whose nonnegative elements have a square root in $\mathbf{R}$. The plane over $\mathbf{R}$ is the natural setting for Euclid's ruler and compass constructions [**15**, §37]: more precisely, a figure (composed of line segments and circular arcs) can be constructed by ruler and compass in the real Euclidean plane if and only if its vertices and sides exist in the subplane coordinatized by $\mathbf{R}$. For example, since $\sqrt[3]{2}$, π, and cos 20° are not in $\mathbf{R}$, the cube cannot be duplicated, the circle cannot be squared, and a 60° angle cannot be trisected (see [**10**, §3.16] for a proof that avoids the sophisticated techniques of modern algebra).

In contrast with Examples 1 and 2, all lines in the plane over $\mathbf{R}$ are comparable, as are all circles. In fact, in this plane all of Euclid's theorems (but, of course, not all *Euclidean theorems*) are valid without modifications.

4. Euclid I.1. Critics of Euclid point out that he skipped over some details. Consider, for example, his Proposition I.4 (side-angle-side): If 2 triangles have 2 sides equal to 2 sides, respectively, and have the angles contained by the equal sides equal, they will also have their third sides equal, and their remaining angles equal respectively. It is proved without the use of any postulate or previous proposition. This is no problem for us, since I defined angle equality in such a way that Euclid's argument will work.

Perhaps more fundamental is the criticism of Euclid's treatment of his I.1, where he constructs an equilateral triangle on AB as base: draw circles having centers A and B, and equal radii $|AB|$. But how do we know that these circles will intersect? To answer this question it is common to increase the list of postulates (as did Hilbert [**15**]) to include notions of order and continuity. I prefer to amend the statement of any proposition that makes such claims of existence to "If a required point exists, here's how to construct it." As in Example 3 of the previous section, proofs of existence often depend on the properties of $\mathbf{R}$; it turns out, for example, (by the Pythagorean theorem, Euclid's I.47) that equilateral triangles exist if and only if $\sqrt{3} \in \mathbf{R}$. Although the entire *Elements* is based on I.1, usually all that is required is the existence of an isosceles triangle on AB as base. Now this is easily proved: Let $A'B'$ be any 2 points on $|z|=1$ but not on the same diameter; then $\sigma(A'B';AB)$ takes the origin to the apex of an isosceles triangle on AB as base.

Similarly, I.9 can be modified to "to bisect a given angle *whose sides are comparable lines*" (comparable in the sense of 3.2 (a). Heath in [**14**] points out when Euclid has omitted a detail. It is then a routine matter to complete the proof.

As examples of the power of the methods of Euclid, consider III.21, III.22, and their converses, which can be combined into

4.1. *Two triangles on the same base have equal or supplementary vertical angles if and only if the circle passing through the extremities of the base and the vertex of one triangle will pass through the vertex of the other triangle also* (see Fig. 4).

The proof given by Euclid depends on most of his first 32 propositions; it remains valid in our more general setting after the minor modifications suggested above.

4.2. Philosophical Aside. Perhaps I may be permitted to repeat a comment that has been made many times before [**9**], [**14**, p. vii]: Every mathematician can benefit from studying Euclid's *Elements* as Euclid presented it; his work is timeless in the same way as the work of Shakespeare or the Bible. Euclid knew what is fundamental and significant, and how it should be presented for future generations to reinterpret as they see fit.

5. The past 2277 years. On the other hand, there have been many lovely theorems and useful techniques discovered since Euclid. To see how Principle 1.1 can be used, let's look at the inversion, a transformation invented by Jakob Steiner about 1824 (according to J. J. Burckhardt, the translator of [**5**] into German). For its definition we adjoin a new point P_∞ to the plane, and the corresponding element ∞ to $\mathbf{C}$.

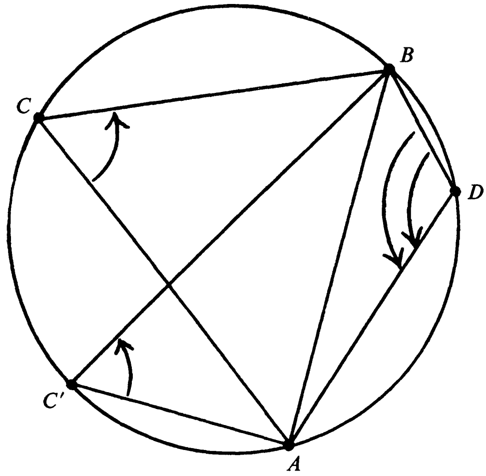

FIG. 4. $\angle ACB + \angle BDA = \angle AC'B + \angle BDA =$ two right angles.

5.1. DEFINITION. *Inversion in the circle* $|z-a|^2 = r$ is the transformation of the extended plane for which the image z' of the point z is given by

$$z' = \begin{cases} a + r/(\bar{z} - \bar{a}) & z \neq a, \infty \\ \infty & z = a \\ a & z = \infty \end{cases}$$

5.2. NOTATION. Denote by **J** the group generated by inversions. As in the case of the real inversive plane, an element of **J** is either an element of **E** or the product of an inversion and an isometry [**5**, 6.71]. An algebraic description of **J** may be found in [**8**, §6.4].

It is useful to regard a line as a kind of circle, namely, a circle that contains the special point P_∞. Thus, without exception, 3 points determine a circle of the extended plane. Furthermore, standard arguments suffice to show

5.3. (a) *The image of a circle of the extended plane under an inversion is a circle* (synthetic and analytic proofs both can be found in [**5**, §6.3 and §9.7] and [**22**, Ex. 21.3 and §21.1]).

(b) *Inversion preserves the* (*undirected*) *angle between intersecting circles* [**5**, §6.4], [**22**, §22.2].

Among other things, the group **J** can be used to simplify problems involving lines, circles, and angles by sending a suitable point to P_∞. As a simple example, let's prove

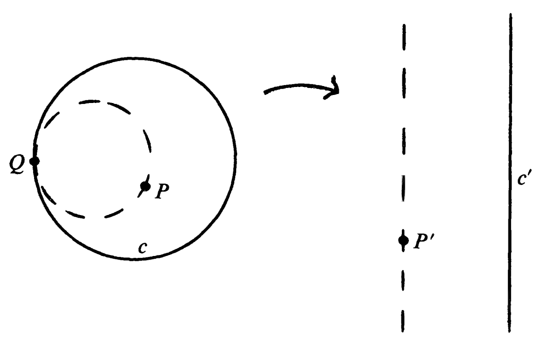

FIG. 5A. The proof of 5.4.

5.4. *Given a circle c with the points* $Q \in c$ *and* $P \notin c$, *then there is one and only one circle through P tangent to c at Q* (Fig. 5A).

Proof. Inversion in any circle whose center is Q transforms Q to P_∞, c to a line c', and P to some point $P' \notin c'$. The problem is transformed to: "There is one and only one line through P' parallel to c'," which is Postulate 5. ■

Less simple examples of how conformal maps simplify results such as the Riemann mapping theorem of complex analysis can be found in [**33**, §3.2 and 17.2]. Other examples from geometry are in [**5**, Chap. 6], [**10**, Chap. 6], and [**22**]. Let's content ourselves here with a proof of a theorem that is fundamental to the study of the modern theory of inversive (or Möbius) geometry [**8**, 6.1.5], [**3**, III.2].

5.5. THEOREM (Miquel). *Let* $c_i (i = 1,2,3,4)$ *be 4 circles, no 3 with a common point, but* $c_i \cap c_{i+1} = \{A_i, B_i\}$ *(where subscripts are taken mod 4). Then the points* A_i *are on the same circle if and only if the points* B_i *are on the same circle (Fig. 5B).*

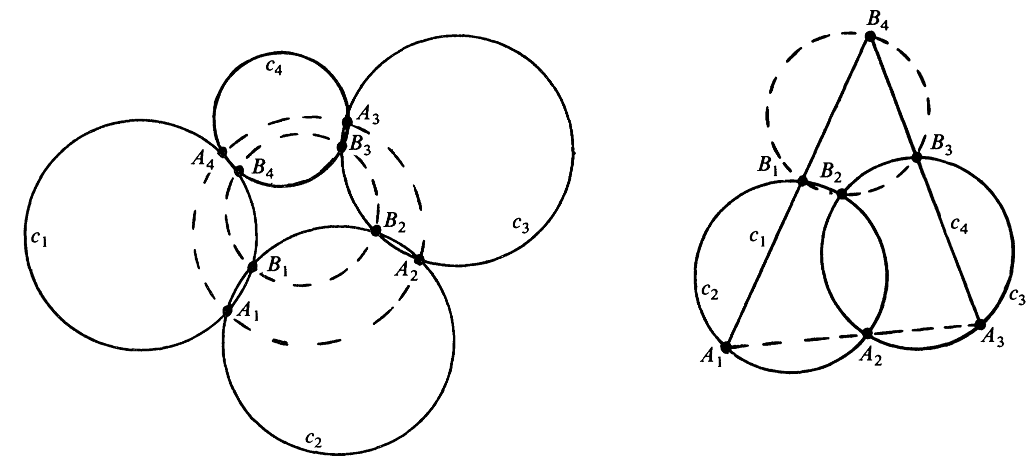

FIG. 5B. Two illustrations of Theorem 5.5 (Miquel's Theorem).

Proof. Miquel's theorem is equivalent by inversion in a circle with a center A_4 to the pivot theorem [**11**, p. 17]: Given $\Delta A_1A_3B_4$ with points $A_2 \in A_1A_3, B_3 \in A_3B_4, B_1 \in A_1B_4$, then if B_2 is the second point of intersection of the circles $c_2 = A_1A_2B_1$ and $c_3 = A_2A_3B_3$, the circle $B_1B_3B_4$ passes through B_2 (the right half of Fig. 5B). It suffices (by 4.1) to show that $\angle B_4B_3B_2 \approx \angle B_2B_1B_4$, where I temporarily adopt the symbol $\approx$ to mean *equal or supplementary*. Then the hypotheses imply via Definition 2.6 and Proposition 4.1 that

$$\angle B_4B_3B_2 \approx \angle B_2B_3A_3 \approx \angle A_3A_2B_2 \approx \angle B_2A_2A_1 \approx \angle A_1B_1B_2 \approx \angle B_2B_1B_4. \qquad ■$$

5.6. REMARK. This same proof remains valid when **R** is any field having a separable quadratic extension [**25**, p. 210]. In particular, it works over all finite fields.

6. Applications. Here is a brief summary of various more rigorous treatments of the study of Euclidean properties in a more general setting, together with an indication of how they have been applied.

6.1. THE FOUNDATIONS OF GEOMETRY. Hilbert's influential *Grundlagen der Geometrie* of 1899 stimulated the study of axiom systems for geometry that conform to a modern standard of rigor. In the supplements to the tenth edition [**15**], there is a survey of some of the vast literature

concerned with how these axioms are interrelated and what happens when some are modified or omitted. For Euclid's geometry as defined in §2, one needs to modify only slightly Hilbert's axioms of order (cf. [**23**, Chap. 9] or [**32**, II.2]) and congruence (cf. [**32**, II.3]), and to omit his axioms of continuity. Much of this program has been carried out explicitly for finite planes by Ilkka [**16**]. Also relevant here is a fundamental work of Tarski [**31**], even though his emphasis is on logic rather than geometry. This work has stimulated new research exploiting the idea that without continuity Euclidean geometry becomes a fruitful first order theory. Compare Principle 1.1 with Tarski's metatheorem that his theory of elementary geometry is complete; roughly, what he showed is that the geometry over any real closed field would serve as a model for his axioms, a sentence being valid (and therefore holding in all models) if and only if it holds in the geometry over the reals. As a special bonus, his theory is also decidable. There is an extensive bibliography in [**30**].

6.2. THE CIRCLE GEOMETRIES OF BENZ. In [**3**] Benz develops a number of powerful algebraic tools to study Möbius geometry in a very general setting, as well as the geometries of Laguerre and of Minkowski. Evidence of the applicability of these methods can be found in the work of Schröder [**24**], where there is a detailed study of angle measure.

6.3. METRIC AFFINE PLANES. There have been two important characterizations of the metric properties of Euclid's geometry. Their conclusion: Any affine plane that has certain desirable metric properties is coordinatizable by a field that admits a separable quadratic extension and has a metric induced by an irreducible quadratic form. Lingenberg [**21**, §6.5] characterizes "Euclidean" planes (and various other metric planes) in terms of the reflection geometry of Bachmann [**2**]. The characterization begun by Kustaanheimo (see [**32**]) and extended by Karzel and Stanik [**18**], [**19**], makes use of an axiom system reminiscent of Hilbert's; the authors employ isometries as a major tool, but also make use of arguments directly from Euclid. An elementary discussion of metric geometry over $GF(2^n)$ (independent of the above works) is given in [**1**].

6.4. FINITE APPROXIMATIONS TO THE REAL WORLD. A school of applied mathematicians in Finland argues that finite geometry might be the basis for explaining the discreteness in the phenomena of modern physics [**20**]. Their approach has led to significant results in pure mathematics (sample theorem (Segre-Kustaanheimo): *Every oval of* $PG(2,q)$, *q odd, is a conic* [**26**, §175]). In addition, they have discussed intriguing and useful models for such phenomena as the gravitational interaction of discrete particles. There is a bibliography in [**32**].

6.5. TRIGONOMETRY OVER FINITE FIELDS. D. W. Crowe in [**6**] showed that the trigonometry over $GF(2^n)$ has much in common with the trigonometry of the usual Euclidean plane. He then applied his results to the study of finite hyperbolic geometry. A study of the trigonometry over $GF(q)$ can be found in [**24**], [**12**], and [**16**].

Finally, let's look at how Principle 1.1 has been used to discover new theorems. Although not particularly significant, Theorem 6.7 will serve nicely as a simple prototype.

The recently developed theory of 2-graphs is a far-reaching theory that brings together concepts common to group theory, geometry, and combinatorics [**28**]. Some technical terminology is required in the statement of Theorem 6.7, but nothing is used in its proof beyond the theory that I have already developed.

6.6. DEFINITIONS. Let Ω be a finite set. An *n-subset* is a subset of Ω containing n elements. A 2-*graph* is an ordered pair of sets (Ω,Δ) where Δ is a set of 3-subsets (or triples) of Ω with the property that every 4-subset of Ω contains an even number of triples of Δ.

We shall use Euclid's geometry to obtain a description of an important family of 2-graphs. Toward this end we require some further notation:

Let Ω be the q^2+1 points of the inversive plane (of §5) over the finite field $\mathbf{R}=GF(q)$, where

q is an *odd* prime power. The group $\mathbf{J}$ generated by inversions partitions the circles of the plane into 2 orbits $\mathbf{O}_1, \mathbf{O}_2$ of $(q^3+q)/2$ circles [**8**, 6.4.3]. In fact these are the same orbits described in 3.3 (this follows from 3.2 (b), 3.3 (b), (c), and 5.2). Hence, 2 circles of the inversive plane over $\mathbf{R}=\mathrm{GF}(q)$ are in the same orbit if and only if they have common tangent circles. Then

6.7. THEOREM. *If Ω is the set of points of an inversive plane over* $\mathrm{GF}(q), q$ *odd, and a triple is in Δ if and only if it determines a circle of* $\mathbf{O}_1$, *then (Ω, Δ) is a 2-graph.*

6.8. ASIDE. This (Ω, Δ) turns out to be isomorphic to the Paley 2-graph [**34**]. It was described in quite a different form by Delsarte [**7**]. See [**28**, §13] for further details.

Proof of Theorem 6.7. Let $A, B, C, D \in \Omega$. We must show that of the four 3-subsets of these points, either 0, 2, or 4 are in Δ (i.e., determine a circle of $\mathbf{O}_1$). Assume that none of A, B, C are at infinity, and let d be the circle determined by these 3 points. Then, if $D \in d$, all four 3-subsets determine the same circle, and 4 triples or none are in Δ according as $d \in \mathbf{O}_1$ or not. So suppose $D \notin d$. Since an inversion in a circle with center D preserves $\mathbf{O}_1$ (thus preserving the triples of Δ), we can assume without loss of generality that D is at P_∞; hence 3 of the 3-subsets determine the sides a, b, c of ΔABC, while the fourth determines the circumcircle d. Let d' be the tangent to d at A. Since d' is in the same orbit as d (by the sentence before the statement of the theorem), it suffices to show that 0, 2, or 4 of a, b, c, d' are in $\mathbf{O}_1$. But the angle determined by d' and c equals or is supplementary to $\angle ACB$ (Euclid III.32; see Fig. 6). This implies that there is a similarity that takes d' to b, and c to a. This similarity will either preserve the 2 orbits, or interchange them. Thus c and d' are in distinct orbits if and only if a and b are also, as desired. ∎

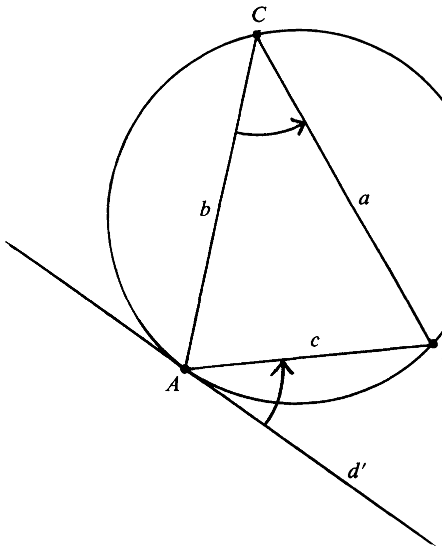

FIG. 6. Euclid III.32.

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MISCELLANEA

21. Where it all began. Professor [J. L.] Coolidge pointed out that before the time of Benjamin Peirce [1809–1880] it never occurred to anyone that mathematical research "was one of the things for which a mathematical department existed."

R. C. Archibald, this MONTHLY 32 (1925) 11.

THE AIRY TRANSFORM

D. V. WIDDER

1. Introduction. It is perhaps appropriate to give the above designation to the transform

$$u(x,t)=\int_{-\infty}^{\infty} K(x-y,t)\phi(y)\,dy, \tag{1.1}$$

where

$$K(x,t)=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{xs+ts^3}ds, \qquad c>0. \tag{1.2}$$

For this kernel $K(x,t)$ can be expressed in terms of the Airy function

$$\mathrm{Ai}(x)=\frac{1}{\pi}\int_0^{\infty}\cos\left(xy+\frac{y^3}{3}\right)dy. \tag{1.3}$$

See [**1**, p. 447]. As we shall see, the transform (1.1) bears much the same relation to the equation

$$\frac{\partial^3 u}{\partial x^3}=\frac{\partial u}{\partial t} \tag{1.4}$$

as the Poisson transform does to the heat equation. See [**3**, p. 32].

2. The kernel. We show first that the integral (1.2) converges absolutely for $c>0$ and is independent of c. If $s=\sigma+i\tau$, then for $t>0, c>0$

$$\int_{c-i\infty}^{c+i\infty} e^{xs+ts^3}ds \ll \int_{-\infty}^{\infty} e^{xc+tc^3-3tc\tau^2}d\tau<\infty.$$

This shows that the integral (1.2) converges absolutely for $-\infty<x<\infty, 0<t, 0<c$. Now apply Cauchy's theorem to the integrand (1.2), an entire function of s, integrating over the rectangle with vertices $s=\pm iR, c\pm iR$. As $|R|$ becomes infinite, the integrals over the horizontal sides approach zero:

$$\int_{\pm iR}^{c\pm iR}\exp[xs+ts^3]\,ds \ll \int_0^c \exp[x\sigma+t\sigma^3-3t\sigma R^2]\,d\sigma$$

$$\leqslant \exp\{|x|c+tc^3\}\int_0^{\infty} e^{-3t\sigma R^2}d\sigma=\frac{\exp[|x|c+tc^3]}{3tR^2}=o(1), |R|\to\infty.$$

Consequently the integral (1.2) is independent of c and indeed it converges (not absolutely) when $c=0$. Thus

$$K(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[ix\tau-it\tau^3]\,d\tau=\frac{1}{\pi}\int_0^{\infty}\cos(x\tau-t\tau^3)\,d\tau. \tag{2.1}$$

Comparing integrals (1.3) and (2.1), we see that

$$K(x,t)=\frac{1}{(3t)^{1/3}}\mathrm{Ai}\left(\frac{-x}{(3t)^{1/3}}\right), \tag{2.2}$$

thus justifying the title of the paper.

The author received his Ph.D. from Harvard under the direction of G. D. Birkhoff in 1924. He taught at Bryn Mawr, and from 1931 until he retired, at Harvard. His principal interests are integral transforms and the heat equation; his books include *The Laplace Transform*, 1941; *Advanced Calculus*, 1947 and 1961; *The Convolution Transform* (with I. I. Hirschman), 1955; *An Introduction to Transform Theory*, 1971; *The Heat Equation*, 1975.

The title of this article reminds us of Sir George Airy, Astronomer Royal, a noted mathematical physicist, who is not memorialized in William Allingham's lines "Up the airy mountain, down the rushy glen," but who is unfortunately most often remembered as having neglected in 1846 to search for the planet Neptune where J. C. Adams had predicted that it ought to be. "Herostratus lives that burnt the Temple of Diana; he is almost lost that built it." (Sir Thomas Browne, 1658)—*Editors*

From (1.2), we have immediately

$$|K(x,t)| \leqslant \frac{1}{2\pi}\exp[xc+tc^3]\int_{-\infty}^{\infty} e^{-3tc\tau^2}\,d\tau = \exp[xc+tc^3]/2\sqrt{3\pi ct}\,. \tag{2.3}$$

We summarize these results in

THEOREM 1. *The integral* (1.2) *is independent of* c, *converges absolutely for* $-\infty<x<\infty, 0<t<\infty, 0<c<\infty$. *Moreover, for fixed* $t>0$,

$$K(x,t)=O(e^{xc}) \qquad x\to\pm\infty. \tag{2.4}$$

By use of (2.2) and by reference to [**1**, p. 448], a more precise asymptotic estimate than (2.4) is available:

$$K(x,t)\sim\frac{1}{\sqrt{\pi}\,(3tx)^{1/4}}\cos\left(\frac{2}{3}\left(\frac{x^3}{3t}\right)^{1/2}-\frac{\pi}{4}\right) \qquad x\to\infty, \tag{2.5}$$

$$\sim\frac{1}{\sqrt{\pi}\,(3t|x|)^{1/4}}\exp\left[-\frac{2}{3}\frac{|x|^{3/2}}{(3t)^{1/2}}\right] \qquad x\to-\infty. \tag{2.6}$$

For our present purposes (2.4) is adequate.

3. Laplace transform of the kernel.

THEOREM 2. *For* $0<t<\infty, 0<\sigma<\infty$

$$e^{ts^3}=\int_{-\infty}^{\infty} e^{-sx}K(x,t)\,dx, \tag{3.1}$$

the integral converging absolutely.

By (2.4)

$$\int_0^{\infty} e^{-sx}K(x,t)\,dx \ll \int_0^{\infty} e^{-\sigma x}O(e^{cx})\,dx<\infty \qquad \sigma>c,$$

$$\int_{-\infty}^{0} e^{-sx}K(x,t)\,dx \ll \int_{-\infty}^{0} e^{-\sigma x}O(e^{cx})\,dx<\infty \qquad \sigma<c.$$

Since c is arbitrary (>0), the absolute convergence of (3.1) in the designated region is established. The result is also clear from (2.5) and (2.6). Now invert (1.2), considered as a Fourier transform:

$$e^{-xc}K(x,t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ixy}e^{t(c+iy)^3}\,dy,$$

$$e^{t(c+iy)^3}=\int_{-\infty}^{\infty} e^{-ixy}e^{-xc}K(x,t)\,dx. \tag{3.2}$$

See, for example [**2**, p. 42]. The use of principal values, usually required for the inversion, is not here needed since the integral (3.2) is known to converge. Since (3.2) is clearly equivalent to (3.1), the theorem is proved.

4. The Huygens or semigroup property. Like the source solution of the heat equation, $K(x,t)$ has the Huygens property. Compare [**3**, p. 155].

THEOREM 3. *If* $0<t_1\leqslant t_2, -\infty<x<\infty$, *then*

$$K(x,t_1+t_2)=\int_{-\infty}^{\infty} K(x-y,t_1)K(y,t_2)\,dy=K(x,t_1)*K(x,t_2). \tag{4.1}$$

This is an immediate consequence of the product theorem for bilateral Laplace transforms,

[4, p. 258]. It yields

$$e^{t_1s^3}\cdot e^{t_2s^3}=\int_{-\infty}^{\infty}e^{-sy}K(y,t_1)*K(y,t_2)\,dy \qquad \sigma>0.$$

But from (3.1)

$$e^{(t_1+t_2)s^3}=\int_{-\infty}^{\infty}e^{-sy}K(y,t_1+t_2)\,dy.$$

Now by the uniqueness theorem for Laplace transforms, [4, p. 258], equation (4.1) follows for almost all values of x, at least. But by use of (2.4), or otherwise, it may be seen that the integrals (4.1) and (1.2) converge uniformly for $|x|\leqslant R$, every $R>0$. Hence both sides of (4.1) are continuous functions of x, so that equality holds for all x.

5. The equation $u_{xxx}=u_t$. The kernel $K(x,t)$ satisfies this equation. From (1.2)

$$K_{xxx}=K_t=\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\exp[xs+ts^3]s^3\,ds \tag{5.1}$$

if differentiation under the integral sign is valid. When $0<\delta\leqslant t\leqslant R$, $|x|\leqslant R$, the integral (5.1) is dominated by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp[cR+c^3R-3\delta c\tau^2](c^2+\tau^2)^{3/2}\,d\tau<\infty.$$

Hence the integral (5.1) converges uniformly for $\delta\leqslant t\leqslant R, |x|\leqslant R$, so that equation (5.1) holds for all x and all positive t. It is thus reasonable to expect that the Airy transform (1.1) will represent solutions of equation (1.4) when it converges.

6. The Cauchy problem. This is the problem of finding a solution of (1.4) which reduces to a given function when $t=0$. From analogy with the theory of heat conduction, [3, p. 60], we would expect the transform (1.1) to provide a solution. We show that this is indeed the case for a restricted class of initial functions $\phi(x)$.

THEOREM 4. *If*

(1) $\phi(x)$ *is continuous and of bounded variation,* $-\infty<x<\infty$.

(2) $f(s)=\int_{-\infty}^{\infty}e^{-sx}\phi(x)\,dx$ *converges absolutely,* $\sigma=c_1$ (6.1)

(3) $\int_{-\infty}^{\infty}|f(c_1+i\tau)|\,d\tau<\infty$

(4) $u(x,t)=\int_{-\infty}^{\infty}K(x-y,t)\phi(y)\,dy,$ (6.2)

then

$$\begin{aligned}&\text{A.}\quad u_{xxx}=u_t, && -\infty<x<\infty, 0<t<\infty,\\ &\text{B.}\quad u(x,0+)=\phi(x), && -\infty<x<\infty.\end{aligned}$$

Observe first that $u(x,t)$ is continuous. For, by (2.3) we have for $0<\delta\leqslant t\leqslant R$, $|x|\leqslant R, 0<c<\infty$,

$$\int_{-\infty}^{\infty}K(x-y,t)\phi(y)\,dy\ll\frac{e^{Rc+Rc^3}}{2\sqrt{3\pi c\delta}}\int_{-\infty}^{\infty}e^{-cy}|\phi(y)|\,dy.$$

The dominant integral converges by hypothesis (2) if we choose $c=c_1$. That is, integral (6.2) converges uniformly for $\delta\leqslant t\leqslant R, |x|\leqslant R$, and $u(x,t)$ is continuous for $0<t<\infty$, $-\infty<x<\infty$.

Again using the product theorem for the two generating functions (3.1) and (6.1), we obtain

$$f(s)e^{ts^3}=\int_{-\infty}^{\infty}e^{-sx}u(x,t)\,dx, \tag{6.3}$$

the integral converging absolutely for $\sigma=c_1$. Now invert (6.3), [4, p. 241],

$$u(x,t)=\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} e^{xs+ts^3}f(s)\,ds. \tag{6.4}$$

Since this integral clearly converges absolutely there is no need for principal values. Now conclusion A follows at once if differentiation under the integral sign (6.4) is valid. It is so since for $0<\delta\leqslant t\leqslant R, |x|\leqslant R$;

$$\int_{c_1-i\infty}^{c_1+i\infty} e^{xs+ts^3}s^3f(s)\,ds \ll Me^{c_1R+c_1^3R}\int_{-\infty}^{\infty} e^{-3c_1\delta\tau^2}(c_1^2+\tau^2)^{3/2}\,d\tau<\infty, \tag{6.5}$$

$$M=\int_{-\infty}^{\infty} e^{-c_1x}|\phi(x)|\,dx.$$

That is, the first integral (6.5) converges uniformly for x,t in the designated intervals. The integral (6.4) for fixed x converges uniformly for $0\leqslant t\leqslant 1$ since

$$\int_{c_1-i\infty}^{c_1+i\infty} e^{xs+ts^3}f(s)\,ds \ll e^{xc_1+c_1^3}\int_{-\infty}^{\infty}|f(c_1+i\tau)|\,d\tau<\infty. \tag{6.6}$$

Hence

$$u(x,0+)=\frac{1}{2\pi i}\int_{c_1-i\infty}^{c_1+i\infty} e^{xs}f(s)\,ds.$$

This integral converges (by the continuity of $\phi(x)$) and equals $\phi(x)$ by the classical inversion theory, [4, pp. 68, 241]. This concludes the proof of B.

The hypotheses of Theorem 6 are very restrictive and will be broadened in the following section. The method of inverting (6.2) here used is a standard one when the Laplace transform of the kernel is known. The applicability of the Airy transform for solving equation (1.4) is necessarily more restricted than the Poisson transform for solving the heat equation. From (2.5) it is clear that the Airy transform of $\phi(x)$ diverges even when $\phi(x)$ is a polynomial, and of course the present hypothesis (2) fails then. An example in which all hypotheses are satisfied is provided by the pair $\phi(x)=e^{-x^2}$, $f(s)=\sqrt{\pi}\,e^{s^2/4}$.

7. Cauchy problem, continued. As noted above, Theorem 6 provides no solution to the Cauchy problem unless the boundary function $\phi(x)$ is very small at ∞. The integral (6.1) diverges even for so simple a function as $\phi(x)=x$. To treat such functions, we adopt a method similar to one used by Titchmarsh for divergent Fourier integrals, [2, p. 4]. We break the integral (6.1) into two unilateral Laplace transforms, one convergent in a right half-plane, the other in a left half-plane. We discuss here an interesting case in which one of these transforms extends analytically into the negative of the other.

In the following theorem, the statement that $\phi(x)$ has growth $\{1,\rho\}$ means that it is of order <1 or of order $=1$ and of type $\leqslant\rho$. Compare [3, p. 185] or [4, p. 94]. The two paths of integration Γ and Γ' are, respectively, the line $\sigma=c$ from $\tau=-\infty$ to $\tau=+\infty$, and the same path except that the segment from $c-ic$ to $c+ic$ is replaced by a detour around the square S with vertices $\pm c\pm ic$. (See Fig. 1.)

THEOREM 5. *If*

(1) $\phi(x)$ *is entire of growth* $\{1,\rho\}$

(2) $f(s)=\int_0^\infty e^{-sy}\phi(y)\,dy, \qquad \sigma>\rho,$ (7.1)

(3) $f(s)$, *extended analytically, has a finite number of poles as its only singularities*, $|s|<\infty$

(4) $\lim_{s\to\infty} sf(s)=0$

(5) $u(x,t)=\dfrac{1}{2\pi i}\int_\Gamma e^{xs+ts^3}f(s)ds-\dfrac{1}{2\pi i}\int_{\Gamma'} e^{xs+ts^3}f(s)\,ds, \qquad c>\rho,$ (7.2)

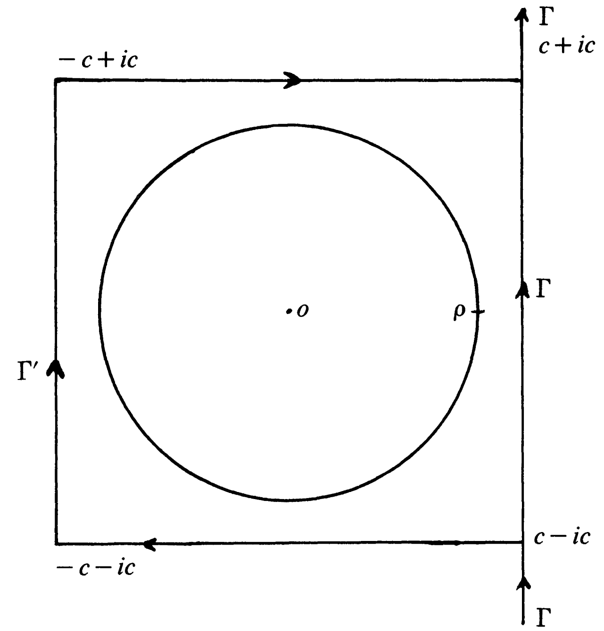

FIG. 1

then

A. $u_{xxx} = u_t$, $\quad 0 < t < \infty, -\infty < x < \infty$

B. $u(x,0+) = \phi(x)$, $\quad -\infty < x < \infty$.

If $\phi(y) = \sum_{n=0}^{\infty} b_n y^n / n!$, then by [3, p. 202], and hypothesis (4)

$$f(s) = \sum_{n=1}^{\infty} \frac{b_n}{s^{n+1}}, \qquad \sigma > \rho. \tag{7.3}$$

This power series provides an analytic continuation of $f(s)$ outside the circle $|s| = \rho$. Now

$$\int_{-\infty}^{0} e^{-sy}\phi(y)\,dy = \int_0^{\infty} e^{sy} \sum_{n=1}^{\infty} \frac{b_n(-y)^n}{n!}\,dy = \sum_{n=1}^{\infty} \frac{(-1)^n b_n}{(-s)^{n+1}}, \qquad \sigma < -\rho. \tag{7.4}$$

Consequently, the integral (7.4) is equal to $-f(s)$ for $\sigma < -\rho$ and hence also for $|s| > \rho$ when the function defined by the integral (7.4) is extended analytically. From (7.3) it is clear that $f(s) = O(|s|^{-2}), |s| \to \infty$, so that for $c > \rho$

$$f(c + i\tau) = O(|\tau|^{-2}), \qquad |\tau| \to \infty. \tag{7.5}$$

Both integrals (7.2) converge absolutely since

$$\exp\left[t(c+i\tau)^3\right] = O(e^{-3c\tau^2}), \qquad c > 0, |\tau| \to \infty.$$

Note that Γ' could not be replaced by the whole line $\sigma = -c$. Obvious estimates, similar to (6.5), show that conclusion A is here valid also. And estimates similar to (6.6) also apply here in the presence of (7.5), to show that integrals (7.2) both converge uniformly for $0 \leqslant t \leqslant 1$; so that

$$u(x,0+) = \frac{1}{2\pi i}\int_\Gamma e^{xs} f(s)\,ds - \frac{1}{2\pi i}\int_{\Gamma'} e^{xs} f(s)\,ds. \tag{7.6}$$

Since the paths Γ and Γ' coincide away from S, it is clear that

$$u(x,0+) = \frac{1}{2\pi i}\int_S e^{xs} f(s)\,ds = \Sigma\,\mathrm{Res} f(s) e^{xs} \tag{7.7}$$

(the sum of all the residues of $e^{xs}f(s)$). But the integrals over the horizontal sides of S approach zero as the sides recede to infinity:

$$\int_{-c}^{c} \exp[x(\sigma+iR)]f(\sigma+iR)\,d\sigma \ll \int_{-c}^{c} e^{|x|c}O(R^{-2})\,d\sigma = o(1), \qquad |R|\to\infty.$$

Thus (7.6) becomes

$$u(x,0+) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{xs}f(s)\,ds - \frac{1}{2\pi i}\int_{-c-i\infty}^{-c+i\infty} e^{xs}f(s)\,ds.$$

The first term is the inverse of (7.1); the second (minus sign included), of (7.4). That is

$$u(x,0+) = \begin{cases} \phi(x) & +0 & x>0 \\ \phi(0+)/2 & +\phi(0-)/2 & x=0 \\ 0 & +\phi(x) & x<0, \end{cases}$$

so that conclusion B is also established.

COROLLARY 5. *If hypothesis* (4) *is omitted* $u(x,t) = \sum \operatorname{Res} e^{xs+ts^3}f(s)$.

Without hypothesis (4) relation (7.5) no longer holds, so that integrals (7.6) may diverge. However, we still have $f(s) = O(|s|^{-1})$ and integrals (7.2) converge. Thus

$$u(x,t) = \frac{1}{2\pi i}\int_{S} e^{xs+ts^3}f(s)\,ds$$

by the same argument used to establish (7.7).

8. Examples. If $\phi(x) = x^n, n=1,2,\ldots,$ then $f(s) = n!/s^{n+1}$ and all conditions of Theorem 5 are satisfied. If $n=0$, Corollary 5 is applicable. In either case

$$u(x,t) = \operatorname{Res}_{s=0} e^{xs+ts^3}/s^{n+1}.$$

To compute this multiply together the two exponential series

$$e^{xs} = \sum_{k=0}^{\infty} \frac{x^k s^k}{k!}, \qquad e^{ts^3} = \sum_{k=0}^{\infty} \frac{t^k s^{3k}}{k!}$$

and seek the coefficient of s^n in the product series. It is

$$\sum_{k=0}^{[n/3]} \frac{t^k}{k!}\frac{x^{n-3k}}{(n-3k)!},$$

so that $u(x,t)$ is this series multiplied by $n!$. The result is also suggested by use of the symbolic operator e^{tD^3}:

$$u(x,t) = e^{tD^3}x^n = \sum_{k=0}^{\infty} \frac{t^k}{k!}\frac{d^{3k}}{dx^{3k}}x^n = n!\sum_{k=0}^{[n/3]} \frac{t^k}{k!}\frac{x^{n-3k}}{(n-3k)!}.$$

As another example take

$$\phi(x) = e^{ax}, \qquad f(s) = 1/(s-a).$$

Here $\phi(x)$ has growth $\{1,|a|\}$, but hypothesis (4) of Theorem 5 is not satisfied. By Corollary 5

$$u(x,t) = \operatorname{Res}_{s=a} e^{xs+ts^3}/(s-a) = e^{ax+a^3t}.$$

As special cases take $\phi(x) = \sin ax$ or $\cos ax$ to obtain $u(x,t) = \sin(ax-a^3t)$ or $\cos(ax-a^3t)$, respectively.

Note that the example of §6 cannot be treated by Theorem 5. Neither Theorem 5 nor Theorem 6 generalizes the other.

Observe further that equation (7.2) is formally equivalent to

$$u(x,t)=\int_0^{\infty} K(x-y,t)\phi(y)\,dy+\int_{-\infty}^{0} K(x-y,t)\phi(y)\,dy.$$

We have employed here only one of many possible methods of treating divergent transforms to broaden their applicability. A deeper study is indicated.

I am grateful to Professor Garrett Birkhoff for suggesting this problem. Without the suggestion, I may have had less pleasant things to think about during a recent hospitalization.

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SILVERMAN'S GAME ON INTERVALS

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1. Introduction. *Silverman's game* is played by two players on a given set $\mathbb{S}$ of positive real numbers. At the outset, a "threshold" T and a "penalty" ν are given, where $0<\nu<\infty$, $1<T<\infty$. Each player secretly selects an element of $\mathbb{S}$. The player with the larger number (call him A) wins the amount 1 from his opponent B unless A's number is at least T times as large as B's, in which case B wins the amount ν from A. Equal numbers draw. Thus, if $\nu>1$, there is a ν-fold penalty for choosing too large a number.

By symmetry, the value of this game is 0, i.e., neither player has a strategy which will enable him to expect positive winnings. We are interested in finding an optimal strategy, that is, a strategy which will yield a value of 0 or more against every opposing strategy. Our notion of strategy is made precise in §2.

In the related game of Mendelsohn [3, p. 212], [4], the optimal strategy is unexpectedly elegant, and this is true of Silverman's game in many cases. For example, for the game played on the set of natural numbers with $T=3$ and $\nu=2$, the strategy of selecting 1, 2, and 5 with probabilities $1/4$, $1/2$, and $1/4$, respectively, is optimal. Indeed, there is a unique optimal strategy for the game with $\nu\geqslant 1$ played on any discrete set (i.e., any subset of $(0,\infty)$ having no cluster points in $[0,\infty)$). This strategy is given explicitly in [2]. See [1] for an especially simple treatment in the case $\nu=1$.

In this paper, we consider the games played on especially nice nondiscrete sets, namely the open intervals (A,B) with $0<A<B\leqslant\infty$. We restrict our attention to the case $A=1$, since an optimal strategy for the game played on $(1,B/A)$ naturally yields a corresponding optimal strategy for the game played on (A,B). We obtain the following surprising result. There exists an optimal strategy for the game played on $(1,B)$ if and only if, for some integer $n\geqslant 2$, $\nu=\sec(\pi/(n+1))-1$ and $T^{n-1}<B\leqslant\infty$. The optimal strategy is explicitly given in §4, and the nonexistence is proved in §5. The author is indebted to David Silverman (former Problems Editor of the

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Journal of Recreational Mathematics), who first discovered an optimal strategy for the game played on $(1,\infty)$ with $\nu=1$, by looking at limiting cases of discrete games.

2. General strategies. Let $1<B\leqslant\infty$. A *strategy* for Silverman's game on $(1,B)$ is a rule whereby an element X in $(1,B)$ is randomly selected in accord with a probability distribution. Thus there is a Borel measure μ defined on the real line such that, for any Borel set E, the probability that selection X is in E is $\mu(E)$. Associated with μ is the (cumulative) distribution function $F:(-\infty,\infty)\to[0,1]$ defined by $F(x)=\mu(-\infty,x]$. Note that for each subinterval $(a,b]\subset(1,B)$, the probability that selection X is in $(a,b]$ is $F(b)-F(a)$. The distribution function F is right-continuous and nondecreasing, with $F(1)=0$ and $F(B)=\lim_{x\to B}F(x)=1$. The points selected with positive probability are the points of discontinuity of F. There are only countably many such points. For more details, see [**6**, p. 261; p. 100, Problem 4].

The *uniform distribution* on a finite interval (r,s) is given by the distribution function $F(x)=(x-r)/(s-r)$. The uniform distribution will play a role in the optimal strategy defined in §4.

The strategy of selecting one specific element and no others is called a *pure strategy*. It is easy to show (using the fact that the integral of a non-negative integrable function is non-negative) that a strategy is optimal if and only if it ultimately defeats or ties every pure strategy.

3. The sequence $\{p_k\}$. In a first blind attempt at discovering an optimal strategy for the game on $(1,\infty)$, one might try the discrete strategy η of selecting $1,T,T^2,T^3,\dots$ with probabilities $p_0,p_1,p_2,p_3,\dots$, respectively, where of course $p_0=0$ and $\sum_{i=0}^{\infty}p_i=1$. Fix $X\in(1,T)$. If η is to draw with the pure strategy X, then $p_1-\nu(1-p_1)=0$, i.e., $p_1=\nu/(\nu+1)$. If η is to draw with the pure strategy TX, then $-p_1+p_2-\nu(1-p_1-p_2)=0$, i.e., $p_2=2p_1/(\nu+1)-p_0$. If η is to draw with the pure strategy T^2X, then $\nu p_1-p_2+p_3-\nu(1-p_1-p_2-p_3)=0$, i.e., $p_3=2p_2/(\nu+1)-p_1$. Recognizing a pattern, we are motivated to define a sequence $p_0,p_1,p_2,\dots$ recursively as follows:

$$p_0=0,\qquad p_1=\nu/(\nu+1),\qquad p_{k+1}=2p_k/(\nu+1)-p_{k-1}\qquad (k\geqslant 1).\tag{1}$$

Although η does not turn out to be optimal, the numbers p_k indeed play an essential role in the optimal strategy given in §4. We proceed to examine the p_k.

As shown in [**5**, p. 96], the p_k can be evaluated explicitly, namely,

$$p_k=\nu^{1/2}(\nu+2)^{-1/2}(z^k-\bar{z}^k)/2i,$$

where $z=(\nu+1)^{-1}+i(\nu+1)^{-1}(\nu^2+2\nu)^{1/2}$ and $\bar{z}$ is the complex conjugate of z. Note that $|z|=1$ so that $z=e^{i\theta}$ with $0<\theta<\pi/2$ and $\nu=\sec\theta-1$. Straightforward calculations show that for $k\geqslant 1$,

$$p_k=p_1(\sin k\theta)/\sin\theta\tag{2}$$

and

$$p_1+p_2+\cdots+p_k=(p_1+p_k-p_{k+1})/2p_1=1/2-\frac{\cos(\theta/2+\theta k)}{2\cos(\theta/2)}.\tag{3}$$

It follows from (2) that for $k>1$,

$$p_k=0\quad\text{when }\theta=\pi/k\tag{4}$$

and

$$p_1,p_2,\dots,p_k>0\quad\text{when }0<\theta<\pi/k.\tag{5}$$

It follows from (3) that for $k>1$,

$$p_1+p_2+\cdots+p_k=1\quad\text{when}\quad\theta=\pi/k\quad\text{or}\quad\theta=\pi/(k+1),\tag{6}$$

$$p_1+p_2+\cdots+p_k>1\quad\text{when }\pi/(k+1)<\theta<\pi/k,\tag{7}$$

and

$$p_1+p_2+\cdots+p_k<1\quad\text{when }0<\theta<\pi/(k+1).\tag{8}$$

4. The optimal strategy. Suppose that $\nu=\sec(\pi/(n+1))-1$ and $T^{n-1}<B\leqslant\infty$ for some fixed integer $n\geqslant 2$. Let Y be any number such that $1<Y\leqslant T$ and $T^{n-1}Y\leqslant B$. We will show that

the following strategy τ is optimal: choose one of the intervals $(1,Y),(T,TY),\ldots,(T^{n-1},T^{n-1}Y)$ with probability $p_1,p_2,\ldots,p_n$, respectively, and then select from the chosen interval using the uniform distribution. Strategy τ is well defined since $p_1+\cdots+p_n=1$ and each $p_i>0$ by (5) and (6).

Strategy τ is a more likely candidate for an optimal strategy than is a discrete strategy δ (for example, strategy η in §3) in which elements of a discrete set S are selected with certain positive probabilities. One reason is that δ is vulnerable to certain pure strategies in S. We illustrate with the game played on $(1,\infty)$ with $\nu=1$. Let δ be the strategy of selecting randomly from $S=\{T,T^2\}$; this strategy is "almost optimal" since it defeats or ties every pure strategy in $(1,T)\cup(T,\infty)$. However, the pure strategy T vanquishes δ's big gun at T^2 while it ties up the gun at T. Strategy τ is not so vulnerable—the artillery is spread out and no single gun is used with probability exceeding 0.

To show that τ is optimal, it suffices to show that the game value G (i.e., the expected winnings) for τ against any given pure strategy $b\in(1,B)$ satisfies $G\geqslant 0$.

Case 1. $b\in(1,T^n]$.

If $b\in[T^{m-1}Y,\ T^m]$ for some integer m with $1\leqslant m\leqslant n$, then G is not changed when b is replaced by $T^{m-1}Y$. Thus we may assume without loss of generality that $b\in(T^{m-1},T^{m-1}Y]$ for some integer m with $1\leqslant m\leqslant n$. Write $b=T^{m-1}X$ where $1<X\leqslant Y$. Then for $1\leqslant m\leqslant n$,

$$G=\nu(p_1+\cdots+p_{m-2})+\nu\Big(\frac{X-1}{Y-1}\Big)p_{m-1}-\Big(\frac{Y-X}{Y-1}\Big)p_{m-1}-\Big(\frac{X-1}{Y-1}\Big)p_m$$
$$+\Big(\frac{Y-X}{Y-1}\Big)p_m+\Big(\frac{X-1}{Y-1}\Big)p_{m+1}-\nu\Big(\frac{Y-X}{Y-1}\Big)p_{m+1}-\nu(p_{m+2}+\cdots+p_n). \tag{9}$$

(To see that (9) holds in the case $m=n$, note that $p_{n+1}=0$ by (4).) The coefficient $(\nu p_{m-1}-p_m+p_{m+1})$ of $(X-1)/(Y-1)$ in (9) equals the coefficient of $(Y-X)/(Y-1)$ in (9), by (1). Thus (9) becomes

$$G=\nu(p_1+\cdots+p_{m-1})-p_m+p_{m+1}-\nu(p_{m+2}+\cdots+p_n). \tag{10}$$

Replacing $(p_{m+2}+\cdots+p_n)$ in (10) by $1-(p_1+\cdots+p_{m+1})$, we obtain

$$G=2\nu(p_1+\cdots+p_m)+(\nu+1)(p_{m+1}-p_m)-\nu.$$

It thus follows from (3) that $G=0$.

Case 2. $b\in(T^n,B)$.

If $b\geqslant T^nY$, then clearly $G>0$, since b loses to every element which can possibly be selected by strategy τ. If $b\in(T^n,T^nY)$, then an argument as in Case 1 shows that $G>0$. Therefore $G>0$ in Case 2.

5. The nonexistence of an optimal strategy. For some fixed integer $n>1$, either

$$\nu=\sec(\pi/(n+1))-1 \tag{11}$$

or

$$\sec(\pi/(n+1))-1<\nu<\sec(\pi/n)-1. \tag{12}$$

We show in this section that if (12) holds, then there exists no optimal strategy for the game played on $(1,B)$, and that if (11) holds, then there exists no optimal strategy for the game on $(1,B)$ when $B\leqslant T^{n-1}$. This will be accomplished by producing, for an arbitrary strategy λ on $(1,B)$, a strategy σ on $(1,B)$ which defeats λ. Let F be the distribution function associated with λ.

Let $s_0=1$, let $s_1\in(1,T)$, and define $s_i=s_1T^{i-1}$ for $i=1,2,3,\ldots$. Let σ be a discrete strategy on $(1,B)$ in which $s_1,s_2,s_3,\ldots$ are selected with probabilities $q_1,q_2,q_3,\ldots$, respectively. Thus the q_i are non-negative numbers whose sum is 1, and $q_i=0$ when $s_i\geqslant B$. Set $q_0=0$. Let H_σ denote the game value for σ against λ. It suffices to show that the s_i and q_i can be chosen so as to determine a strategy σ for which $H_\sigma>0$.

Assume that

$$s_i \text{ is selected by } \lambda \text{ with only zero probability, for } i=1,2,3,\ldots. \tag{13}$$

Then

$$H_\sigma = \sum_{i \geqslant 0} D_i(F(s_{i+1}) - F(s_i)), \tag{14}$$

where

$$D_i = \nu(q_1 + \cdots + q_{i-1}) - q_i + q_{i+1} - \nu(q_{i+2} + q_{i+3} + \cdots),$$

because D_i is the game value for σ against a pure strategy in the interval $(s_i, s_{i+1}]$. Note that by replacing $(q_{i+2} + q_{i+3} + \cdots)$ above by $1-(q_1 + \cdots + q_{i+1})$, we obtain

$$D_i = 2\nu(q_1 + \cdots + q_i) + (\nu+1)(q_{i+1} - q_i) - \nu. \tag{15}$$

Case 1. $B \leqslant T^{n-1}$.

For some k with $1 \leqslant k \leqslant n-1$, we have $T^{k-1} < B \leqslant T^k$. We proceed to choose s_i and q_i so as to define the strategy σ of selecting $s_1, \ldots, s_k$ with probabilities $q_1, \ldots, q_k$, respectively. Let $M = p_1 + \cdots + p_k$. By (5), $p_1, p_2, \ldots, p_n > 0$, and hence $0 < M < 1$ by (8). Choose $q_i = p_i/M$ for $1 \leqslant i \leqslant k$, and $q_i = 0$ for $i > k$. Note that $q_1 + \cdots + q_k = 1$. We now choose an appropriate s_k, after which all of $s_1, s_2, \ldots$ are automatically specified. As noted in §2, there are only countably many points which can be selected by λ with positive probability. Thus there exist points s_k arbitrarily close to B such that (13) holds. Since $F(x) \to 1$ as $x \to B$, we can and do choose $s_k \in (T^{k-1}, B)$ close enough to B such that

$$(1 - F(s_k))|\nu - (\nu+1)q_k - \nu(M^{-1} - 1)| < \nu(M^{-1} - 1) \tag{16}$$

and such that (13) holds. Strategy σ is now determined.

We now show that $H_\sigma > 0$. Since (13) holds, (14) can be applied to evaluate H_σ; thus,

$$H_\sigma = \sum_{i=0}^{k-1} D_i(F(s_{i+1}) - F(s_i)) + D_k(1 - F(s_k)). \tag{17}$$

By (15), $D_k = \nu - (\nu+1)q_k$. By (15) and (3), $D_i = \nu(M^{-1} - 1)$ when $0 \leqslant i \leqslant k-1$. Therefore, by (17),

$$\begin{aligned} H_\sigma &= F(s_k)\nu(M^{-1} - 1) + (1 - F(s_k))(\nu - (\nu+1)q_k) \\ &= \nu(M^{-1} - 1) + (1 - F(s_k))(\nu - (\nu+1)q_k - \nu(M^{-1} - 1)), \end{aligned}$$

so that $H_\sigma > 0$ by (16).

Case 2. $T^{n-1} < B$ and (12) holds.

In this case we will choose the s_i and q_i so as to define the strategy σ of selecting $s_1, \ldots, s_n$ with probabilities $q_1, \ldots, q_n$, respectively. Let $N = p_1 + \cdots + p_n$. By (7), $N > 1$. Choose $q_i = p_{n+1-i}/N$ for $1 \leqslant i \leqslant n$ and $q_i = 0$ for $i > n$. Note that $q_1, \ldots, q_n > 0$ by (5), and $q_1 + \cdots + q_n = 1$. We now choose an appropriate s_1, after which all of $s_1, s_2, \ldots$ are automatically specified. Since F is right-continuous, $F(x) \to 0$ as $x \to 1$. Arguing as in Case 1, we can and do choose s_1 close enough to 1 such that $s_n < B$ and

$$F(s_1)|(\nu+1)q_1 - \nu - (\nu - \nu/N)| < (\nu - \nu/N), \tag{18}$$

and such that (13) holds. Strategy σ is now determined.

We now show that $H_\sigma > 0$. Replacing $(q_1 + \cdots + q_i)$ in (15) by $1 - (q_{i+1} + \cdots + q_n)$, we see that for $i \geqslant 0$,

$$D_i = \nu + (\nu+1)(q_{i+1} - q_i) - 2\nu(q_{i+1} + \cdots + q_n). \tag{19}$$

It follows from (19) and the definition of q_i that for $1 \leqslant i \leqslant n$

$$D_i = (\nu - \nu/N) - N^{-1}(2\nu(p_1 + \cdots + p_{n-i}) + (\nu+1)(p_{n+1-i} - p_{n-i}) - \nu).$$

Hence, by (3),

$$D_i=(\nu-\nu/N) \quad \text{for } 1\leqslant i\leqslant n. \tag{20}$$

By (15),

$$D_0=(\nu+1)q_1-\nu \quad \text{and} \quad D_i=\nu \quad \text{for } i>n. \tag{21}$$

Using (20) and (21) in (14), we have

$$\begin{aligned} H_\sigma &= F(s_1)((\nu+1)q_1-\nu)+(F(s_{n+1})-F(s_1))(\nu-\nu/N)+(1-F(s_{n+1}))\nu \\ &= F(s_1)((\nu+1)q_1-\nu-(\nu-\nu/N))+(\nu-\nu/N)+(1-F(s_{n+1}))\nu/N, \end{aligned}$$

so $H_\sigma>0$ by (18).

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THE SOLUTIONS OF AN INEQUALITY FOR THE nTH ITERATE OF A FUNCTION

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Introduction. The continuous functions which satisfy the functional equation

$$\varphi^n(x)=g(x) \tag{1}$$

(where φ^n denotes the nth iterate of φ), or the functional inequality

$$\psi^n(x)\leqslant g(x), \tag{2}$$

where in each case g is a given continuous function, have been studied by several authors ([1]; [2]; [3, Chap. 15]; [4]; [5]). In this paper I determine the continuous functions ψ that satisfy (2) under the assumption that ψ and g commute.

This problem has applications in chemical engineering. Suppose that x represents the concentration of a substance and that ψ stands for a chemical process. Then $\psi^n(x)$ represents the concentration if we start with concentration x and apply the process n times. Hence a solution of (2) represents a process that, starting from concentration x, yields after n applications a concentration no greater than $g(x)$.

Throughout this paper we shall assume that

(H) The function g is continuous and strictly increasing in the interval $[a,b]$, the points a and b are fixed points of the function g, and

$$g(x)<x \quad \text{for } x\in(a,b). \tag{3}$$

Now, following M. Kuczma [3, Chap. 15, §3], we construct the general continuous solution of (1). This construction will be utilized in §2.

Construction. Take an arbitrary $x_0\in(a,b)$ and let $x_{n-1},x_{n-2},\ldots,x_1\in(a,b)$ satisfy the inequalities

$$g(x_0)<x_{n-1}<\cdots<x_1<x_0. \tag{4}$$

Put

$$x_{\nu+n}=g(x_\nu) \quad \text{for } \nu=0,\pm 1,\pm 2,\cdots. \tag{5}$$

(The sequence $x_{-\nu}$ ($\nu \geqslant 0$) is increasing and converges to b, the sequence x_ν ($\nu \geqslant 0$) is decreasing and converges to a). Let $\varphi_1,\varphi_2,\ldots,\varphi_{n-1}$ be arbitrary strictly increasing continuous functions defined on the intervals $[x_1,x_0],[x_2,x_1],\ldots,[x_{n-1},x_{n-2}]$, respectively, and satisfying the following conditions:

$$\varphi_i(x_{i-1})=x_i, \qquad i=1,2,\ldots,n-1. \tag{6}$$

Put

$$\varphi_{\nu+n}(x)=g(\varphi_{\nu+1}^{-1}(\varphi_{\nu+2}^{-1}(\cdots(\varphi_{\nu+n-1}^{-1}(x))\cdots)))$$
$$\text{for } x\in[x_{\nu+n},x_{\nu+n-1}], \qquad \nu=0,\pm 1,\pm 2,\ldots. \tag{7}$$

Then the function $\varphi:[a,b]\to R$, defined as follows:

$$\varphi(x)=\varphi_\nu(x) \quad \text{for } x\in[x_\nu,x_{\nu-1}], \qquad \nu=0,\pm 1,\pm 2,\ldots,$$
$$\varphi(a)=a,\ \varphi(b)=b \tag{8}$$

is a continuous solution of equation (1) in $[a,b]$. Taking all possible systems of points $x_1,x_2,\ldots,x_{n-1}$ satisfying (4) and all possible systems of strictly increasing continuous functions $\varphi_1,\varphi_2,\ldots,\varphi_{n-1}$ satisfying conditions (6), we get all continuous solutions of equation (1).

1. In this section we give the general continuous solution of (2) in the class of functions ψ which commute with some continuous solution φ of (1)

We shall need a lemma from [**4**]; it is easily proved inductively.

LEMMA 1. *Let continuous functions $\psi,\phi:[a,b]\to[a,b]$ satisfy the inequalities*

$$\psi(x)<x,\ \phi(x)<x \quad \textit{for } x\in(a,b), \tag{9}$$
$$\phi(x)\leqslant\psi(x) \quad \textit{for } x\in[a,b]. \tag{10}$$

If either ψ or ϕ is increasing in $[a,b]$, then

$$\phi^k(x)\leqslant\psi^k(x) \quad \textit{for } x\in[a,b], \qquad k=1,2,\cdots. \tag{11}$$

REMARK 1. It is easy to prove that if hypothesis H is assumed, every continuous solution ψ of (2) and every continuous solution φ of (1) satisfy (9).

THEOREM 1. *Let the function g satisfy hypothesis* (H) *and let φ be a continuous solution of* (1) *in $[a,b]$. If a continuous function $f:[a,b]\to[a,b]$ satisfies the inequality*

$$f(x)\leqslant x \quad \textit{for } x\in[a,b], \tag{12}$$

and commutes with φ, then the function

$$\psi(x)=\varphi(f(x)) \quad \textit{for } x\in[a,b] \tag{13}$$

is a continuous solution of (2) *in $[a,b]$.*

Proof. Since φ is a continuous solution of (1) in $[a,b]$ it is a strictly increasing map of the interval $[a,b]$ onto itself and $\varphi(x)<x$ for $x\in(a,b)$ (see [**3**, Chap. 15, §3]). Hence, by virtue of Remark 1, Lemma 1, and the obvious inequality $\psi(x)\leqslant\varphi(x)$ for $x\in[a,b]$, we have

$$\psi^n(x)\leqslant\varphi^n(x)=g(x) \quad \text{for } x\in[a,b].$$

Since f commutes with φ, we have for $x\in[a,b]$

$$\psi(\varphi(x))=\varphi(f(\varphi(x)))=\varphi(\varphi(f(x)))=\varphi(\psi(x)),$$

and the proof is complete.

THEOREM 2. *Let hypothesis* (H) *be satisfied and let the continuous function $\psi:[a,b]\to[a,b]$ be an increasing solution of* (2) *in $[a,b]$. If there exists a continuous solution φ of* (2) *which commutes with ψ in $[a,b]$, then there exists a continuous function $f:[a,b]\to[a,b]$ satisfying* (12) *and such that* (13) *holds.*

Proof. We define the function f by $f(x) := \psi(\varphi^{-1}(x))$ for $x \in [a,b]$. It is obvious that $f:[a,b] \to [a,b]$ and that f is an increasing function in $[a,b]$. Suppose that $f(x_0) > x_0$ for some point $x_0 \in [a,b]$. From the monotonicity of f we obtain $f^2(x_0) \geqslant f(x_0) > x_0$, and repeating this reasoning k times we have

$$f^k(x_0) > x_0 \qquad \text{for } k = 1,2,\ldots. \tag{14}$$

It is easy to verify that a function which commutes with φ also commutes with φ^{-1}. This together with (14) implies

$$x_0 < f^n(x_0) = \psi(\varphi^{-1}(\psi(\varphi^{-1}(\cdots(\psi(\varphi^{-1}(x_0)))\cdots)))) =$$
$$\psi^n(\varphi^{-n}(x_0)) = \psi^n(g^{-1}(x_0)) \leqslant g(g^{-1}(x_0)) = x_0,$$

which is impossible. Hence, for every $x \in [a,b]$, we have (12) and the proof is complete.

REMARK 2. Theorems 1 and 2 give us the general continuous solution of (2) in $[a,b]$ in the class of increasing functions which commute with a solution φ of (1) (each function ψ may commute with a different solution φ of (1)). If ψ commutes with φ then ψ is given by (13).

2. In this section we state the form of the continuous solutions of (2) in the class of strictly increasing functions that commute with g. The proof, which depends on Theorem 3 (which is proved in [**5**]), will be given elsewhere.

THEOREM 3. *Let the function g satisfy hypothesis* (H) *and let the function $\psi:[a,b]\to[a,b]$ be a strictly increasing solution of* (2) *commuting with g. If there exists a point $x_0 \in (a,b)$ such that $\psi^n(x_0) = g(x_0)$, then there exists a continuous solution φ of* (1) *such that*

$$\varphi(x) = \psi(x) \quad \textit{for } x \in [\psi^{n-1}(x_0), x_0] \tag{15}$$

and

$$\psi(x) \leqslant \varphi(x) \quad \textit{for } x \in [a,b]. \tag{16}$$

THEOREM 4. *Let the hypothesis* (H) *be satisfied and let $\psi:[a,b]\to[a,b]$ be a strictly increasing continuous solution of* (2), *commuting with g. If there exists a point $x_0 \in (a,b)$ such that*

$$\psi^n(x_0) = g(x_0) \tag{17}$$

then there exist continuous solutions $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$ of (1) *such that*

$$\psi(x) = \min\{\varphi_0(x), \varphi_1,(x), \ldots, \varphi_{n-1}(x)\} \quad \textit{for } x \in [a,b]. \tag{18}$$

REMARK 3. It is easy to show that for functions f_1, f_2 commuting with the strictly increasing function g the inequality

$$f_1(x) \leqslant f_2(x) \text{ implies } f_1(g(x)) \leqslant f_2(g(x)).$$

This implies that the function $\min\{\varphi_0(x), \varphi_1(x), \ldots, \varphi_{n-1}(x)\}$ for $x \in [a,b]$, where $\varphi_0, \varphi_1, \ldots, \varphi_{n-1}$ are continuous solutions of (1), commutes with g (because every solution of (1) commutes with g); it also satisfies inequality (2) (see Lemma 1).

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PROGRESS REPORTS

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It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

Progress Reports is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses, what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal: usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

NONLINEAR CONSERVATION EQUATIONS

CATHLEEN S. MORAWETZ

Nowadays an existence theorem for a problem in partial differential equations comes when the mathematician grasps firmly in his hands the nettles of rigor. Very occasionally when the nettles are released there lies not only illumination for the problem but a method for solving it. Such an occasion is provided by Glimm's proof [1] of the existence of a vector U that equals a given function $u(x)$ for $t=0$ and satisfies for $t \geqslant 0$, and $-\infty < x < \infty$,

$$\partial U/\partial t + \partial F(U)/\partial x = 0,$$

where F is a given vector function of U.

Such equations govern airplane flight, explosions and combustion, motion in interstellar gases, plasmas in the fusion process (sometimes), shallow water, and so on. Usually t is time, and having one space variable x is a simplification. The difficulties of this equation were first met with, well over a hundred years ago, by Earnshaw and Stokes in the study of sound. Although a great deal is known today in special cases, or believed in general cases, very little is proved. It took a long time for the right questions to be formulated. Glimm's work rests on foundations laid by, among others, Riemann, Rankine, Hugoniot, Rayleigh, Hadamard, Friedrichs, Burgers, and Lax.

What we are talking about here is a *weak* solution of a *genuinely nonlinear hyperbolic* system of *conservation* equations satisfying an *entropy* condition.

To understand the italicized technical terms we begin with the easiest, *conservation*: This simply means that we have an equation of the form $\operatorname{div} W(U)=0$. Then Green's theorem says $\oint W \cdot n\, ds = 0$ where the integral is around any region. If one of the variables is time and if, as so often happens, the terms from large $|x|$ vanish, then the integral of the first component $\int W_1\, dx$ is independent of time; i.e., it is *conserved*.

Secondly, *weak*: The solution does not have to be smooth or even continuous. In fact, the situation is much worse: a solution that starts out perfectly smooth generally becomes so

singular that at a finite time the obvious derivatives in the equation do not exist. Only an integrated form of the equation holds, i.e.,

$$-\int (U\cdot \partial\chi/\partial t + F(U)\cdot \partial\chi/\partial x)\,dx\,dt + \int u\cdot\chi\,dx|_{t=0} = 0$$

for all smooth χ that vanish for $|x|$ and t large and positive. (To get the first equation back, integrate by parts.) The term on $t=0$ vanishes if $\chi(x,0)=0$; but, if $\chi(x,0)\neq 0$, it pulls in the initial value data u. As an example, suppose that U is scalar and $F(U)=U^2$; then $U=\pm k$ ($K>0$) is a weak solution. But it can jump up and down anywhere you like (see Fig. 1). That's too much freedom. The only legitimate jump should be one that comes as a limit from a heat equation (this is where the *entropy* condition comes in). Here it is for our example:

$$\partial W/\partial t = \partial W^2/\partial x + \epsilon\partial^2 W/\partial x^2.$$

Then $U=\pm k$ is the limit as $\epsilon\to 0$ of the solution $W=-k\tanh k(x-a)/\epsilon$ so that U can *jump exactly once* from $+k$ to $-k$, at an arbitrary point a. This is called an *entropy* condition. For the vector case there are as many entropy conditions as are needed to make the piecewise constant solution (the steady shock) coming from $F(U)=$constant have exactly one discontinuous solution, and they come about from adding dissipative, that is heat-equation-like, terms.

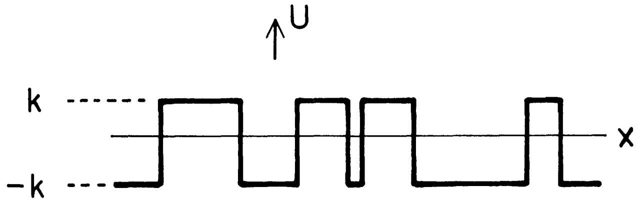

FIG. 1

We come now to the more familiar word *hyperbolic*: Linearize the system by setting $U=k+\delta U$ so that to lowest order δU satisfies

$$\partial\delta U/\partial t + \partial(F_U(k)\delta U)/\partial x = 0,$$

and $\delta U=\exp i\omega(ct-x)$ is a solution if c is any eigenvalue of the matrix $F_U=\{\partial F_i/\partial U_j\}$ at $U=k$. If the c's are all real and generate a complete system of right eigenvectors r, the system is hyperbolic and the c's are speeds of propagation of waves. If the system is linear, F_U is a constant matrix and δU is a solution. For the system to be *genuinely nonlinear* there is the obvious possibility that F_U is nonconstant; but it is the speeds that matter and the right requirement is that the eigenvalues c of F_U each satisfy $r\cdot\nabla c\neq 0$.

Having unwrapped the labels, we now assert that Glimm's theorem proves constructively that *for all initial values u of bounded variation sufficiently small, there exists a solution U for our problem which is of bounded variation*. The smallness is yielded by the proof and no one expects it to be necessary.

So much for the nettle. The rest requires special solutions other than constants. First, *traveling shock waves*: $U=G(st-x+a)$, where s an eigenvalue of something and G discontinuous at the origin. From χ roughly a constant in the *weak* paragraph we find the eigenvalue relation to be, after some integration by parts, s (jump in U)−jump in $F(U)=0$. Once again the entropy condition pins it down. Secondly, *centered simple waves*: $U=H(c)$, $(x-a)/t=c$, an eigenvalue of F_U as before, $\dot{H}$ is a right eigenvector r. So U is constant on $c=$constant. With these two cases we can build solutions by a theorem of Lax: Any two constant states, not too different, can be patched together locally and uniquely by centered simple waves and traveling shocks. This is called solving a Riemann problem.

Let us illustrate by the example where U is a scalar, $F=U^2$, $a=0$ with the two states, left and

right of zero, given by $U=10$, $U=-1$. First U is decreasing so a shock might work. From the jump condition $-11s+99=0$ or $s=9$. So the shock lies on $dt/dx=1/9$. If the two states, left and right, are given instead by $U=-1$, $U=10$, shock is ruled out by the entropy condition. We try a centered simple wave, $x/t=c=2U$ and find $U=-1$ for $x/t \leqslant -2$, $U=x/2t$ for $-2 \leqslant x/t \leqslant 20$, and $U=10$ for $20 \leqslant x/t$. Some algebra will confirm the uniqueness and Figure 2 shows what is meant by patching two constant states.

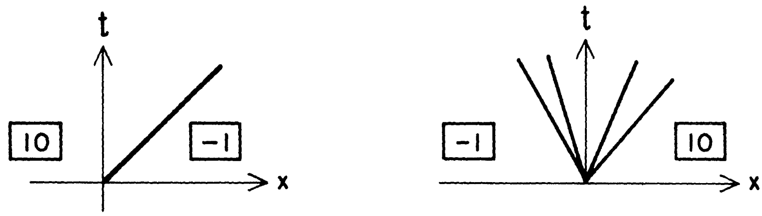

FIG. 2

A better description is that they are divided by a *fan* wave, a term we shall use. In the vector case, fancier combinations of shocks and centered simple waves emanating from a point still work and we again call such a combination a fan wave.

Next step: U can be found for any piecewise constant initial value function for a small time. Figure 3 tells all—the cut-off time is when one fan hits another.

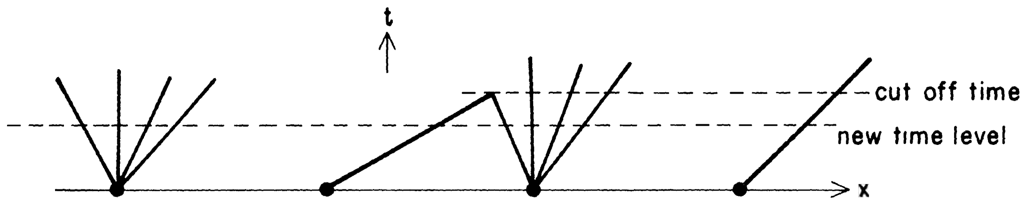

FIG. 3

Logic for constructing a solution: Put an equal mesh on the line. Replace by something piecewise continuous that jumps on the mesh points. Go forward for a short enough time by fan waves. Start over again at a new time level with the same position of discontinuities and repeat indefinitely. To do better, take a finer approximation to start with. The limiting case is to give the actual solution. Possible trouble: Cut-off times which fix the new time level could get indefinitely shorter but the "small variation" takes care of this. Real trouble: How to pick up constant states at each new time level. (Godunov was the first to try this approach and had this trouble.)

Illumination: Any fixed method, such as always approximating by values at a given set of mesh points, is too biased. Shocks are more often wrong than right if this is tried.

Method: You must sweep out "almost all" the points in each mesh interval as you move forward in time. Glimm makes, at each new time level, a random choice of a point in each mesh interval and picks the value of U there as the new constant state.

Question: If we let the mesh size go to zero, do we get a solution to the original initial value problem? Answer: Mostly yes. More precisely, almost every sequence U_n constructed by Glimm's recipe has a limit point which is a solution (n going to infinity corresponds to mesh size going to zero). So we have existence. But also, if you proceed to compute a solution by refining the mesh and using Glimm's recipe, the chances are negligible that you will not come arbitrarily close to a solution.

The proof comes via the central limit theorem. Later work simplifies this (see Liu [2]), but the crucial philosophy is unaltered.

Why be so tricky? Could one not go to a singular perturbation and add the term $-\epsilon(\partial^2 U/\partial x^2)$ as we did in the scalar case? So far, no one has succeeded much beyond what has been done by Oleinik in the scalar case and a special two-dimensional one. Numerically one can succeed by adroit differencing. However, Glimm's random choice has proved best in the complications of combustion, treated numerically by Chorin, et al., where chemical reactions take place and fooling around with dissipative terms can be catastrophic (in the original sense of the word).

The future? Uniqueness. Higher dimensions. Finite regions. Boundary conditions. Fewer nettles in the proof.

References

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MATHEMATICAL NOTES

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A CHARACTERIZATION OF CIRCLES WHICH CONTAIN RATIONAL POINTS

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A natural algebraic partition of the plane E^2 consists of the sets:

$$\mathcal{R} = \{(x,y)\colon x \in \mathbf{R} \text{ and } y \in \mathbf{R}\}; \qquad [\mathbf{R} \equiv \text{rational numbers}]$$

$$\mathcal{I} = \{(x,y)\colon x \in \mathbf{I} \text{ and } y \in \mathbf{I}\}; \qquad [\mathbf{I} \equiv \text{irrational numbers}]$$

$$\mathcal{M} = E^2 - (\mathcal{R} \cup \mathcal{I}).$$

The distribution of these points throughout the plane has been widely studied and described in many ways; in particular, each set is everywhere dense, 0-dimensional, etc. A natural question concerning the distribution of these points is how these points are situated on circles centered at the origin. For example, if $C(r)$ denotes the circle of radius r centered at the origin, do there exist radii r, such that $C(r)$ contains only finitely many points of one of these three sets, and for which values of r is each of the sets dense on $C(r)$? In what follows, we completely characterize the distribution of the sets $\mathcal{R}$, $\mathcal{I}$, and $\mathcal{M}$ on circles centered at the origin using only elementary techniques from calculus, number theory, and the theory of Diophantine equations. Our main results are listed below as Theorems 1, 2, and 3.

THEOREM 1. *If r is rational, then $\mathcal{R}, \mathcal{I}$, and $\mathcal{M}$ are dense on $C(r)$.*

THEOREM 2. *If r^2 is irrational, then $\mathcal{I}$ and $\mathcal{M}$ are dense on $C(r)$, but $\mathcal{R} \cap C(r) = \varnothing$.*

THEOREM 3. *If r is irrational, but $r^2=p/q$ is rational, then $\mathcal{G}$ and $\mathfrak{M}$ are dense on $C(r)$, and*

(a) *$\mathfrak{R}\cap C(r)=\varnothing$ if pq is not the sum of two square integers,*

(b) *$\mathfrak{R}$ is dense on $C(r)$ if pq is the sum of two square integers.*

An interesting corollary is the following:

COROLLARY 4. If C is any circle centered at the origin, then $\mathfrak{R}$ is dense on C if $\mathfrak{R}\cap C\neq\varnothing$.

To verify these results we prove a series of lemmas; the first two of these lemmas are obvious and are listed without proof.

LEMMA 1. *Given any positive real number r, $\mathcal{G}$ is dense on $C(r)$.*

LEMMA 2. *If r^2 is irrational, then $C(r)\cap\mathfrak{R}=\varnothing$ and consequently both $\mathcal{G}$ and $\mathfrak{M}$ are dense on $C(r)$.*

If a,b,c are natural numbers with $a^2-b^2=c^2$, then (a,b,c) is called a *Pythagorean triple.* The set $P=\{b/a:(a,b,c)$ is a Pythagorean triple$\}$ and is called the set of *Pythagorean ratios.*

LEMMA 3. *P is dense on* $[0,1]$.

Proof. If (a,b,c) is a Pythagorean triple, then a,b, and c can be represented parametrically as

$$a=r^2+s^2$$
$$b=2rs$$
$$c=r^2-s^2$$

where r and s are relatively prime integers and $r>s$. It follows that

$$b/a=2rs/(r^2+s^2)$$
$$=2(s/r)/(1+(s/r)^2).$$

As the function $f(x)=2x/(1+x^2)$ for $0\leqslant x\leqslant 1$ is continuous and has $[0,1]$ as its range, the lemma follows by noting that $\{s/r:r>s,r$ and s are relatively prime$\}$ is dense on $[0,1]$.

LEMMA 4. *If r is rational, then $\mathfrak{R}$ and $\mathfrak{M}$ are dense on $C(r)$.*

Proof. In order to find points of $\mathfrak{R}$ on $C(r)$, we will suppose that the abscissa of a point on $C(r)$ is rational and obtain conditions on the ordinate that it too might be rational. Let $x=p/q$ and suppose $r=m/n$ where $p,q,m,n\in\mathbf{Z}$. ($\mathbf{Z}\equiv$integers.) If the point (x,y) is on $C(r)$, then

$$y^2=r^2-x^2=m^2/n^2-p^2/q^2=(q^2m^2-n^2p^2)/(n^2q^2). \tag{1}$$

The numerator of (1) will be a perfect square if $p=bm$ and $q=an$ where (a,b,c) is a Pythagorean triple $(a^2-b^2=c^2)$. In this case,

$$q^2m^2-n^2p^2=m^2n^2(a^2-b^2)=m^2n^2c^2, \text{ and hence } y=\pm mc/q. \tag{2}$$

It follows, then, from Lemma 3 that $\mathfrak{R}$ is dense on $C(r)$.

Under closer inspection of (2) we note that if $p=bm$ and $q=an$ where (a,b,c) is not a Pythagorean triple then y is not rational and consequently, $(x,y)\in\mathfrak{M}\cap C(r)$. Further, for a given rational number a, the number of Pythagorean triples with a as first entry is at most $\sqrt{a}$, and $\operatorname{limit}_{a\to\infty}\sqrt{a}/a=0$. It follows that $\{b/a:a^2-b^2$ is not a perfect square$\}$ is dense on $[0,1]$, and hence $\mathfrak{M}$ is dense on $C(r)$.

Theorems 1 and 2 now follow from the first four lemmas which characterize all cases except when the radius of $C(r)$ is the square root of a rational number. This final investigation, interestingly enough, splits into two subcases depending on the form of the rational number whose square root is the radius in question.

LEMMA 5. *If $r=\sqrt{p/q}$ and pq is not the sum of two squares, then $C(r)\cap\mathfrak{R}=\varnothing$.*

Proof. The proof hinges on the following elementary result from the theory of Diophantine equations:

The integer n is the sum of two squares if and only if, in the prime factorization of n, primes of the form $4k+3$ *have even powers.* (*)

Now, suppose $(x, y) \in \mathfrak{R} \cap C(r)$, then

$$x^2+y^2=r^2=p/q \quad \text{or} \quad (qx)^2+(qy)^2=pq. \tag{3}$$

It follows, by multiplying (3) by the square of the least common denominator of x and y, that there is an integral solution to

$$X^2+Y^2=(pq)Z^2. \tag{4}$$

As pq is not the sum of squares, it follows from (*) that there is a prime of the form $4k+3$ which divides pq an odd number of times. Consequently, the prime factorization of the right hand side of (4) contains a prime of the form $4k+3$ raised to an odd power. However, (*) also entails that the prime factorization of the left hand side of (4) contains primes only of the form $4k+3$ raised to even powers. This contradicts the fact that $(x, y) \in \mathfrak{R} \cap C(r)$ and the lemma is proved.

LEMMA 6. *If* $r=\sqrt{p/q}$ *and pq is the sum of two squares, then* $\mathfrak{R}$ *and* $\mathfrak{M}$ *are dense on* $C(r)$.

Proof. In order to show $\mathfrak{R} \cap C(r) \neq \varnothing$ we must find rational numbers x and y such that

$$x^2+y^2=r^2=p/q \quad \text{or} \quad (qx)^2+(qy)^2=pq. \tag{5}$$

Now, let $m=pq$ and suppose $x=u/v$ is rational. Then

$$y^2=(v^2m-u^2)/v^2, \tag{6}$$

and (6) has a rational solution for y if and only if v^2m-u^2 is a perfect square. Consequently, we're led to find integral solutions of the equation

$$X^2+Y^2=mZ^2. \tag{7}$$

As m is the sum of two squares, $m=n_1^2+n_2^2$ $(n_1<n_2)$ and we let

$$X=n_1r+n_2s, \text{ and } Y=n_2r-n_1s. \tag{8}$$

Then,

$$X^2+Y^2=(n_1^2+n_2^2)r^2+(n_1^2+n_2^2)s^2=m(r^2+s^2). \tag{9}$$

It follows immediately that (9), and consequently (7), has a solution if and only if r and s are the "legs" of a Pythagorean triple. Parametrically then, let

$$\begin{aligned} &r=a^2-b^2 \text{ and } s=2ab, \text{ so that} \\ &x=n_1(a^2-b^2)+n_2(2ab) \\ &Y=n_2(a^2-b^2)-n_1(2ab) \\ &Z=a^2+b^2. \end{aligned} \tag{10}$$

Then if $u=X$ and $v=Z$, the x value of u/v yields the rational y value of $\pm\sqrt{mq^2-p^2}\,/q$. This, of course, shows that $x^2+y^2=m$ contains an infinitude of rational solutions and, hence, $C(r) \cap \mathfrak{R}$ is infinite. To show that $\mathfrak{R}$ is dense on $C(r)$ we use the technique introduced in Lemma 3. Let

$$f(x)=n_1+2(n_2x-n_1)/(x^2+1) \qquad 0 \leqslant x \leqslant 1. \tag{11}$$

Then as the set of Pythagorean ratios P is dense on $[0,1]$, $f(P)$ is dense in the range of f. As f is continuous, its range is a closed interval whose left endpoint is easily seen to be less than zero. Using elementary calculus to compute the maximum value of f on $[0,1]$ we find that this maximum is $\sqrt{m}$ and occurs at $x^*=(n_1+\sqrt{m}\,)/n_2$. That is, $f(P)$ is dense on $[0, \sqrt{m}\,]$. But, if

$a/b \in P$, then

$$\begin{aligned} f(a/b) &= n_1 + 2(n_2(a/b) - n_1)/((a/b)^2 + 1) \\ &= n_1 + 2b(n_2 a - n_1 b)/(a^2 + b^2) \\ &= [n_1(a^2 - b^2) + n_2(2ab)]/(a^2 + b^2) \\ &= u/v. \end{aligned} \tag{12}$$

And these u/v are precisely the x values which yield rational y values. It follows, from this and the symmetry of $C(r)$, that $\mathfrak{R}$ is dense on $C(r)$.

The proof that $\mathfrak{M}$ is also dense on $C(r)$ is strictly analogous to the proof given in Lemma 4. This completes the proof of Lemma 6.

The last of our main results, Theorem 3, follows directly from Lemmas 5 and 6 and our task is complete.

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DEPENDENT RANDOM VARIABLES WITH INDEPENDENT SUBSETS

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Introduction. Let $X_1, \ldots, X_q$ be q ($\geqslant 3$) random variables. If they are independent, then proper subsets of them are also independent. But if *all* proper subsets of them are independent, it does not necessarily follow that they are independent. For the discrete case and $q = 3$, there are many counterexamples. Essentially, they are modifications of the example constructed by S. N. Bernstein (see [1] and [2]).

In this note, we shall extend Bernstein's example to $q \geqslant 3$ and also to a continuous case. We shall exhibit $q \geqslant 3$ continuous as well as discrete random variables $X_1, \ldots, X_q$ such that every subset of them is mutually independent. Bernstein's example is a special case if $q = 3$ and the variables are of the discrete type. It is interesting to note the relation between the discrete case and the continuous case in this example. Recently, Joffe [3] has extended Bernstein's example in another direction. His construction is analytical, while ours is geometrical.

The discrete case. Let $p \geqslant 2$ and $\mathbf{A} \subseteq R^p$ be defined by

$$\mathbf{A} = \{\mathbf{a} = (\alpha_1, \ldots, \alpha_p) : \alpha_i = 1 \quad \text{or} \quad 0, \qquad i = 1, \ldots, p\}. \tag{1}$$

Write $I(\mathbf{a}) = \sum_{i=1}^{p} \alpha_i$, so that $I(\mathbf{a})$ takes values $0, 1, \ldots, p$. Map $\mathbf{A}$ onto $\mathbf{B} \subseteq R^{p+1}$ by

$$T: \begin{array}{ll} \mathbf{a} \to \mathbf{b} = (\alpha_1, \ldots, \alpha_p, 1) & \text{if} \quad I(\mathbf{a}) = \text{even}, \\ \mathbf{a} \to \mathbf{b} = (\alpha_1, \ldots, \alpha_p, 0) & \text{if} \quad I(\mathbf{a}) = \text{odd}. \end{array} \tag{2}$$

It follows from (2) that T is 1-1 onto, $|\mathbf{A}| = |\mathbf{B}| = 2^p$ and $\mathbf{B}$ is permutation invariant.

Define T_j^{-1} which maps $\mathbf{B}$ onto $\mathbf{A}^{(j)} \subseteq R^p$ by

$$T_j^{-1} : \mathbf{b} \to \mathbf{a}^{(j)} = (\beta_1, \ldots, \beta_{j-1}, \beta_{j+1}, \ldots, \beta_{p+1}). \tag{3}$$

That is, $\mathbf{a}^{(j)}$ is obtained from $\mathbf{b}$ by eliminating the jth component of $\mathbf{b}$. Let $\mathbf{A}^{(j)} = T_j^{-1}(\mathbf{B})$. Clearly, $\mathbf{A}^{(p+1)} = \mathbf{A}$. Since $\mathbf{B}$ is permutation invariant, we have $\mathbf{A}^{(j)} = \mathbf{A}^{(p+1)}$ for all $j = 1, \ldots, p$, and hence $\mathbf{A}^{(j)} = \mathbf{A}$ for all $j = 1, \ldots, p+1$.

Let $\mathbf{X} = (X_1, \ldots, X_{p+1})$ be a discrete random vector with probability distribution function (pdf) defined by

$$\begin{aligned} f(\mathbf{x}) &= 2^{-p}, \quad && \text{if} \quad \mathbf{x} \in \mathbf{B}, \\ &= 0, && \text{elsewhere}. \end{aligned} \tag{4}$$

Let $\mathbf{X}^{(j)} = T_j^{-1}(\mathbf{X}) = (X_1,\dots,X_{j-1},X_{j+1},\dots,X_{p+1})$. Since T_j^{-1}'s are 1-1 from $\mathbf{B}$ onto $\mathbf{A}$, the pdf of the $\mathbf{X}^{(j)}$ are

$$g(\mathbf{x}^{(j)}) = 2^{-p}, \quad \text{if } \mathbf{x}^{(j)} \in \mathbf{A}, \qquad = 0, \quad \text{elsewhere.} \tag{5}$$

It then follows from (5) that the one-dimensional marginals of the X_i are

$$h(x_i) = 2^{-1}, \quad \text{if } x_i = 0 \text{ or } 1, \qquad = 0, \quad \text{elsewhere.} \tag{6}$$

From (4), (5) and (6), it is obvious that $X_1,\dots,X_{p+1}$ are not independent but all proper subsets of them are independent with identical one-dimensional marginal pdf (6).

The continuous case. Let $\mathbf{A}$ and $\mathbf{B}$ be as defined in the last section and let $\mathbf{U} = \times_{i=1}^{p}[0,1]$ be the unit hypercube in R^p. Define the intervals $I_0 = [0,\frac{1}{2})$ and $I_1 = [\frac{1}{2},1]$. For $\mathbf{a} = (\alpha_1,\dots,\alpha_p)$ and $\mathbf{b} = (\beta_1,\dots,\beta_{p+1})$, define $\mathbf{U}_\mathbf{a} = \times_{i=1}^{p} I_{\alpha_i}$ and $\mathbf{U}_\mathbf{b} = \times_{i=1}^{p+1} I_{\beta_i}$ to be hypercubes in R^p and R^{p+1}, respectively, and $\mathbf{U}_\mathbf{B} = \bigcup_{\mathbf{b}\in\mathbf{B}} \mathbf{U}_\mathbf{b}$. Clearly, $\bigcup_{\mathbf{a}\in\mathbf{A}} \mathbf{U}_\mathbf{a} = \mathbf{U}$, $\mathbf{U}_\mathbf{a} \cap \mathbf{U}_{\mathbf{a}'} = \phi$, if $\mathbf{a} \neq \mathbf{a}'$, $\mathbf{U}_\mathbf{b} \cap \mathbf{U}_{\mathbf{b}'} = \phi$, if $\mathbf{b} \neq \mathbf{b}'$, and the volumes of $\mathbf{U}_\mathbf{a}$ and $\mathbf{U}_\mathbf{b}$ are 2^{-p} and 2^{-p-1}, respectively, for all $\mathbf{a} \in \mathbf{A}$ and $\mathbf{b} \in \mathbf{B}$. Since there are 2^p $\mathbf{U}_\mathbf{b}$'s, the volume of $\mathbf{U}_\mathbf{B} = 2^{-1}$.

Let $\mathbf{X} = (X_1,\dots,X_{p+1})$ be a continuous random vector with pdf

$$f(\mathbf{x}) = 2, \quad \text{if } \mathbf{x} \in \mathbf{U}_\mathbf{B}, \qquad = 0, \quad \text{elsewhere.} \tag{7}$$

To calculate the pdf of $\mathbf{X}^{(j)} = T_j^{-1}(\mathbf{X}) = (X_1,\dots,X_{j-1},X_{j+1},\dots,X_{p+1})$, we proceed as follows: Let $\mathbf{B}_j^{(0)} = \{\mathbf{b} : \mathbf{b} \in \mathbf{B}, j\text{th component of } \mathbf{b} = 0\}$ and $\mathbf{B}_j^{(1)} = \mathbf{B} \setminus \mathbf{B}_j^{(0)}$. Then

$$\mathbf{U}_\mathbf{B} = \left[\bigcup_{\mathbf{b}\in\mathbf{B}_j^{(0)}} \mathbf{U}_\mathbf{b} \right] \cup \left[\bigcup_{\mathbf{b}\in\mathbf{B}_j^{(1)}} \mathbf{U}_\mathbf{b} \right], \tag{8}$$

and the $\mathbf{U}_\mathbf{b}$ are mutually disjoint hypercubes in R^{p+1}. Let $\chi_\mathbf{b}$ denote the indicator function of $\mathbf{U}_\mathbf{b}$; then we can write (7) as

$$f(\mathbf{x}) = \sum_{\mathbf{b}\in\mathbf{B}_j^{(0)}} 2\chi_\mathbf{b} + \sum_{\mathbf{b}\in\mathbf{B}_j^{(1)}} 2\chi_\mathbf{b}, \quad \text{for } j = 1,\dots,p+1. \tag{9}$$

Fix $\mathbf{x}^{(j)} \in \mathbf{U}$. Then $\mathbf{x}^{(j)} \in \mathbf{U}_\mathbf{a}$ for exactly one $\mathbf{a} \in \mathbf{A}$ and the set $S = \{\mathbf{x} : \mathbf{x} \in \mathbf{U}_\mathbf{B}, T_j^{-1}(\mathbf{x}) = \mathbf{x}^{(j)}\} \subseteq \mathbf{U}_{\mathbf{b}'}$ for exactly one $\mathbf{b}' \in \mathbf{B}$. It then follows from (9) that if the jth component of $\mathbf{b}'$ is 0,

$$g(\mathbf{x}^{(j)}) = \int_0^{\frac{1}{2}} 2\,dx_j = 1; \tag{10}$$

if it is 1,

$$g(\mathbf{x}^{(j)}) = \int_{\frac{1}{2}}^{1} 2\,dx_j = 1. \tag{11}$$

Consequently, the pdf of $\mathbf{X}^{(j)}$ is

$$g(\mathbf{x}^{(j)}) = 1, \quad \text{if } \mathbf{x}^{(j)} \in \mathbf{U}, \qquad = 0, \quad \text{elsewhere.} \tag{12}$$

From (12), we obtain the one-dimensional pdf of X_i, $i = 1,\dots,p+1$, namely,

$$g(x_i) = 1, \quad \text{if } 0 \leqslant x_i \leqslant 1, \qquad = 0, \quad \text{elsewhere.} \tag{13}$$

Thus the p random variables $X_1,\dots,X_{j-1},X_{j+1},\dots,X_{p+1}$, for $j = 1,\dots,p+1$, are jointly uniformly distributed over the unit hypercube $\mathbf{U}$ with each one-dimensional marginal distribution $U(0,1)$. Consequently, these, and also any further subset of these, are independent random variables. But from (7) and (12), it is manifest that the $p+1$ random variables $X_1,\dots,X_{p+1}$ are not mutually independent.

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RESEARCH PROBLEMS

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In this department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.

A PROBLEM IN HILBERT SPACE THEORY ARISING FROM THE QUANTUM THEORY OF MEASUREMENT

ABNER SHIMONY AND HOWARD STEIN

The following problem arises in the quantum mechanical analysis of the measuring process: Do there exist a unitary operator $U: L^2(\boldsymbol{R}^2) \to L^2(\boldsymbol{R}^2)$ and a non-zero function $w \in L^2(\boldsymbol{R})$ such that

(i) $U[f(x)w(y)]$ has its support in $(0,\infty)\times(0,\infty)$ whenever $f \in L^2(\boldsymbol{R})$ has its support in $(0,\infty)$,

(ii) $U[f(x)w(y)]$ has its support in $(-\infty,0)\times(-\infty,0)$ whenever $f \in L^2(\boldsymbol{R})$ has its support in $(-\infty,0)$,

(iii) for all $r \in \boldsymbol{R}$ the shift operator T_r commutes with U, where

$$(T_r h)(x,y) = h(x+r, y+r)?$$

Chapter 6 of von Neumann's classical book on quantum mechanics [1] describes the measuring process in the following idealized manner. Suppose that one wishes to measure an observable quantity $\mathfrak{Q}$ of system I, using system II as the measuring apparatus. Let H_1 and H_2, respectively, be the Hilbert spaces of the state vectors of systems I and II, let vectors in $H_1 \otimes H_2$ represent states of the composite system which consists of systems I and II in interaction, and let Q be a self-adjoint operator on H_1 which represents $\mathfrak{Q}$. If $\mathfrak{Q}$ has a discrete set of possible values $\{\lambda_i\}$ (which from an experimental standpoint does not diminish generality, because of practical limitations upon discrimination), then $\mathfrak{Q}$ will have a complete set of orthonormal eigenvectors $\{u_i\}$ with respective eigenvalues $\{\lambda_i\}$. According to von Neumann, system II is suitable as an apparatus for measuring $\mathfrak{Q}$ if there exist a vector w of H_2, a self-adjoint operator A on H_2, a set (not necessarily complete) of orthonormal eigenvectors $\{v_i\}$ of A with respective eigenvalues $\{\mu_i\}$, and a time-interval Δt such that, if U is the time-evolution operator for Δt, we have for all i:

$$U[u_i \otimes w] = u_i \otimes v_i, \tag{1}$$

$$\lambda_i \neq \lambda_j \text{ implies } \mu_i \neq \mu_j. \tag{2}$$

(The time-evolution operator U for Δt is the unitary operator which takes any vector representing the state of the composite system at a time t to a vector representing the state at time $t+\Delta t$.) If the process (1) occurs with the experimentalist initially ignorant of the value λ_i, then because of (2) he can unequivocally infer this value from a determination of the eigenvalue μ_i associated with v_i. In experimental terms, an observable quantity of the apparatus serves as a reliable index of the value of $\mathfrak{Q}$. A more general scheme of measurement, still in the spirit of von Neumann's formulation, consists in the replacement of (1) by the following:

$$U[u_i \otimes w] = \Sigma' c_{jk} u_j \otimes v_k, \text{ with } \Sigma' |c_{jk}|^2 = 1, \tag{1'}$$

where the summation is restricted to values of j such that $\lambda_j = \lambda_i$ and values of k such that $\mu_k = \mu_i$. Note that if a measurement yields a certain value of $\mathfrak{Q}$, then (1′) is a necessary and sufficient condition that an immediate re-measurement will yield the same value.

Wigner [2] and Araki and Yanase [3] proved the following theorem: *A necessary condition for $\mathfrak{Q}$ to be measurable in accordance either with* (1) *or with* (1′) *is that Q commute with every bounded self-adjoint operator M_1 on H_1 for which there exists a bounded self-adjoint operator M_2 on H_2 with the property that $M_1 \otimes 1 + 1 \otimes M_2$ commutes with the time-evolution operator U.* If the conditions of boundedness could be removed in this theorem, one would have a very stringent limitation upon the possibility of measuring $\mathfrak{Q}$ in accordance with the schemes (1) and (1′) above, since the principles of conservation of linear and of angular momentum ensure that there are at least six independent unbounded self-adjoint operators M_1 on H_1 such that no matter what apparatus is devised there will be an M_2 with the property that $M_1 \otimes 1 + 1 \otimes M_2$ commutes with U. The question of the removability of the conditions of boundedness has not been fully settled. Stein and Shimony [4] have shown that the theorem is valid for unbounded operators if scheme (1) is assumed, but they obtained only partial results for the scheme (1′).

It is possible that the following situation will provide a counterexample to the assertion obtained by omitting the conditions of boundedness in the theorem of Wigner, Araki, and Yanase. Let systems I and II both be (spinless) particles confined to one dimension, which we just think of as $\boldsymbol{R}$. In the "position representation," then, $H_1 = H_2 = L^2(\boldsymbol{R})$ and $H_1 \otimes H_2 = L^2(\boldsymbol{R}^2)$. Let $Q = A$ be the self-adjoint operator such that $Qf = f$ if $f \in L^2(\boldsymbol{R})$ and the support of f lies in $(0, \infty)$, while $Qf = -f$ if $f \in L^2(\boldsymbol{R})$ and the support of f lies in $(-\infty, 0)$. Physically, Q and A represent the observable quantities which have the value 1 if system I or system II, respectively, is located to the right of the origin and -1 if to the left of the origin. The linear momentum operator P_1, which can be defined as $-i\ d/dx$, does not commute with Q, but $P_1 \otimes 1 + 1 \otimes P_2$ (which can be expressed as $-i\ d/dx - i\ d/dy$) does commute with U because of the conservation of linear momentum. It has not been established, to our knowledge, that $\mathfrak{Q}$ cannot be measured according to scheme (1′), and the problem is to determine whether or not this is possible. Since the shift operator defined in the first paragraph can be expressed by

$$T_r = \exp\,[(d/dx + d/dy)r],$$

the commutation of U with $P_1 \otimes 1 + 1 \otimes P_2$ is equivalent to the commutation of U with all the T_r. Consequently, the problem just stated is equivalent to the problem posed in the first paragraph.

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CLASSROOM NOTES

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THE APPLICATION OF A CLASSICAL INEQUALITY TO POPULATION MATHEMATICS

DAVID H. ANDERSON

1. As will be shown in paragraph 2, the following analysis has an application in population mathematics, but its main theorem may be of interest for its own sake. Consider an equation of the form

$$p(x)\equiv x^k-\sum_{i=1}^{n} a_i x^{n-i}=0, \tag{1}$$

where $0<x<\infty$, $n\geqslant 2$, k is a real number, $k>n-1$, and $a_i\geqslant 0$ for all $i=1,2,\ldots,n$. Let

$$q(x)\equiv 1-p(x)/x^k=\sum_{i=1}^{n} a_i/x^{k-n+i}\geqslant 0.$$

Then b is a root of $p(x)=0$ if and only if $q(b)=1$. The function q has a positive vertical asymptote at $x=0$, approaches zero as $x\to\infty$, and is strictly decreasing on $(0,\infty)$. Therefore there exists a unique $\lambda>0$ such that $q(\lambda)=1$. Hence $p(x)=0$ has a unique positive root λ. The purpose of this paper is to show that a lower bound, and in certain important cases a lower *estimate*, for λ can be found by a direct application of the arithmetic-geometric means inequality.

A lower bound on the root λ of equation (1) is now constructed via the arithmetic-geometric means. To begin, it is observed that the sum appearing in $p(\lambda)$ has nonnegative coefficients a_i. This suggests that the a_i can be made into nonnegative "weights" w_i which sum to unity. Let

$$R_0\equiv\sum_{i=1}^{n} a_i,\qquad R_1\equiv\sum_{i=1}^{n} i a_i,$$

and define, for $i=1,2,\ldots,n$,

$$w_i\equiv a_i/R_0\geqslant 0,\qquad x_i\equiv\lambda^{n-i}>0.$$

Then (1) is equivalent to the normalized equation

$$0=(\lambda^k/R_0)-\sum_{i=1}^{n} w_i x_i, \tag{2}$$

where $w_1+w_2+\cdots+w_n=1$. The arithmetic mean-geometric mean inequality for (x_i) and (w_i) is

$$\sum_{i=1}^{n} w_i x_i\geqslant\prod_{i=1}^{n} x_i^{w_i}, \tag{3}$$

where equality occurs if and only if $x_1=x_2=\cdots=x_n$. Let us assume $R_0\neq 1$ (if $R_0=1$, then $\lambda=1$ since $p(1)=1-R_0=0$). Then $\lambda\neq 1$ and the x_i's are all distinct. Hence (3) is a strict inequality. From (2) and (3) one obtains

$$\begin{aligned}0&<(\lambda^k/R_0)-\prod_{i=1}^{n}(\lambda^{n-i})^{a_i/R_0}\\&=(\lambda^k/R_0)-\lambda^{n-R_1/R_0}.\end{aligned}$$

This inequality can immediately be solved for λ to get the following theorem. While this result is

known, the proof given in [**1**, p. 709] is much more complicated than that presented here.

THEOREM. The unique positive root λ of equation (1) is greater than

$$\lambda^* \equiv R_0^{1/(k-n+R_1/R_0)} \tag{4}$$

provided $R_0 \neq 1$.

Using any upper bound on λ [**1**, pp. 703–707], the difference $\lambda - \lambda^*$ can always be estimated. When λ is close to unity, however, the difference *is* in fact small. Intuitively, this is so because the range of values $x_1, x_2, \ldots, x_n$ should be small and thus (3) will be a tight inequality. To substantiate the claim, let $\lambda = 1+\varepsilon$ where ε is small. Substituting $x = 1+\varepsilon$ into (1) and expanding both sides in powers of ε gives

$$R_0 = 1 + (k-n+R_1)\varepsilon + O(\varepsilon^2); \ |\varepsilon n| \ll 1.$$

Then

$$\lambda - \lambda^* = 1 + \varepsilon - (1 + (k-n+R_1)\varepsilon + O(\varepsilon^2))^c$$

where

$$c = (k-n+R_1)^{-1} + R_1\varepsilon + O(\varepsilon^2); \ |(k-n)(k-n+R_1)\varepsilon| \ll 1.$$

Thus

$$\lambda - \lambda^* = 1 + \varepsilon - (1 + \varepsilon + O(\varepsilon^2)) = O(\varepsilon^2).$$

As a consequence, the *bound* λ^* will be small in error as an *estimate* of λ provided λ is near one.

2. Two deterministic age-specific population models that are often presented in a mathematical modeling course and are also of practical demographic value are the discrete time model due to P. H. Leslie [**5**], [**4**], [7], [2], [**8**] and the continuous time model first introduced by A. J. Lotka [**6**], [**4**], [7], [3]. In the analysis of the asymptotic behavior of either model, there is a real parameter, called the *intrinsic growth rate*, to be estimated. This constant is the unique positive root λ of an equation of the form (1). It is well known that the form of the characteristic equation of the Leslie model [7, p. 37], [**1**, p. 702] is (1) when $k = n$. The Lotka model is also served by (1). To see this, consider the characteristic equation

$$\int_\alpha^\beta f(t)\,dt \equiv \int_\alpha^\beta b(t)l(t)e^{-rt}\,dt = 1 \tag{5}$$

of the Lotka model which a real parameter r must satisfy [**3**, p. 298], [**4**, p. 100]. Here $b(t)$ and $l(t)$ are the birth and survival rates of the population and are empirically obtained from census data. To estimate r, the integral in (5) can be approximated by finite sums,

$$\int_\alpha^\beta f(t)\,dt \simeq \sum_{i=1}^n c_i f(t_i),$$

where the c_i's are nonnegative constants determined by the method of quadrature. When the new positive parameter $\lambda \equiv \exp(r\tau)$ is introduced to replace r(τ a known positive constant), equation (5) is of the form given in (1).

The exponential form in (4) has an interesting connection with Lotka's work [**6**, p. 70]. Due to the shape and distribution of data, Lotka replaced the product $b(t)l(t)$ in the equation (5) by a normal distribution with mean μ and variance of σ^2. He then deduced an approximate value for r of

$$r^* \equiv (\ln R_0)/\mu.$$

In [**1**, p. 709] it is shown that Lotka's r^* corresponds to the value λ^* in (4) via the transformation $\lambda^* = \exp(r^*\tau)$. Among population analysts probably the most widely used estimate of r is r^* (or equivalently, λ^*), because this number is easily calculated and *appears* to be small in error [**4**, p. 142]. That the error is small can be confirmed. For, r is normally quite close to zero and thus λ is

near one. Hence the arguments given above now apply and indicate that the difference $\lambda-\lambda^*$ must be small.

To demonstrate the theorem, consider the population data reported in [4, p. 29] for females in the United States, 1964: $k=9.5$, $n=8$, and $a_1,a_2,\ldots,a_8$ are 0.00205, 0.17265, 0.51983, 0.42250, 0.24328, 0.11627, 0.03173, and 0.00182, respectively. Then $R_0=1.5101$, $R_1=5.7475$, and the theorem yields a lower bound on the growth rate λ of $\lambda^*=1.0808$. The computed true value is $\lambda=1.0816$ [1, p. 718].

Acknowledgments. I would like to thank Professors G. M. Wing and H. Pollard for their very helpful comments on this paper.

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AN ELEMENTARY PROOF OF A TRIGONOMETRIC IDENTITY

ROBERT M. YOUNG

The fundamental identity

$$\frac{1}{\sin^2 t}=\sum_{-\infty}^{\infty}\frac{1}{(t-n\pi)^2}$$

is generally established either by quoting the Mittag–Leffler theorem [1, p. 186] or by appealing directly to the calculus of residues [2, p. 113]. In either case, contour integration plays a fundamental role and, moreover, the requisite details are not entirely trivial. The purpose of this note is to point out that the above identity is an immediate consequence of Parseval's equality.

Indeed, whenever A is real and not equal to an integer, the nth Fourier coefficient of the function e^{iAx}, taken over the interval $[-\pi,\pi]$, is easily computed to be

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{iAx}e^{-inx}\,dx=\frac{\sin\pi(A-n)}{\pi(A-n)}.$$

Parseval's relation then yields

$$\sum_{-\infty}^{\infty}\left[\frac{\sin\pi(A-n)}{\pi(A-n)}\right]^2=1,$$

and the result follows by setting $A=t/\pi$.

The validity of the identity for complex values of t can now be established as an immediate consequence of the uniqueness theorem for analytic functions.

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AN ELEMENTARY PROOF OF THE ONE-DIMENSIONAL DENSITY THEOREM

LUDĚK ZAJÍČEK

Let λ stand for the Lebesgue measure in R. A point $x \in R$ is, by definition, a point of density for a measurable set $E \subset R$ if

$$\lim_{h \to 0_+} \frac{\lambda((x-h, x+h) \cap E)}{2h} = 1.$$

The well-known one-dimensional density theorem can be stated in the following form.

THEOREM. *Almost all points of an arbitrary measurable set $E \subset R$ are points of density for E.*

It was proved originally by H. Lebesgue [**1**] as a consequence of his theorem concerning integration of Dini derivates. There exist also direct proofs, which usually use the Vitali covering theorem or similar means (e.g., [**2**], [**3**]). We present a simple proof which uses no covering theorem.

First we shall formulate some simple facts and prove a well-known lemma.

If X, A are measurable subsets of R and $0 < \lambda A$, then we define the mean density of X in A as

$$D_m(X, A) = \frac{\lambda(A \cap X)}{\lambda A}.$$

The following properties of the mean density are obvious:

(a) If A_α, $\alpha \in C$, is a countable system of pairwise disjoint measurable subsets of R, $X \subset R$ a measurable set, $c \in R$ and $D_m(X, A_\alpha) > c (\geqslant c, < c, \leqslant c)$ for all α, then $D_m(X, \bigcup_{\alpha \in C} A_\alpha) > c (\geqslant c, < c, \leqslant c)$.

(b) Let $A \subset R$ be a measurable set, $a \in R$ and $h > 0$. Then the function $p(x) = D_m(A, (x, x+h))$ is continuous in R and $q(x) = D_m(A, (a, x))$ is continuous in (a, ∞).

If $M \subset R$ is a measurable set and $x \in R$, then the right lower density of M in x is defined as

$$D_+(M, x) = \liminf_{h \to 0_+} D_m(M, (x, x+h)).$$

The left lower density $D_-(M, x)$ is defined similarly. The following fact is obvious.

(c) If $M \subset R$ is measurable and $x \in R$, then x is a point of density for M if and only if $D_+(M, x) = D_-(M, x) = 1$.

LEMMA *Let $M \subset R$ be a measurable set. Then the function $g(x) = D_+(M, x)$ is measurable.*

Proof. From the definition of $D_+(M, x)$ it is easy to see that

$$g(x) = \lim_{n \to \infty} (\inf\{D_m(M, (x, x+h)); 0 < h < 1/n\}).$$

Using (b) we have

$$g_n(x) = \inf\{D_m(M, (x, x+h)); 0 < h < 1/n\} = \inf\{D_m(M, (x, x+h)); 0 < h < 1/n, h \text{ is rational}\}.$$

Therefore, $g_n(x)$, as the infimum of a countable family of continuous functions, is measurable, and $g(x) = \lim_{n \to \infty} g_n(x)$ is measurable as well.

Proof of Theorem. Put $A^+=\{x\in E; D_+(E,x)<1\}$ and $A^-=\{x\in E; D_-(E,x)<1\}$. Suppose that the statement of the Theorem is false. By the Lemma A^+ and A^- are measurable and by (c) $\lambda A^+>0$ or $\lambda A^->0$. Without loss of generality we can suppose that $\lambda A^+>0$. Putting $A_n^+=\{x\in E; D_+(E,x)<1-1/n\}$, we see that $A^+=\bigcup_{n=1}^{\infty}A_n^+$ and by the Lemma all sets A_n^+ are measurable. Therefore there exists a positive integer n such that $\lambda A_n^+>0$. Let $F\subset A_n^+$ be a closed set such that $0<\lambda F<\infty$. Choose an open set $G\supset F$ such that $D_m(F,G)>1-1/n$. By (a) there exists an interval $(a,b)\subset G$ such that $D_m(F,(a,b))>1-1/n$. Put

$$S=\{a\}\cup\{y\in(a,b\rangle; D_m(F,(a,y))\leqslant 1-1/n\}$$

and $s=\sup S$. We have $b\not\in S$ and $s\in S$, since S is closed by (b). Therefore $a\leqslant s<b$, and we see that there exists $s<t<b$ such that $D_m(F,(s,t))\leqslant 1-1/n$. In fact, if $s\not\in F$, the existence of t is obvious, since F is closed. If $s\in F$, then $s\in A_n^+$ and by the definition of A_n^+ there exists $s<t<b$ such that $D_m(F,(s,t))\leqslant D_m(E,(s,t))<1-1/n$. By (a) we obtain that $D_m(F,(a,t))\leqslant 1-1/n$ and therefore $t\in S$. This is a contradiction and the proof is complete.

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MATHEMATICAL EDUCATION

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CUPM ANNOUNCEMENT

Minimal Mathematics for College Graduates. The Committee on the Undergraduate Program in Mathematics (CUPM) of the Mathematical Association of America has established a panel to consider the question: "What should every graduate of an American college or university know of mathematics?" It is hoped that the ultimate recommendations of this panel will provide welcome guidance to colleges, universities, and such bodies as state boards of education, many of which are already actively considering such questions in the current wave of renewed interest in core curricula and general education.

The panel, relying extensively on surveys of informed opinion, wishes to arrive at a list of minimum mathematical competencies for all college graduates, where "mathematical" is meant to include statistics, computing, etc., as well as mathematics in the narrow sense. Its report should contain, besides this list, a reasoned statement about why every college graduate should have acquired some understanding of mathematical thought; suggestions about courses in which the minimal competencies might be acquired; and general observations about such matters as interinstitutional coordination in furthering mathematical literacy.

The panel has begun to collect information and opinions on the problem and will welcome

contributions from readers of this announcement. Facts about other local, regional, or national efforts (past or present) in the area of the panel's charge, personal views about the general issue or specific aspects, and copies of, or references to, pertinent documents are among the things the panel would be glad to receive. They may be sent to the panel in care of its chairman, D. Bushaw, Department of Pure and Applied Mathematics, Washington State University, Pullman, WA 99164.

MODEL FORMULATION USING INTERMEDIATE SYSTEMS

RICHARD L. RUBIN

Introduction. A curious aspect of the attention being paid to applied topics in the mathematics curriculum is the lack of instruction in techniques of model formulation. While discussions of mathematical modeling are becoming an established part of the curriculum, emphasis is often placed on the philosophical aspects of models (their relation to "reality", etc.), on the history and uses of certain models, or on the analysis of already-formulated models. These aspects are important, as is recognition of the complex interactive character of the modeling process; nevertheless, a major purpose of this paper is to lobby for increased emphasis on instruction in one aspect of the process: formulating mathematical models.

The conventional wisdom concerning model construction holds that "formulating mathematical models is an art best learned by experience." The idea seems to be that, since model formulation involves creativity, systematic instruction is inappropriate or ineffective. At best, in the traditional approach, a student is provided with a set of maxims to keep in mind during his or her creative agony. Often the result of this attitude is that students in beginning modeling courses either do very little model construction (as opposed to model manipulation) or become discouraged with applied mathematics and retreat into more structured areas. In this paper I will describe an approach I have used to teach formulation to undergraduates with no modeling experience. The technique is simple and can be considered an outgrowth of the traditional views of teaching model construction by trial and error. Instruction is not based on a new method of formulating mathematical models but rather on the idea that patterns of behavior of successful modelers can be used to construct a technique helpful to beginners. Although it has met with enthusiastic responses, the technique described here is only a small first step toward developing an efficient and effective method of teaching mathematical model formulation to beginners. It is my hope that this report will help to stimulate the development of improved methods.

Intermediate models. Instruction in model formulation using intermediate systems is based, in part, on the observation that the construction of a mathematical model can be viewed as a sequence of translations between various physical, linguistic, mathematical, or logical systems. A typical model construction problem consists of a set of statements in some system which we denote by P (often of a hybrid nature, combining linguistic, mathematical, and graphical elements), which must be represented in another system (usually mathematical or logical) which we denote by S. Only rarely does the modeler directly construct a representation of the specified statements of P into S. The common procedure is to produce a sequence of representations using intermediate systems which are usually hybrids consisting of progressively larger subsystems of S and smaller subsystems of P.

The above observations lead to a technique of instruction in model formulation based on the following ideas.

1. Since mathematical models are diverse and modeling is a creative activity, frequent practice in model formulation is important. Traditional techniques address this need by giving students lists of case studies to model. I adopt this procedure, although I emphasize that case studies should be chosen to illustrate not only the application of mathematical techniques but

also the process of problem formulation. Some case studies should contain extraneous information which the student must disregard, and some should require additional information which the student must assume. (A major emphasis of the techniques being discussed in this article is to help to isolate relevant data and to identify any need for additional assumptions.)

2. If average undergraduate students are to learn to formulate models, rather than just to manipulate established models, some structure must be provided to diminish the frustration and inefficiency of the trial-and-error method. The need for a middle road is where the technique of using intermediate systems enters. Students are required to present their work modeling a case study as a sequence of models culminating in what they believe is an acceptable mathematical formulation. The first models must explicitly identify the structural components of the situation being studied (e.g., decision variables, constraints, states, stages, etc.). These models are required to be largely linguistic. Separating the identification of the components of a model from the problem of expressing certain ideas mathematically reduces the confusion many beginners feel, especially if their ideas about model formulation were formed by seeing a case study juxtaposed with an elegant solution.

3. One difficulty beginners sometimes encounter is that during the formulation process they do not have a clear idea of what they are trying to construct. This leads to two types of problems. When a student has no target model form in mind, the role of various components of the model may become confused, e.g., the roles of independent and dependent variables may be interchanged. On the other hand, if a student decides on a form before the structure of the situation is discovered, much time may be spent trying to fit a case study into an incompatible mathematical formulation. The intermediate systems technique requires students to identify the components of the problem in the first few models and then to decide on a tentative target model form. As work progresses, the target form may be changed; however, explicit reasons for the change must be given.

4. Another part of the intermediate systems technique is the requirement that each intermediate model be a complete model: "intermediate" refers to the system in which the model is formulated (logical, mathematical, linguistic, or some hybrid), not to the scope of the model. This requirement helps to diminish the discouragement sometimes felt by beginners when modeling a complex situation. Each intermediate model demonstrates progress, thus motivating continued efforts; comparison of the model with the target form indicates where further work is needed.

5. A related aspect of the method is that, when one component of a model is changed, all the other components must be updated to reflect the change. This emphasis on the interrelatedness of the entire model allows a formulation technique to be discovered in its most obvious context and transferred to other settings. In a sense, the problem is naturally modeled in the least difficult manner.

Classroom implementation. This section contains an outline of the general procedures used to implement the above ideas. The students are given a sequence of case studies, all of which can be modeled using a particular class of mathematical formulations. I use deterministic decision problems which lead to certain constrained optimization or mathematical programming formulations. The students are shown a few examples of model construction and are asked to formulate models for the case studies. They encounter problems similar to those described in the preceding section. The technique of using intermediate models is described and an appropriate intermediate system for the class of deterministic decision problems is introduced. After the students have become reasonably proficient in modeling using this system, they are given other lists of case studies which require different types of intermediate structures; depending upon the students' mathematical background, some appropriate classes of problems include probability models based on stochastic processes or models of systems obeying conservation laws which lead to formulations using partial differential equations. Students are required to develop their own intermediate systems to model these case studies. They present their work to the class,

which synthesizes an intermediate system from the ideas discussed. This consensus system is used by each class member to model the appropriate case studies. As students progress, the requirement that each stage of the modeling process be documented is relaxed, and only the final formulation, accompanied by commentary explaining notation and assumptions, must be submitted for evaluation.

An example. I shall describe the manner in which an example would be presented to students with little previous modeling experience. The presentation is based on the following case study.

A university is opening an extension program consisting of 40 classes to be offered at an off-campus location. Due to construction delays, the classroom building at the extension center will not be ready for the upcoming semester. The university will therefore have to rent classrooms in the community. A survey shows that 5 classrooms in an elementary school located 1 mile from the extension center site and 3 classrooms in a high school located $1\frac{1}{2}$ miles from the extension center site are available at a cost of \$18 per classroom per week. In addition, 2 religious organizations have classroom buildings located 2 miles from the extension center site consisting of 10 classrooms each, which are available on weekdays at a cost of \$20 per classroom per week. All of the classrooms seat at least 40 students, which is more than the anticipated enrollment in any of the classes.

The university offers classes from 8:00 AM to 4:00 PM (an 8-hour time period) according to 4 possible schedules:

Schedule	Class Length	Days of Meeting
1	1 hr.	M, T, Th, F
2	1 hr. (M, F) 2 hrs. (W)	M, W, F
3	2 hrs.	T, Th
4	4 hrs.	W

Ten of the 40 classes must use schedule 1, 4 must use schedule 2, 6 must use schedule 3, and 2 must use schedule 4. All but 2 of the remaining classes may each use schedule 1, 2, or 3. The 2 other classes must each use schedule 3 or 4. Construct a mathematical model which will yield specific recommendations for the university with respect to classroom rentals.

To begin the modeling process, I ask the class to write a few sentences describing the problem to be solved. Almost immediately, we agree that we wish to determine which classrooms to rent in order to satisfy the scheduling constraints described in the case study. After a bit more discussion, the class decides that this description is incomplete; an objective must be used to guide the selection of classrooms, e.g., minimizing the total cost of classroom rentals or minimizing the total distance of the classrooms from the extension center site. The students decide that having as many classes as possible in the same building should take precedence over the other possible goals, so we make the objective to rent classrooms in the minimum number of buildings. We write the first intermediate model: The university wishes to rent classrooms in 4 possible locations so as to minimize the number of classroom locations while at the same time satisfying the university scheduling requirements and classroom availability restrictions.

Next, I give the students a general description of the second intermediate model used for deterministic decision processes: The second model is an elaboration of the first, consisting of the following six components. The *decision-maker*: the person or persons who make the relevant decisions. The *objective*: the precise goal the decision-maker wishes to achieve. The *decision variables*: *all* the variables which correspond to the processes the decision-maker can control. The *objective variable*: the variable which measures the results of the decision-maker's actions. The *objective function*: the explicit mathematical relationship which assigns to each value of the decision variables a specific value of the objective variable. The *constraints*: the restrictions on the decision-maker's possible choices, i.e., the list of restrictions on the possible values of the decision variables. Each of these restrictions should be capable of being expressed by a set of

expressions of the form $g_i(x_1,\dots,x_n)\#0$, $i=1,\dots,m$, where $x_1,\dots,x_n$ are the decision variables; g_i, $i=1,\dots,m$, are functions of the decision variables; and # represents an equal sign or an inequality symbol. This component of the model determines the properties of the decision variables, so every condition which the decision variables must obey should be determined by this set of expressions.

We formulate the second model for our case study. *Decision-maker*: Advisor to the university concerning classroom rentals and scheduling. *Objective*: To minimize the number of classroom buildings used. *Decision variables*: r_i = number of rooms rented in location i, $i=1,2,3,4$; x_i = number of classes using schedule i, $i=1,2,3,4$. *Objective variable*: L = number of classroom buildings used. At this point we must establish a functional relationship between L and x_i and r_i. After some discussion, we decide to revise the r_i as follows:

$$r_i=\begin{cases}1 \text{ if location } i \text{ is used}\\ 0 \text{ if location } i \text{ is not used}\end{cases}$$

This allows us to write: *Objective function*: $L=\sum_{i=1}^4 r_i$. *Constraints*:

(i) (Minimum number of classes using the various schedules) $x_1\geqslant 10$; $x_2\geqslant 4$; $x_3\geqslant 6$; $x_4\geqslant 2$.
(ii) While 22 classes have specified schedules, 16 others may each use schedule 1, 2, or 3.
(iii) The remaining 2 classes may each use schedule 3 or 4.
(iv) $\sum_{i=1}^4 x_i=40$.
(v) Classes at a given time at location $i\leqslant c_ir_i$, $i=1,2,3,4$, where $c_1=10$, $c_2=10$, $c_3=5$, $c_4=3$.
(vi) $r_i=0$ or 1, $i=1,2,3,4$.
(vii) x_i are integers, $i=1,2,3,4$.

At this stage we decide upon a target model form. Since we have a deterministic decision model, we must guess whether to work with a linear, integer, mixed-integer, or quadratic programming format, or other optimization model. I list the general form of some of these models for the students. The class decides at once to use an integer programming format. This target form defines the system S described earlier in the paper.

We observe that the second intermediate model is written in S except for constraints (ii), (iii), and (v), each of which must be expressed as a linear equality or inequality involving the x_i's and the r_i's.

Considering (ii), we note that the total $x_1+x_2+x_3$ must include at least 16 classes in addition to the required 10 for x_1, 4 for x_2, and 6 for x_3. In other words: $\sum_{i=1}^3 x_i\geqslant 36$. We attempt to formulate (iii) in a similar manner but do not succeed. Leaving (iii) in linguistic form, we go to (v). After some discussion we decide that the number of classes using a location depends upon the schedules employed during each day, which leads us to reformulate (v) to read: classroom hours used on a given day $\leqslant$ total classroom hours available in various locations on that day. This translates to:

$$(\text{Monday-Friday})\ x_1+x_2\leqslant 8(10r_1+10r_2+5r_3+3r_4);$$
$$(\text{Tuesday-Thursday})\ x_1+2x_3\leqslant 8(10r_1+10r_2+5r_3+3r_4);$$
$$(\text{Wednesday})\ 2x_2+4x_4\leqslant 8(10r_1+10r_2+5r_3+3r_4).$$

We return to constraint (iii) and begin to symbolize the relation. Let y_i = the number of classes chosen from the 2 additional which use schedule i, $i=3,4$. Then (iii) is equivalent to $y_3+y_4=2$. Moreover, it is easy to see that the other constraints can be reformulated using both the x_i's and y_i's. Therefore, we revise the decision variables to include y_3, y_4.

After some discussion, we observe that because of the 1-, 2-, and 4-hour lengths of the class periods and the 8-hour length of the university day, any allocation of classes to schedules 1 through 4 satisfying the above constraints can be implemented. I point out that we may need to use more decision variables and more-complicated constraints to model problems with less compatible time periods.

The results of this revision yield the next intermediate model. The Decision-maker, Objective, Objective Variable, and Objective Function remain the same as in the second intermediate model.

For convenience we designate as regular classes all those except the 2 which must each use schedule 3 or 4. We have:

Decision Variables:

$$r_i = \begin{cases} 1 \text{ if location } i \text{ is used} \\ 0 \text{ if location } i \text{ is not used} \end{cases} \quad i = 1,2,3,4$$

$$x_i = \text{number of regular classes using schedule } i, \qquad i = 1,2,3,4$$

$$y_i = \text{number of special classes using schedule } i, \qquad i = 3,4$$

Constraints:

(i) $x_1 \geqslant 10,\ x_2 \geqslant 4,\ x_3 \geqslant 6,\ x_4 = 2$

(ii) $\sum_{i=1}^{3} x_i = 36$

(iii) $\sum_{i=3}^{4} y_i = 2$

(iv) $\sum_{i=1}^{4} x_i = 38$

(v) $x_1 + x_2 - 80r_1 - 80r_2 - 40r_3 - 24r_4 \leqslant 0$

$x_1 + 2x_3 + 2y_3 - 80r_1 - 80r_2 - 40r_3 - 24r_4 \leqslant 0$

$2x_2 + 4x_4 + 4y_4 - 80r_1 - 80r_2 - 10r_3 - 24r_4 \leqslant 0$

(vi) $r_i = 0, 1$

(vii) x_i, y_j integer; $i = 1,2,3,4, \quad j = 3,4$

(viii) $y_j \geqslant 0, \quad j = 3,4$

We now have a model written in the format of S. We observe that since $x_4 = 2$, equation (iv) can be omitted and the third inequality in (v) can be rewritten as

$$2x_2 + 4y_4 - 80r_1 - 80r_2 - 10r_3 - 24r_4 \leqslant -8.$$

Eliminating common factors in the inequalities in (v) yields the final mathematical formulation.

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OPPORTUNITIES IN MATHEMATICS: REFERENCES, PEOPLE, COURSES

S. G. TELLMAN

1. Introduction. The Department of Mathematics at the University of Arizona distributes a Student Career Guide, "Opportunities in Mathematics: Material to Read, People to See, Courses to Take." The Guide provides a range of career opportunities for a person with an undergraduate background in mathematics—careers which the student, and perhaps the student's adviser, will in many cases be quite unaware of or uninformed about—and a means of taking first steps to explore them in more detail; see [2], in particular (B),(D), and (G). It implicitly highlights the central question of long-term vocational goals and the directly related question of the extent to which present course work is consistent with those goals; see [3]. We hope that it contributes to the creation of an open, frank, positive departmental atmosphere in which the student will be encouraged, and assisted, to face the often difficult, painful decisions to which it is addressed.

The purpose of this report is to call attention to the stimulating, helpful role that such organized career information can play and to provide a model for others who may wish to make similar material available.

2. Description of the Student Career Guide. The material for each career field is divided into three parts.

Courses to take. A list of recommended "Professional Courses" (and suggested electives) and recommended supporting "Mathematics Courses" (and suggested electives).

People to see. A list of quantitatively-oriented people, usually in the field in question, people who are qualified—in terms of training and, perhaps even more important, active, involved interest—and willing to talk about career opportunities in their own areas for a person with an undergraduate major in mathematics.

Material to read. A list of career pamphlets and articles relevant to the field. A list of more general references follows the material on specific fields.

The Career Guide has been organized, after an appropriate introduction, into the following roughly indicative broad areas.

Part 1: Opportunities in Applied Mathematics.

A. Government and Business (Operations Research/Management Science, Accounting, Actuarial Science, Econometrics/Business Statistics, Management Information Systems)

B. Renewable Natural Resources, Agriculture, and Health Sciences (Biometrics/Biostatistics, Medical Computer Science, Pest Management and Applied Mathematical Ecology, Soil Physics, Hydrology/Watershed Management)

C. Engineering (Industrial Engineering/Systems Analysis, Reliability Engineering/Quality Control, Mathematical Engineering, Biomedical Engineering, Nuclear Engineering)

D. Applied Statistics

E. Computer Science, Numerical Analysis, and Programming

Part 2: Opportunities in Research and Teaching.

A. Social Sciences (Econometrics/Mathematical Economics, Mathematical Sociology)

B. Natural Sciences (Mathematical Biology, Genetics, Geophysics, Atmospheric Physics/ Meteorology, Optical Science, Mathematical Physics/Chemistry)

C. Secondary and High School Teaching

D. Junior College Teaching

E. College and University Research and Teaching

The following comments may be of assistance to others who wish to prepare a guide.

Comment 1. On "Courses to Take": The course lists were approved by qualified people in the field concerned; an attempt was made to enable mathematics courses to replace other stated prerequisities.

Comment 2. On "People to See": Each entry has the general form "Name; Address (or University Office); Phone; Brief description of interests (optional); List of Principal Mathematical Tools (optional—coded by the letters A-L)." For example: "T. L. Tinvence; Aerospace and Mechanical Engineering Laboratory 110; 884–2325 (application of optimal control theory and differential game theory to problems in aerospace engineering and environmental management. B, D, E)." The coded "Tools" are Linear Programming, Game Theory, Partial Differential Equations, Ordinary Differential Equations, Applied Analysis, Statistics, Applied Probability, Modern Algebra, Matrix Theory, Graph Theory, Numerical Techniques, Computer. The approximately 150 individuals listed—some in more than one area—were drawn from the department, the university, and the local teaching and business community; an attempt was made to list many areas and to obtain several people within each area. Initial contacts were made, for example, on the basis of recommendations of department heads; these initial contacts often could suggest several other suitable names. Recruitment was not a problem; most people reacted positively to the idea and only one declined to have his name listed. We stressed that this program would probably involve no more than three student visits a year. Only two or three unhurried calls a day are advised—conversations have a way of becoming extended and this should be anticipated. A number of interested persons might split up the areas among them. The entire assemblage process—although at times tedious and strangely tiring—may well prove to be a most educative, warm experience.

Comment 3. On "Material to Read": A special collection on career opportunities in mathematics has been established in the department library. The references within each area were ordered by first giving references to career description literature and then giving references to literature on recommended undergraduate training for that area. The references were drawn primarily, but not solely, from standard MAA, CUPM, professional, and government sources.

Our Career Guide is distributed through undergraduate advisers and from a "Free–Take" dispenser (about 50 per semester are distributed locally in this way). Updated copies are sent to local high schools.

Readers may obtain copies of the Guide from the author; in turn, he would appreciate receiving copies of any similar materials that have been assembled.

References

1. I. E. Block, PRIME 80 yields proposals for applications-oriented math curricula, SIAM News, 11 (1978).
2. Report of the Committee on New Priorities for Undergraduate Education in the Mathematical Sciences, this MONTHLY, 81 (1974) 984–988.
3. E. Spanier, The undergraduate program in mathematics, this MONTHLY, 77 (1970) 752–755.

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MISCELLANEA

22. How not to do applied mathematics. Mr. Morrison, for example, was our expert on the three-dimensional concentrations of stress around riveted plate joints.... Every two years or so he produced a paper . . . full of the most complicated mathematics proving to the aeroplane designer what he knew already from his own experience. Mr. Fox-Marvin was another of them.... [He] had been working since 1935 on the torsional instability of struts, with Miss Bucklin aiding and abetting him for much of the time.... At the end of all those years they had got the unstabilized eccentrically loaded strut of varying section just about buttoned up, regardless of the fact that unstabilized struts are very rare today in any aircraft structure.

Neville Shute, *No Highway*, London, 1948, pp. 1–2.

PROBLEMS AND SOLUTIONS

EDITED BY A. P. HILLMAN

The editors receive more problems than they can use, but they are always on the lookout for interesting and novel problems. The task of selection will be much easier if proposers will keep the following guidelines in mind:

Send all problems (both elementary and advanced) to A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131, in duplicate if possible. The editors urge proposers to include any solutions or information that will help the editors, including reasons why the problem is interesting. Problems in well-known textbooks and results that appear in generally accessible sources are not acceptable.

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution. If you submit a problem without a solution, you should tell the editors whether you know (or somebody else knows) how to solve the problem. If you are uncertain, the problem might be more suitable for the Research Problems section; see the instructions given there.

Proposers are asked to aim for the same audience as for the rest of the MONTHLY: *a rule of thumb is to think of people who have had at least a year of graduate work in mathematics, very likely some years ago. Proposals that can be both understood and solved by high school students, or by those in the first two years of college, are usually more appropriate for the Mathematics Magazine or the Two-Year College Mathematics Journal. No precise line can be drawn between elementary and advanced problems, but in general an elementary problem will involve notions no more difficult than those encountered in the first year of graduate work and should be solvable by methods common at the same level. An advanced problem may deal with more advanced concepts or require more sophisticated techniques for its solution.*

A problem that takes half a page or more to state, or that involves several ad hoc definitions, is usually unsuitable. The statement of a problem should avoid specialized jargon and symbols whenever the English language is more readily understood: for example, "f is a continuous function" is preferable to "$f \in C$."

Before submitting a problem, be sure that it says exactly what you intend it to say. Solvers are not telepathic and will usually solve what was proposed, not what you had in mind.

PROBLEMS DEDICATED TO EMORY P. STARKE

Solutions of the problems in this issue dedicated to Professor Emory P. Starke should be mailed to Prof. A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, N.M. 87131 (U.S.A.) before August 31, 1979. To facilitate consideration, solutions should be typed (with double spacing).

S 9. *Proposed by M. S. Klamkin and A. Liu, University of Alberta*

(a) Determine all positive integers n such that $\gcd(x,n)=1$ implies that $x^2\equiv 1 \pmod n$.

(b) Determine all positive integers n such that $xy+1\equiv 0 \pmod n$ implies that $x+y\equiv 0 \pmod n$.

S 10. *Proposed by Richard K. Guy and J. L. Selfridge, University of Calgary*

When n-in-a-row (the generalization of tic-tac-toe) is played on a large enough board, it is easy to see that the first player has a winning strategy if $n=1,2,3$, or 4. There is a folk-theorem that Go-Moku ($n=5$) is also a first-player win, but nothing has been proved for $5\leqslant n\leqslant 8$. Show that the second player can force a draw if $n\geqslant 9$, no matter how large the board is.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA (USA) 94303. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (in duplicate with double spacing) and should be mailed before August 31, 1979. Please enclose a self-addressed card or label (for acknowledgment).

E 2767. *Proposed by James W. Burgmeier, University of Vermont*

Let f be a function with sufficiently many derivatives and let D_n be the determinant

$$D_n=\begin{vmatrix} f' & f & 0 & 0 & \cdots & 0 & 0 \\ \frac{f''}{2!} & f' & f & 0 & \cdots & 0 & 0 \\ \frac{f'''}{3!} & \frac{f''}{2!} & f' & f & \cdots & 0 & 0 \\ \vdots & & & & & & \\ \frac{f^{(n)}}{n!} & \frac{f^{(n-1)}}{(n-1)!} & & & \cdots & & f' \end{vmatrix}.$$

Show that

$$D_{n+1}=f'D_n-\frac{1}{n+1}fD_n'.$$

E 2768. *Proposed by Jim Fickett, University of Colorado, Boulder*

Is there a subset E of $[0,1]$ with E and $[0,1]\setminus E$ homeomorphic?

E 2769. *Proposed by Harry D. Ruderman, Hunter College, Manhattan*

Let λ and λ' be (not necessarily coplanar) lines in space. On each of these lines, set up a real number coordinate system, with possibly different units of length. Let XX' be the line segment joining a point X on λ to the point X' on λ' with the same coordinate. Describe how to obtain X such that XX' has minimal length for all such segments.

E 2770. *Proposed by Warren Page, New York City Community College*

Let n and N be fixed positive integers and let

$$S_k=\sum_{m=1}^{n} m^k \text{ for } k=1,2,\ldots,N.$$

Prove

$$\text{(a)}\ \sum_{h=1}^{N}\sum_{k=1}^{h}\binom{h+1}{k}S_k=\frac{n+1}{n}\left[(n+1)^{N+1}-(N+1)n-1\right],$$

and

$$\text{(b)}\ \sum_{h=1}^{N}\sum_{k=1}^{h}(-1)^h\binom{h+1}{k}S_k=\begin{cases}\dfrac{n+1}{n+2}\left[(n+1)^{N+2}+1\right] \text{ for odd } N,\\[2ex] \dfrac{(n+1)^2}{n+2}\left[(n+1)^N-1\right] \text{ for even } N.\end{cases}$$

E 2771. *Proposed by Robert Breusch, Amherst College*

Let p be a prime and $p \not\equiv 1 \pmod 8$. Prove that the equation $x^{2p}+y^{2p}=z^{2p}$ has no solution in positive integers x,y,z with $xyz \not\equiv 0 \pmod p$.

E 2772. *Proposed by Robert B. McNeill, Northern Michigan University*

Let m be a positive integer. Find all ordered pairs of positive integers (a,b) for which $(a+b)|(a^{2m}+b^{2m})$.

SOLUTIONS OF ELEMENTARY PROBLEMS

An Invertible Incidence Matrix

E 2690 [1978, 48]. *Proposed by Anthony J. Quinzi, Temple University.*

Let $S_1, S_2, \ldots, S_k$ be a list of all non-empty subsets of $\{1,2,\ldots,n\}$. Thus $k=2^n-1$. Let $a_{ij}=0$ if $S_i \cap S_j = \varnothing$ and $a_{ij}=1$ otherwise. Show that the matrix $A=(a_{ij})$ is non-singular.

Solution by Stanley F. Cohen, University of Pennsylvania. We may assume that the list has been chosen so that

$$i = \sum_{j \in S_i} 2^{j-1}, \qquad 1 \leqslant i \leqslant k. \tag{1}$$

If $i+j>k$ then it is clear from (1) that $S_i \cap S_j \neq \varnothing$, i.e., $a_{ij}=1$. If $i+j=k$ then S_i and S_j are complementary subsets and so $a_{ij}=0$. Subtracting the last row of A from other rows and then the last column from other columns one obtains a matrix B of the form

$$\begin{bmatrix} & & & & -1 & 0 \\ * & & & -1 & & 0 \\ & & \cdot^{\cdot^{\cdot}} & & & \vdots \\ & -1 & & 0 & & 0 \\ -1 & & & & & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 \end{bmatrix}$$

Hence, $\det A = \det B = \pm 1$ and so A is non-singular. In fact, if $n \geqslant 2$ then $k(k-1)/2$ is odd and $\det A = -1$.

Also solved by fifty-one other readers.

A Rational Approximation to Arctan

E 2693 [1978, 48]. *Proposed by Alexandru Lupas, ITC Cluj-Napoca, Romania*

Find a rational function $f(x)=P(x)/Q(x)$, where $P(x)$ and $Q(x)$ are polynomials with integral coefficients of degree at most 6, which is a good approximation to $\arctan x$ on $[0,1]$. More precisely we want $g(x)=\arctan x - f(x)$ to satisfy $0 \leqslant g(x) < \varepsilon$ for $x \in [0,1]$ and ε to be small. (Such an approximation exists if $\varepsilon = 0.000033$.)

Remark by the editor. The required approximation is to be found in Y. L. Luke, Special Functions and Their Approximations, vol. 2, table 57, p. 392 (at the line $n=3$). It is a convergent, namely,

$$\frac{x}{1+}\,\frac{x^2}{3+}\,\frac{4x^2}{5+}\,\frac{9x^2}{7+}\,\frac{16x^2}{9+}\,\frac{25x^2}{11},$$

to the continued fraction expansion. The approximation deviates the most at $x=1$. If the last fraction, $25x^2/11$, is replaced by $1.88x^2 \equiv 47x^2/25$, the error ε at $x=1$ is reduced markedly (by

80 percent), with very little change in the error for other values of x. The point of the problem is that an approximation of this sort can be used to minimize the number of keystrokes needed to calculate arctan x with a hand-held computer. Hence the requirements that the numbers be small integers and the number of operations be small.

Also solved by the proposer.

An Inequality

E 2695 [1978, 116]. *Proposed by Eliyahu Beller, Bar-Ilan University, Israel.*

Prove or disprove the following conjecture: For $a>1$ and $x>0$, show that

$$-\log(1-(1-e^{-x})^a)<x^a.$$

Solution by Otto G. Ruehr, Michigan Technological University. Set $y=1-e^{-x}$. Then $0<y<1$ and we have

$$\begin{aligned}-\log(1-(1-e^{-x})^a)&=-\log(1-y^a)\\&=y^a\sum_{n\geq 0}\frac{y^{na}}{n+1}<y^a\sum_{n\geq 0}\frac{y^n}{n+1}\\&<y^a\left(\sum_{n\geq 0}\frac{y^n}{n+1}\right)^a=(-\log(1-y))^a=x^a.\end{aligned}$$

Also solved by Theodore Bolis, Aage Bondesen (Denmark), Paul Bracken (Canada), Robert Breusch, Andreas Brunnschweiler (Germany), David Carlson, David Cohoon, Peter Ehlers (Canada), Bernd Eifrig (Germany), Piotr Grabowski (Poland), Gustaf Gripenberg (Finland), Richard Groeneveld, Emil Grosswald, G. A. Heuer, Joel Levy, Peter Lindstrom, A. Meir (Canada), N. Miku (Netherlands), John Myers, M. P. Ojha, O. P. Lossers (Netherlands), St. Olaf Problem Solving Group, Joachim Suck (Germany), and John Utz.

Toroidal n-Queens Problem

E 2698 [1978, 116]. *Proposed by Paul Monsky, Brandeis University.*

Let A_n be an n by n chessboard. The n queens problem (placing n counters on A_n so no two lie in any row, column, or diagonal) admits solutions for all $n\neq 2$ or 3.

Let B_n be the "chessboard" obtained from A_n by identifying opposite sides so that the resulting surface is a torus. (Now, every diagonal of B_n consists of n squares.)

(1) For which values of n does there exist a solution of the n queens problem on B_n?

(2)* If n satisfies (1) then a solution of (1) gives, by cyclic permutation, n superimposable solutions to the n queens problem on A_n. Do there exist n superimposable solutions (for A_n) for any other values of n?

Solution of part (1) *by Richard Z. Goldstein, State University of New York at Albany.* We label rows and columns of B_n by integers modulo n, in the obvious way. All summations will be over integers modulo n and all congruences are modulo n. We claim that the n-queens problem on B_n has a solution iff $(n, 6)=1$.

Assume that we have a solution of this problem with the queens occupying the positions $(i, \sigma(i))$. Clearly σ is a permutation, but moreover $\sigma+\iota$ and $\sigma-\iota$ are also permutations (ι is the identity permutation). Hence $\Sigma(\sigma(i)+i)\equiv\Sigma i$ and consequently $\Sigma i=n(n+1)/2\equiv 0$. This implies that n is odd. From $\Sigma(\sigma(i)+i)^2\equiv\Sigma(\sigma(i)-i)^2$ we obtain $4\Sigma i\sigma(i)\equiv 0$, and so $\Sigma i\sigma(i)\equiv 0$. On the other hand, $\Sigma(\sigma(i)+i)^2\equiv\Sigma i^2$ implies that $2\Sigma i\sigma(i)\equiv-\Sigma i^2$. It follows that

$$-\frac{n(n+1)(2n+1)}{6}\equiv 0$$

and consequently n is not divisible by 3.

Conversely, if $(n, 6)=1$, then we can define σ by $\sigma(i)\equiv 2i$ and we obtain a solution of the n-queens problem on B_n.

Part (1) was also solved by the proposer.

Remarks. O. P. Lossers and Kenneth Rosen inform us that a solution to (1) has been also published by T. Kløve, The modular n-queen problem, Discrete Math. 19 (1977)289–291. According to the proposer the answer to (2) is negative if $n=8$. For this he refers to the Messenger of Math. vol. 44, p. 48.

Linear Independence Modulo Zero Sequences

E 2699 [1978, 117]. *Proposed by Emile Haddad and Peter Johnson, American University of Beirut, Lebanon.*

Suppose that $1=\theta_0>\theta_1>\cdots>\theta_k>0$, and that

$$\sum_{i=0}^{k} a_i \cos n\theta_i\pi \to 0$$

as $n\to\infty$ through the integers.

Does it follow that $a_i=0$ for all i?

Solution by O. P. Lossers, Eindhoven University of Technology, Netherlands. For $-1<t<1$ we have

$$\sum_{r=0}^{k} a_r \sum_{n\geqslant 0} (t\exp i\theta_r\pi)^n = \sum_{r=0}^{k} a_r/(1-t\exp i\theta_r\pi). \tag{1}$$

Taking real parts and dividing by 2 we find

$$f(t)=\sum_{n\geqslant 0} c_n t^n = \sum_{r=0}^{k} \frac{a_r(1-t\cos\theta_r\pi)}{1-2t\cos\theta_r\pi+t^2}, \tag{2}$$

where $c_n=\sum_{r=0}^{k} a_r\cos\theta_r\pi$.

Clearly (2) is valid for complex z satisfying $|z|<1$.

Assume that some $a_r\neq 0$ and let $z_0=\exp i\theta_r\pi$. The second expression for $f(t)$ in (2) shows that z_0 is a simple pole of $f(z)$. On the other hand, if $\varepsilon>0$, then since $c_n\to 0$ $(n\to\infty)$, there exists N such that $|c_n|<\varepsilon$ for $n>N$. We have

$$|(z-z_0)f(z)|\leqslant |z-z_0|\sum_{n=0}^{N}|c_n|+\frac{|z-z_0|}{1-|z|}\varepsilon, \qquad |z|<1. \tag{3}$$

If $z=\rho z_0$ $(0<\rho<1)$ then the right-hand side of (3) is $<2\varepsilon$ for ρ close to 1. This is a contradiction since $\varepsilon>0$ was chosen arbitrarily and z_0 is a simple pole of $f(z)$.

Therefore all $a_r=0$.

Note that this result can be interpreted as saying that the infinite sequences $(\cos n\theta_r\pi)$, $0\leqslant r\leqslant k$, are linearly independent over the reals modulo zero-sequences (i.e., sequences converging to zero).

Also solved by N. Miku (Netherlands) and James Smith.

Complemented Finite Lattices

E 2700 [1978, 117]. *Proposed by Richard Stanley, M.I.T.*

Let L be a finite lattice with minimal element 0 and maximal element 1. Suppose that for all

$x \neq 0$ in L, the interval $[0, x]$ contains an even number of elements. Show that L is complemented, i.e., for all x in L there is a y in L such that $x \wedge y = 0$ and $x \vee y = 1$.

Solution by Ralph Freese, University of Hawaii, and the University of Wyoming Problem Group (independently). We can clearly proceed by induction on $|L|$. Since the claim is obvious for $|L| = 2$, we assume that $|L| > 2$. Let M be the set of maximal elements (coatoms) of L. Since $[0, x] \cap [0, y] = [0, x \wedge y]$, we have by the inclusion-exclusion principle

$$|L| - 1 = \Big|\bigcup_{x \in M} [0, x]\Big| \equiv \sum_{\varnothing \neq S \subset M} [0, \wedge S] \pmod 2.$$

Hence, since $|L| - 1$ is odd, $\wedge M = 0$.

Thus, if $x \neq 0$ there exists $m \in M$ such that $x \vee m = 1$. If $y = x \wedge m$ then by the induction hypothesis applied to $[0, m]$ there exists z such that $y \wedge z = 0$ and $y \vee z = m$. Then $x \wedge z = x \wedge m \wedge z = y \wedge z = 0$ and $x \vee z = x \vee y \vee z = x \vee m = 1$. Hence L is complemented.

Also solved by Anders Björner (Sweden), and the proposer.

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to Professor Roger C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109. To facilitate their consideration, solutions of Advanced Problems in this issue should be typed (in duplicate with double spacing) and should be mailed before August 31, 1979.

6264. *Proposed by William C. Waterhouse, The Pennsylvania State University.*

Despite the commonly repeated story, it appears that Kummer never actually made the mistake of presuming unique factorization in rings of cyclotomic integers. In a paper withdrawn before publication, however, he did once assume that when he had two elements with no common factor he could write 1 as a linear combination of them. [See H. M. Edwards, Arch. Hist. Exact Sci. 14 (1975) 219–236, and 17 (1977) 381–394.] Show that for a Noetherian integral domain this assumption implies unique factorization.

6265. *Proposed by John H. Cook & David Sanders, Metropolitan Life, New York City.*

Prove or disprove the following assertion: If $x = s_n$ is the solution to the equation

$$e^{-x}\left(1 + x + \frac{1}{2}x^2 + \cdots + \frac{1}{n!}x^n\right) = 1/2$$

then $s_n - n$ approaches $2/3$ as n goes to ∞.

6266. *Proposed by Leopoldo Nachbin, University of Rochester.*

It is easily shown that every countable set S has the following property:

(P) Given any function f: $S \times S \to R_+$, there exists a function g: $S \to R_+$ such that $f(x,y) \leqslant g(x)g(y)$ for all x, y in S.

It can be shown that (P) fails if the cardinal number of S is at least equal to that of the continuum. Can it be shown without the Continuum Hypothesis that (P) fails when S is uncountable?

SOLUTIONS OF ADVANCED PROBLEMS

Counterexample

6168 [1977, 659]. *Proposed by Edmond Dale Dixon, Tennessee Technological University*

The following theorem appears in several recent books, e.g., C. G. Cullen, *Matrices and*

Linear Transformations; J. D. Gilbert, *Elements of Linear Algebra*; and Ben Noble, *Applied Linear Algebra.* "*Let A be a diagonalizable matrix with eigenvalues* $\lambda_1,\lambda_2,\ldots,$ *such that* $|\lambda_1|>|\lambda_2| \geqq \cdots$, *and let X be any vector not in the subspace spanned by the eigenvectors associated with* $\lambda_2,\lambda_3,\ldots$. *Let* E_i *be the vector with* 1 *in the* ith *position and zeros elsewhere. Then* $E_i\cdot A^{n+1}X/E_i\cdot A^nX\to\lambda_1$, *for each i, where the denominators are nonzero.*"

Find a counterexample.

Editorial note. No individual solver is singled out due to the number of similar solutions. Most of the counterexamples were in the following pattern:

$$A=\begin{pmatrix}2&0\\0&1\end{pmatrix},\qquad X=\begin{pmatrix}1\\1\end{pmatrix}.$$

Elsner refers to a paper of his, on circumstances where the indicated algorithm fails, presented at the Conference on Numerical Methods, Keszthely, Hungary, 1977 (Proceedings to appear, ed. P. Rozsa, Hungarian Acad. Sci.).

Solved by Jimmy Arnold & Monte B. Boisen, Jr., Marcia Ascher, Paul Cull, Richard J. Driscoll, L. Elsner (West Germany), O. Hajek, A. A. Jagers (Netherlands), David C. Lantz, Detlef Laugwitz (West Germany), R. W. K. Odoni & A. E. Stratton (England), Donald W. Robinson, Robert Singleton, Walter Taylor, Ken Yocom, and the proposer.

Trace of a Product of Matrices

6171 [1977, 660]. *Proposed by R. W. K. Odoni and J. B. Wilker, the University of Exeter, England.*

Let F be a field and let n and d be positive integers, each $\geqslant 2$. Let σ be any permutation of $\{1,2,\ldots,n\}$ and let σ_0 be the n-cycle $j\to j+1 \pmod n$. Prove that σ is a power of σ_0 if and only if for every sequence of n $d\times d$ matrices over F, trace $\Pi_{j=1}^n M_j=$ trace $\Pi_{j=1}^n M_{\sigma(j)}$.

All solutions used the same idea. Equality of the traces of $M_1\cdots M_n$ and $M_i\cdots M_nM_1\cdots M_{i-1}$ follows from trace $AB=$ trace BA and induction. For the converse, let $E_{i,j}$ be the matrix with 1 in the (i,j)-th place and 0 elsewhere. Let $M_1=E_{1,2},\ldots,M_{n-1}=E_{n-1,n}$, $M_n=E_{n,1}$. Then $M_i\cdots M_nM_1\cdots M_{i-1}=E_{i,i}$, with trace 1, while $M_{\sigma(1)}\cdots M_{\sigma(n)}=0$ for every σ that is not a power of σ_0.

Solutions by Marguerite Gerstell, Eric Grinberg, A. A. Jagers (Netherlands), David C. Lantz, L. E. Mattics, Eric Rosenthal, Walter Taylor, Paul J. Zwier, and the proposers.

A Characterization of sin x

6173 [1977, 660]. *Proposed by Otomar Hájek, Case Western Reserve University.*

For C^2 functions $f\neq 0$ vanishing at 0 and π, consider the functional $\inf_{(0,\pi)}f''/f$ (ignore undefined values). Show that its maximum -1 is attained only by $\sin x$ and its multiples.

Solution by J. G. Mauldon, Amherst College. Suppose $f\in C^2$, $\inf_{(0,\pi)}f''/f\geqslant -1$, and that there exist p,c,q, with $0\leqslant p<c<q\leqslant\pi$, such that $f(p)=f(q)=0$ and $f(c)\neq 0$. Then, without invoking the continuity of f'', we shall prove that $p=0,q=\pi$, and $f(x)=A\sin x$.

Writing $\pm f$ for f, we may take $f(c)>0$, and we let (a,b) be a maximal subinterval of (p,q) throughout which $f>0$. Then $f''\geqslant -f$ throughout (a,b), and we observe that $f(a)=f(b)=0$, $f'(a)\geqslant 0$ and $f'(b)\leqslant 0$.

We now write

$$g(x)=f'(x)\sin x-f(x)\cos x \tag{1}$$

so that

$$g(a)=f'(a)\sin a\geqslant 0,$$

$$g'(x)=(f''+f)\sin x \geqslant 0 \quad \text{for} \quad x\in(a,b),$$
$$g(b)=f'(b)\sin b \leqslant 0.$$

These three inequalities imply that $g(x)=0$ for all $x\in[a,b]$ and so (1) yields $f(x)=A\sin x$ $(A\neq 0)$.

Hence $(a,b)=(0,\pi)$, proving as required that $(p,q)=(0,\pi)$.

Also solved by Richard J. Driscoll, A. A. Jagers (Netherlands), L. E. Mattics, Peter Sarnak, Walter Taylor, and the proposer.

More on Converses to Uniform Integrability

6174 [1977, 743]. *Proposed by Bartell W. Huff, Queen's University, Kingston, Ontario*

A family $\mathcal{F}=\{X_\lambda|\lambda\in\Lambda\}$ of random variables is said to be uniformly integrable if

$$\lim_{\alpha\to\infty} \sup_\lambda E|X_\lambda|\cdot I_{[|X_\lambda|\geqslant\alpha]}=0,$$

where I_A is the indicator function of the event A. One sufficient condition for uniform integrability is that there exists a random variable Y such that $|X_\lambda|\leqslant Y$ a.s.,$\forall\lambda$,and $EY<\infty$. Problem 6085 [1976, 292] asks whether the converse is true.

A weaker sufficient condition is that there exists a nonnegative random variable Y such that $P[|X_\lambda|\geqslant\alpha]\leqslant P[Y\geqslant\alpha], \forall\alpha>0, \forall\lambda$ and $EY<\infty$ (Billingsley, *Convergence of Probability Measures*, p. 32). Is the converse to the weaker condition true?

Solution by L. E. Clarke, University of East Anglia, Norwich, England. It will be shown that the converse is false.

Let $\Lambda=(e,\infty)$ and, for $\lambda\in\Lambda$, let X_λ be the random variable which takes the values λ and 0 with corresponding probabilities $1/(\lambda \log \lambda)$ and $1-1/(\lambda \log \lambda)$. Then

$$E(|X_\lambda|I_{\{|X_\lambda|\geqslant\alpha\}})=0 \text{ if } \lambda<\alpha,$$

and$=1/(\log \lambda)$ if $\lambda\geqslant\alpha$. Therefore $\sup_\lambda E(|X_\lambda|I_{\{|X_\lambda|\geqslant\alpha\}})=1/(\log \alpha)$ for $\alpha>e$, and so $\mathcal{F}$ is uniformly integrable.

Now suppose that Y is a nonnegative random variable such that

$$P(|X_\lambda|\geqslant\alpha)\leqslant P(Y\geqslant\alpha) \ (\alpha>0;\lambda\in\Lambda).$$

Then, for $\alpha>e$,

$$P(Y\geqslant\alpha)\geqslant P(X_\alpha\geqslant\alpha)=1/(\alpha \log \alpha).$$

Since $E(Y)=\int_0^\infty P(Y\geqslant\alpha)d\alpha, E(Y)=\infty$.

Also solved by Columbia University Problem Group, Ellen Hertz, Theodore Hill, James Inglis, Terence Shore, Lajos Takács, and the proposer.

Random Polynomials

6175 [1977, 743]. *Proposed by Ignacy I. Kotlarski, Oklahoma State University*

Let $(X_1,X_2,\ldots,X_n)$ be an n-dimensional real random vector. Consider the random polynomial of order $n(n=2,3,\ldots)$ on the complex plane,

$$P_n(\lambda)=(\lambda-X_1)(\lambda-X_2)\cdots(\lambda-X_n),\lambda\in Z$$

and define

$$Z_n=\frac{1}{i}\frac{P_n(i)-(-1)^nP_n(-i)}{P_n(i)+(-1)^nP_n(-i)}, n=2,3,\ldots.$$

Show that if one of the X_k is independent from the others and follows the Cauchy distribution

$$P(X_k \leqslant x) = \frac{1}{2} + \frac{1}{\pi} \text{ arc tan } x, \, x \in R,$$

then all the Z_n are real random variables having the same Cauchy distribution.

Solution by R. R. Miller, Toronto, Ontario. Let the random variable $W_k = \arctan X_k$ for $k = 1,2,\ldots,n$. Define the random variable $S_n = W_1 + W_2 + \cdots + W_n$ reduced modulo π to the interval $[-\pi/2,\pi/2)$. Then an easy calculation shows that the given random variable $Z_n = \tan S_n$. Now, a random variable X has the given Cauchy distribution if and only if arctan X has the uniform distribution on $[-\pi/2,\pi/2)$, and so the given conditions imply that W_k is independent of the remaining W_i's and has the uniform distribution on $[-\pi/2,\pi/2)$. A trite calculation shows that the sum of two independent random variables, each of which is concentrated on $[-\pi/2,\pi/2)$ and one of which has the uniform distribution, has, when reduced modulo π to the interval $[-\pi/2,\pi/2)$, the uniform distribution. Hence S_n has the uniform distribution on $[-\pi/2,\pi/2)$ and thus Z_n has the given required Cauchy distribution.

Also solved by L. E. Clarke (England), Paul S. Bruckman, Columbia University Problem Group, Richard A. Groeneveld, A. A. Jagers, J. H. B. Kemperman, L. E. Mattics, Michael Skalsky, Lajos Takács, and the proposer.

Simple Groups of Square Order

6176 [1977, 744]. *Proposed by Morris Newman and Daniel Shanks, National Bureau of Standards*

Prove that for the most common type of simple group, which is designated $\mathrm{PSL}_2(p^n)$, its order N is never a perfect square. Find at least one simple group that does have square order.

Solution by Lorraine L. Foster, California State University, Northridge. (1) A more general result is proved.

THEOREM. *The order of* $\mathrm{PSL}_m(p^n)$ *is not a square for any* $m \geqslant 2$, $n \geqslant 1$, *and* p *a prime.*

We begin with four easy lemmas. All letters below (except $i = \sqrt{-1}$) will represent elements of the set Z of rational integers.

LEMMA 1. $x^3 - 1 = 4\square$ *implies* $x = 1$.

Proof. Suppose that $x^3 = 4y^2 + 1 = (2y + i)(2y - i)$. Then $2y + i = (a + bi)^3 = a(a^2 - 3b^2) + ib(3a^2 - b^2)$ for some a, b $(\in Z)$. (For, $Z[i]$ is a U.F.D. in which units are cubes. Further, since $(2y + i)(-y) + (2y - i)(y + i) = 1$, $2y + i$ and $2y - i$ are relatively prime.) Then, $b(3a^2 - b^2) = 1$, $3a^2 - 1 = \pm 1$, $3a^2 = 0$ or 2. Thus $a = 0, y = 0$, $x = 1$. ■

LEMMA 2. $x^3 - 1 = 2\square$ *implies* $x = 1$.

Proof. Suppose $x^3 = 2y^2 + 1 = (1 + \sqrt{-2}\, y)(1 - \sqrt{-2}\, y)$. Then $1 + \sqrt{-2}\, y = (a + b\sqrt{-2}\,)^3 = a(a^2 - 6b^2) + b(3a^2 - 2b^2)\sqrt{-2}$. (For $Z[\sqrt{-2}\,]$ is a U.F.D. Further, since $(1 + \sqrt{-2}\, y)(1 - \sqrt{-2}\, y - y^2) - y^2(1 - \sqrt{-2}\, y) = 1$, $1 + \sqrt{-2}\, y$ and $1 - \sqrt{-2}\, y$ are relatively prime.) Thus, $a(a^2 - 6b^2) = 1$, $1 - 6b^2 = \pm 1$, $6b^2 = 0$, 2. Hence $b = 0, y = 0$, $x = 1$. ■

LEMMA 3. $x^4 + 1 = 2\square$ *implies* $x = \pm 1$.

Proof. Suppose $x^4 + 1 = 2y^2$. Then x is odd and

$$(x^4 - 1)^2 = (x^4 + 1)^2 - 4x^4 = 4(y^4 - x^4), \qquad y^4 - x^4 = u^2 \; (u = (x^4 - 1)/2 \in Z).$$

By Theorem 114, p. 227, [**1**], $uxy = 0$. It follows easily that $x = \pm 1$. ■

LEMMA 4. *Let* $g = (a^{2^N} + 1, a^M - 1)$, $N \geqslant 1$, $2^{N+1} \nmid M$. *Then* $g = 1$ *or* 2.

Proof. Suppose $p|g$, p an odd prime. Then $a^{2^N}\equiv -1$, $a^M\equiv 1 \pmod p$. Clearly the order of a mod p is 2^{N+1}. Thus $2^{N+1}|M$, a contradiction. Since $a^{2^N}+1\equiv 1$ or $2 \pmod 4$ we are done. ■

Proof of Theorem. Let $f(m,x)=(x^{m(m-1)/2}\prod_{j=2}^{m}(x^j-1))/d$, where $d=(m,x-1)$, $m\geqslant 2$. Then $f(m,p^n)=|\mathrm{PSL}_m(p^n)|$ (p. 708, [**2**]). We shall prove in fact that $f(m,x)\neq\square$ for $x\neq\pm 1, 0$. Thus we suppose $f(m,x)=\square$ and examine several cases.

$m=2$: $x((x^2-1)/d)=\square$. Then $x=z^2$, $(z^2)^2-1=d\square$, $d=1$ or 2. If $d=1$ then $|z|=1$, $|x|=1$. Now $z^4-y^4=2w^2$ implies $wyz=0$ (Theorem 116, p. 230, [**1**].) Hence $d=2$ implies $|z|=1$, $|x|=1$.

$m=3$: $(x(x-1))^2x(x+1)z=\square$ where $z=(x^2+x+1)/d\in Z$. (For, in general, $x\equiv 1$, $x^{n-1}+\cdots+x+1\equiv n\equiv 0 \pmod d$.). Hence $x(x+1)z=\square$. Since x, $x+1$, z are pairwise relatively prime we have: $x=y^2$, $(y^2)^2+1=\square$, $y=0$, $x=0$.

$m=4$: $(x^2+1)((x^3-1)/d)=\square$. If x is even, then $d=1$, $x^2+1=\square$ (by Lemma 4), $x=0$. If x is odd, then $d=2$ or 4 and $x^2+1=2\square$. (For $2|x^2+1$, $4\nmid x^2+1$ so that $2|(x^3-1)/d$. Hence, by Lemma 4, $(x^2+1, (x^3-1)/d)=2$.) Thus $(x^3-1)/d=2y^2$, $x^3-1=(2y)^2$ or $2(2y)^2$, $x=1$, from Lemmas 1, 2. ■

$m=5$: $f(5,x)$ is of the form $B^2A(x^2+x+1)(x^2+1)$, where $A=(x^4+x^3+x^2+x+1)/d\in Z$. Hence, by Lemma 4, $x^2+1=\square$, $x=0$. (For A, x^2+x+1 are odd divisors of x^5-1, x^3-1, respectively.)

$m=6$, 7: $f(6,x), f(7,x)$ are of the form $B^2(A/d)x(x^2+1)$, where $A/d\in Z$ and A is a product of terms of the form x^M-1, $4\nmid M$. Hence, by Lemma 4, $x+y^2$, $y^4+1=\square$ or $2\square$. In the former case, $y=0=x$ and in the latter case, by Lemma 3, $|y|=1=|x|$.

$2^k\leqslant m<2^{k+1}$, $k\geqslant 3$. $f(m,x)$ is of the form $x^t(A/d)(x^{2^{k-1}}+1)$, where A is a product of terms of the form x^M-1, $2^k\nmid m$. Thus $(x^{2^{k-3}})^4+1=x^{2^{k-1}}+1=\square$ or $2\square$, $|x|=0$ or 1 again. ■

References

1. T. Nagell, Introduction to Number Theory, John Wiley, New York, 1951.
2. J. F. Hurley and A. Rudvalis, Finite simple groups, this MONTHLY, 84 (1977)693–714.

(2) From the known orders of the groups $\mathrm{PS}p_4(q)$ it follows that the order of such a group can be a square if and only if q^2+1 is twice a perfect square. This condition is satisfied by the following primes: 7, 41, 239, and 9369319.

Also solved by the Columbia University Problem Group, Paul Eitner, Marguerite Gerstell, J. E. Humphreys, A. A. Jagers (Netherlands), Kenneth Klinger, Catherine M. Murphy, Nikolay Williams, J. Benjamin Zipperer, and the proposers. All examples with square orders were groups $\mathrm{PS}p_4(q)$, as above, and Humphreys asks if $\mathrm{PS}p_4(7)$ is the smallest simple group of square order.

Newman, Shanks, and H. C. Williams, in a paper to appear in Acta Arithmetica, report that extensive computer calculation has shown that, for $p=3$, 5, 7, 19, 29, 47, 59, 163, 257, 421, 937, and 947, the integer

$$q=\left[(1+\sqrt{2})^p+(1-\sqrt{2})^p\right]/2$$

is a prime such that q^2+1 is twice a square.

MISCELLANEA

23. From the point of view of logic we may say that the apparently simple is most often very complicated, and even if it is not so, symbolism will make it seem so . . . (continued on p. 318)

important pedagogical drawbacks. Many students will already tend to conceive of infinity as just a large number, and the presentation will tend to reinforce their feeling that ∞ (which is depicted) *is* a number. Moreover, although the hotel rooms are shown numbered $1,2,3,\ldots$, we are not told that the arriving salesmen can be ordered in like manner. In particular, the pairing of salesmen with rooms will fail if the "infinite number" of salesmen happens to be uncountable.

The second serious error in the treatment is more transparent: "The second infinite number is the number of points in the entire universe..." (frame 62). The booklet makes it clear that the authors have indeed got it wrong: "The set of real numbers form a higher infinite set that Cantor called aleph-one... ."

The strip *Probability Paradoxes* also has a couple of lapses. At one point (frame 18), the implication is that because calculated probabilities add to 1, the corresponding events are exhaustive; whether the fractions calculated are the *correct* probabilities or not is left begging: "If our probabilities are correct, they should add to one. They do. This tells us it is certain that one of the three splits will occur." The discussion of the Gamow–Stern elevator paradox begins by hypothesizing that the elevators "average the same waiting time on each floor" (frame 25). The teacher seeking elucidation will find that the booklet adds that they must have constant speeds, but says they have "the same average waiting time on each floor." Is the waiting time referred to the one experienced by would-be passengers or is it just the length of the pause the elevator makes at each floor? And what are we to make of *either* hypothesis about averages, much less what the authors *meant* to say?

The filmstrips differ to some degree in the maturity that they presume. Perhaps the hardest collection of paradoxes to digest are those about time, which are of a significantly more philosophical turn of mind than the others. In addition, the explanation of one of the episodes in the strip (frames 20–22) hinges on divergence of the harmonic series. The paradoxes involving probability and statistics are at the other end of the spectrum; they are closer to the actual material that students will be studying, or have studied, on those topics. As might be expected, however, motivation is probably enhanced more for students who have had some smattering of the concepts of probability or statistics than for those who have not, although the curiosity of the latter is bound to be aroused. The remaining filmstrips fall more toward a middle ground.

PAUL J. CAMPBELL, Beloit College
PAUL JORGENSEN, Carleton College
RICHARD SCOTT, Northfield, Minnesota, High School

(continued from p. 315) . . . and thus draw attention to what might easily be overlooked.

P. E. B. Jourdain, The Philosophy of Mr. B*rtr*nd R*ss*ll, London, 1918, p. 28.

REVIEWS

Edited by J. Arthur Seebach, Jr., and Lynn A. Steen
with the assistance of the mathematics departments of
St. Olaf, Carleton, and Macalester Colleges
Collaborating Editor for Films: Seymour Schuster, Carleton College

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FILMS

The Paradox Box. Set of six filmstrips with accompanying audio cassettes and booklets: *Geometry Paradoxes, Logic Paradoxes, Number Paradoxes, Probability Paradoxes, Statistics Paradoxes, Time Paradoxes*. Scientific American, Inc., New York, N.Y., 1975. Concept and script: Martin Gardner and Bob Tappay. Illustrations: Jim Glen. Sound track: Bernard Cowan. Produced by Bob Tappay. Price: $135.

"The Paradox Box is a set of filmstrips on mathematical paradoxes in the fields of logic, probability, numbers [including set theory and transfinite numbers], geometry, statistics and time. . . . The word 'paradox' is used in a broad sense to include any mathematical result so contrary to intuition and common sense that it arouses an immediate emotion of surprise. Such paradoxes are of three main types: 1. An assertion that seems clearly false but actually is true. 2. An assertion that seems clearly true but actually is false. 3. A line of reasoning that seems impeccable but which leads to a logical contradiction." (Taken from the introductory material appearing in each booklet.)

The six filmstrips vary from 55 to 67 frames each (plus head and tail). In this review, reference to a specific frame will be made by using the initial letter of the title of the filmstrip—e.g., "P" for "*Probability Paradoxes*"—plus the number of the frame, lead-in and title frames being disregarded.

All of the illustrations from which the filmstrips are photographed are watercolors by the same artist. They are simple and uniform, the style being an amusing surrealism, so that the filmstrips resemble cartoon strips.

Each filmstrip is accompanied by an audio cassette tape of between 17 and 21 minutes of narration, background music, and sound effects. The same audio track occurs on both sides of the tape: on one side with audible tones to indicate filmstrip advance, and on the other with inaudible tones for automatic projectors.

The filmstrip and cassette are augmented by an 18-page guidebook which reproduces the slides and narration and also provides further commentary and references.

The authors' objectives were "to motivate student interest in mathematics; to introduce the viewer to significant mathematical ideas; to initiate enrichment activities; to give insight into the process of problem solving; and to give students an appreciation of the beauty, variety, and even the humor of modern mathematics." The authors recommend that each filmstrip first be shown from beginning to end without interruption, followed by a subsequent showing during which interruption and discussion may take place. The series is designed for secondary and junior college audiences.

Although we will express reservations and criticism below about some aspects of the series, our overall reaction to it is overwhelmingly positive. It is much better than most mathematical films, especially those for secondary school students. The artwork is eye-catching and the audio is dynamic, so that the combined media should hold the students' attention. The questions raised are thought-provoking, and the language in general is understandable. We recommend the series highly, not least as a means to increase motivation and stimulate curiosity. It can

provide a refreshing vacation from the customary presentation of mathematics in finished form.

A teacher should be careful about the manner in which the materials are treated and used. Most important, they are not designed to deliver material or substitute for teacher explanation of conventional topics. The authors make this clear, as when they state of the statistics filmstrip that it is "in no sense an introduction to statistics. . . . The filmstrip, by itself, does not teach the elements of statistics."

We strongly disagree with the authors' recommendation that a filmstrip be introduced by showing it uninterrupted in its entirety. Although each filmstrip presentation lasts only 20 minutes or so, the pace is brisk (no repetition or recapping) and a large number of different questions are treated. Each filmstrip consists of from 10 to 19 "episodes," or groups of frames. There are so many ideas presented on each strip that students can lose interest if shown the entire strip at once; the questions and answers come so quickly that it makes the mathematics seem too easy or too magical. Moreover, students conditioned to conceive of mathematics as a calculational activity which leads to exactly one "correct" answer per problem will be unsettled by the ambiguity resulting from the several paradoxes that are raised but *not* decisively resolved. Zipping through a few of these may erode their confidence in mathematics; while a rapid succession of quickly parried paradoxes may tend to reinforce their conviction that, whatever there may be to mathematics, it is certainly beyond them.

We suggest instead that the teacher unfold a little of a filmstrip at a time. Such a gradual unveiling is facilitated by the division of the strip into episodes. If the students are to derive more than entertainment benefit from the materials, it is absolutely essential that there be an immediate opportunity for thorough discussion and debate of each paradox as it comes up. To be fully prepared to facilitate the discussion, the teacher must carefully preview filmstrip and tape, read the accompanying booklet, and devote some time to pondering the paradoxes.

The series will have difficulty meeting secondary-school adoption standards in many states, as the strips do not afford proportional representation of blacks or other minority groups. Although an effort seems to have been made at some points to show women in status roles equal to men (e.g., S, 22: a woman attorney is referred to), at other places the presentation appeals to stereotyped male-female roles (P, 9 and 10: boy kittens are depicted with a football, girl kittens with a doll). Some will no doubt take offense at the portrayal of a worker as dumb in comparison with his knowledgeable boss (S, 3–10), the parallel use of "man" and "girl" (L, 53 and 54), and a statistician named Miss Lonelyhearts with a cutesy-pie voice who confesses at the end of the episode, "But I *still* don't understand it!" (S, 39–49).

Occasional slip-ups occur in the filmstrip frames. In *Number Paradoxes*, frame 50, the numerical coding of letters and symbols is presented in an incomplete and confusing manner, and subsequent frames reveal that the numbers given for comma and period are wrong. Moreover, the booklet fails to provide any clarification. In *Geometry Paradoxes*, frame 65 is reproduced correctly on the filmstrip but mirror-reversed in the booklet, while frame 10 is incorrectly drawn (the balls' reflections appear in the wrong place, and the balls should be perpendicular to the mirror). In *Probability Paradoxes*, the last two frames appear in correct order on the filmstrip but transposed in the booklet.

A few of the words on the tapes will likely be vocabulary unfamiliar to the audience, and the teacher will find it useful to bring out their meanings: "black hole" (N, 55: physicists will object to its misleading—though fanciful—use here); "matrix" (N, 35: "square array" or other terminology would be better, since no use is made of the matrix representation); "cosmologists" (G, last frame); "mass" (N, 48); and "unbounded" (G, 57). A couple of words are pronounced in an unorthodox fashion: "kilometer" (S, 15) and "Omega" (L, 48); and there is no good reason for employing "decimeter" (G, 19).

Mention must be made of a small number of mistakes in the mathematical content. The exposition of cardinal numbers in *Number Paradoxes* (frames 55–63) is seriously marred by two errors. One lies in the mention of a hotel with an "infinite number" of rooms, which is confronted with an "infinite number" of arriving salesmen. The use of this terminology has

important pedagogical drawbacks. Many students will already tend to conceive of infinity as just a large number, and the presentation will tend to reinforce their feeling that ∞ (which is depicted) *is* a number. Moreover, although the hotel rooms are shown numbered $1,2,3,\ldots$, we are not told that the arriving salesmen can be ordered in like manner. In particular, the pairing of salesmen with rooms will fail if the "infinite number" of salesmen happens to be uncountable.

The second serious error in the treatment is more transparent: "The second infinite number is the number of points in the entire universe..." (frame 62). The booklet makes it clear that the authors have indeed got it wrong: "The set of real numbers form a higher infinite set that Cantor called aleph-one... ."

The strip *Probability Paradoxes* also has a couple of lapses. At one point (frame 18), the implication is that because calculated probabilities add to 1, the corresponding events are exhaustive; whether the fractions calculated are the *correct* probabilities or not is left begging: "If our probabilities are correct, they should add to one. They do. This tells us it is certain that one of the three splits will occur." The discussion of the Gamow–Stern elevator paradox begins by hypothesizing that the elevators "average the same waiting time on each floor" (frame 25). The teacher seeking elucidation will find that the booklet adds that they must have constant speeds, but says they have "the same average waiting time on each floor." Is the waiting time referred to the one experienced by would-be passengers or is it just the length of the pause the elevator makes at each floor? And what are we to make of *either* hypothesis about averages, much less what the authors *meant* to say?

The filmstrips differ to some degree in the maturity that they presume. Perhaps the hardest collection of paradoxes to digest are those about time, which are of a significantly more philosophical turn of mind than the others. In addition, the explanation of one of the episodes in the strip (frames 20–22) hinges on divergence of the harmonic series. The paradoxes involving probability and statistics are at the other end of the spectrum; they are closer to the actual material that students will be studying, or have studied, on those topics. As might be expected, however, motivation is probably enhanced more for students who have had some smattering of the concepts of probability or statistics than for those who have not, although the curiosity of the latter is bound to be aroused. The remaining filmstrips fall more toward a middle ground.

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(continued from p. 315) . . . and thus draw attention to what might easily be overlooked.

P. E. B. Jourdain, The Philosophy of Mr. B*rtr*nd R*ss*ll, London, 1918, p. 28.

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, L. *Women and Mathematics, Science and Engineering.* Else Høyrup (Roskilde U Center, Bogevej 8, DK 3500 Vaerløse, Denmark), 1978, vi + 62 pp, (P). A bibliography by "a mathematician who has switched to work with sociological and psychological problems in relation to mathematics." In addition to psychological and sociological studies, the bibliography does include historical references for a number of the best-known women mathematicians. JAS

GENERAL, S? *Games, Tricks, and Puzzles for a Hand Calculator.* Wallace Judd. Dilithium Pr, 1978, 91 pp, $4.95 (P). [ISBN: 0-918398-20-7] Originally published by Dymax (TR, February 1975). Miscellaneous collection of things that can be done with a hand calculator, marred by numerous typographical errors. RSK

GENERAL, S. *A Book on Casino Gambling, Written by a Mathematician and a Computer Expert, Revised, Expanded, Second Edition.* Virginia L. Graham, C. Ionescu Tulcea. Van N-Rein, 1978, x + 156 pp, $6.95 (P). [ISBN: 0-442-25829-1] Non-mathematical description of how to play craps, roulette, blackjack, baccarat, slot machines, and keno, including odds and blackjack strategies. RSK

GENERAL, S(13-16), L. *Mathematics: An Introduction to Its Spirit and Use.* Morris Kline. Freeman, 1979, 249 pp, $6 (P). [ISBN: 0-7167-0369-6] A new *Scientific American* reader, a sequel to Kline's 1968 *Mathematics in the Modern World,* prefaced by an eloquent apologia by Kline. Some articles in this one were also in the 1968 volume; many others are reprints of Martin Gardner columns; few are recent. (Beware confusion with Saaty and Weyl's book of similar purpose and title: *The Spirit and Uses of the Mathematical Sciences,* McGraw, 1969.) LAS

GENERAL, S(10-13). *Games With the Pocket Calculator.* Sivasailam Thiagarajan, Harold D. Stolovitch. Dilithium Pr, 1978, 47 pp, $3.95 (P). [ISBN: 0-918398-20-7] Reprinting of 1976 Dymax original edition (TR, June 1976). LAS

GENERAL, P, L. *PRIME-80: Proceedings of a Conference on Prospects in Mathematics Education in the 1980's.* MAA, 84 pp, $3.50 (P). [ISBN: 0-88385-429-5] A multifaceted assessment of the current state of collegiate mathematics and its future. The recommendations may not be surprising, but they provide an important guide for the MAA and college educators. Throughout there is an emphasis on the importance of students' pre-college preparation: "There is much evidence available it is extremely difficult, if not impossible, for colleges to make up in any reasonable amount of time for academic deficiences of entering college students." PJC

PRECALCULUS, T(13: 1). *Trigonometry, An Analytic Approach, Third Edition.* Irving Drooyan, Walter Hadel, Charles C. Carico. Macmillan, 1979, ix + 373 pp, $13.95. [ISBN: 0-02-330240-2] This *Third Edition* (*Second Edition,* TR, April 1973) includes exercises for use with calculators. Topics rearranged from earlier editions. LLK

PRECALCULUS, T(13: 2), S. *Background for Calculus.* Ada Peluso. Kendall/Hunt, 1978, 604 pp, $14.95 (P). [ISBN: 0-8403-1944-4] Excellent precalculus text or supplement for students in calculus. Treatment is informal but complete. Ample exercises designed to prepare for calculus. LLK

FOUNDATIONS, T(18), P. [illegible]. Thomas Jech. Pure and Appl. Math., V. 79. Acad Pr, 1978, xi + 621 pp, $53. [ISBN: 0-12-[illegible]] [illegible]ear encyclopedic coverage of axiomatic set theory, with a ponderous wealth of material on forcing, large cardinal theory and descriptive set theory. Treatment is concise throughout, with no apologies to the timid. Random browsing revealed a careless flaw (prefaced by "it is clear that") in proof of Ramsey's theorem; hopefully an isolated lapse. Great numbers of pertinent exercises lend the book to graduate or professional study. Bibliography and historical notes. GHM

FOUNDATIONS, T(17-18), P. *Recursion-Theoretic Hierarchies.* Peter G. Hinman. Springer-Verlag, 1978, xii + 480 pp, $44. [ISBN: 0-387-07904-1; 3-540-07904-1] Synthetic exposition of generalized recursion theory and descriptive set theory as they deal with problems of definability, i.e., classification of mathematical objects by complexity of their definition. Excellent coverage of classical and recent results on the analytical and projective hierarchies and on recursion in higher types. Exercises (with hints) extend the results of the text. A well-written challenge to the "logic cabal" (see preface) with many rewards for the diligent reader (including strawberries and rum!). GHM

COMBINATORICS, S(16), P, L. *Studies in Combinatorics.* Ed: Gian-Carlo Rota. Stud. in Math., V. 17. MAA, 1978, xi + 262 pp, $14. [ISBN: 0-88385-117-2] A wide sample of current trends in combinatorics is represented by these seven survey articles. Topics include combinatorial matrix theory, proof techniques for extremal problems on finite sets, Ramsey theory, the use of generating functions in combinatorial problems, non-constructive or probabilistic methods for problems in discrete mathematics, matroids and combinatorial geometries, and the construction of combinatorial systems and designs. Each article comes with a significant list of references. CEC

COMBINATORICS, P, L. *Percy Alexander MacMahon. Collected Papers, Volume I: Combinatorics.* Ed: George E. Andrews. MIT Pr, 1978, xxix + 1438 pp, $75. [ISBN: 0-262-13121-8] "In all mathematics it would be hard to find a more blatant instance of [historical injustice] than the theory of symmetric functions... Each generation rediscovers them and presents them in the latest jargon as the *dernier cri*. Today it is K-theory, yesterday it was categories and functors, and the day before group representations. Behind these and several other attractive theories stands one immutable source: symmetric functions. ...Anyone who pretends familiarity with [them]...cannot avoid wading through the papers of Major Percy A. MacMahon." --from the Preface by G.C. Rota. This volume contains the first 12 of 20 chapters, organized to parallel MacMahon's *Combinatory Analysis*, with extensive commentary to place his work in historical context. LAS

NUMBER THEORY, T*(14: 1), S, L. *A Computer [illegible] Manual for Number Theory, Student Manual.* Donald G. Malm. COMPUTe (Dartmouth College, Hanove[illegible]), 1978, xii + 256 pp, $6 (P); *Instruc-[illegible]* [illegible]i + 455 pp, $10 (P). A solid one-semester course in experimental number theory re-[illegible] of Basic. The *Instructor's Supplement* gives sample programs for all the exercises. Topics include factorization, congruence, diophantine equations, distribution of primes, quadratic congruences, continued fractions, partitions and magic squares. Is this the book of number theory of the future? CEC

NUMBER THEORY, T(17-18: 2), P. *Cyclotomic Fields.* Serge Lang. Grad. Texts in Math., V. 59. Springer-Verlag, 1978, xi + 253 pp, $19.80. [ISBN: 0-387-90307-0; 3-540-90307-0] A cyclotomic field is a subfield of Q(ω), ω a root of unity. This is a modern account of their theory, which goes back to Kummer but was revived in the 1950's by Iwasawa, Leopoldt and others. JD-B

NUMBER THEORY, T(14: 1), S, L. *An Introduction to Number Theory.* Harold M. Stark. MIT Pr, 1978, x + 347 pp, $8.95 (P). [ISBN: 0-262-69060-8] Corrected reprint of a 1970 publication (TR, June 1970; ER, October 1971). This book is noteworthy because of its outstanding problems, its geometrical approach to continued fractions and its introduction to quadratic fields. Contains all the material usually covered in a one semester introductory course in number theory except for the intentional omission of the law of quadratic reciprocity. CEC

NUMBER THEORY, S(18), P. *Simultaneous Approximations in Transcendental Number Theory.* A. Bijlsma. Math. Centre Tracts, No. 94. Math Centrum, 1978, vii + 101 pp, Dfl. 12 (P). [ISBN: 90-6196-162-9] This monograph catalogues recent results which generalize theorems like Hermite-Lindemann and Gel'fond-Schneider to the simultaneous approximation form. The author's results of this type are given in their entirety, together with consideration of the p-adic case. Includes a bibliography. CEC

LINEAR ALGEBRA, S(14-16). *Exercices et problèmes résolus d'algèbre, Deuxième édition revue et corrigée.* Lucien Chambadal. Dunod (US Distr: SMPF, 14 E. 60th St., NY 10022), 1972, 213 pp, 48F (P). [ISBN: 2-04-009394-X] Algebra in this case means linear and multilinear algebra, polynomials, and quadratic and hermitian forms. JAS

LINEAR ALGEBRA, S(15-17). *Mathematics for Econometrics.* Phoebus J. Dhrymes. Springer-Verlag, 1978, viii + 136 pp, $18.80 (P). [ISBN: 0-387-90316-X; 3-540-90316-X] Concise definition of matrix algebra used by econometricians. Unappealing style for most economists: no examples, solved problems, or indications of where concepts are used. Possible reference for persons studying econometrics. Limited index. Too expensive for a paperback. WC

ALGEBRA, T*(18: 1), S, P. *Introduction to Categories, Homological Algebra and Sheaf Cohomology.* Jan R. Strooker. Cambridge U Pr, 1978, ix + 246 pp, $28.50. [ISBN: 0-521-21699-0] A text based on course notes of several years before, supported by clear explanations, good references, and an extensive index. The book gives a feeling of relaxed perspective. There are, however, few exercises or problems--possibly not a significant problem for a text at this level. The author consciously omits the recent developments in non-abelian homological algebra and theory of toposes. JAS

FINITE MATHEMATICS, T(13-14), S, P. *Rekursive Verfahren: Ein problemorientierter Eingangskurs.* Roland Stowasser, Benno Mohry. Hermann Schroedel, 1978, 105 pp, (P). [ISBN: 3-507-83201-1] An introduction to finite mathematics, intended for use in German schools before calculus, linear algebra and probability are studied. Should interest those who train mathematics teachers. JD-B

FINITE MATHEMATICS, T*(13: 1, 2). *Mathematics with Applications in the Management, Natural, and Social Sciences, Second Edition.* Margaret L. Lial, Charles D. Miller. Scott F, 1979, 658 pp, $15.95 [ISBN: 0-673-15185-9]; *Study Guide with Computer Problems,* 97 pp, $4.95 (P) [ISBN: 0-673-15187-5]. Changes from first edition (TR, November 1974; ER, June-July 1975) include an expanded algebra review, a new chapter on the mathematics of finance, an improved presentation of linear programming, more calculus, and many more examples and applications. The study guide includes 27 Basic programs, along with explanations and sample runs. TRS

CALCULUS, T(13: 1), P. *Mathematik heute: Grundkurs Analysis 1.* Hermann Athen, Heinz Griesel. Hermann Schroedel, 1978, 105 pp, (P). [ISBN: 3-507-83082-5] An introduction to differential calculus written for use in German schools. Covers only a little material, but carefully and clearly. JD-B

CALCULUS, S(13). *Computer Laboratory Manual for Calculus and Linear Algebra.* K.D. Stroyan. CONDUIT (P.O. Box 388, Iowa City, IA 52240), 1978, iv + 292 pp, $6.50 (P). Exercises, programs, sample printouts for an independent computer lab taught simultaneously with courses in (infinitesimal-based) calculus and linear algebra. "The beauty of the computer is that [students] can experiment with mathematics before [they] reach full mathematical understanding." LAS

REAL ANALYSIS, P. *Theory and Applications of Differentiable Functions of Several Variables. VI.* Ed: S.M. Nikol'skii. Proc. of Steklov Inst. of Math., No. 140. AMS, 1979, iv + 312 pp, $60 (P). [ISBN: 0-8218-3039-2]

REAL ANALYSIS, T*(15-16: 1), S*, P, L*. *An Introduction to Orthogonal Polynomials.* T.S. Chihara. Math. and its Appl., V. 13. Gordon, 1978, xii + 249 pp, $39.50. [ISBN: 0-677-04150-0] A unified, elementary treatment of real orthogonal polynomials: a sequence of polynomials is defined to be orthogonal with respect to an abstract moment functional which is later represented as a Riemann-Stieltjes integral. Topics include Gauss quadrature, moment problems, continued fractions, Krein's theorem. Presumes only elementary real variables and linear algebra. Many, good exercises. An excellent text for undergraduate study. TRS

COMPLEX ANALYSIS, S(18), P. *Lecture Notes in Mathematics-646: Extremum Problems for Bounded Univalent Functions.* Olli Tammi. Springer-Verlag, 1978, vii + 313 pp, $12 (P). [ISBN: 0-387-08756-7; 3-540-08756-7] The aim of this monograph is to give a survey of the main methods concerning univalent functions, i.e., the use of Löwner's functions, Schiffer's differential equations and Grunsky type inequalities. The reference list includes these works which are directly related to the theme. CEC

COMPLEX ANALYSIS, P. *Advances in Holomorphy.* Ed: Jorge Alberto Barroso. Math. Stud., V. 34. North-Holland, 1979, ix + 766 pp, $62.25 (P). [ISBN: 0-7204-1965-4] Proceedings of the *Seminário de Holomorfia* held in Rio de Janeiro in September 1977. 32 research and expository papers on holomorphic functions and their applications. LAS

COMPLEX ANALYSIS, T*(16-17: 1), S, L. *Complex Analysis: The Argument Principle in Analysis and Topology.* A.F. Beardon. Wiley, 1979, xiii + 239 pp, $35. [ISBN: 0-471-99671-8] A novel exposition of complex analysis, which places the concept of angle at the forefront. Well in advance of line integrals, the author introduces winding numbers and uses them to discuss the topology and geometry of the subject without the calculus. The Cauchy theory then appears in a straight-forward, natural way. A rigorous, well-written text with many exercises. TRS

DIFFERENTIAL EQUATIONS, P. *Differential-Delay Equations With Two Time Lags.* Roger D. Nussbaum. Memoirs No. 205. AMS, 1978, vi + 62 pp, $6.80 (P). [ISBN: 0-8218-2205-5] The author studies criteria which guarantee periodic solutions to equations which can be transformed to the form $y'(t) = \alpha f(y(t-1)) - \beta f(y(t-2))$, where f is increasing, bounded below and satisfies $f(0) = 0$. TRS

DIFFERENTIAL EQUATIONS, T(14-15: 1, 2). *Linear Algebra & Differential Equations: An Integrated Approach.* Charles G. Cullen. Prindle, 1979, ix + 414 pp, $15.95. [ISBN: 0-87150-262-3] Well-integrated, up-to-date, classroom-tested treatment of standard fare. Light on applications. Flexible enough for use by students with or without experience in linear algebra. Matrix exponential treated somewhat at length. Non-routine exercises tagged M, T, A or C for motivational, theoretical, application or challenging. Most numerical answers given. JK

DIFFERENTIAL EQUATIONS, P. *Nonlinear Evolution Equations.* Ed: Michael G. Crandall. Acad Pr, 1978, ix + 255 pp, $11. [ISBN: 0-12-195250-9] This volume constitutes the proceedings of the Symposium on Nonlinear Evolution Equations held in Madison, Wisconsin, October 17-19, 1977. Thirteen of the fourteen invited papers appear here; the paper of Moser appears elsewhere. CEC

DIFFERENTIAL EQUATIONS, T(14-15: 1). *Differential Equations.* John A. Tierney. Allyn, 1979, xii + 403 pp, $14.95. [ISBN: 0-205-06167-2] An applications-oriented elementary introduction, emphasizing the role of differential equations in modelling diverse phenomena in the physical, behavioral, and biological sciences. Includes chapters on the Laplace transform, on series solutions, on numerical methods, and on partial differential equations and Fourier series. LAS

DIFFERENTIAL EQUATIONS, T*(16-17: 1), S, L. *Partial Differential Equations, Third Edition.* Fritz John. Appl. Math. Sci., V. 1. Springer-Verlag, 1978, ix + 198 pp, $14.80. [ISBN: 0-387-90327-5; 3-540-90327-5] A completely rewritten new edition. Main content of earlier editions has survived, though in new form, and there are considerable additions in which extensive use is made of Fourier transform techniques, Hilbert space, and finite difference methods. Exercises. (*First Edition,* TR, June-July 1971; ER, November 1972; *Second Edition,* TR, August-September 1975.) TRS

DIFFERENTIAL EQUATIONS, T*(16-17: 1), S, L. *Ordinary Differential Equations, Third Edition.* Garrett Birkhoff, Gian-Carlo Rota. Wiley, 1978, xi + 342 pp, $18.95. [ISBN: 0-471-07411-X] A sophisticated introductory text which smooths the transition from the elementary to the more advanced theory. The discussions of standard topics include many concepts and methods atypical of most texts, e.g., transfer functions, Nyquist diagrams, the method of majorants, the use of Green's functions. The two chapters on numerical methods provide error estimates and are excellent. Abundant exercises. TRS

DIFFERENTIAL EQUATIONS, S(18), P. *Systems of Partial Differential Equations and Lie Pseudogroups.* J.F. Pommaret. Math. and its Appl., V. 14. Gordon, 1978, xiv + 411 pp, $39. [ISBN: 0-677-00270-X] Two major topics are discussed: Part I is a modern version by way of manifold theory and Spencer cohomology of earlier work by Janet on non-linear systems of partial differential equations; Part II treats Lie pseudogroups. Though intended "for any graduate student" it assumes considerable background. Includes exercises, bibliography, index of definitions. JS

NUMERICAL ANALYSIS, P. *Numerical Analysis.* Ed: Gene H. Golub, Joseph Oliger. Proc. of Symp. in Appl. Math., V. XXII. AMS, 1978, viii + 135 pp, $14. [ISBN: 0-8218-0122-8] Papers (in typescript) from AMS short course in Atlanta, January 1978. Not a survey of numerical analysis, but a selective entree to areas of greatest current research interest. LAS

NUMERICAL ANALYSIS, S(17-18), P. *Numerische Behandlung partieller Differentialgleichungen.* Th. Meis, U. Marcowitz. Springer-Verlag, 1978, viii + 452 pp, $20.90 (P). [ISBN: 0-387-08967-5; 3-540-08967-5] An account of modern methods for the numerical solution of initial- and boundary-value problems for linear and nonlinear partial differential equations, and of systems of such equations. Appendix contains six related Fortran programs. JD-B

NUMERICAL ANALYSIS, P. *Applications des modèles numériques en physique*. Claude Jablon, Jean Claude Simon. Birkhäuser, 1978, 283 pp, Fr. 48 (P). [ISBN: 3-7643-1001-4] A study of various numerical problems and the precautions necessary to avoid bizarre or just plain false results when using modern computers. JAS

NUMERICAL ANALYSIS, S(17-18), P. *Differenzenapproximationen partieller Anfangswertaufgaben*. Rainer Ansorge. Teubner, Stuttgart, 1978, 298 pp, DM 29,80 (P). [ISBN: 3-519-02347-4] On extensions to semilinear, quasi-linear and nonlinear initial-value problems of the Lax-Richtmyer theory. JD-B

FUNCTIONAL ANALYSIS, P. *Integral Operators in the Theory of Induced Banach Representations*. Irwin E. Schochetman. Memoirs No. 207. AMS, 1978, vii + 53 pp, $6.80 (P). [ISBN: 0-8218-2207-1]

FUNCTIONAL ANALYSIS, P. *Interpolation Theory, Function Spaces, Differential Operators*. Hans Triebel. Math. Lib., V. 18. North-Holland, 1978, 528 pp, $66.75. [ISBN: 0-7204-0710-9] A formidable and comprehensive account of modern interpolation theory, covering: the general theory in Banach spaces; function spaces of the Sobolev-Besov type in Euclidean n-space and in domains; applications to boundary value problems with elliptic differential operators; aspects of the structure theory of nuclear function spaces. Over 800 references are included. TRS

FUNCTIONAL ANALYSIS, S(17-18), P. *Spectral Theory of Linear Operators*. H.R. Dowson. London Math. Soc. [illegible], No. 12. Acad Pr, 1978, xii + 422 pp, $39. [ISBN: 0-12-220950-8] After providing background on general spectral theory and on compact, Riesz, and Hermitian operators, the author presents an extensive account of recent developments in the theory of prespectral and well-bounded operators. TRS

FUNCTIONAL ANALYSIS, S(18), P*. *Reflexive and Superreflexive Banach Spaces*. D. van Dulst. Math. Centre Tracts, No. 102. Math Centrum, 1978, v + 273 pp, Dfl. 33 (P). [ISBN: 90-6196-171-8] A careful, self-contained presentation of the general geometric theory of infinite dimensional Banach spaces, aimed primarily at many of the beautiful theorems of R.C. James which characterize reflexive (and reflexive-like) spaces in terms of the "rotundity" of their unit spheres. The full details of proof should please all readers. TRS

ANALYSIS, T(17-18: 1, 2), S. *Principles of Advanced Mathematical Physics, Volume I*. Robert D. Richtmyer. Springer-Verlag, 1978, xv + 422 pp, $19.80. [ISBN: 0-387-08873-3; 3-540-08873-3] A physicist presents functional analysis with seasonings from physics, probability theory, differential equations, and quantum mechanics. Rather short on exercises but otherwise a reasonable text for a course on applicable analysis (not applied analysis). JAS

ANALYSIS, S(14-16). *Exercices et problèmes résolus d'analyse*. Lucien Chambadal. Dunod (US Distr: SMPF, 14 E. 60th St., NY 10022), 1973, 187 pp, 48F (P). [ISBN: 2-04-010072-5] Problems concerning elementary topology of Euclidean spaces, functions, integration (not measure), series, differential equations and functions of several variables. JAS

ALGEBRAIC GEOMETRY, T(16), S, L. *Algebraic Curves*. Robert J. Walker. Springer-Verlag, 1978, x + 201 pp, $9.80 (P). [ISBN: 0-387-90361-5; 3-540-90361-5] An introduction to the study of algebraic geometry centering around birational transformations and linear series. Since the first two chapters contain preliminary material on algebra and projective space, the book is almost entirely self-contained. An effort has been made to relate newer algebraic methods with older analytic and geometric methods. Many examples and exercises. JEG

ALGEBRAIC GEOMETRY, P. *Notes on Crystalline Cohomology*. Pierre Berthelot, Arthur Ogus. Princeton U Pr, 1978, vi + 243 pp, $7.50 (P). Notes reproducing a 1974 seminar at Princeton University that develop the basic tools used in the study of crystalline cohomology of algebraic varieties in positive characteristics. JEG

DIFFERENTIAL GEOMETRY, P. *On Uniformization of Complex Manifolds: The Role of Connections*. R.C. Gunning. Princeton U Pr, 1978, ii + 141 pp, $6 (P). Notes based on a 1976 course (including some material from a 1963-64 course) which study "complex analytic pseudogroup structures on complex manifolds, viewed as an extension of the theory of uniformization of Riemann surfaces." JAS

DIFFERENTIAL GEOMETRY, S(18), P. *Exceptional Lie Algebras and the Structure of Hermitian Symmetric Spaces*. Daniel Drucker. Memoirs No. 208. AMS, 1978, iv + 207 pp, $9.20 (P). [ISBN: 0-8218-2208-X] The author determines the orbit structure of all irreducible hermitian spaces by Lie-algebraic means, using a construction and results of Tits, with an important role played by the exceptional algebras. Includes three appendices and a bibliography. JS

DIFFERENTIAL GEOMETRY, T(18), S, P. *Differential Geometry, Lie Groups, and Symmetric Spaces*. Sigurdur Helgason. Pure and Appl. Math., V. 80. Acad Pr, 1978, xv + 628 pp, $27. [ISBN: 0-12-338460-5] "This first volume is an extensive revision of a part of *Differential Geometry and Symmetric Spaces*" (1962) with numerous additions, including new exercises, solutions, and chapters on semi-simple Lie groups and classification of simple Lie algebras and symmetric spaces. A second volume on groups and geometric analysis is planned. JS

GEOMETRY, S(15-16), P*, L*. *Geometric Symmetry*. E.H. Lockwood, R.H. Macmillan. Cambridge U Pr, 1978, x + 228 pp, $24.50. [ISBN: 0-521-21685-0] Non-mathematical Part I reviews basic notions and culminates with methods for constructing new designs. Via a geometrical approach, more technical Part II develops the usual theorems on frieze, wallpaper, and 3-space patterns, ending with colored symmetry. Almost 400 diagrams, from simple to intricate. Helpful charts, tables, and plates throughout, with end-of-book indexes and summaries. A must purchase for all touched by symmetry--and who isn't? JK

GEOMETRY, S(13-16), P, L. *Géométrie 2/espaces euclidiens, triangles, cercles et sphères.* Marcel Berger. CEDIC/Nathan (US Distr: SMPF, 14 E. 60th St., NY 10022), 1977, 214 pp, (P). [ISBN: 2-7124-0702-4; 2-7124-0700-8 set) *Volume 2* of a five-volume treatment of a wide range of topics in geometry. JK

TOPOLOGY, S(14-16). *La topologie.* André Delachet. Pr U France, 1978, 126 pp, (P). One of an extended series (typical titles *Le siècle des Médicis, Le concerto, Le tourisme social*) amounting to extended articles from an Encyclopedia. This volume presents topology from the point of view of analysis. It is a very technical essay, not a popular rubber-sheet-geometry approach, but it does offer some overviews of what the definitions and theorems are all about. JAS

TOPOLOGY, P. *Topology and Algebra.* M.-A. Knus, G. Mislin, U. Stammbach. L'Enseignement Math, 1978, 280 pp, Frs. 75 (P). The papers from the colloquium on topology and algebra held at the ETH in April 1977 in celebration of the sixtieth birthday of Beno Eckmann. JAS

TOPOLOGY, P, L. *Hyperspaces of Sets; A Text with Research Questions.* Sam B. Nadler, Jr. Pure and Appl. Math., V. 49. Dekker, 1978, xvi + 707 pp, SF 118. [ISBN: 0-8247-6768-3] For a metric continuum both the set of non-empty compact subsets and the set of connected subsets comprise metric spaces with the Hausdorff metric. This book, the first devoted entirely to the topic, presents a nearly encyclopaedic study unified by the use of Whitney maps. The first part is sophisticated but leisurely, and of use as a text. In the rest of the book exercises are replaced with research questions and the main body of the current development of the subject is presented with its interrelations to infinite dimensional topology, functional analysis, and convexity theory. The bibliographies and indices are very extensive making the book useful for the forgetful reader. JAS

TOPOLOGY, P. *Lecture Notes in Mathematics-673: Algebraic Topology.* Ed: P. Hoffman, R. Piccinini, D. Sjerve. Springer-Verlag, 1978, vi + 277 pp, $14.30 (P). [ISBN: 0-387-08930-6; 3-540-08930-6] A collection of articles based on talks given at the Workshop and Conference in Algebraic Topology at the University of British Columbia, Vancouver in the summer of 1977. JEG

TOPOLOGY, T(17-18: 1, 2), S, L. *Homology and Cohomology Theory, An Approach Based on Alexander-Spanier Cochains.* William S. Massey. Pure and Appl. Math., V. 46. Dekker, 1978, xiv + 412 pp, $29.75 [ISBN: 0-8247-6662-8] A detailed and careful exposition for beginning graduate students of Čech-Alexander-Spanier cohomology and homology theories. The presentation is in the style of lecture notes: basic bibliography, minimal index, and relatively few problems. Minimal formal prerequisites and readable exposition appear to offer a student a nice alternative treatment of homology theory. JAS

PROBABILITY, P. *Lecture Notes in Mathematics-671: First-Passage Percolation on the Square Lattice.* R.T. Smythe, John C. Wierman. Springer-Verlag, 1978, viii + 196 pp, $12.50 (P). [ISBN: 0-387-08928-4; 3-540-08928-4] A first passage percolation process is a lattice model of the random spread of a fluid in which passage times along edges have a non-negative probability distribution with finite means. These notes provide the first comprehensive treatment of these processes, much of it based on quite recent results. LAS

PROBABILITY, P. *Gaussian Random Processes.* I.A. Ibragimov, Y.A. Rozanov. Trans: A.B. Aries. Appl. of Math., No. 9. Springer-Verlag, 1978, x + 275 pp, $24.80. [ISBN: 0-387-90302-x; 3-540-90302-x] Three problems concerning Gaussian stationary processes: equivalence, spectral measures, and estimation of mean value. Translation of 1970 original Russian edition. LAS

STATISTICS, S(16-17), P. *Quelques problèmes concernant les sondages.* Pierre D. Thionet. Vandenhoeck & Ruprecht, 1978, 137 pp, DM 34 (P). [ISBN: 3-525-11240-8] An overview and history of sampling theory. Contains little technical mathematics but the discussion is at a high level and contains extensive bibliographic references. JAS

STATISTICS, T(13: 1). *The Ways and Means of Statistics.* Leonard J. Tashman, Kathleen R. Lamborn. HarBrace J, 1979, xiii + 527 pp, $15.95. [ISBN: 0-15-595132-7] Presupposes only high school algebra. The usual statistical topics plus material on multiple regression and some non-parametric tests. FLW

STATISTICS, T(16-17: 1, 2), P. *Statistical Analysis of Reliability and Life-Testing Models, Theory and Methods.* Lee J. Bain. Statistics, V. 24. Dekker, 1978, xii + 450 pp, $37.50. [ISBN: 0-8247-6665-2] Treats techniques associated with models of lifetimes or time to failure, based primarily on exponential and Weibull distributions, and to a lesser extent on gamma, extreme value, logistic and other distributions. Much attention to censored sampling, i.e., where certain observations are excluded. Includes a review of basic probability and statistics concepts. Good set of references. RSK

STATISTICS, T(14-17: 1, 2). *Maps and Statistics.* Peter Lewis. Halsted Pr, 1977, xviii + 318 pp, $22.50. [ISBN: 0-470-99094-5] Elementary statistics for geography students. Treatment tends to be terse and would require some sophistication, although the only mathematical prerequisite is algebra. Emphasizes techniques important to geographers, hence includes many nonparametric tests and tests of independence and randomness. RSK

STATISTICS, P. *Stochastic Abundance Models, with Emphasis on Biological Communities and Species Diversity.* S. Engen. Chapman & Hall, 1978, x + 126 pp, $13.95. [ISBN: 0-412-15240-1] One of their Monographs on Applied Probability and Statistics. Roughly 3/4 theory, dealing with "the basic description of populations of classes, abundance models, and the estimation of population parameters," and 1/4 ecological applications. Good set of references. RSK

STATISTICS, T(13-14: 1). *Elementary Statistics in a World of Applications.* Ramakant Khazanie. Goodyear, 1979, vii + 488 pp, [illegible]. [ISBN: 0-87620-242-3] Presupposes only high school algebra. Descriptive statistics, probability, and the usual topics in statistical inference. No Bayesian methods. FLW

STATISTICS, S*, P, L**. *R.A. Fisher, The Life of a Scientist.* Joan Fisher Box. Wiley, 1978, xii + 512 pp, $24.95. [ISBN: 0-471-09300-9] In the Wiley Series in Probability and Mathematical Statistics. Sympathetic biography of Sir Ronald A. Fisher (1890-1962) written by one of his daughters. In addition to illuminating presentations of his scientific ideas and concerns, it provides fascinating insights into his private life. Includes a complete bibliography of the diverse works of this scientific genius and pioneer in the fields of statistics and genetics. RSK

STATISTICS, S**, P, L*. *International Encyclopedia of Statistics.* Ed: William H. Kruskal, Judith M. Tanur. Free Pr, 1978 [ISBN: 0-02-917960-2 set]. *V. 1,* xxi + 666 pp; *V. 2,* 684 pp, $100 set. Contains material first published in the 1968 *International Encyclopedia of the Social Sciences* that has been updated, plus new articles and biographies: 75 articles on statistics proper, 57 biographies, and 42 articles covering areas of application. An invaluable source of information for students and professionals alike. RSK

STATISTICS, T(14: 1), S, L. *Statistical Methods for the Earth Scientist, An Introduction.* Roger Till. Macmillan, 1974, 154 pp, $7.95 (P). [ISBN: 333-15005-8] A non-mathematical introduction which emphasizes tools and examples of special interest to geologists. A final chapter gives a cursory introduction to some multivariate techniques. No exercises. FLW

STATISTICS, T*(13-16: 1), S*, P, L**. *Elementary Statistical Quality Control.* Irving W. Burr. Statistics, V. 25. Dekker, 1979, xii + 413 pp, $14.95. [ISBN: 0-8247-6686-5] "One of a very small number of books on quality control which is both statistically sound and written at a level where an individual without a background in statistics can easily understand it." (From a note in the book by D.B. Owen, the series editor.) FLW

STATISTICS, T(13: 1). *Introductory Statistics.* James A. Twaite, Jane A. Monroe. Scott F, 1979, 592 pp, $14.95. [ISBN: 0-673-15097-6] Presupposes only high school algebra. Pays more attention to non-numerical data than do most elementary texts. Not much development of probabilistic ideas. The usual descriptive and inferential topics. FLW

STATISTICS, S*, P, L*. *Graphical Representation of Multivariate Data.* Ed: Peter C.C. Wang. Acad Pr, 1978, ix + 278 pp, $15. [ISBN: 0-12-734750-X] Papers from a symposium held at the Naval Postgraduate School in Monterey in 1978. Many interesting uses of profiles, stars, Fourier series representations, Chernoff faces, and other devices. FLW

COMPUTER PROGRAMMING, S(13-14). *Tools for Structured Design.* Marilyn Bohl. SRA, 1978, vii + 200 pp, $6.95 (P). [ISBN: 0-574-21170-5] Development of algorithms; system and program flowcharts; top-down, modular, structured system and program design; use of Hierarchy-Input-Process-Output (HIPO) diagrams; system and program verification, documentation procedures. Easy to read and use. Many diagrams, charts. Appendices. Chapter exercises. Responses to selected exercises. RJA

COMPUTER PROGRAMMING. *The Home Computer Book: A Complete Guide for Beginners.* Len Buckwalter. Wallaby Book, 1978, 254 pp, $4.95 (P). [ISBN: 0-671-79029-3] An up-to-date non-technical consumer-oriented survey, complete with program lists, product descriptions and glossary. No scientific or technical information. LAS

COMPUTER SCIENCE, S(12-15), L. *The Best of BYTE, Volume 1.* Ed: David H. Ahl, Carl T. Helmers, Jr. Creative Computing Pr, 1977, vii + 376 pp, $11.95 (P). [ISBN: 0-916688-04-6] Three score and ten articles (most on hardware) from the first twelve issues of the 16-issue first volume (1975-76) of BYTE, now called the "small systems journal." LAS

COMPUTER SCIENCE, T(14-18: 1, 2), S, L. *Fundamentals of Computer Algorithms.* Ellis Horowitz, Sartaj Sahni. Computer Sci Pr, 1978, xiv + 626 pp, $18.95. [ISBN: 0-914894-22-6] Text is organized around a number of fundamental strategies of algorithm design: divide-and-conquer, the greedy method, dynamic programming, search and traversal techniques, backtracking, branch-and-bound, algebraic simplification and transformations. More advanced theoretical topics are lower bound theory, NP-Hard and NP-Complete problems, approximation algorithms for NP-Hard problems. Also includes a chapter on data structures. Chapter references. Chapter exercises. Appendix on programming language SPARKS. Index. RJA

COMPUTER SCIENCE, T(15-18: 1, 2), S, P, L. *Communications Architecture for Distributed Systems.* R.J. Cypser. A-W, 1978, xxii + 711 pp, $20.95. [ISBN: 0-201-14458-1] Text is divided into 4 parts: overview of current teleprocessing systems, their diversity, structural evolution, and major components; description of a specific architecture, the Systems Network Architecture (SNA); description of SNA operations that affect all levels of the system; advanced function of a distributed system. Some chapters contain exercise sets. Chapter references. Appendexes. Index. RJA

COMPUTER SCIENCE, S(14-18), P, L. *Databases: Improving Usability and Responsiveness.* Ben Shneiderman. Acad Pr, 1978, xiii + 431 pp, $21. [ISBN: 0-12-642150-1] Papers from a conference held in Haifa, Israel in August, 1978. FLW

APPLICATIONS (ENGINEERING), P. *Annual Review of Fluid Mechanics, Volume 11.* Ed: Milton van Dyke, J.V. Wehausen, John L. Lumley. Annual Reviews, 1979, 556 pp, $17. [ISBN: 0-8243-0711-9]

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; William Carlson, St. Olaf; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Jay E. Goldfeather, Carleton; Lorraine L. Keller, St. Olaf; Richard S. Kleber, St. Olaf; Joseph Konhauser, Macalester; George H. Mills, St. Olaf; Thomas R. Savage, St. Olaf; John Schue, Macalester; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1529 18th Street, N.W., Washington, D.C. 20036

SUGGESTION BOX

Members of the MAA are encouraged to send in suggestions, questions, etc., about the operations of the Association. Communications will be referred to the appropriate officer of the Association for answering; from time to time, those of general interest may also be answered in one or both of the official journals. Communications should be addressed to: Suggestion Box, Mathematical Association of America, 1529 18th Street, N.W., Washington, D.C. 20036.

PERSONAL ITEMS

University of Toledo: Associate Professors George Kertz and Henry Wente have been promoted to Professors. Assistant Professor Carl Looney has been promoted to Associate Professor.

Trinity College, Hartford, Connecticut: Dr. Gene Sogliero, formerly Assistant Professor at the University of Connecticut, has been appointed Assistant Professor. Assistant Professor D.L. Reiner has left to become an assistant professor at Grinnell College. Professor R.C. Stewart has been named Dana Professor of Mathematics. Assistant Professor D.A. Robbins has been promoted to Professor and Department Chairman.

DePauw University: Associate Professor John R. Anderson has been promoted to Professor. Assistant Professor Carl Singer has been promoted to Associate Professor.

University of Texas, Austin: During the Fall semester of 1978 Professor Larry L. Schumaker was on leave to Germany and Assistant Professor Murray R. Cantor was on leave at the Institute for Advanced Study, Princeton. The following are on leave for the Spring semester 1979: Professor Glenadine Gibb at the University of Georgia and Professor John Cannon at Colorado State University. Instructor John Tripp has accepted a position at Southeast Missouri State University, Cape Girardeau. Professors Joe Jerome, Northwestern University, and Haskell P. Rosenthal, are Visiting Professors. Special Visiting Appointments from Institutions in Texas have been given to Professor Joe Cude, Tarleton State University, Stephenville, to Associate Professor Geraldine Daunis, Baylor University, and to Assistant Professor Ann Petrus, Our Lady of the Lake University, San Antonio.

Western Michigan University: Associate Professor Gerald Sievers has been promoted to Professor. Dr. Joseph McKean, formerly at the University of Texas at Dallas, has been appointed Assistant Professor. Assistant Professor Linda Lesniak-Foster is a Visiting Assistant Professor.

Wright State University, Dayton, Ohio: Dr. Manley Perkel and Dr. Stanley Loltek, Western Illinois University, have been appointed Assistant Professors. Dr. Edgar Rutter, University of Kansas, has been appointed Professor and Chairman. Dr. Robert McFarland, Wright State Computer Science Department, has been appointed Assistant Professor of Mathematics. Assistant Professor Joanne Dombrowski has been promoted to Associate Professor.

Chicago State University: Associate Professor Robert L. Bernhardt has been appointed Chairperson of the Mathematics Department. Associate Professor Richard Johnsonbaugh has been promoted to Professor. Assistant Professor Howard Silver has been promoted to Associate Professor.

Mississippi University for Women, Columbus: Professor and Dean Donald A. King has taken the position of Academic Dean at Presbyterian College, Clinton, South Carolina. Instructor Judy Pate has been promoted to Assistant Professor. Associate Professor Jean Ann Parra has been promoted to Professor.

University of Colorado: Dr. Timothy Carlson, University of Minnesota, and Dr. James Curry, Howard University, have been appointed Assistant Professors. Assistant Professor Kenneth Rosen has left to become Assistant Professor at Ohio State University.

University of Detroit: Associate Professor James F. Lanahan has been appointed Chairman of the Mathematics Department. Professor Michael S. Skaff received the 1978 Barnard Award from the International Association of Assessing Officers for the most outstanding research article in the Assessors Journal.

Miami University, Oxford, Ohio: Dr. Sheldon Davis, Auburn University, and Dr. Clifton Ealy, Northern Michigan University, have been appointed Visiting Assistant Professors. Senior Instructor Dorothy Rowe has retired.

Florida A. and M. University: Assistant Professor C. Andrew Foster has been promoted to Associate Professor. Associate Professors Donald M. Hill, Gladys H. Jones, and Wendell L. Motter, and Assistant Professors Gwendolyn K. Humphrey and H. Carl Whitman have been granted tenure.

University of California, Berkeley: Professor Skoshichi Kobayashi has been appointed Chairperson of the Mathematics Department for 1978-79. Professor Murray H. Protter, has been appointed Director of the Center for Pure and Applied Mathematics for 1978-79. Professor Hale Trotter, Princeton University, is a Visiting Professor for the Winter quarter 1979. Dr. Sy Friedman, M.I.T., and Dr. Bruce Reznick, Stanford University, are NSF Postdoctoral Fellows for 1978-79.

University of Minnesota: Dr. Dennis Hejhal, Columbia University, has been appointed Professor. Assistant Professor Chang-Shing Chen has left to accept other employment. Associate Professors Thomas R. Berger, Jay R. Goldman, and Harvey B. Keynes have been promoted to Professors.

University of Michigan: Dr. L. Gaunce Lewis, University of Chicago, has been appointed T. H. Hildebrandt Research Assistant Professor. Professor Robert C. F. Bartels has retired with the titles of Professor Emeritus of Mathematics and Director Emeritus of the Computing Center.

Southeast Missouri State University: Dr. John C. Tripp, University of Texas, has been appointed Instructor. Associate Professor John E. Young has been promoted to Professor. Associate Professors Clarence H. Dalton and Walter Roth have retired with the titles of Professors Emeriti.

Wake Forest University: Associate Professors John V. Baxley and Fredric T. Howard have been promoted to Professors.

Clemson University: Professor Alfred T. Hind has retired with the title of Professor Emeritus. Dr. Calvin T. Long, Washington State University, Dr. Harold B. Reiter, University of North Carolina, Charlotte, and Ms. Betty B. Reiter, University of North Carolina, Charlotte, are Visiting faculty. Professor John W. Kenelly has been elected President of the MAA Southeastern Section. Professor Thomas G. Proctor has been appointed Chairman of the SIAM Education Committee. Professor Clayton W. Ancoin is a member of the Conference Board of the Mathematical Sciences. Professor Robert F. Ling was named Book Review Associate Editor for the Journal of the American Statistical Association.

Cornell College, Mt. Vernon, Iowa: Associate Professors E. T. Hill and D.F. Bailey have been promoted to Professors. Emeritus Professor Watson M. Davis was honored when the College established one of ten new faculty chairs in his name. Dr. Davis was an assistant editor of the *Monthly* from 1942 to 1945.

Assistant Professor Diane Krebs, Valparaiso University, Valparaiso, Indiana, has been promoted to Associate Professor.

Assistant Professor Jane Malbrock, Kean College of New Jersey Union, New Jersey, has been promoted to Associate Professor.

Dr. Michael W. Ecker has been appointed Assistant Professor at the Worthington Scranton Campus of the Pennsylvania State University.

Assistant Professor W. F. Langford, McGill University, has been promoted to Associate Professor.

Dr. Coreen L. Mett has been appointed Assistant Professor at Radford College, Radford, Virginia.

Dr. Michael Petty has been appointed Assistant Professor at Texas Wesleyan College, Fort Worth.

Associate Professor Robert Matulis, Millersville State College, Millersville, Pennsylvania, has been promoted to Professor.

Mr. George M. Sweet has been appointed Instructor at Winthrop College, Rock Hill, South Carolina.

Dr. Peter G. Anderson has been appointed Professor and Department Chairman, Computer and Information Sciences, Seton Hall University, South Orange, New Jersey.

Professor Carlton E. Lemke, Stanford University, shared the 1978 John von Neumann Prize awarded jointly by the Operations Research Society of America and the Institute of Management Sciences.

Assistant Professor Jay I. Miller has left his position at Marquette University to accept non-academic employment.

Assistant Professor Melvin A. Nyman, Manchester College, North Manchester, Indiana, has been promoted to Associate Professor.

Professor Charles E. Kelley, Central Missouri State University, Warrensburg, has retired.

Assistant Professor William Hinds, Midwestern State University, Wichita Falls, Texas, has been promoted to Professor.

Assistant Professor Richard T. Goller, Edgecliff College, Cincinnati, Ohio, has been promoted to Associate Professor.

Professor Dale E. Varberg, Hamline University, St. Paul, Minnesota, is on sabbatical leave and has an office at the Massachusetts Institute of Technology.

Dr. Sheldon O. Sickler has been appointed Associate Professor at Point Loma College, San Diego, California.

Associate Professor Paul Hessler, Wittenberg University, Springfield, Ohio, has been appointed Chairman of the Department of Mathematics. Assistant Professor Donald Peeples left to become Assistant Professor at Dalton Junior College, Dalton, Georgia.

Assistant Professor David Bauer, Virginia Commonwealth University, Richmond, has been promoted to Associate Professor.

Dr. Dennis D. Pence has been appointed Assistant Professor at the University of Wisconsin Mathematical Research Center.

Assistant Professor Michael L. Kovacic, Colorado State University, Fort Collins, has retired.

Assistant Professor James J. Buckley, University of Alabama in Birmingham, has been promoted to Associate Professor.

Associate Professor Stephen V. Ullom, University of Illinois, has been promoted to Professor.

Dr. Gregory A. Fredricks, Southern Utah State College, is a Visiting Lecturer at Texas Tech University, Lubbock.

Dr. Edward C. Svendsen has been appointed Assistant Professor at the University of Illinois.

Professor Burrowes Hunt, Reed College, Portland, Oregon, has retired with the title of Professor Emeritus.

Dr. William J. Browning and Dr. Bruce E. Scranton have been promoted to Senior Associates of Daniel H. Wagner, Associates, Paoli, Pennsylvania.

Associate Professor Gerald Sievers, Western Michigan University, Kalamazoo, has been promoted to Professor.

Dr. Linda Allegri, Peoria, Illinois, died on September 25, 1977. She was a member of the Association for twenty two years.

D. Michael Bazar, Mandeville, Louisiana, died on August 22, 1978, at the age of thirty six. He was a member of the Association for seventeen years.

Stephen W. Borden, Vancouver, British Columbia, died on January 29, 1978, at the age of forty four. He was a member of the Association for seventeen years.

Edward T. Frankel, Schenectady, New York, died on January 21, 1978, at the age of eighty six. He was a member of the Association for sixty one years!

Associate Professor George B. Grunwald, Ball State University, died on August 20, 1978, at the age of forty six. He was a member of the Association for eighteen years.

Professor Elaine S. Johnson, Jamestown Community College, Falconer, New York, died March 4, 1978, at the age of fifty four. She was a member of the Association for fifteen years.

Professor A.F. Nickl, U.S. Merchant Marine Academy, died on April 11, 1977, at the age of fifty six. He was a member of the Association for thirty years.

Professor Gustav Kuerti, Cleveland, Ohio, died on July 10, 1978. He was a member of the Association for seven years.

Dr. Joseph B. Reynolds, Erie, Pennsylvania, died (date not known). He was a member of the Association since 1915!

Mr. Sidney G. Roth, Washington, D.C., died on May 16, 1978. He was a member of the Association for forty five years.

MATHEMATICAL ASSOCIATION OF AMERICA

1978 CONTRIBUTING MEMBERS AND SPECIAL GIFTS*

The Association is deeply indebted to the generosity of the 109 members listed below who have elected to be Contributing Members, Sponsors, or Patrons for 1978 by making contributions beyond the normal dues.

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The Association also acknowledges with thanks an anonymous contribution of $30.

By June 15, 1978 approximately 1250 members of the Association had made gifts, pledges, and advance dues payments totaling over $332,000 to the MAA Building Fund and to special memorial funds to be used to assist in the purchase of the new MAA Headquarters. These gifts will be individually acknowledged elsewhere.

*This note was originally scheduled for publication in the October 1978 issue of the MONTHLY. We regret the delay.

MATHEMATICAL ASSOCIATION OF AMERICA
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February 1, 1979

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Terms of members expire, except where otherwise noted, at the Annual Meeting in January following the last year of service listed below. For temporary committees, no terms are listed since they are automatically discharged at the expiration of the President's term of office, which is the Annual Meeting in January, 1981.

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(All terms expire December 31, 1981)

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MATHEMATICS MAGAZINE
(All terms expire December 31, 1980)

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(All terms expire December 31, 1983)

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CALENDAR OF FUTURE MEETINGS

Fifty-ninth Summer Meeting, University of Minnesota, Duluth, August 21–23, 1979.

Sixty-third Annual Meeting, San Antonio, Texas, January 5–7, 1980.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, Westminster College, New Wilmington, Pennsylvania, April 27–28, 1979.

FLORIDA, early March. Deadline for paper titles 2 weeks before meeting.

ILLINOIS, Northern Illinois University, De Kalb, April 27–28, 1979.

INDIANA, Butler University, Indianapolis, April 21, 1979.

INTERMOUNTAIN, Idaho State University, Pocatello, May 4–5, 1979.

IOWA, Cornell College, Mt. Vernon, April 20–21, 1979.

KANSAS, Johnson County Community College, Overland Park, April 7, 1979.

KENTUCKY, Morehead State University, Morehead, April 6–7, 1979.

LOUISIANA–MISSISSIPPI, Friday–Saturday before February 20. Deadline for papers 3 months before meeting.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK, Adelphi University, May 5, 1979.

MICHIGAN, University of Detroit, May 4–5, 1979.

MISSOURI, late March/early April. Deadline for papers January 31.

NEBRASKA, Nebraska Wesleyan University, Lincoln, April 20–21, 1979.

NEW JERSEY, Monmouth College, West Long Branch, April 28, 1979.

NORTH CENTRAL, College of St. Teresa, Winona, Minnesota, April 27–28, 1979.

NORTHEASTERN, University of Maine, Orono, June 22–23, 1979.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO, Miami University, Middletown, April 20–21, 1979.

OKLAHOMA–ARKANSAS, (approximately) Friday and Saturday of first weekend in April. Deadline for papers 3 weeks before meeting.

PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 15–16, 1979.

PHILADELPHIA, Drexel University, Philadelphia, November 17, 1979.

ROCKY MOUNTAIN, University of Denver, Denver, April 27–28, 1979.

SEAWAY, SUNY, College at Oneonta, May 4–5, 1979.

SOUTHEASTERN, University of Tennessee, Chattanooga, April 6–7, 1979.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, University of Texas, El Paso, April 6–7, 1979.

TEXAS, Texas Tech University, Lubbock, April 6–7, 1979.

WISCONSIN, Friday and Saturday between mid-April and first week in May. Deadline for papers 6 weeks before meeting.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, San Francisco, January 3–8, 1980.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES

AMERICAN MATHEMATICAL SOCIETY, University of Minnesota, Duluth, August 22–25, 1979.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION

ASSOCIATION FOR COMPUTING MACHINERY, Plaza Hotel, Detroit, Michigan, October 29–31, 1979.

ASSOCIATION FOR SYMBOLIC LOGIC, New York City, December 28–29, 1979.

ASSOCIATION FOR WOMEN IN MATHEMATICS

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS, Washington, D.C., August 13–16, 1979.

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Boston, Massachusetts, April 18–21, 1979.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Hyatt Regency Hotel, New Orleans, Louisiana, April 29–May 1, 1979.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Radisson Muehlbach, Kansas City, Missouri, November 1–3, 1979.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Royal York Hotel, Toronto, Canada, June 11–13, 1979.

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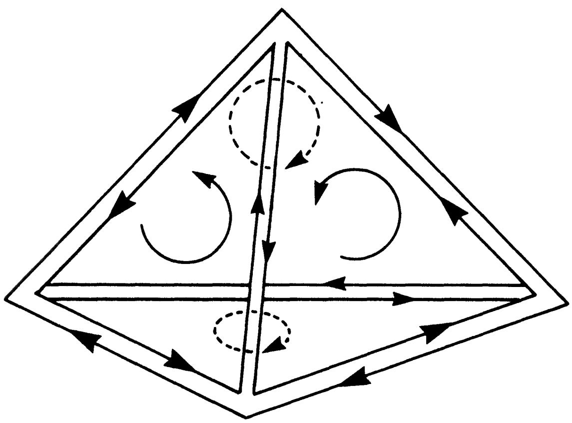
by S. S. Chern



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1979

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FROM TRIANGLES TO MANIFOLDS

SHIING-SHEN CHERN

1. Geometry. I believe I am expected to tell you all about geometry; what it is, its developments through the centuries, its current issues and problems, and, if possible, a peep into the future. The first question does not have a clear-cut answer. The meaning of the word *geometry* changes with time and with the speaker. With Euclid, geometry consists of the logical conclusions drawn from a set of axioms. This is clearly not sufficient with the horizons of geometry ever widening. Thus in 1932 the great geometers O. Veblen and J. H. C. Whitehead said, "A branch of mathematics is called geometry, because the name seems good on emotional and traditional grounds to a sufficiently large number of competent people" **[1]**. This opinion was enthusiastically seconded by the great French geometer Elie Cartan **[2]**. Being an analyst himself, the great American mathematician George Birkhoff mentioned a "disturbing secret fear that geometry may ultimately turn out to be no more than the glittering intuitional trappings of analysis" **[3]**. Recently my friend André Weil said: "The psychological aspects of true geometric intuition will perhaps never be cleared up. At one time it implied primarily the power of visualization in three-dimensional space. Now that higher-dimensional spaces have mostly driven out the more elementary problems, visualization can at best be partial or symbolic. Some degree of tactile imagination seems also to be involved" **[4]**.

At this point it is perhaps better to let things stand and turn to some concrete topics.

2. Triangles. Among the simplest geometrical figures is the triangle, which has many beautiful properties. For example, it has one and only one inscribed circle and also one and only one circumscribed circle. At the beginning of this century the nine-point circle theorem was known to almost every educated mathematician. But its most intriguing property concerns the sum of its angles. Euclid says that it is equal to 180°, or π by radian measure, and deduces this from a sophisticated axiom, the so-called *parallel axiom*. Efforts to avoid this axiom failed. The result was the discovery of non-Euclidean geometries in which the sum of angles of a triangle is less or greater than π, according as the geometry is hyperbolic or elliptic. The discovery of hyperbolic non-Euclidean geometry, in the eighteenth century by Gauss, John Bolyai, and Lobatchevsky, was one of the most brilliant chapters in human intellectual history.

The generalization of a triangle is an n-gon, a polygon with n sides. By cutting the n-gon into $n-2$ triangles, one sees that the sum of its angles is $(n-2)\pi$. It is better to measure the sum of the exterior angles! The latter is equal to 2π, for all n-gons, including triangles.

3. Curves in the plane; rotation index and regular homotopy. By applying calculus we can consider smooth curves and closed smooth curves in the plane, i.e., curves with a tangent line everywhere and varying continuously. As a point moves along a closed smooth (oriented) curve C once, the lines through a fixed point O and parallel to the tangent lines of C rotate through an angle $2n\pi$ or rotate n times about O. This integer n is called the rotation index of C. (See Fig. 1.) A famous theorem in differential geometry says that if C is a simple curve, i.e., if C does not intersect itself, $n=\pm 1$.

Clearly, there should be a theorem combining the theorem on the sum of exterior angles of an n-gon and the rotation index theorem of a simple closed smooth curve. This is achieved by considering the wider class of simple closed sectionally smooth curves. The rotation index of

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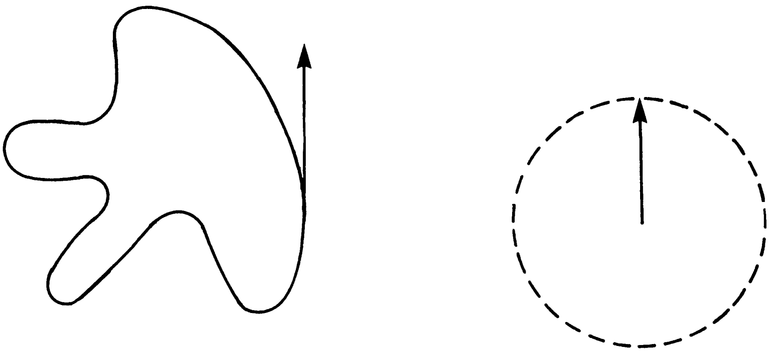


FIG. 1

such a curve can be defined in a natural way by turning the tangent at a corner an amount equal to the exterior angle. (See Fig. 2.) Then the rotation index theorem above remains valid for simple closed sectionally smooth curves. In the particular case of an n-gon formed by straight segments, this reduces to the statement that the sum of its exterior angles is 2π.

This theorem can be further generalized. Instead of simple closed curves we can allow closed curves to intersect themselves. A generic self-intersection can be assigned a sign. Then, if the curve is properly oriented, the rotation index is equal to one plus the algebraic sum of the number of self-intersections. (See Fig. 3.) For example, the figure 8 has the rotation index zero.

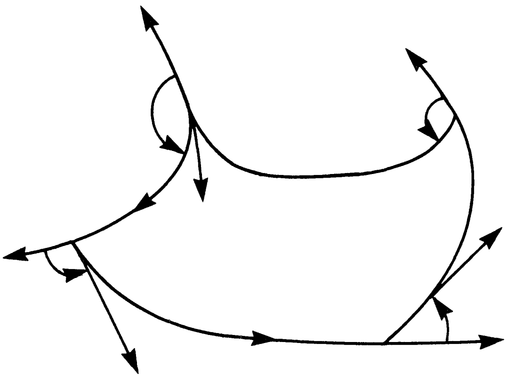
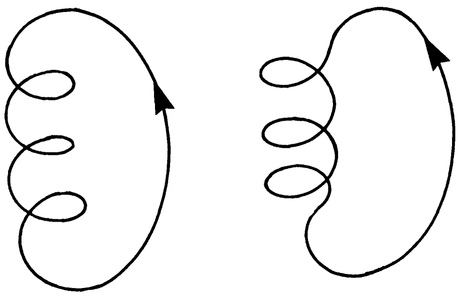


FIG. 2



$n=4$ $n=-2$

FIG. 3

A fundamental notion in geometry, or in mathematics in general, is *deformation* or *homotopy*. Two closed smooth curves are said to be *regularly homotopic* if one can be deformed to the other through a family of closed smooth curves. Since the rotation index is an integer and varies continuously in the family, it must remain a constant; i.e., it keeps the same value when the curve is regularly deformed. A remarkable theorem of Graustein-Whitney says that the converse is true [5]: Two closed smooth curves with the same rotation index are regularly homotopic.

It is a standard practice in mathematics that in order to study closed smooth curves in the plane it is more profitable to look at all curves and to put them into classes, the regular homotopy classes in this case being an example. This may be one of the essential methodological differences between theoretical science and experimental science, where such a procedure is impractical. The Graustein-Whitney theorem says that the only invariant of a regular homotopy class is the rotation index.

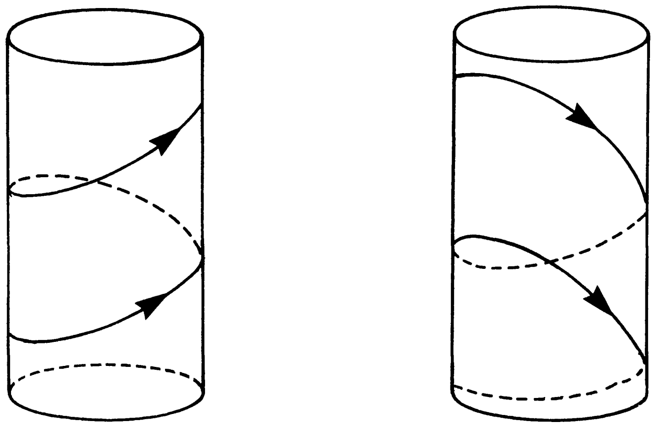


FIG. 4

4. Euclidean three-space. From the plane we pass to the three-dimensional Euclidean space where the geometry is richer and has distinct features. Perhaps the nicest space curve which does not lie in a plane is a circular helix. It has constant curvature and constant torsion and is the only curve admitting ∞^1 rigid motions. There is an essential difference between right-handed and left-handed helices (See Fig. 4), depending on the sign of the torsion; a right-handed helix cannot be congruent to a left-handed one, except by a mirror reflection. Helices play an important role in mechanics. From a geometrical viewpoint it may not be an entire coincidence that the Crick-Watson model of a DNA-molecule is double-helical. A double helix has interesting geometrical properties. In particular, by joining the end points of the helices by segments or arcs, we get two closed curves. In three-dimensional space they have a linking number. (See Fig. 5.)

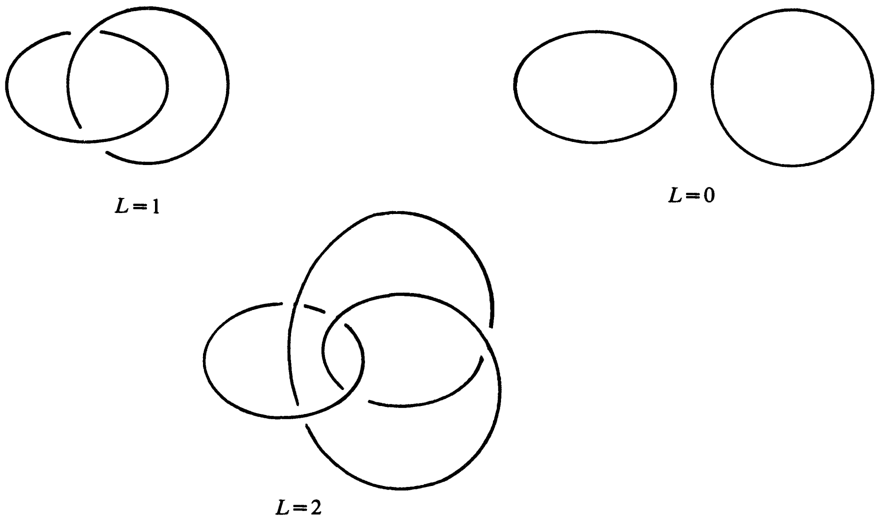


FIG. 5

A recent controversial issue in biochemistry, raised by the mathematicians William Pohl and George Roberts, is whether the chromosomal DNA is double-helical. In fact, if it is, it will have two closed strands with a linking number of the order of 300,000. The molecule is replicated by

separation of the strands and formation of the complementary strand of each. With such a large linking number Pohl and Roberts showed that the replication process would have severe mathematical difficulties. Thus the double-helical structure of the DNA molecule, at least for the chromosome, has been questioned [6]. (Added January 26, 1979: A number of recent experiments have shown that some of the mathematical difficulties for the double helical structure of the DNA-molecule can be overcome by enzymatic activities (cf. F. H. C. Crick, Is DNA really a double helix? preprint, 1978).)

The linking number L is determined by the formula of James H. White [7]:

$$T+W=L, \tag{1}$$

where T is the total twist and W the writhing number. The latter can be experimentally measured and changes by the action of an enzyme. This formula is of fundamental importance in molecular biology. Generally DNA molecules are long. In order to store them in limited space, the most economical way is to writhe and coil them. These discussions could indicate the beginning of a stochastic geometry, with the main examples drawn from biology.

In a three-dimensional space surfaces have far more important properties than curves. Gauss's fundamental work elevated differential geometry from a chapter of calculus to an independent discipline. His *Disquisitiones generales circa superficies curvas* (1827) is the birth certificate of differential geometry. The main idea is that a surface has an intrinsic geometry based on the measure of arc length alone. From the element of arc other geometric notions, such as the angle between curves and the area of a piece of surface, can be defined. Plane geometry is thus generalized to any surface Σ based only on the local properties of the element of arc. This localization of geometry is both original and revolutionary. In place of the straight lines are the geodesics, the "shortest" curves between any two points (sufficiently close). More generally, a curve on Σ has a "geodesic curvature" generalizing the curvature of a plane curve and geodesics are the curves whose geodesic curvature vanishes identically.

Let the surface Σ be smooth and oriented. At every point p of Σ there is a unit normal vector $\nu(p)$ which is perpendicular to the tangent plane to Σ at p. (See Fig. 6.) The vector $\nu(p)$ can be viewed as a point of the unit sphere S_0 with center at the origin of the space. By sending p to $\nu(p)$ we get the Gauss mapping

$$g:\Sigma\to S_0. \tag{2}$$

The ratio of the element of the area of S_0 by the element of area of Σ under this mapping is called the Gaussian curvature. Gauss's "remarkable theorem" says that the Gaussian curvature depends only on the intrinsic geometry of Σ. In fact, in a sense it characterizes this geometry. Clearly the Gaussian curvature is zero if Σ is the plane.

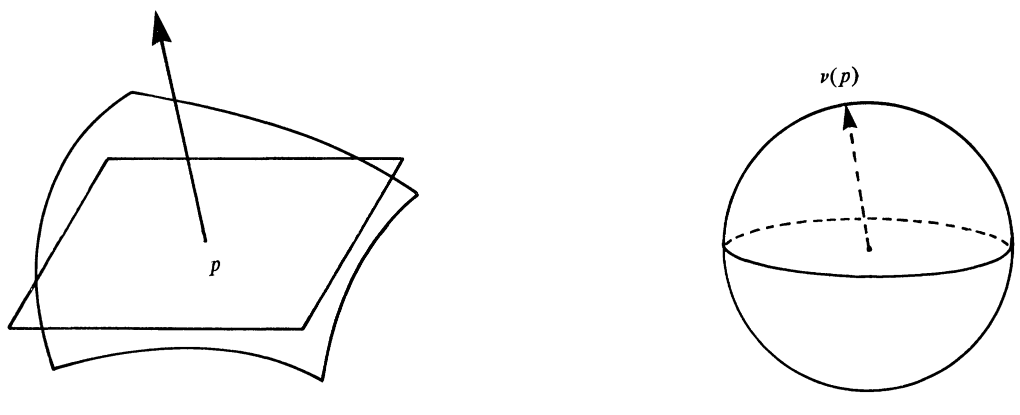


FIG. 6

As in plane geometry we consider on Σ a domain D bounded by one or more sectionally smooth curves. D has an important topological invariant $\chi(D)$, called its Euler characteristic, which is most easily defined as follows: Cut D into polygons in a "proper way" and denote by v, e, and f the number of vertices, edges, and faces, respectively. Then

$$\chi(D)=v-e+f. \tag{3}$$

(Euler's polyhedral theorem was known before Euler, but Euler seems to have been the first one to recognize explicitly the importance of the "alternating sum.")

The Gauss-Bonnet formula in surface theory is

$$\Sigma \text{ ext angles} + \int_{\partial D} \text{geod curv} + \int\int_D \text{Gaussian curv} = 2\pi\chi(D), \tag{4}$$

where ∂D is the boundary of D. For a plane domain the Gaussian curvature is zero. If in addition the domain is simply connected, we have $\chi(D)=1$. Then this formula reduces to the rotation index theorem discussed in §3. We are indeed a long way from the sum of angles of a triangle.

Generalizing the geometry of closed plane curves we can consider closed oriented surfaces in space. The generalization of the rotation index is the degree of the Gauss mapping g in (2). The precise definition of the degree is sophisticated. Intuitively it is the number of times that the image $g(\Sigma)$ covers S_0, counted with sign. Unlike the plane, where the rotation index can be any integer, the degree d is completely determined by the topology of Σ; it is equal to

$$d=\frac{1}{2}\chi(\Sigma). \tag{5}$$

For the imbedded unit sphere this degree is $+1$ independently of its orientation. A surprising result of S. Smale [8] says that the two oppositely oriented unit spheres are indeed regularly homotopic or, more intuitively, that the unit sphere can be turned inside out through a regular homotopy. It is essential that at each stage of the homotopy the surface has a tangent plane everywhere, but is allowed to intersect itself.

5. From coordinate spaces to manifolds. It was Descartes who in the seventeenth century revolutionized geometry by using coordinates. Quoting Hermann Weyl, "The introduction of numbers as coordinates was an act of violence" [9]. From now on, paraphrasing Weyl, figure and number, like angel and devil, fight for the soul of every geometer. In the plane the Cartesian coordinates of a point are its distances, with signs, from two fixed perpendicular lines, the coordinate axes. A straight line is the locus of all points whose coordinates x,y satisfy a linear equation

$$ax+by+c=0. \tag{6}$$

The result is the translation of geometry into algebra.

Once the door was opened for analytic geometry, other coordinate systems came into play. Among them are polar coordinates in the plane and spherical coordinates, cylindrical coordinates in space, and elliptic coordinates in the plane and in space. The latter are adapted to the confocal quadrics and are particularly suited to the study of the ellipsoids, which include our earth.

There is also a need for higher dimensions. For even if we start with a three-dimensional space, the theory of relativity calls for the inclusion of time as a fourth dimension. On a more elementary level, to record the motion of a particle, including its velocity, requires six coordinates (the hodograph). All the continuous functions in one variable form an infinite-dimensional space. Those which are square-integrable form a Hilbert space, which can be coordinatized by an infinite sequence of coordinates. Such a viewpoint, of considering all functions with prescribed properties, is fundamental in mathematics.

From the proliferation of coordinate systems it is natural to have a theory of coordinates.

General coordinates need only the property that they can be identified with points; i.e., there is a one-to-one correspondence between points and their coordinates—their origin and meaning are inessential.

If you find it difficult to accept general coordinates, you will be in good company. It took Einstein seven years to pass from his special relativity in 1908 to his general relativity in 1915. He explained the long delay in the following words: "Why were another seven years required for the construction of the general theory of relativity? The main reason lies in the fact that it is not so easy to free oneself from the idea that coordinates must have an immediate metrical meaning" **[10]**.

After being served by coordinates in the study of geometry, we now wish to be free from their bond. This leads to the fundamental notion of a *manifold.* A manifold is described locally by coordinates, but the latter are subject to arbitrary transformations. In other words, it is a space with transient or relative coordinates (principle of relativity). I would compare the concept with the introduction of clothing to human life. It was a historical event of the utmost importance that human beings began to clothe themselves. No less significant was the ability of human beings to change their clothing. If geometry is the human body and coordinates are clothing, then the evolution of geometry has the following comparison.

Synthetic geometry	Naked man
Coordinate geometry	Primitive man
Manifolds	Modern man

A manifold is a sophisticated concept even for mathematicians. For example, a great mathematician such as Jacques Hadamard "felt insuperable difficulty . . . in maintaining more than a rather elementary and superficial knowledge of the theory of Lie groups" **[11]**, a notion based on that of a manifold.

6. Manifolds; local tools. With coordinates practically meaningless there is a need for new tools in studying manifolds. The key word is invariance. Invariants are of two kinds: local and global. The former refer to the behavior under a change of the local coordinates, while the latter are global invariants of the manifold, examples being the topological invariants. Two of the most important local tools are the exterior differential calculus and Ricci's tensor analysis.

An exterior differential form is the integrand of a multiple integral, such as

$$\iint_D Pdydz + Qdzdx + Rdxdy, \tag{7}$$

in (x,y,z)-space, where P, Q, R are functions in x,y,z and D is a two-dimensional domain. It is observed that a change of variables in D (supposed to be oriented) will be taken care of automatically if the multiplication of differentials is anti-symmetric, i.e.,

$$dy \wedge dz = -dz \wedge dy, \text{ etc.}, \tag{8}$$

where the symbol $\wedge$ is used to denote exterior multiplication. It is also more suggestive to introduce the exterior two-form

$$\omega = Pdy \wedge dz + Qdz \wedge dx + Rdx \wedge dy \tag{9}$$

and to write the integral (7) as a pairing (D,ω) of the domain D and the form ω.

For if the same is done in n-space, then Stokes's theorem can be written

$$(D,d\omega) = (\partial D,\omega), \tag{10}$$

where D is an r-dimensional domain and ω is an exterior $(r-1)$-form; ∂D is the boundary of D and $d\omega$ is the exterior derivative of ω and is an r-form. Formula (10), the fundamental formula in multi-variable calculus, shows that ∂ and d are adjoint operators. The remarkable fact is that, while the boundary operator ∂ on domains is global, the exterior differentiation operator d on forms is local. This makes d a powerful tool. When applied to a function ($=0$-form) and a

1-form, it gives the gradient and the curl, respectively. All the smooth forms, of all degrees ($\leqslant$ dim of manifold), of a differentiable manifold constitute a ring with the exterior differentiation operator d. Elie Cartan used the exterior differential calculus most efficiently in local problems of differential geometry and partial differential equations. The global theory was founded by G. de Rham, after initial work of Poincaré. This will be discussed in the next section.

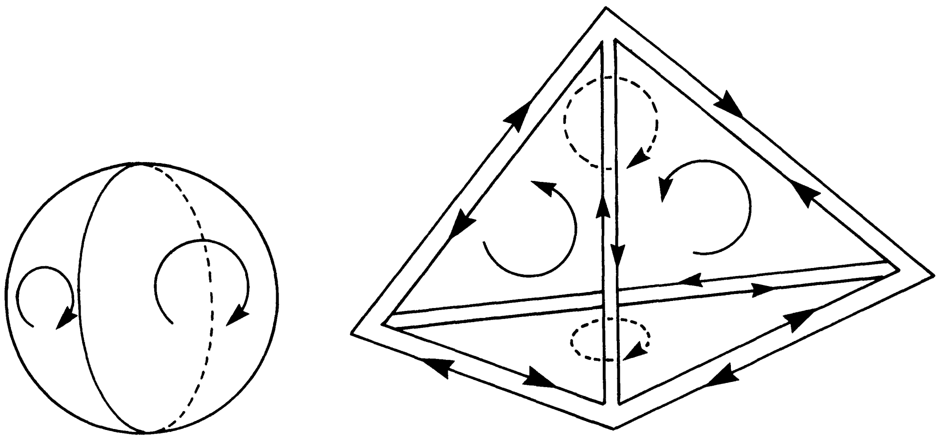
In spite of its importance the exterior differential calculus is inadequate in describing the geometrical and analytical phenomena on a manifold. A broader concept is Ricci's tensor analysis. Tensors are based on the fact that a manifold, being smooth, can be approximated at every point by a linear space, called its tangent space. The tangent space at a point leads to associated tensor spaces. Differentiation of tensor fields needs an additional structure, called an affine connection. If the manifold has a Riemannian or Lorentzian structure, the corresponding Levi-Civita connection will serve the purpose.

7. Homology. Historically a systematic study of the global invariants of a manifold began with combinatorial topology. The idea is to decompose the manifold into cells and see how they fit together. (The decomposition satisfies some mild conditions, which we will not specify.) In particular, if M is a closed manifold of dimension n and α_k denotes the number of k cells of the decomposition, $k=0,1,\ldots,n$, then, as a generalization of (3), the Euler-Poincaré characteristic of M is defined by

$$\chi(M)=\alpha_0-\alpha_1+\cdots+(-1)^n\alpha_n. \tag{11}$$

The basic notion in homology theory is that of a boundary. A chain is a sum of cells with multiplicities. It is called a cycle if it has no boundary, i.e., if its boundary is zero. The boundary of a chain is a cycle (see Fig. 7). The number of linearly independent k-dimensional cycles modulo k-dimensional boundaries is a finite integer b_k, called the kth Betti number. The Euler-Poincaré formula says

$$\chi(M)=b_0-b_1+\cdots+(-1)^n b_n. \tag{12}$$



(a) (b)

FIG. 7

The Betti numbers b_k, and hence $\chi(M)$ itself, are topological invariants of M, that is, they are independent of the decomposition and remain invariant under a topological transformation of M. This and more general statements could be considered the fundamental theorems of

combinatorial topology. After the path-breaking works of Poincaré and L. E. J. Brouwer, combinatorial topology blossomed in the U.S. in the 1920's under the leadership of Veblen, Alexander, and Lefschetz.

While this is an effective way in deriving topological invariants, the danger in cutting a manifold is that it might be "killed." Precisely, this means that by using a combinatorial approach we may lose sight of the relations of the topological invariants with local geometrical properties. It turns out that, while homology theory depends on the boundary operator ∂, there is a dual cohomology theory based on the exterior differentiation operator d, the latter being a local operator.

The resulting de Rham cohomology theory can be summarized as follows: The operator d has the fundamental property that, when applied repeatedly it gives the zero form; that is, for any k-form α, the exterior derivative of the $(k+1)$-form $d\alpha$ is zero. This corresponds to the geometrical fact that the boundary of any chain (or domain) has no boundary. (See (10).) A form α is called closed, if $d\alpha = 0$. It is called a derived form, if there exists a form β, of degree $k-1$, such that it can be written $\alpha = d\beta$. Thus a derived form is always closed. Two closed forms are called cohomologous if they differ by a derived form. All the closed k-forms which are cohomologous to each other constitute the k-dimensional cohomology class. The remarkable fact is that, while the families of k-forms, closed k-forms, derived k-forms are immensely large, the k-dimensional cohomology classes constitute a finite-dimensional linear space whose dimension is the kth Betti number b_k.

De Rham cohomology is the forerunner of sheaf cohomology, which was founded by J. Leray [**12**] and perfected and applied with great success by H. Cartan and J.-P. Serre.

8. Vector fields and generalizations. On a manifold M it is natural to consider continuous vector fields, i.e., the attachment of a tangent vector to each point, varying in a continuous manner. If the Euler-Poincaré characteristic $\chi(M)$ is not zero, there is at least one point of M at which the vector vanishes. In other words, when the wind blows there is at least one spot on earth with no wind (for the Euler characteristic of the two-dimensional sphere is equal to 2). More precisely, at an isolated zero of a continuous vector field, an integer, called the index, can be defined, which describes to a certain extent the behavior of the vector field at the zero, i.e., whether it is a source, a sink, or otherwise. No matter what the vector field is, so long as it is continuous and has only a finite number of zeros, then the theorem of Poincaré-Hopf says that the sum of its indices at all the zeros is a topological invariant which is precisely $\chi(M)$.

This is a statement on the tangent bundle of M, i.e., the collection of the tangent spaces of M. More generally, a family of vector spaces parametrized by a manifold M and satisfying a local product condition is called a vector bundle over M.

A fundamental question is whether such a bundle is globally a product. The above discussion shows that the tangent bundle is not a product if $\chi(M) \neq 0$; for if it were a product, there would exist a continuous vector field which is nowhere zero. The existence of a space which is locally but not globally a product, such as the tangent bundle of a manifold M with $\chi(M) \neq 0$, is not easy to visualize; geometry thus enters a more sophisticated phase.

To describe the global deviation of a vector bundle from a product space the first invariants are the so-called characteristic cohomology classes. The Euler-Poincaré characteristic is the simplest of the characteristic classes.

The Gauss-Bonnet formula (4) in §4 takes the particularly simple form

$$\int\int K dA = 2\pi\chi(\Sigma) \tag{4a}$$

when the surface Σ has no boundary. In this formula K is the Gaussian curvature and dA is the element of area. Formula (4a) is of paramount importance because it expresses the global invariant $\chi(\Sigma)$ as the integral of a local invariant, which is perhaps the most desirable relationship between local and global properties. This result has a wide generalization.

Let

$$\pi : E \rightarrow M \tag{13}$$

be a vector bundle. The generalization of a tangent vector field on M is a section of the bundle, i.e., a smooth mapping $s : M \rightarrow E$, such that the composition $\pi \circ s$ is the identity. Since E is only locally a product, the differentiation of s needs an additional structure, usually called a connection. The resulting differentiation, called covariant differentiation, is generally not commutative. The notion of curvature is a measure of the noncommutativity of covariant differentiation. Suitable combinations of the curvature give rise to differential forms which represent characteristic cohomology classes in the sense of the de Rham theory, of which the Gauss-Bonnet formula (4a) is the simplest example **[13]**. I believe that the concepts of vector bundles, connections, and curvature are so fundamental and so simple that they should be included in any introductory course on multivariable calculus.

9. Elliptic differential equations. When M has a Riemannian metric, there is an operator $*$ sending a k-form α to the $(n-k)$-form $*\alpha, n = \dim M$. It corresponds to the geometrical construction of taking the orthogonal complement of a linear subspace of the tangent space. With $*$ and the differential d we introduce the codifferential

$$\delta = (-1)^{nk+n+1} * d * \tag{14}$$

and the Laplacian

$$\Delta = d\delta + \delta d. \tag{15}$$

Then the operator δ sends a k-form to a $(k-1)$-form and Δ sends a k-form to a k-form. A form α satisfying

$$\Delta \alpha = 0 \tag{16}$$

is called *harmonic*. A harmonic form of degree 0 is a harmonic function in the usual sense.

The equation (16) is an elliptic partial differential equation of the second order. If M is closed, all its solutions form a finite dimensional vector space. By a classical theorem of Hodge this dimension is exactly the kth Betti number b_k. It follows by (12) that the Euler characteristic can be written

$$\chi(M) = d_e - d_0, \tag{17}$$

where d_e (respectively, d_0) is the dimension of the space of harmonic forms of even (respectively odd) degree. The exterior derivative d is itself an elliptic operator and (17) can be regarded as expressing $\chi(M)$ as the index of an elliptic operator. The latter is, for any linear elliptic operator, equal to the dimension of the space of solutions minus the dimension of the space of solutions of the adjoint operator.

The expression of the index of an elliptic operator as the integral of a local invariant culminates in the Atiyah-Singer index theorem. It includes as special cases many famous theorems, such as the Hodge signature theorem, the Hirzebruch signature theorem, and the Riemann-Roch theorem for complex manifolds. An important by-product of this study is the recognition of the need to consider pseudo-differential operators on manifolds, which are more general than differential operators.

Elliptic differential equations and systems are closely enmeshed with geometry. The Cauchy-Riemann differential equations, in one or more complex variables, are at the foundation of complex geometry. Minimal varieties are solutions of the Euler-Lagrange equations of the variational problem minimizing the area. These equations are quasi-linear. Perhaps the "most" non-linear equations are the Monge-Ampère equations, which are of importance in several geometrical problems. Great progress has been made in these areas in recent years **[14]**. With this heavy intrusion of analysis George Birkhoff's remark quoted above sounds even more disturbing. However, while analysis maps a whole mine, geometry looks out for the beautiful

stones. Geometry is based on the principle that not all structures are equal and not all equations are equal.

10. Euler characteristic as a source of global invariants. To summarize, the Euler characteristic is the source and common cause of a large number of geometrical disciplines. I will illustrate this relationship by a diagram. (See Fig. 8.)

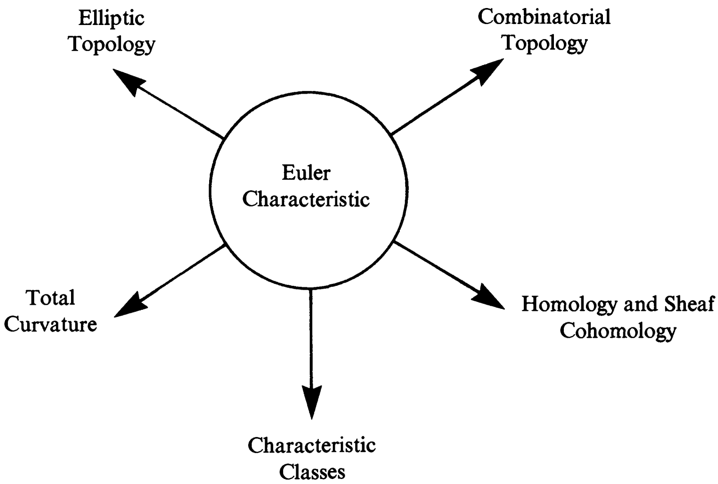


FIG. 8

11. Gauge field theory. At the beginning of this century differential geometry got the spotlight through Einstein's theory of relativity. Einstein's idea was to interpret physical phenomena as geometrical phenomena and to construct a space which would fit the physical world. It was a gigantic task and it is not clear whether he said the last word on a unified field theory of gravitational and electromagnetic fields. The introduction of vector bundles described above, and particularly the connections in them with their characteristic classes and their relations to curvature, widened the horizon of geometry. The case of a line bundle (i.e., when the fiber is a complex line) furnishes the mathematical basis of Weyl's gauge theory of an electromagnetic field. The Yang-Mills theory, based on an understanding of the isotopic spin, is the first example of a nonabelian gauge theory. Its geometrical foundation is a complex plane bundle with a unitary connection. Attempts to unify all field theories, including strong and weak interactions, have recently focused on a gauge theory, i.e., a geometrical model based on bundles and connections. It is with great satisfaction to see geometry and physics united again.

Bundles, connections, cohomology, characteristic classes are sophisticated concepts which crystallized after long years of search and experimentation in geometry. The physicist C. N. Yang wrote [**15**]: "That nonabelian gauge fields are conceptually identical to ideas in the beautiful theory of fiber bundles, developed by mathematicians without reference to the physical world, was a great marvel to me." In 1975 he mentioned to me: "This is both thrilling and puzzling, since you mathematicians dreamed up these concepts out of nowhere." This puzzling is mutual. In fact, referring to the role of mathematics in physics, Eugene Wigner spoke about the unreasonable effectiveness of mathematics [**16**]. If one has to find a reason, it might be expressed in the vague term "unity of science." Fundamental concepts are always rare.

12. Concluding remarks. Modern differential geometry is a young subject. Not counting the strong impetus it received from relativity and topology, its developments have been continuous.

I am glad that we do not know what it is and, unlike many other mathematical disciplines, I hope it will not be axiomatized. With its contact with other domains in and outside of mathematics and with its spirit of relating the local and the global, it will remain a fertile area for years to come.

It may be interesting to characterize a period of mathematics by the number of variables in the functions or the dimension of the spaces it deals with. In this sense nineteenth century mathematics is one-dimensional and twentieth century mathematics is n-dimensional. It is because of the multi-variables that algebra acquires paramount importance. So far most of the global results on manifolds are concerned with even-dimensional ones. In particular, all complex algebraic varieties are of even real dimension. Odd-dimensional manifolds are still very mysterious. I venture to hope that they will receive more attention and substantial clarification in the twenty-first century. Recent works on hyperbolic 3-manifolds by W. Thurston **[17]** and on closed minimal surfaces in a 3-manifold by S. T. Yau, W. Meeks, and R. Schoen have thrown considerable light on 3-manifolds and their geometry. Perhaps the problem of problems in geometry is still the so-called Poincaré conjecture which says that a closed simply connected 3-dimensional manifold is homeomorphic to the 3-sphere. Topological and algebraic methods have so far not led to a clarification of this problem. It is conceivable that tools in geometry and analysis will be found useful.

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STUDENT DAYS—1930

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Alexander Woolcott turned a pretty phrase when he described maturity as "anecdotage." The golden age of mathematical discovery is now—but as always, we draw on the training provided many years ago to our teachers by *their* teachers. These precursor years of the 1920's and 1930's are the subject of these reminiscences—these anecdotes.

State of Mathematics in the 30's. In 1952, N. Jacobson asserted (surely incorrectly) that the only schools of graduate mathematics before the war were Harvard, Princeton, and Chicago. This judgment neglects such pioneers as D. N. Lehmer at Berkeley; H. F. Blichfeldt and W. A. Manning at Stanford; J. Pierpont at Yale; A. A. Bennett and J. D. Tamarkin at Brown; H. F. Ettlinger, R. L. Moore, and H. S. Vandiver at Texas.

But it is true that the big three attracted the most graduate students, and that they furnished the large and small universities with faculty. In fact, Princeton had fewer graduate students than either of the other two.

Oskar Bolza had come to Chicago at the beginning of the century; J. J. Sylvester, to Johns Hopkins even earlier. Wedderburn discovered his structure theorems in the 1920's; the Institute for Advanced Study was founded at about that time; J. W. Alexander was a charter member. He wrote an important early paper on the theory of knots [**2**]. Lefschetz, at Princeton, was originating algebraic topology [**14**]. Morse was inventing calculus of variations in the large [**16**].

A. A. Albert was commencing a study of algebras [**1**]. The Göttingen school had been active in abstract algebra for several years. Almost-periodic functions had been defined by a famous athlete, H. Bohr [**4**]. Differential geometry was still one of the chief fields of research.

At that time, L. S. Pontrjagin was a topologist. One of his papers [**18**] gave counterexamples to "theorems" of four preeminent topologists; two of these were Alexander and van Dantzig.

In the early 1930's the ergodic theorem was discovered by Koopman, von Neumann, and Birkhoff. M. H. Stone wrote his book on Hilbert space operators [**19**]. Later he elucidated the structure of Boolean rings.

Courant, Friedrichs, and Lewy [**7**] had already shown how to derive an existence theorem for partial differential equations by discretization. Milne and others were studying discrete methods. Osgood had given the first example of a Jordan curve with positive area [**17**]. Lusin was studying analytic sets [**15**]. Potential theory was flowering [**13**], but the theory of theta functions was in eclipse.

There were no computers; desk calculators and logarithm tables were ubiquitous in astronomical laboratories. Exactly one mathematics professor at Harvard (E. V. Huntington) owned a calculator. Coolidge did many calculations. When they were too extensive for ordinary paper, he used wallpaper.

The Harvard scene. *The faculty*. The mathematics faculty at Harvard included G. D. Birkhoff, W. F. Osgood, O. D. Kellogg, H. M[arston] Morse, W. C. Graustein, E. V. Huntington, J. L. Walsh, J. L. Coolidge, H. W. Brinkmann, M. H. Stone, and later D. V. Widder. R. Beatley was in the School of Education, but taught mathematics also. Benjamin Peirce fellows included W. Seidel, J. J. Gergen, and others. H. Whitney came soon, and inspired many students.

Undergraduates. Undergraduates at Harvard came from both public and private schools. The

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public schools of Boston were superb at that time. In fact, the Phi Beta Kappa chapter at Harvard initiated a competition among the preparatory schools, based on comparative scores on entrance examinations. The competition was to extend over seven years. The prize would then be awarded to that school, or to those schools, that had the highest examination scores. (The score was to be the composite of the top seven scores from each school.) It was naturally supposed that, perhaps for one of the seven years, or at most two or three, a single school would take first place.

Aside from Boston Public Latin School, all the high-scoring schools were private ones—Exeter, Andover, Hotchkiss. The Latin School had the highest score for each of the first three years. Also in the fifth year the Latin School won the competition, and thus gained preemptive possession of the Phi Beta Kappa cup.

Graduates. Many of the able undergraduate and graduate mathematics students of the time entered from the Boston Public Latin School. These included A. E. Heins, M. H. Heins, S. B. Myers, W. Kaplan. Those that came from other schools included C. B. Morrey, R. P. Boas, G. B. Price, A. Sard, M. R. Hestenes, and J. L. Doob. There were also many from the Orient. At that time, too, an attempt was made to find places for European mathematicians who had been displaced. Some of the younger ones spent some time in, or near, the Harvard graduate school.

The school was so active, with its teaching, research, and seminars, that graduate students from other universities came to spend a year or longer. A. W. Tucker and N. E. Steenrod were among these.

The Harvard faculty had the duty to choose which students should be awarded scholarships, and which ones should be encouraged to enter the Ph.D. curriculum. This curriculum included (a) course work, (b) library study on an assigned subject, evidenced by a "minor thesis," (c) the usual major thesis and oral examination.

Although the faculty was conscientious in its responsibilities, G. D. Birkhoff told a story, going back to the early 1920's, to illustrate how difficult it is to select a promising student. He remembered, at Princeton, attending an important conference with Henry Norris Russell, the astronomer, and one other professor. The issue was whether to encourage Robert A. Millikan to do graduate work in physics. "They chewed over the matter with very long faces" is the way GDB put it. Then he pointed out that Millikan was currently president of the California Institute of Technology, and was the first, with his oil-drop experiment, to measure the charge on the electron.

The curriculum. As late as 1926, there was a course (Mathematics D) on Descartes' rule of signs, solution of polynomial equations in one variable, and a few other topics from the point of view of Hall and Knight's algebra book—binomial coefficients and counting formulas. Mathematics "proper" began with course 1, analytic geometry and trigonometry.

One of the most spectacular indications that the style of mathematics has changed is, indeed, the syllabus of the time. Aside from a course or two in number theory, Harvard had no formal course in algebra. There was no course (or even part of a course) in linear algebra; no course in group theory, formal or applied; no course in the structure of algebras. (Brinkmann did organize a course in the theory of Lie groups, emphasizing geometric aspects.) Nowadays, there are more courses in this field.

The classes in mechanics were numbered in geometric progression. Mathematics 2 included elementary kinematics. Mathematics 4 was dynamics; course 8 included the theory of Hamilton–Jacobi; course 16 was G. D. Birkhoff's course on relativity. Brinkmann once proposed a special course, to be numbered 32—but only in jest.

Undergraduate class activities. The professors taught freshman and other undergraduate courses regularly. A graduate student who had advanced through the ranks of the undergraduate curriculum could be well acquainted, in the classroom, with several professors. Coolidge, Brinkmann, and Beatley had the most individual of undergraduate teaching styles. To explain passing to the limit, Coolidge once stated, "The logarithm function approaches infinity with the

argument, but very reluctantly." Another time, during analytic geometry class, Coolidge's watch suffered an accident. He had the habit of twirling his watch chain around his index finger—back and forth. At the far end of the chain was a gold watch; it was undoubtedly a valuable one. At the beginning of a vicious swing, to emphasize an important point, the chain broke, and the watch sailed across the room. It landed safely on the sill of a window. The sill, in old Sever Hall, was wide and accommodating. Coolidge accepted the lesson the watch had imparted. "That, gentlemen," he intoned, "was a perfect parabola."

Coolidge's paper "The Gambler's Ruin" [5] was one of the first investigations of the effect of finite stakes on the prospects of a visitor to Las Vegas. He proved that (under his assumptions) the best strategy is to bet the entire stake available on the first turn of a fair coin. "It is true," Coolidge concluded, "that a man who does this is a fool. I have only proved that a man who does anything else is an even bigger fool."

H. W. Brinkmann taught "everything," often all during the same semester. One of his graduate students was J. S. Frame. The assistants usually attended the lectures; Frame was also Coolidge's assistant in the gold watch class, so he had the opportunity to absorb a large dose of humor.

Brinkmann's classes were among the most brilliantly organized. He introduced his subjects logically, and raised questions in just the right way. He encouraged the class to participate, but often he answered questions just before they were asked. His timing was superb.

Most of his jokes were based on his sense of timing, so it is hard to recall them. Once, during a presentation, Brinkmann reached the end of an argument. He wanted to say, "An obvious equality will now finish the proof," but, with an unintended use of hysteron proteron, he said, "and the obvious fact is here..." and paused. His syntax was unavoidably tangled, but he recovered and concluded with "obvious." A student later cagily remarked that he was glad to have an admission that an "obvious" fact can be sometimes otherwise described.

In another undergraduate class, Brinkmann mentioned a book by Pascal, *Essai sur les passions que se font dans l'amour*, but hesitated to recommend that the students borrow the book from the library. "It's probably out," whispered H. Wexler. (Wexler was later chief of the U.S. Weather Bureau.)

Brinkmann had been a Stanford undergraduate. In that role, he had helped Manning with his book *Primitive Groups*. He came to Harvard when his application was accepted for graduate work. As far as I know, he is the only graduate student that Harvard had to prompt. His telegraphed reply was "Yes"—no more.

Brinkmann's field was group characters. In the 1920's he calculated the characters of $PSL(3, q)$. No other mathematician attempted this until 50 years later, when Frame and Simpson [9] made the calculation, verifying Brinkmann's unpublished results.

Few other lecturers had Brinkmann's facility for keeping a large class attentively on edge, and of being able to field spontaneous questions. E. P. Kohler taught an elementary chemistry class of 400 students, which he conducted as a recitation and lecture. Dawson taught an introductory course in zoology. During the final lecture, he agreed to answer written questions. One student submitted the question, "Is sterility heritable?" Dawson must have heard this one before. He replied, "There was once a blind man in a dark room, looking for a black cat that wasn't there." He didn't even fall into the trap of rephrasing the question so it made sense: "Is fertility heritable?"

Extracurricular diversions. A minority of the mathematics faculty owned their living quarters. Graustein and Birkhoff both rented apartments. Admittedly, Birkhoff's was a grand one, overlooking the Charles River. Whitney, young as he was, also rented. Morse lived in the Commodore Hotel. Osgood owned a house, and those that visited it did much better if they were immune to cigar smoke.

There was never smoking in the regular classrooms, but there was a more relaxed attitude in the graduate mathematics club that met once a month. Coolidge often brought his large Airedale, Sandy, to those sessions. Sandy was the most relaxed soul present. Just after the

introduction was made, he chose a position on the rug before the blackboard. He lay there quietly during the talk. The speaker had free use of both the far left and the far right ends of the blackboard; the center portion was preempted.

Professor and Mrs. Graustein gave annual parties for the graduate students; they were gracious hosts. Their example was later followed by Morse, and eventually (when he became master of Lowell House) by Coolidge. The younger faculty members mixed with students informally. Brinkmann and Whitney played musical instruments; Widder was an expert squash player. Gergen and his wife visited and attended restaurants with graduate students. Seidel was a chess player of nearly, if not quite, master caliber.

Finding positions for graduates. When teaching positions were scarce, some graduate students derived support from post-Ph.D. traveling fellowships. This solution to the unemployment situation was only temporary. When the fellowship terminated, the student still had to be placed. S. B. Myers accepted a fellowship, after being assured that he could be placed later. But it was touch and go; he related that one of the faculty members was so embarrassed that he crossed the street to avoid a head-to-head meeting. Myers was a professor at Michigan before his untimely death at the end of the war.

The solution of taking a position in industry was tacitly assumed to be tantamount to treason. M. M. Slotnick was the only Harvard Ph.D. in mathematics who worked outside academia. He was a much-loved mathematician in the geological research branch of a Texas seismic exploration company. He *did* explain his work to his colleagues; that was one of the main reasons they admired him.

Osgood took a personal role in assigning positions in the teaching field to Harvard graduate students. In many cases, a letter from a school would be routed to the Harvard employment agency. Osgood visited the office of the agency regularly, and assigned students to the available positions. (He was chairman of the department at the time.)

Colloquia. Colloquia were held every month, or oftener. Many graduate students attended these. From outside the university, faculty members came from M.I.T., Dartmouth, and Brown. Wiener's contributions were already so numerous and varied that, it seemed, every talk was related in some way to his work. Morse gave one of the first expositions of *calculus of variations in the large* at a colloquium. Many of the talks were presented by visiting faculty members, some of whom came from a distance.

Occasionally, a visitor spoke at a regular classroom lesson. L. Fejér gave a lecture in Walsh's class (on real variables). Osgood attended this lecture. Fejér took occasion to preface his talk by a statement that he had learned the theory of complex variables from Osgood's book.

German relations. Osgood was a known Germanophile. His book was written in German. One of his daughters married a professor of German at Harvard. (I have met the man, and I am sure it was a happy match.)

But the closest spiritual connection between Harvard and the Göttingen tradition was probably through Garrett Birkhoff, who became a junior prize fellow in 1933, and joined the faculty later. He studied group theory, and his 1933 paper, "On the Combination of Subalgebras" [3], was fully in the spirit of modern algebraic structure theory.

Domestic relations. The parochialism of some Harvard professors was understandable. The school had its own state charter, and was (I think) one of the oldest universities in the country. A student who cited, much less used, a proof from a book by Pierpont did so at his peril. It was unlikely that the proof would be accepted. It was even possible that the proof would not be read.

On the other hand, some Harvard professors traveled widely, and some had even taught at other schools before coming to Harvard. The preferred temporary assignments at the time were Princeton and Berkeley. D. R. Curtiss was a former, and a brilliant, graduate of Harvard who had served time at all three locations before he departed for Northwestern University. Within two years, he was head of the mathematics department there. He was managing editor of the *Bulletin of the A.M.S.* for eleven years.

I believe the *Bulletin* was the only outlet in the U.S. in those times, for short research papers.

Indeed, the total number of mathematics research journals in North America hardly exceeded four. The Duke journal was founded in 1935. Later, it nearly collapsed. (There are more than 50 research journals in mathematics published in North America now.)

Bell Telephone Laboratories employed a few mathematicians as early as 1920. Their duties were primarily connected with statistics. By 1930, it had been suggested that the country was in dire need of more mathematicians for commercial and industrial positions. It was asserted that over the next ten years the deficit might rise to nine or ten Ph.D.'s.

Academic Shades. Other departments at Harvard had their share of illustrious teachers and scholars. There were G. L. Kittredge, the foremost Shakespearian scholar of the day ("Who would examine me?"); K. D. Lake, a professor of biblical literature; A. L. Lowell, the president, who had been a professor of history and government; and J. B. Conant, who was studying the chemistry of hemoglobin. W. Piston was an active teacher in the first flush of fame as a musical composer.

One of the standard jokes of the day concerned a person who came to see Mr. Lowell. "The President is not here," he was informed. "He has gone to Washington to see Mr. Coolidge." (When U.S. President Coolidge died, Dorothy Parker inquired, "How can they be sure?")

Although the atmosphere was not rural—Kittredge could be seen, white beard flowing, brandishing an umbrella in futile attempts to slow traffic at the crosswalk—it was rather common for professors, and even for Mr. Lowell, to take walks through the campus (the "yard"). Sometimes he would walk across Cambridge Street to the next campus, that of the law school. This venerable institution was presided over by Dean Roscoe Pound (never referred to as simply Dean Pound, even by his wife). Pound lent color and glamour to the annual commencement activities by appearing in his scarlet robe and cap; this costume was authorized by Oxford, from which Dean Pound held a degree.

Al Smith spoke at one Harvard Commencement. He first received an honorary degree; then he sang for his supper. He began, "Honored guests and fellow alumni!" After the ensuing chuckles, he continued, "You see, it doesn't take me long to get acclimated."

To return to the other departments. The chemistry department was operating under the shadow of T. W. Richards, who had received a Nobel prize for his accurate determination of the atomic weights of several elements. But it was already certain that (at least most) elements existed in several isotopic forms—radioactivity was discovered in 1896. Aston, in England, had invented the mass spectrometer. With that machine, isotopic analysis was faster and more accurate than by Richards' method.

Nevertheless, some chemists at Harvard persisted in devoting time to accurate chemical methods. They proved that minerals from certain parts of the world contained radio lead, rather than primordial lead. Eventually, however, the power of physical chemistry as a science became apparent, and the need to bolster that section of the chemistry department was clear. G. Kistiakowsky was invited to Harvard, and accepted an assistant professorship. From then onward, test tube chemistry began its downhill course. Kistiakowsky was later scientific advisor to the White House (under Eisenhower).

Great chemical discoveries (compounds of xenon and of boron; structure of proteins; synthesis of organic polymers) were later made at other laboratories. But during the 30's, chemistry was still almost synonomous with Harvard. Among those who enrolled as students of chemistry, even though briefly, were Stone and several others who later became mathematics Ph.D.'s. I am told that von Neumann began his academic career as a chemistry student also.

Mathematical anecdotes. A few mathematical anecdotes will be interesting. Once, G. B. Price was sitting in Birkhoff's office when a call came from the geology department. "Can you tell me offhand, please, the formula for the volume of a spherical cap?" Neither Birkhoff nor anybody else had memorized this formula. Birkhoff hung up, worked out the formula by using a check of

dimensions, and by determining the correct proportionality factor (using a hemisphere)—all in a few seconds—and then reported the answer to his questioner. Price was impressed, not so much by the accomplishment as by the straightforward method used. What was needed was the answer, not the reason for it.

Using analytic sets, Currier proved a theorem concerning second derivatives of a function of two real variables [6]. (His result was valid without any assumption concerning continuity of the second derivatives.) He then expounded his work, along with Lusin's book, in a course. The course was attended by two students, both of whom were auditors. Measure theory was not a popular subject.

In principle, a Benjamin Peirce instructor could be admitted to the rarified rolls of a permanent appointment. But the decision to admit was apparently made on tenuous grounds. Instead, the B. P. instructors were given high recommendations and shipped off after the period of tenure of the instructorship. Seidel was an exceptional scholar and a superb lecturer. During one lecture, he mentioned that nobody had determined the dual structure of a nonseparable space. A student immediately called attention to a paper of Hildebrandt in a current issue of the *Transactions* [12]. Seidel studied the paper, and lectured on it during the next class period. This was all the more remarkable, since he had also to learn about the Kolmogorov integral on which it depended. But perhaps he was already familiar with the entire encyclopedia of mathematical literature.

In 1934, G. H. Hardy gave his lectures on Ramanujan's work [10]. The audience was so large that Hardy would have needed television to present the formulas as he referred to them. But television did not exist. Hardy's solution was to have these formulas numbered and printed in advance; the sheets were passed out like programs at a concert. This logical method still has its adherents.

A favorite story, said to have originated with B. O. Peirce, and repeated by J. L. Coolidge, goes like this. The numbers e,π are important numbers in mathematics. They are connected by a mysterious formula. Everyone believes it is important. It reads

$$\sqrt[\sqrt{-1}]{-1} = e^{\pi}$$

(see also [20]).

Huntington took pride in teaching students to obtain a correct numerical result, not one 32 times too large. Osgood, on the other hand, emphasized a "check of dimensions." He did not agree with Huntington's assertion that mass and W/g are equal. To emphasize his point, he sent Huntington a Christmas card reading, "Merry X-W/g."

One of Huntington's interests was the apportionment of representatives in Congress [11]. The method that he advocated was eventually adopted.

When Courant came to NYU in 1934 as chairman of the mathematics graduate school, some people wondered how the (private) school could afford to pay him even a modest salary. The Depression was very deep. In just a few years, his center received government grants of many millions of dollars, putting to rest the doubts of any fiscal conservatives.

Mathematics was a recondite subject; besides this, travel to attend a distant meeting was uncommon. In 1940, total attendance at one West Coast meeting of the A.M.S. was less than 40.

In an after-dinner speech at an earlier Chicago meeting, D. R. Curtiss proposed that mathematics might well be publicized. Automobile trailers were just coming into vogue. He suggested that a trailer be equipped with two desks and chairs and parked on the grounds of the World's Fair. A sign on the trailer was to identify the occupants as "Mathematicians at Work."

L. E. Dickson and G. A. Miller were both alive and fairly active at that time. Miller remained approachable. Dickson, however, was immersed (with many students) in just one subject—the Waring problem (see [8]). He seemed to reject any other. In fact, during a discussion period that followed one paper at an A.M.S. meeting, Dickson criticized the choice of topic. "It is a lucky thing that newspaper reporters do not attend these meetings," he opined. "If they did, they

would see how little our activities are related to the real needs of society." Fifteen minutes later, he outlined a proof that every sufficiently large integer can be written as a sum, not of 1140 tenth powers (the best previous result), but of 1046 tenth powers.

Acknowledgment. R. P. Boas has acted as my Maxwell Perkins. Some of the anecdotes are his. The errors are all mine.

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EULER SUBDUES A VERY OBSTREPEROUS SERIES

E. J. BARBEAU

The task of evaluating the infinite series $1-1!+2!-3!+\cdots$ caused Euler to clarify his ideas on the meaning of assigning a sum to a series, even one which, in modern eyes, is divergent. In this article, we summarize these ideas and outline four ingenious approaches of Euler to evaluate

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the above series. The consistency of these approaches is discussed, with reference to summability methods, extrapolation, continued fractions, and infinite differential operators.

1. Assigning a value to a divergent series. In the late seventeenth and early eighteenth centuries, mathematicians were busily developing what promised to be a significant body of powerful techniques consequent to the creation of the calculus. However, there was considerable uncertainty about the best formulation of the underlying concepts. Probably no better example of this can be found than in the discussion of the meaning of the sum or value of an infinite series. Since sums of monomials can be easily differentiated and integrated, the discovery by Newton and his contemporaries that a great many functions could be developed as power series meant that calculus had quite wide applicability. Consequently, the question of attaching a sum to a series attracted much interest and controversy.

Although mathematicians of this period were aware intuitively that, for some series, the sum could be regarded as the limit of the partial sums, in their view this did not adequately cover the matter. Even when this limit did not exist, many series nevertheless seemed to possess a natural value. Their attitude was influenced in part by their notion of a function as an analytic expression defined over the widest possible domain, including complex numbers and quantities infinitely great or small. Not being in possession of pathological counterexamples, they considered that two analytic expressions agreeing on a continuous set must agree everywhere. Thus, for example, if $(1+x)^c$ is synonymous with its binomial expansion for $|x|<1$, then $(1+x)^c$ must be the value of that expansion for all x, except possibly for obvious singularities.

These opinions were buttressed by experience. It was generally found that, where there were several ways of determining the value of an infinite series, they gave the same result. Moreover, in computations, the practice of interchanging an infinite series with its value did not appear to cause trouble.

Euler's paper, "De seriebus divergentibus" **[12]**, published in 1760, illuminates this spirit well. It can be split into two parts. The first subtly treats the question of assigning a value to a series. The second is devoted to evaluating "Wallis' hypergeometric series"

$$1-1!+2!-3!+4!-5!+\cdots.$$

Here we have a somewhat different approach to mathematical acceptability than that of today. Euler's concern is to put his result beyond all reasonable doubt, and this he does by arriving at it by a number of routes. It is consistency, as much as logical argument, which puts its stamp of approval on the mathematics. (See **[15]** for a wider discussion of this issue.)

Although the modern investigator would quarrel with details of the work of Euler or of his contemporaries, it nevertheless displays a compelling consistency and usually leads to results demonstrably correct according to today's standards of rigour. Consequently, unusual methods of assigning a value to an infinite series have not been disdained during the past century, but rather formalized, studied in detail, compared, and extended. In situations where normal convergence fails, it is possible to find an alternative definition of "sum" which retains many of the properties associated with the usual concept (and, indeed, agrees with it for series convergent in the normal sense) and which will assign to a given infinite series the value of the function which generates it. This can be done for the binomial expansion and other power series beyond the circle of convergence, as well as for Fourier series, witness Fejér's theorem on the Cesàro-summability of the Fourier expansion of a continuous function. A discussion of summability from the modern point of view can be found elsewhere **[5**, pp. 5–10], **[20]**, **[23]**.

2. Euler's general outlook. The prospectus to his paper **[12]** declares Euler's intention "to clarify a concept causing up to now the greatest difficulties." While he would not accept that mathematics is necessarily free of controversy, he is confident that mathematical disputes, unlike those in other areas, can be completely resolved once the evidence has been thoroughly weighed. So it is with assigning values to infinite series. Infinite series can be divided into four categories

according as the terms are positive or alternating, bounded or unbounded. Examples of the four groups are

$$\begin{array}{ll} \text{I.} & 1+1+1+1+\cdots \\ \text{II.} & 1-1+1-1+\cdots \\ \text{III.} & 1+2+4+8+\cdots \\ \text{IV.} & 1-2+4-8+\cdots \end{array}$$

Series in group I present no difficulty. Either they converge to a finite sum in the modern sense, or they diverge to the infinite sum, $a/0$. More contentious are the series of group II. Euler bases his discussion on the expansion $1-a+a^2-a^3+\cdots$ of the fraction $1/(1+a)$. While no one would deny that these two expressions agree when $|a|<1$, one might object to assigning the fraction as the sum of the series when $|a|\geqslant 1$ on the grounds that the remainder term $\pm a^{n+1}/(1+a)$ in the equation

$$\frac{1}{1+a}=1-a+a^2-a^3+\cdots\mp a^n\pm\frac{a^{n+1}}{1+a}$$

cannot be neglected. Some of those who support the fraction as sum counter that, for infinite n, the ambiguous sign makes the remainder indeterminate, so that the remainder should be forgotten. In any case, they say, when you sum to infinity, you never reach the place where the remainder has to be inserted. Euler reserves his own position until later.

Those who would assign sums to divergent series appear to be in deep trouble with series in group III. Although it might seem appropriate to assign for these series, as for those of group I, an infinite sum, there occur situations in which the sum indicated by analysis is not only finite but negative. For example, substituting -3 for a in the expansion of $1/(1+a)$ yields the paradoxical equation

$$-\frac{1}{2}=1+3+9+27+81+\cdots.$$

Here one is in the absurd position of adding together positive terms to get a negative sum. Nevertheless, explaining this is a mere challenge to the ingenuity! To resolve the paradox (says Euler), some try to distinguish between two types of negative numbers, those that are less than zero and those that are greater than infinity. An example of the first type is the difference between an integer and its successor: $-1=n-(n+1)$. An example of the second type is $-1=1/-1$, since it fits naturally into the "increasing" sequence

$$\ldots,1/3,1/2,1/1,1/0,1/-1,1/-2,1/-3,\ldots.$$

Euler disapproves of this distinction on the grounds that it "does violence to the certitude of analysis" to have two different concepts of -1. However, he is prepared to accept that "the same quantities which are less than zero can be considered to be greater than infinity."

Series in group IV can sometimes be handled as those in group II, already treated. For example, from the expansion of $1/(1+1)^2$, it is found that $\frac{1}{4}=1-2+3-4+5-6+\cdots$. Euler says very little about this type in general, except to remark that it "is usually burdened with problems of its very own."

Euler affirms that the real justification for assigning a value to a divergent series does not rest in any of the specious arguments given above, but rather in a substitution principle. If an infinite expansion can be replaced in a calculation by the expression which generates it without any ensuing error, then this replacement should be considered valid. One has only to be careful that the rules for doing this are properly investigated. As for the techniques to determine exactly what the value of a given series is, their power can be demonstrated by treating the particularly violent specimen which occupies the rest of the paper.

3. Euler's treatment of Wallis' series. Euler's attribution of the series $1-1!+2!-3!+\cdots$ to Wallis is a mystery. While Wallis had much to say about summing progressions, I have found no reference in his work to this particular series. His interest in the factorial function lay in interpolating its values for non-integral arguments. He discusses this question in the Scholium to Proposition 190 in his *Arithmetica infinitorum* (1655) and, again, in a letter to Leibniz dated January 16, 1699 [**14**, p. 59], where he seeks a formula for $n!$ which makes sense for nonintegral n comparable to the formula $\frac{1}{2}(n^2+n)$ for the sum $1+2+3+\cdots+n$. The adjective "hypergeometric" simply signifies that each term is obtained from its predecessor by multiplying by a factor which varies (presumably in some regular way) from term to term. This is in contrast to a "geometric" progression in which the multiplying factor remains the same. However, with the great interest in the factorial function, it is likely that the problem of summing "Wallis'" series was formulated in the correspondence of the early eighteenth century. Euler himself discussed it in at least two letters to Nicholas Bernoulli [**13**, pp. 538, 543, 546] before publishing his findings in the paper under discussion. The series gets brief mention in the books by Kline [**6**, pp. 451, 1114], Bromwich [**2**, p. 323] and Hardy [**5**, pp. 26–29].

Euler evaluates the series by four different methods. In the first he is content to get a rough numerical approximation by exploiting the fact that the series is alternating. To motivate his approach, let us first consider the sum of an alternating series which converges: $1-1/2+1/3-1/4+1/5-1/6+\cdots=\log 2$. An upper bound for the sum is any partial sum whose last term is positive—for example, $1-1/2+1/3-1/4+1/5=47/60$; a lower bound is any partial sum whose last term is negative—for example, $1-1/2+1/3-1/4=7/12$. Apply the same reasoning to Wallis' series. The partial sums are $1,0,2,-4,20,-100,620,\ldots$. The odd partial sums give upper bounds for the value of the series; the even partial sums give lower bounds. However, because the general term does not tend to zero, we do not obtain a very good estimate. Indeed, all that can be said is that the value lies between 0 and 1. Consequently, we would like to transform the series into an equivalent series which is alternating but which is capable of giving a better estimate.

To see how this might be done, notice that the summing of the alternating series $a_1-a_2+a_3-a_4+a_5-a_6+\cdots$ can be achieved by evaluating the power series

$$a_1x-a_2x^2+a_3x^3-a_4x^4+a_5x^5-a_6x^6+\cdots$$

at $x=1$. We effect a change of variables to produce the required transformation. Introduce y by the equation

$$x=y(1-y)^{-1}=y+y^2+y^3+y^4+\cdots.$$

After substitution and some formal manipulation, the power series becomes

$$a_1y-(\Delta a_1)y^2+(\Delta^2a_1)y^3-\cdots+(-1)^{k-1}(\Delta^{k-1}a_1)y^k+\cdots$$

where Δ is the forward difference operator defined by

$$\Delta^0a_i=a_i,\qquad \Delta^1a_i=\Delta a_i=a_{i+1}-a_i$$

$$\Delta^ka_i=\Delta^{k-1}a_{i+1}-\Delta^{k-1}a_i=\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}a_{i+j}$$

for $i\geqslant 1$, $k\geqslant 2$. Since $x=1$ corresponds to $y=\frac{1}{2}$, we can evaluate $a_1-a_2+a_3-a_4+\cdots$ by evaluating the y-series at $y=\frac{1}{2}$:

$$\tfrac{1}{2}a_1-\tfrac{1}{4}(\Delta a_1)+\tfrac{1}{8}(\Delta^2a_1)-\tfrac{1}{16}(\Delta^3a_1)+\cdots.$$

Euler applies this to obtaining the value

$$A\equiv 1-1+2-6+24-120+720-5040+40320-\cdots.$$

First remove the first two terms, $1-1$, which cancel, and divide by 2 to get

$$\begin{array}{lccccccccccccccc}
\frac{A}{2}= & 1 & & -3 & & +12 & & -60 & & +360 & & -2520 & & +20160 & & -181440\\
 & & 2 & & 9 & & 48 & & 300 & & 2160 & & 17640 & & 161280 & \\
 & & & 7 & & 39 & & 252 & & 1860 & & 15480 & & 143640 & & \\
 & & & & 32 & & 213 & & 1608 & & 13620 & & 128160 & & & \\
 & & & & & 181 & & 1395 & & 12012 & & 114540 & & & & \\
 & & & & & & 1214 & & 10617 & & 102528 & & & & & \\
 & & & & & & & 9403 & & 91911 & & & & & & \\
 & & & & & & & & 82508 & & & & & & &
\end{array}$$

The rows under the series give, for the absolute values of its terms, differences of the first, second, third, etc., orders, respectively. Applying the transformation, we find that

$$\frac{A}{2}=\frac{1}{2}-\frac{2}{4}+\frac{7}{8}-\frac{32}{16}+\frac{181}{32}-\frac{1214}{64}+\frac{9403}{128}-\frac{82508}{256}+\cdots .$$

Cancelling the first two terms and multiplying by 2 gives

$$A=\frac{7}{4}-\frac{32}{8}+\frac{181}{16}-\frac{1214}{32}+\frac{9403}{64}-\frac{82508}{128}+\cdots .$$

It can be seen that not much progress has been made! However, Euler continues transforming the series, to get at the next turn of the crank,

$$A=\frac{7}{8}-\frac{18}{32}+\frac{81}{128}-\frac{456}{512}+\frac{3123}{2048}-\frac{24894}{8192}+\cdots .$$

Now the second term has smaller magnitude than the first. From the first two partial sums, A must be between $7/8$ and $5/16$. After one more application of the transformation, Euler is prepared to say that A is about 0.580.

The difference operator intervenes also in Euler's second attack on the series. His strategy is to define a sequence whose zeroth term is formally Wallis' series and then to compute this zeroth term numerically. This requires Newton's method of extrapolation, which will be briefly described. For a given sequence, $(a_1,a_2,\ldots,a_n)$, observe that

$$\begin{aligned}
a_{m+1}&=a_m+(a_{m+1}-a_m)=a_m+\Delta a_m\equiv(1+\Delta)a_m\\
a_{m+2}&=a_{m+1}+\Delta a_{m+1}=(1+\Delta)a_{m+1}=(1+\Delta)(1+\Delta)a_m\\
&=a_m+2\Delta a_m+\Delta^2 a_m
\end{aligned}$$

and, for any positive integers m and k,

$$\begin{aligned}
a_{m+k}&=(1+\Delta)^k a_m\\
&\equiv a_m+k\Delta a_m+\frac{k(k-1)}{2}\Delta^2 a_m+\frac{k(k-1)(k-2)}{6}\Delta^3 a_m+\cdots .
\end{aligned}\tag{3.1}$$

For k other than a nonnegative integer, the right side of (3.1) still makes sense, so that (3.1) can be used to represent "terms" of the sequence corresponding to indices other than natural numbers.

Euler considers the sequence (P_n) whose terms are given by $P_1=1$, $P_2=2$, $P_3=5$, $P_4=16$, $P_5=65$, and, generally,

$$P_{n+1}=nP_n+1 \quad \text{for } n=2,3,4,\ldots, . \tag{3.2}$$

From the fact that $\Delta^i P_1=i!$ $(i=0,1,2,3,\ldots,)$, the formula (3.1) with $m=1$, $k=n-1$, yields a formula for P_n:

$$P_n=(1+\Delta)^{n-1}P_1=P_1+(n-1)\Delta P_1+\binom{n-1}{2}\Delta^2P_1+\binom{n-1}{3}\Delta^3P_1+\cdots$$
$$=1+(n-1)+(n-1)(n-2)+(n-1)(n-2)(n-3)+\cdots.$$

Further, substituting 0 for n gives

$$P_0=1-1!+2!-3!+4!-\cdots.$$

How can a numerical value for P_0 be found?

Euler next applies (3.1) with $m=1$, $k=-1$ to the sequences whose general terms are $a_n=1/P_n$ and $a_n=\log_{10}P_n$. In the first case, the zeroth term is found to be

$$1-(-1/2)+(1/5)-(-3/80)+(-36/1040)-(11271/220376)+\cdots$$
$$=1+0.5+0.2+0.0375-0.0364154-0.0511444+\cdots$$
$$=1.651740 \text{ (Euler's figure).}$$

Taking 1.651740 as $1/P_0$, we have that $P_0=0.60542$. Analysis of the second sequence, $(\log_{10}P_n)$, corroborates this determination of P_0 quite well. The zeroth term of the sequence is

$$0-0.3010300+0.0969100-0.0103000-0.0128666-0.0053006+\cdots$$

and this Euler, using the transformation procedure of his first method, computes as $\bar{1}.7779089$. Thereupon, $P_0=0.59966$.

This method raises two interesting questions. First, are the series obtained for $1/P_0$ and $\log_{10}P_0$ actually convergent? Second, if the terms of one sequence are a certain function of the corresponding terms of another, how reasonable is it to expect that the functional relation will persist to the extrapolated terms as well? This does not always happen; if, for positive integers n, $a_n=n$, $b_n=f(a_n)$ with $f(z)=\sin\pi z/(\pi z)$, $f(0)=1$, then Newton's extrapolation procedure yields $a_0=b_0=0$; but $f(a_0)=1$. One suspects that it is not enough for f to be analytic but that it should have less than exponential growth at infinity as well.

The last two approaches of Euler hinge on finding a closed expression for a power series in x, which, for $x=1$, produces Wallis' series. In the third method, he observes that the power series

$$s(x)=x-1x^2+2x^3-6x^4+24x^5-120x^6+\cdots \tag{3.3}$$

formally satisfies the differential equation

$$s'+(s/x^2)=1/x. \tag{3.4}$$

This first order equation can be solved in the usual way; the solution which vanishes for $x=0$ is

$$s(x)=e^{1/x}\int_0^x\frac{e^{-1/t}}{t}\,dt. \tag{3.5}$$

Using the substitution $v=\exp(1-1/t)$, $t=1/(1-\log v)$, $dt/t=dv/v(1-\log v)$, Euler transforms (3.5) to

$$s(x)=e^{(1/x-1)}\int_0^{e^{1-1/x}}\frac{dv}{1-\log v}. \tag{3.6}$$

For future reference, we record here that, making the substitution $s=1/t-1/x$, the integral can be rendered

$$s(x)=\int_0^\infty\frac{xe^{-s}}{1+xs}\,ds. \tag{3.7}$$

These three integrals yield the following alternative forms for the value of Wallis' series:

$$s(1)=e\int_0^1\frac{e^{-1/t}}{t}\,dt=\int_0^1\frac{dv}{1-\log v}=\int_0^\infty\frac{e^{-s}}{1+s}\,ds, \tag{3.8}$$

of which Euler computes the approximate values of the first and second by the trapezoidal rule.

Euler checks that the second integral of (3.8) ought to give Wallis' series by substituting $y=1$ into the expansion (obtained by integrating by parts),

$$\int_0^y \frac{dv}{1-\log v}=\frac{y}{1-\log y}-\frac{1\cdot y}{(1-\log y)^2}+\frac{1\cdot 2\cdot y}{(1-\log y)^3}-\frac{1\cdot 2\cdot 3\cdot y}{(1-\log y)^4}+\cdots. \tag{3.9}$$

This integral allows for an alternative computation of the value of Wallis' series which Euler does not mention in the paper but which he confides in a letter to Bernoulli [**13**, p. 546]. The left side of (3.9) is expanded in ascending powers of $(1-v)$, and the series is integrated term by term. Upon substitution of 1 for y, there results

$$1-1+2-6+24-\cdots=1-1/2+1/6-1/12+1/30-7/360+19/2520-3/560+\cdots. \tag{3.10}$$

Euler's fourth approach is to obtain a "continued fraction" expansion for the power series

$$u(x)=s(x)/x=1-x+2x^2-6x^3+24x^4-120x^5+\cdots. \tag{3.11}$$

This has the form $1/(1+B)$ where

$$B=\frac{x-2x^2+6x^3-\cdots}{1-x+2x^2-6x^3+\cdots}.$$

In turn, B can be put in the form $x/(1+C)$, with

$$C=\frac{x-4x^2+18x^3-96x^4+\cdots}{1-2x+6x^2-\cdots}.$$

So far, we have found that

$$u(x)=\cfrac{1}{1+\cfrac{x}{1+C}},$$

which, for short, we denote by $u(x)=1/1+x/1+C$, with the convention that each slash incorporates everything which follows. This process is continued: C is written as $x/(1+D)$, D as $2x/(1+E)$, and so on. Carrying on indefinitely, Euler finds

$$u(x)=1/1+x/1+x/1+2x/1+2x/1+3x/1+3x/1+4x/1+4x/\cdots. \tag{3.12}$$

The value of $u(x)$ can be approximated by the convergents obtained by stopping the continued fraction (3.12) at any slash. These are $p_1(x)/q_1(x)=1/1+x$; $p_2(x)/q_2(x)=1/1+x/1+x$; $p_3(x)/q_3(x)=1/1+x/1+x/1+2x$. The nth convergent is $p_n(x)/q_n(x)$, where, for small values of n, $p_n(x)$ and $q_n(x)$ are given in the following table:

n	$p_n(x)$	$q_n(x)$
1	1	$1+x$
2	$1+x$	$1+2x$
3	$1+3x$	$1+4x+2x^2$
4	$1+5x+2x^2$	$1+6x+6x^2$
5	$1+8x+11x^2$	$1+9x+18x^2+6x^3$
6	$1+11x+26x^2+6x^3$	$1+12x+36x^2+24x^3$
7	$1+15x+58x^2+50x^3$	$1+16x+72x^2+96x^3+24x^4$.

In general, for $n\geqslant 1$, these relations hold:

$$\begin{aligned} p_{2n+1}(x)&=p_{2n}(x)+(n+1)xp_{2n-1}(x) & p_{2n+2}(x)&=p_{2n+1}(x)+(n+1)xp_{2n}(x)\\ q_{2n+1}(x)&=q_{2n}(x)+(n+1)xq_{2n-1}(x) & q_{2n+2}(x)&=q_{2n+1}(x)+(n+1)xq_{2n}(x). \end{aligned} \tag{3.13}$$

The sum of Wallis' series ought to be $u(1)$. For $x=1$, the convergents of the continued fraction (3.12) are 1/2, 2/3, 4/7, 8/13, 20/34, 44/73, 124/209, 300/501,... forming an apparently convergent sequence.

Euler has another, somewhat curious, way of evaluating the continued fraction at $x=1$. He writes

$$A=1/1+1/1+1/1+2/1+2/1+3/1+3/1+4/1+4/1+5/\cdots/1+10/1+10/1+p$$

$$\text{where}\qquad p=11/1+11/1+12/\cdots/1+15/1+15/1+q,$$

$$\text{where}\qquad q=16/1+16/1+17/\cdots/1+20/1+20/1+r, \text{ and}$$

$$\text{where}\qquad r=21/1+21/1+22/1+22/1+23/1+23/\ldots.$$

From this

$$A=\frac{491459820+139931620p}{824073141+234662231p}$$

$$p=\frac{2381951+649286q}{887640+187440q}$$

$$q=\frac{11437136+2924816r}{3697925+643025r}.$$

The calculation depends on determining r. If, in the definition of r, we replace the numbers $22,23,24,25,\ldots$ all by 21, we obtain the approximate equation $r=21/(1+r)$, which is satisfied by

$$r=\tfrac{1}{2}(\sqrt{85}-1)=4.10977\ldots. \tag{3.14}$$

Euler has a second way of finding r. We have that

$$r=21/1+21/1+s=(21+21s)/(22+s),$$

where

$$s=22/1+22/1+t=(22+22t)/(23+t)\quad\text{and}\quad t=23/1+23/1+24/1+24/\cdots.$$

Euler assumes that r, s, and t are in arithmetic progression, so that $r+t=2s$. Since $t=(23s-22)/(22-s)$, he finds that

$$r+t=(2s^2+925s-22)/(484-s^2)=2s,$$

whence $2s^3+2s^2-43s-22=0$. This is solved by an approximate method (Newton's) to obtain $s=4.423$, from which $r=4.31$, $q=3.71645446$, $p=3.0266600163$, $A=0.5963473621372$ (Euler's accuracy). Euler notes that close rational approximations can be obtained from the convergents of the simple continued fraction expansion of A,

$$1/1+1/1+1/2+1/10+1/1+1/1+1/4+1/2+1/2+1/13+1/4+\cdots.$$

Euler's ingenuity has brought forth a variety of ways of handling the seemingly impossible problem of attaching a value to Wallis' series. If these all lead to the same numerical result, then it will reinforce the conclusion that Wallis' series has a natural value and that, within computational error, we have found it. Let us make the test. Euler's first method is crude, but does give the value 0.580. His second gives the values 0.60542 (from $1/P_0$) and 0.59966 (from $\log P_0$). The trapezoidal rule for approximate integration with ten subintervals gives, respectively, 0.59637255 and 0.58734359 for the first and second integrals of (3.8). Formula (3.10) gives about 0.59940472. Using the convergents 124/209 and 300/501 of the expansion (3.12) for $u(1)$ puts the value between 0.5933 and 0.5988. Using (3.14) for r gives $A=0.59634738$, and using $r=4.31$ gives 0.596347362. It can be fairly concluded that these results are consistent. Whatever differences arise seem to reflect the accuracy or efficiency of the method. What should the answer be? Hardy [5, p. 26], by computing (3.7), obtains

$$1-1!+2!-3!+\cdots=-e\left(\gamma-1+\frac{1}{2.2!}-\frac{1}{3.3!}+\cdots\right)$$

where $\gamma=0.577215664901533\ldots$ is Euler's constant. The value obtained is about 0.59635. In the remainder of this article, we will explore the compatibility of these methods in more detail.

4. Remarks on Euler's evaluations. Euler's definition of the summation of $a_1-a_2+a_3-a_4+\cdots$ in terms of the summation of $\frac{1}{2}a_1-\frac{1}{4}(\Delta a_1)+(1/8)(\Delta^2 a_1)-\cdots$ is the progenitor of the (E,q) summability method based on the transformation $x=y(1-qy)^{-1}$ [5, Ch. 8]. In his discussion of Wallis' series [5, pp. 28, 196], Hardy points out that no application of Euler's method will convert the series into a convergent one, although by making sufficiently many transformations we can make the error introduced very small by stopping the resulting series at a suitable point. By making an adjustment to this method, Hardy shows how to obtain a value for Wallis' series, $\Sigma_n(-1)^n n!$, which agrees with Euler's third method. Since,

$$n!=\int_0^2 e^{-t}t^n\,dt+\int_2^5 e^{-t}t^n\,dt+\cdots+\int_{2^p+2^{p-1}-1}^{2^{p+1}+2^p-1} e^{-t}t^n\,dt+\cdots,$$

we write formally

$$\sum_n(-1)^n n!=\sum_n\int_0^2(-1)^n e^{-t}t^n\,dt+\sum_n\int_2^5(-1)^n e^{-t}t^n\,dt+\cdots+\sum_n\int_{2^p+2^{p-1}-1}^{2^{p+1}+2^p-1}(-1)^n e^{-t}t^n\,dt+\cdots.$$

Each of the series on the right side is (E,q)-summable for q sufficiently large. However, the size of q required to evaluate the series becomes arbitrarily large with p. Putting in the values obtainable in this way, and adding, we obtain for the right side

$$\int_0^2\frac{e^{-t}}{1+t}\,dt+\int_2^5\frac{e^{-t}}{1+t}\,dt\cdots=\int_0^\infty\frac{e^{-t}}{1+t}\,dt, \tag{4.1}$$

which is the third integral of (3.8).

Euler's third method amounts to assigning the value (3.7) to the power series (3.3), or equivalently the integral $\int_0^\infty e^{-s}(1+xs)^{-1}ds$ to (3.11). His judgment can be vindicated in a number of ways. For example, expanding the integrand we find that

$$\int_0^\infty\frac{e^{-s}}{1+xs}\,ds=1-x+2!x^2-3!x^3+\cdots+(-1)^n n!x^n$$
$$+(-1)^{n+1}x^{n+1}\int_0^\infty\frac{e^{-s}s^{n+1}}{1+xs}\,ds. \tag{4.2}$$

For positive values of x, the integral in the remainder term is dominated by $\int_0^\infty e^{-s}s^{n+1}ds=(n+1)!$, so that the remainder term is of the same sign as and of less magnitude than the term $(-1)^{n+1}(n+1)!x^{n+1}$. While the ratio test reveals that the series (3.11) does indeed diverge for all positive values of x, nevertheless, from (4.2), for each fixed n

$$\int_0^\infty\frac{e^{-s}}{1+xs}\,ds=1-x+2!x^2-3!x^3+\cdots+(-1)^n n!x^n+O(x^{n+1})\quad\text{as } x\to 0.$$

(See [4] for further discussion of this type of behavior.)

Alternatively, an independent evaluation of the power series (3.11) can be made by a modification of Borel's integral method of summability [5, pp. 182, 192], which is quite powerful. Recall that, to sum the series $a_0+a_1+a_2+a_3+\cdots$, we define the function

$$U(s)=\sum_{k=0}^\infty\frac{a_k}{k!}s^k. \tag{4.3}$$

Assuming an analytic determination of $U(s)$ for $0\leqslant s<\infty$, the Borel or, more properly, (B^*) sum is defined to be

$$\int_0^\infty e^{-s}U(s)\,ds. \tag{4.4}$$

For the series (3.11), $a_k = (-1)^k k! x^k$, so that

$$U(s) = \sum_{k=0}^{\infty} (-1)^k (xs)^k.$$

When $|xs| < 1$, $U(s)$ is equal to $(1+xs)^{-1}$; we take this as an evaluation of $U(s)$ for all nonnegative s. Then the (B^*) sum of (3.11) will be

$$\int_0^{\infty} e^{-s}(1+xs)^{-1}\, ds. \tag{4.5}$$

Another treatment appears in a 1941 paper of Hardy **[16]**. In effect, he defines the value of $\Sigma_{n=0}^{\infty}(-1)^n n! x^n$ to be the limit, as $\delta \to 0$, of

$$\sum_{n=0}^{\infty} (-1)^n e^{-\delta\lambda_n} n! x^n \tag{4.6}$$

where $\lambda_0 = \lambda_1 = \lambda_2 = 0$ and $\lambda_n = n \log n \log\log n$ for $n \geqslant 3$. Through the residue theorem, (4.6) can be expressed as a sum of two contour integrals which tends, as $\delta \to 0$, to the limit

$$\int_C \frac{\Gamma(s+1)}{e^{2\pi i s}-1}(-x)^s\, ds,$$

for any complex number x not lying on the negative real axis, where C is a suitable contour. Using the integral form of $\Gamma(s+1)$ and effecting an inversion of the order of integration, Hardy obtains from this integral the required quantity (4.5).

The consistency between Euler's third method, using the differential equation for $s(x)$, and his fourth method, using the continued fraction for $u(x)$, was established by both Laguerre **[18]** and Stieltjes **[22]**. Both do this by showing that a suitable integral can be developed as a continued fraction. Laguerre treats $\int_x^{\infty} e^{-t}/t\, dt$ which, for $z = 1/x$, $t = 1/s$, becomes $\int_0^z e^{-1/s}/s\, ds$ (cf. (3.5)). On the other hand, Stieltjes substitutes $z = 1/x$ into (3.12) to get a continued fraction of the form

$$1/a_1 z + 1/a_2 + 1/a_3 z + 1/a_4 + 1/a_5 z + \cdots . \tag{4.7}$$

The convergents of this continued fraction are rational functions possessing partial fraction decompositions of the type $\Sigma M_i/(z+x_i)$ with both M_i and x_i positive. Stieltjes writes the nth convergent as a special kind of integral

$$\int_0^{\infty} \frac{d\phi_n}{z+y}$$

where ϕ_n is an increasing jump function, $\phi_n(0) = 0$ and $\lim_{y\to\infty} \phi_n(y) = 1/a_1$. For this particular case, as n increases, the functions $\phi_n(y)$ tend to a limit $\Phi(y)$, an increasing function determinable by solving the moment problem

$$\int_0^{\infty} t^k\, d\Phi(t) = c_k \qquad (k = 0, 1, 2, \cdots)$$

where the c_k are derivable from the known quantities a_i. Thus, (4.7) can be written as $\int_0^{\infty} d\Phi(t)/(z+t)$ from which (3.12) can be expressed as

$$\int_0^{\infty} \frac{e^{-t}}{1+xt}\, dt. \tag{4.8}$$

However, I will take a more direct approach in connecting the continued fraction for $u(x)$ to its differential equation. Since $u(x) = s(x)/x$ and since s satisfies (3.4), the differential equation for u is

$$x^2 u' + (1+x)u = 1 \tag{4.9}$$

(which can be seen to have the solution (4.8) with $u(0) = 1$). Our task is to show, first, that for

$0<x<\infty$, the sequence $(p_n(x)/q_n(x))$ of convergents of (3.12) converges, and, second, that the limit is a solution of (4.9).

With the help of (3.13), it is straightforward to establish that, for $n \geqslant 1$:

(1) $p_{2n}, p_{2n+1}, q_{2n-1}, q_{2n}$ are polynomials of degree n whose coefficients are all nonnegative and whose constant terms are equal to 1.

(2) The leading coefficient of both q_{2n} and q_{2n+1} is $(n+1)!$.

(3) $p_n q_{n+2} - p_{n+2} q_n = p_n q_{n+1} - p_{n+1} q_n$.

(4) Let

$$w_n = p_n q_{n+1} - p_{n+1} q_n. \tag{4.10}$$

Then

$$w_{2n} = -(n+1)xw_{2n-1} \quad \text{and} \quad w_{2n+1} = -(n+1)xw_{2n},$$

so that

$$\begin{aligned} w_{2n} &= (n+1)(n!)^2 x^{2n+1} \\ w_{2n-1} &= -(n!)^2 x^{2n}. \end{aligned} \tag{4.11}$$

(5)

$$\begin{aligned} \frac{p_{2n}}{q_{2n}} - \frac{p_{2n+2}}{q_{2n+2}} &= \frac{(n+1)(n!)^2 x^{2n+1}}{q_{2n+2} q_{2n}} \\ \frac{p_{2n+1}}{q_{2n+1}} - \frac{p_{2n-1}}{q_{2n-1}} &= \frac{(n!)^2 x^{2n}}{q_{2n+1} q_{2n-1}} \\ \frac{p_{2n}}{q_{2n}} - \frac{p_{2n+1}}{q_{2n+1}} &= \frac{(n+1)(n!)^2 x^{2n+1}}{q_{2n+1} q_{2n}} \\ \frac{p_{2n}}{q_{2n}} - \frac{p_{2n-1}}{q_{2n-1}} &= \frac{(n!)^2 x^{2n}}{q_{2n} q_{2n-1}}. \end{aligned}$$

Because of (1) and (2), for all positive values of x, the right sides of all four equations are positive and of the last two are each less than $1/(n+1)$. Therefore, it follows that, on the positive real axis, the odd convergents increase and the even convergents decrease to a common limit function $u(x)$.

In order to check that the limit of the convergents satisfies (4.9), we examine the expression $x^2u' + (1+x)u - 1$, with u replaced by p_n/q_n. This is a fraction, whose numerator is

$$x^2(p_n' q_n - p_n q_n') + r_n q_n, \tag{4.12}$$

where $r_n = (1+x)p_n - q_n$ is a polynomial satisfying the same recursion relations (3.13) as p_n and q_n.

It is readily conjectured that, for $n = 1, 2, \ldots$ and $q_0 = 1$,

$$\begin{aligned} q_{2n+1}' &= (n+1)^2 q_{2n-1} \\ q_{2n}' &= n(n+1) q_{2n-2}, \end{aligned} \tag{4.13}$$

which an induction argument shows to be indeed true. This suggests that we should examine $p_{2n+1}' - (n+1)^2 p_{2n-1}$ and $p_{2n}' - n(n+1)p_{2n-2}$. Again, an induction argument reveals that, for $n = 1, 2, \ldots$ and $p_0 = 0$,

$$\begin{aligned} x^2(p_{2n+1}' - (n+1)^2 p_{2n-1}) + r_{2n+1} &= 0 \\ x^2(p_{2n}' - n(n+1)p_{2n-2}) + r_{2n} &= 0. \end{aligned} \tag{4.14}$$

Inserting (4.13) and (4.14) into (4.12), we find that, for every positive integer n,

$$x^2(p_n'q_n-p_nq_n')+(1+x)p_nq_n-q_n^2=w_n,$$

where w_n is given by (4.10). Therefore,

$$\left(\frac{p_n}{q_n}\right)'+\left(\frac{1+x}{x^2}\right)\left(\frac{p_n}{q_n}\right)-\frac{1}{x^2}=\frac{w_n}{x^2q_n^2} \tag{4.15}$$

for $x>0$. From the above facts (1), (2), and (4),

$$\frac{w_{2n}}{x^2q_{2n}^2}\leqslant\frac{(n+1)(n!)^2x^{2n+1}}{x^2((n+1)!)^2x^{2n}}=\frac{1}{(n+1)x}$$

so that, as n tends to infinity through even values, the right side of (4.10) converges uniformly to zero on compact subsets of $(0,\infty)$. Since p_n/q_n tends uniformly to $u(x)$ on compact subsets of $(0,\infty)$, from (4.15), $\lim_{n\to\infty}(p_{2n}/q_{2n})'$ exists and must be $u'(x)$ for $x>0$. Therefore, $u'+(1+x)x^{-2}u-x^{-2}=0$, so that u satisfies (4.9).

Actually, the convergents give an efficient approximation to $u(x)$ in the sense that the series expansion of $p_n(x)/q_n(x)$ reproduces the terms in the series (3.11) up to degree n, which is about twice the degree of the numerator or the denominator. Indeed, $q_n(x)u(x)=p_n(x)+$(terms of degree exceeding n), so that the convergents are among the Padé approximants of $u(x)$ [**10**, Ch. 20]. The properties of these convergents also give some insight into the dismal failure of (3.11) to converge while at the same time $u(x)$ can be evaluated for all nonnegative real x. Consider the real zeros of the polynomials $q_n(x)$. It is clear that they must all be negative. By inspection, we note that q_1 and q_2 each have a negative zero. Suppose that for $k\geqslant 2$ it has been established that for $i=1,2,3,\ldots,k$, q_i has at least one negative zero and that, if b_i is the largest real zero of q_i, then $b_{i-1}<b_i$ for $2\leqslant i\leqslant k$. Thus, if $1\leqslant i\leqslant j\leqslant k$, $q_i(x)>0$ for $b_j<x\leqslant 0$. Then, from (3.13), we can deduce that $q_{k+1}(b_k)<0$, so that q_{k+1} has a zero in the interval $(b_k,0)$. In particular, b_{k+1} exists and $b_k<b_{k+1}<0$. Therefore, $\lim b_n$ exists. We now show that the limit is 0. Substituting $x=-1/(n+1)$ into the second and fourth equations of (3.11), we find that $q_{2n+2}(-1/(n+1))=-q_{2n-1}(-1/(n+1))$. With a little arguing, it follows that $b_{2n+2}>-1/(n+1)$, so that indeed $\lim b_n=0$. Consequently, the radius of convergence of the power series expansion of $p_n(x)/q_n(x)$ becomes arbitrarily small for n sufficiently large, while at the same time the singularities keep away from the positive axis. It is not surprising that while (3.11) diverges, the convergents p_n/q_n can be used to assign a value to (3.9) for nonnegative x. In the next section, we will take up Euler's second method.

5. Extrapolation and factorial series. Euler's second method of evaluation requires the value of P_0 given that, for each positive integer n, $P_n=1+(n-1)+(n-1)(n-2)+(n-1)(n-2)(n-3)+\cdots$. This could be obtained if a natural extension function $h(z)$ were found which is in some sense regular and defined on a region of the complex plane containing the nonnegative integers, and for which $h(n)=P_n$ for positive integers n.

In fact, there is a great deal of latitude in choosing the function $h(z)$, even if we require it to be entire. By the Weierstrass Theorem [**1**, p. 194], there is an entire function $f(z)$ with simple zeros at $1,2,3,\ldots$. The Mittag-Leffler Theorem [**1**, p. 185] then provides a meromorphic function $g(z)$ whose only poles occur at $1,2,3,\ldots$ and whose singular part at $z=n$ is

$$\frac{P_n}{f'(n)(z-n)}.$$

If $h(z)=f(z)g(z)$, then $h(z)$ is entire and $h(n)=P_n$ $(n=1,2,3,\ldots)$. If this seems too arbitrary, we can also require that the recursion relation (3.2) satisfied by P_n also extends:

$$h(z+1)=zh(z)+1 \quad \text{for } z\in C. \tag{5.1}$$

Even this will not pin h down. For, as pointed out by Lorne Campbell, if $h(z)$ is one such function, then $c\Gamma(z)\sin 2\pi z+h(z)$ is another for any constant c. We may well ask whether there

is a "right" extension for P_n in the sense that the gamma function is the "right" extension of $n!$.

Following Euler, we try to get one by Newtonian extrapolation. Let the sequence $(a_1, a_2, a_3, \ldots)$ be given and let the ith order difference at the first term be u_i:

$$u_i = \Delta^i a_1 \qquad (i = 0, 1, 2, 3, \ldots). \tag{5.2}$$

Newton's extrapolation formula for a function $f(z)$ with $f(n) = a_n$ for positive integral n is

$$\begin{aligned} f(z) &= (1+\Delta)^{z-1} a_1 \\ &= u_0 + u_1(z-1) + \frac{u_2}{2!}(z-1)(z-2) + \cdots \\ &\quad + \frac{u_k}{k!}(z-1)(z-2)(z-3)\cdots(z-k) + \cdots . \end{aligned} \tag{5.3}$$

This is a factorial series of the second kind, whose theory is expounded in great detail by Nörlund [8]. For such a series, there are two real numbers θ_0 and θ_1, both finite or both infinite, which, when both finite, satisfy $0 \leqslant \theta_1 - \theta_0 \leqslant 1$, such that the series converges for $\operatorname{Re} z > \theta_0$ and converges absolutely for $\operatorname{Re} z > \theta_1$. Convergence is uniform on compact subsets of the domain of convergence, so that on such subsets $f(z)$ is analytic. This extrapolation has some agreeable properties. If, for some polynomial p of degree m, $a_n = p(n)$, then $u_k = 0$ for $k \geqslant m+1$ and $f(z) = p(z)$. In particular, the factorial series corresponding to the sequence $(1, 1, 1, \ldots)$ is simply 1. Moreover, suppose that $f(z)$ and $g(z)$ are the factorial series arising from the sequences (a_n) and (b_n). Then clearly the factorial series from (ca_n) is $cf(z)$ and from $(a_n + b_n)$ is $f(z) + g(z)$. Not so clear, but also true, is the fact that the factorial series from the product sequence $(a_n b_n)$ is the product $f(z)g(z)$ expanded in ascending factorial powers $1, (z-1), (z-1)(z-2), (z-1)(z-2)(z-3), \ldots$. From this, it follows that, if $b_n = 1/a_n$, then $f(z)g(z) = 1$. More generally, if ϕ is a rational function for which $b_n = \phi(a_n)$, then $g(z) = \phi \circ f(z)$. In the light of this, it is entirely reasonable that Euler should extend (P_n) to P_0 by Newton's method and should use the inverse sequence $(1/P_n)$ to compute "$1/P_0$." Is his use of the sequence $(\log_{10} P_n)$ equally justifiable?

Unfortunately, for $a_n = P_n$, the series (5.3) converges nowhere except at the positive integers, where it terminates. Therefore we will come to an assessment of the consistency of Euler's second method with his others by summing (5.3) by various natural methods. For the modified method of Borel discussed above, the function $U(s)$ of (4.3) is

$$U(s) = \sum_{k=0}^{\infty} \frac{(z-1)(z-2)\cdots(z-k)}{k!} s^k.$$

For $|s| < 1$, this is equal to $(1+s)^{z-1}$, and this expression can be used to extend $U(s)$ along the positive real axis. Then, from (4.4), we find that

$$h(z) = \int_0^{\infty} e^{-s}(1+s)^{z-1} ds. \tag{5.4}$$

(The value of $(1+s)^{z-1}$ derives from the real logarithm of $(1+s)$.) The integral converges for all complex values of z and $h(z)$, so determined, is entire. Integrating by parts reveals that $h(z)$ satisfies (5.1); if $z = n$, a positive integer, then $h(n) = P_n$. Finally, for $z = 0$, h takes the value (3.8). We can recover the factorial series $1 + (z-1) + (z-1)(z-2)(z-3) + (z-1)(z-2)(z-3) + \cdots$ from (5.4) by expanding the binomial in the integrand and integrating (invalidly) term by term. This relationship between function and factorial series can be made less tenuous. Define the incomplete gamma function

$$\Gamma(z, x) = \int_x^{\infty} e^{-t} t^{z-1} dt = e^{-1} \int_{x-1}^{\infty} e^{-t}(1+t)^{z-1} dt$$

for nonnegative real x and complex z. Then $h(z) = e\Gamma(z, 1)$. Successive integration by parts yields

$$\Gamma(z,x)=e^{-x}x^{z-1}\left\{1+\frac{(z-1)}{x}+\frac{(z-1)(z-2)}{x^2}+\cdots\right.$$
$$\left.+\frac{(z-1)(z-2)\cdots(z-k+1)}{x^{k-1}}\right\}+R_k(z,x)$$

where $R_k(z,x)=(z-1)(z-2)\cdots(z-k)\int_x^\infty e^{-t}t^{z-k-1}dt$. For fixed z, this remainder becomes arbitrarily small for large x. In the language of asymptotic expansions [4, 9], we have

$$e\Gamma(z,x)\sim e^{1-x}x^{z-1}\sum_{k=0}^{\infty}\frac{(z-1)(z-2)\cdots(z-k)}{x^k}\quad \text{as } x\to\infty,$$

the right side of which, for $x=1$, gives the factorial series for $h(z)$.

There are other methods, somewhat ad hoc but nevertheless convincing, of obtaining the same closed expression for $h(z)$. An amusing way begins with the observation that the sum of the first n terms of the series $e=1+1/1!+1/2!+\cdots$ is $P_n/(n-1)!$. This suggests that it might be worthwhile to consider the x-polynomial

$$q(n,x)=1+x+x^2/2!+x^3/3!+\cdots+x^{n-1}/(n-1)!.$$

We have that $q(n,0)=1$, $P_n=\Gamma(n)q(n,1)$, and $q(n,x)$ satisfies

$$\frac{\partial}{\partial x}q(n,x)-q(n,x)=-x^{n-1}/(n-1)!.$$

Define the extension $q(z,x)$ of $q(n,x)$ for z complex (but not a nonpositive integer) as that function which satisfies the differential equation

$$\frac{\partial}{\partial x}q(z,x)-q(z,x)=-x^{z-1}/\Gamma(z)\qquad (x\geqslant 0)$$

subject to the initial condition $q(z,0)=1$. This first order equation has the solution

$$q(z,x)=\frac{e^x}{\Gamma(z)}\int_x^\infty e^{-t}t^{z-1}dt.$$

The resulting extension of P_n is thus

$$\Gamma(z)q(z,1)=e\int_1^\infty e^{-t}t^{z-1}dt=\int_0^\infty e^{-s}(1+s)^{z-1}ds,$$

which agrees with (5.4), when analytically continued to all of C.

The foregoing approach is peculiar to the sequence (P_n). We put forward a final method which is more general and involves infinite differential operators (cf. [3]). Let (a_n) be a given sequence, and, as above, let $u_i=\Delta^i a_1$ $(i=0,1,2,\ldots)$. Define $\chi_k(t)=t^k$ for $k=0,1,2,\ldots$. Observe the similarity between the expansion of $f(z)$ in (5.3) and the binomial expansion

$$(1+t)^{z-1}=\sum_{k=0}^{\infty}\frac{(z-1)(z-2)\cdots(z-k)}{k!}\chi_k(t).$$

We search for a linear functional, L_a (the subscript indicates its dependence on the sequence (a_n)), defined on a linear space which contains all the functions $\chi_k(t)$ as well as $(1+t)^{z-1}$ for complex values of z, for which

$$L_a(\chi_k)=u_k\quad \text{for } k=0,1,2,\ldots.$$

Then we might take as an evaluation of the series (5.3)

$$f(z)=L_a\big((1+t)^{z-1}\big).$$

We should choose L_a in such a way that, if for some polynomial p, $a_n=p(n)$, then we should obtain the evaluation $f(z)=p(z)$.

Let us denote differentiation with respect to t by D or by $'$. Observe that $D^k\chi_r(0)=0$ when

$r \neq k$ while $D^r\chi_r(0)=r!$. Consequently, for suitable $w(t)$, define

$$L_a(w(t)) = u_0 w(0) + \frac{u_1}{1!}w'(0) + \frac{u_2}{2!}w''(0) + \cdots + \frac{u_r}{r!}w^{(r)}(0) + \cdots$$

where the series can be evaluated somehow. Three examples will illustrate the process.

EXAMPLE 1. Let p be any polynomial and let $a_n = p(n)$. Then $u_i = \Delta^i p(1)$ and

$$L_a((1+t)^{z-1}) = p(1) + \frac{\Delta p(1)}{1!}(z-1) + \frac{\Delta^2 p(1)}{2!}(z-1)(z-2) + \cdots + 0 + 0 + 0 + \cdots$$

which is the factorial expansion of the polynomial $p(z)$.

EXAMPLE 2. Let $a_n = 1/n$. Then $u_i = (-1)^i/(1+i)$, so that

$$\begin{aligned} L_a(w(t)) &= w(0) - w'(0)/2! + w''(0)/3! - w'''(0)/4! + \cdots \\ &= \left(1 - \frac{D}{2!} + \frac{D^2}{3!} - \frac{D^3}{4!} + \cdots\right)w(0) \\ &= \frac{1}{D}(1-e^{-D})w(0). \end{aligned}$$

By Taylor's Theorem, formally,

$$\begin{aligned} (1-e^{-D})w(t) &= w(t) - w(t) + w'(t) - w''(t)/2! + \cdots \\ &= w(t) - w(t-1). \end{aligned}$$

Thus,

$$\begin{aligned} (1 - D/2! + D^2/3! - D^3/4! + \cdots)w(t) &= D^{-1}(1-e^{-D})w(t) \\ &= D^{-1}(w(t) - w(t-1)) \\ &= \int_{t-1}^{t} w(s)\,ds + C, \end{aligned}$$

where D^{-1} can be interpreted as integration and C is the constant of integration. In the case that $w(t) = \chi_k$,

$$\begin{aligned} &(1 - D/2! + D^2/3! - D^3/4! + \cdots)\chi_k \\ &\quad = t^k - kt^{k-1}/2! + k(k-1)t^{k-2}/3! + \cdots + 0 + 0 + 0 + \cdots \\ &\quad = (k+1)^{-1}\left((k+1)t^k - \binom{k+1}{2}t^{k-1} + \cdots + (-1)^{r-1}\binom{k+1}{r}t^{k-r+1} + \cdots\right) \\ &\quad = \frac{t^{k+1} - (t-1)^{k+1}}{k+1} = \int_{t-1}^{t} s^k\,ds. \end{aligned}$$

Agreement with the previous calculation demands that we take the constant C to be zero. This we do for arbitrary $w(t)$. We are in a position to evaluate $f(z)$ by taking $w(t) = (1+t)^{z-1}$. Since

$$D^{-1}(1-e^{-D})(1+t)^{z-1} = \int_{t-1}^{t}(1+s)^{z-1}ds = \frac{(1+t)^z - t^z}{z},$$

we find that $f(z) = 1/z$, as we would wish.

EXAMPLE 3. Let the sequence be (P_n). Since $\Delta^i P = i!$, $L_P(w(t)) = (1 + D + D^2 + \cdots + D^k + \cdots)w(0) = (1-D)^{-1}w(0)$. Let $y(t) = (1-D)^{-1}w(t)$. Then $(1-D)y(t) = w(t)$, i.e., y is a solution of the differential equation $y' - y = -w$. Thus,

$$y(t) = e^t\int_t^{\infty} e^{-s}w(s)\,ds + Ce^t.$$

For the particular case $w(t)=\chi_k(t)$, we find that

$$(1+D+D^2+\cdots+D^k+\cdots)w(0)=k! \quad \text{while } y(0)=k!+C.$$

Again, this indicates that 0 is the appropriate choice for the constant C. Hence, the extension for P_n is

$$L_P((1+t)^{z-1})=\int_0^\infty e^{-s}(1+s)^{z-1}ds,$$

which agrees with (5.4).

This means of summation might be compared with that of Hughes [17] for summing a factorial series of the form $\Sigma v_n/z(z+1)(z+2)\cdots(z+n)$, in that integration plays a role similar to that of differentiation here.

The particular form of $h(z)$ suggests that we consider a linear functional defined by integrating over the positive real half-line. Thus, given a sequence (a_n), we might seek out a measure μ for which

$$\int_0^\infty t^k d\mu=\Delta^k a_1 \quad \text{for } k=0,1,2,\ldots.$$

This is the Stieltjes moment problem; it is always solvable, the solution not being unique [25]. To get an extension function $f(z)$ with $f(n)=a_n$ for positive integers n, we choose any μ for which $(1+t)^{z-1}$ is integrable, and compute $f(z)=\int_0^\infty(1+t)^{z-1}d\mu$. The problem is to find a systematic way of choosing the "right" measure; how could we be led to the choice of $e^{-t}dt$ in the case of (P_n)? Going back to the polynomials does not seem to be of much help here, involving as it does an inversion of the partial Mellin transform

$$p(z)=\int_0^\infty(1+t)^{z-1}d\mu_p$$

for the measure μ_p.

We conclude this section with one quick observation on the infinite series (5.3) evaluated at $z=0$. If b_n is the sum of the first n terms, it can be shown that, in terms of the given sequence (a_n),

$$b_n=\sum_{i=1}^{n}(-1)^{i-1}\binom{n}{i}a_i.$$

If Q is the infinite matrix implementing the transformation from (a_n) to (b_n), it is not hard to show that Q^2 is the identity matrix. Is this indicative of some deeper structure? Can this be exploited to justify Euler's consideration of $(1/P_n)$ and $(\log_{10}P_n)$ in computing P_0?

6. Closing Remarks. While finding a sum for Wallis' series is hardly of great mathematical significance, there is some fascination attached to the problem. Doubtless, Euler's analysis can be the starting point for a deeper excursion into mathematical interrelationships in a variety of areas—asymptotic expansions, continued fractions, summability, moment problems, factorial series, rational function approximations, infinite differential operators. In this, as in Euler's other investigations, the breadth and ingenuity justify study in something close to the original form by mathematical students.

It could be pointed out that the difficulty of showing that Wallis' series has a value is a result of the field in which we chose to operate. For any prime p, it is clear that Wallis' series converges in the p-adic completion of the rationals, and to an integer, too!

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MATHEMATICAL NOTES

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SUMS OF RECIPROCALS OF INTEGERS MISSING A GIVEN DIGIT

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The harmonic series $\Sigma_{n=1}^{\infty} 1/n$ diverges. If we omit those terms for which n has, say, at least one "9" in its base 10 representation, then the remaining series converges [6]. In fact, this result holds for any base $b \geqslant 2$ and any digit m, $0 \leqslant m \leqslant b-1$. (See [4, Theorem 144, p. 120].) Various

estimates for these sums are obtained in [1]–[7], although nowhere is a method discussed that easily produces accurate values of the sums.

This note gives a simple technique for accurately summing these series. We also give, to twenty decimal places, the values of the ten sums where the "missing digit" ranges from 0 through 9, base 10. (See Table 1.) This method easily generalizes to other bases.

Choose the missing digit m, $0 \leqslant m \leqslant 9$. Let S be the set of positive integers that have no digit equal to m, and let S_i be the set of integers in S which have exactly i digits. Define

$$s(i,j) = \sum_{x \in S_i} 1/x^j \qquad (j \geqslant 1). \tag{1}$$

The sum we are interested in is

$$t_m = \sum_{i=1}^{\infty} s(i,1), \tag{2}$$

the sum of the reciprocals of all positive integers missing the digit m.

Now t_m cannot be computed from (2) because the convergence is much too slow: the remainder after summing 10^{27} terms still exceeds one! We can, however, derive a recursion formula which gives $s(i+1,j)$ from $s(i,j+n)$, $n \geqslant 0$, and for small i, $s(i,j+n)$ can be computed directly.

The recursion formula is based on the observation that

$$S_{i+1} = \bigcup_{x \in S_i} \left[\{10x, 10x+1, \ldots, 10x+9\} \backslash \{10x+m\}\right].$$

Therefore,

$$s(i+1,j) = \sum_x \left[\sum_k{}' (10x+k)^{-j} \right] \tag{3}$$

where the outer sum extends over all x in S_i, and Σ' is over $k = 0, 1, \ldots, 9, k \neq m$.

Let us make these definitions:

$$c(j,n) = (-1)^n \binom{j+n-1}{n} \qquad (j > 0, n \geqslant 0)$$

$$b_n = 0^n + 1^n + \cdots + 9^n - m^n \qquad (n > 0), b_0 = 9$$

$$a(j,n) = b_n c(j,n)/10^{j+n}.$$

Next, rearrange (3) using the binomial theorem to expand $(10x+k)^{-j}$:

$$\begin{aligned} s(i+1,j) &= \sum_x \left\{ (10x)^{-j} \sum_k{}' \left[\sum_{n=0}^{\infty} c(j,n)(k/10x)^n \right] \right\} \\ &= \sum_x \left\{ (10x)^{-j} \left[\sum_{n=0}^{\infty} b_n c(j,n)(10x)^{-n} \right] \right\} \\ &= \sum_x \left[\sum_{n=0}^{\infty} a(j,n) x^{-(j+n)} \right]. \end{aligned}$$

According to (1), we can write this last sum as

$$s(i+1,j) = \sum_{n=0}^{\infty} a(j,n) s(i,j+n). \tag{4}$$

The needed $s(i,j)$ for $i \leqslant 4$ can be computed explicitly. (4) is then used with at most 10 terms to get $s(i,1)$ for $5 \leqslant i \leqslant 30$. For $i \geqslant 31$, we use the estimate

$$\sum_{i=31}^{\infty} s(i,1) \sim 9 \cdot s(30,1).$$

This comes from using only the first term of (4). It can be shown that the round-off and truncation errors will not affect the twentieth decimal place of t_m.

It seems plausible that the t_m are irrational.

TABLE 1.

m	t_m			
0	23.10344	79094	20541	61603
1	16.17696	95281	23444	26657
2	19.25735	65328	08072	22453
3	20.56987	79509	61230	37107
4	21.32746	57995	90036	68663
5	21.83460	08122	96918	16340
6	22.20559	81595	56091	88416
7	22.49347	53117	05945	39817
8	22.72636	54026	79370	60283
9	22.92067	66192	64150	34816

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A GENERALIZATION OF ZOLOTAREV'S THEOREM

PATRICK MORTON

In 1872 Zolotarev [4] published the following theorem:

THEOREM 1. *Let p be an odd prime and let a be an integer not divisible by p. Let π_a be the permutation on the group of reduced residues* mod p *given by*

$$\pi_a(k) \equiv ak(\operatorname{mod} p),\ k = 1,\ldots,p-1.$$

Then π_a is an even or an odd permutation according as a is a quadratic residue or nonresidue mod p; *in symbols*

$$\operatorname{sgn}\pi_a = \left(\frac{a}{p}\right),$$

where sgn *is the signum of the permutation and* $\left(\frac{a}{p}\right)$ *is the Legendre symbol.*

Several proofs of this theorem have been given [1], [2], [3]. A group-theoretic proof can be constructed by considering the map $a \to \operatorname{sgn} \pi_a$ as a homomorphism from the group of reduced residues onto the group $\{\pm 1\}$ and showing that the kernel is exactly the subgroup of squares mod p. This argument leads to the following generalization.

THEOREM 2. *Let G be a finite group, and let G^+ be the subgroup of G generated by the squares in G. For a in G let π_a be the permutation on G given by*

$$\pi_a(k) = ka,\ k \text{ in } G.$$

Then if the order of G is even and its 2-Sylow subgroups are cyclic,

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$$\operatorname{sgn}\pi_a=\begin{cases}+1, \text{ if } a \text{ is in } G^+,\\ -1, \text{ if not.}\end{cases}$$

If the order of G is odd or its 2-Sylow subgroups are not cyclic, then $\operatorname{sgn}\pi_a=+1$ *for all a in G.*

Proof. For a in G let $\chi(a)=\operatorname{sgn}\pi_a$. Then χ is a homomorphism from G into $\{\pm 1\}$. If G satisfies the conditions of the first assertion, this map is onto. For let T be a cyclic 2-Sylow subgroup of G, and let g generate T. Then π_g is a product of $[G:T]$ cycles of length $|T|$:

$$\pi_g=(1 \quad g \quad g^2\ldots g^{|T|-1})(k \quad kg\ldots kg^{|T|-1})\ldots,$$

where each cycle moves all and only those elements of a left coset of T in G. Since the length of each cycle is even and $[G:T]$ is odd, $\chi(g)=-1$. Thus χ is onto $\{\pm 1\}$, and its kernel K has index 2 in G. It remains to show that $K=G^+$. From $\chi(a^2)=\chi(a)^2=+1$ it is clear that $G^+\leqslant K$.

Now G has a subgroup H of odd order and index $|T|$. We prove this by induction on $|G|$. This is clear if $|G|=2$. Suppose that the assertion holds for all groups of even order $<|G|$ with cyclic 2-Sylow subgroups. By the remark above the kernel K of χ has index 2 in G. If $|K|$ is odd, let $H=K$. Otherwise the 2-Sylow subgroups of K are cyclic and have order $|T|/2$. By induction K has a subgroup H of odd order and index $|T|/2$ in K. Since $[G:H]=[G:K][K:H]=|T|$, H works for G as well. This proves the assertion.

Now let T^+ be the subgroup of T of squares in T. Since T is cyclic, $|T^+|=|T|/2$. Since every element of H is a square, we have

$$HT^+\leqslant G^+\leqslant K.$$

From the fact that the orders of H and T^+ are relatively prime it follows that $|HT^+|=|H||T^+|$, and hence

$$|G|/2=|H||T^+|\leqslant|G^+|\leqslant|K|=|G|/2.$$

Thus equality holds throughout and so $K=G^+$.

To prove the second assertion let a be any element of G, let e be its order and $f=|G|/e$. Then π_a is a product of f cycles of length e, where each cycle moves the elements of a left coset of the cyclic group generated by a. Hence $\chi(a)=(-1)^{(e+1)f}$. If the order of G is odd, $e+1$ is even. If the 2-Sylow subgroups of G are not cyclic, f is even, since in this case a cannot have an order divisible by the maximal power of 2 dividing $|G|$. In both cases $\chi(a)=+1$, and this proves the theorem.

Theorem 2 shows as a special case that if m is an integer $\geqslant 2$, $G=G_m$ is the group of prime residues $\bmod m$, and $(a,m)=1$, then

$$\operatorname{sgn}\pi_a=\begin{cases}\left(\dfrac{a}{p}\right), \text{if } m=p^n,\ 2p^n,\ n\geqslant 1, p \text{ an odd prime},\\ (-1)^{(a-1)/2}, \qquad m=4,\\ +1, \text{ otherwise.}\end{cases}$$

This follows from the fact that the only m for which G_m has a nontrivial cyclic 2-Sylow subgroup are $m=4$, $p^n, 2p^n$.

Added in proof: The above-mentioned corollary of Theorem 2 has also been proved by Emma Lehmer in her paper "Generalizations of Gauss's Lemma," which appears in *Number Theory and Algebra,* Hans Zassenhaus, ed., New York, 1977, 187–194.

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ON PRODUCTS OF TRANSPOSITIONS AND THEIR GRAPHS

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1. Introduction. The original purpose of this paper was to give a short, purely permutational proof of Cauchy's well-known result that a permutation cannot be expressed as the product of both an even number and of an odd number of transpositions. We were led to it by simplifying the argument recently given in Herstein and Kaplansky [**9**, pp. 110–113]. Only after having finished most of our work did we find that essentially the same simplification had occurred to others (see Spitznagel [**12**] and Fraleigh [**8**, pp. 48–49]). However, the reduction algorithm for products of transpositions we had obtained allows some interesting applications to so-called minimal products and may even deserve to be studied for its own sake. Our main result is a characterization of minimal products in terms of the graphs formed by considering each transposition in a given product as an (undirected) edge linking the two elements occurring in it.

2. Terminology and notation. All permutations considered below act from the *right* on a fixed finite nonempty set Ω, of cardinality n.

By a *product* will be meant a product of transpositions; and by its *length*, the number of factors in it. Two products will be called *equivalent* if they produce the same permutation. A product is *minimal* if it is not equivalent to any product of strictly shorter length. Clearly, every product is equivalent to at least one minimal product.

Letters such as a, b, c, d will denote *distinct* elements of Ω; the letters σ and τ, transpositions; and the letters Π and Σ, products. The empty product will be denoted by I, the length of a product Π by $|\Pi|$, and if Π and Σ are equivalent we shall write $\Pi\sim\Sigma$.

For any product Π, define the *a-reduced form* Π^a of Π by induction on $|\Pi|$ as follows. If $|\Pi|\leqslant 1$, put $\Pi^a=\Pi$. If $|\Pi|\geqslant 2$, say $\Pi=\sigma\tau\Sigma$, put

$$\Pi^a=(\sigma\tau\Sigma)^a=\begin{cases}\sigma(\tau\Sigma)^a \text{ if } a \text{ does not occur in } \sigma & (\alpha)\\ \Sigma^a \text{ if } \sigma=\tau=(ab) & (\beta)\\ (bc)[(ab)\Sigma]^a \text{ if } \sigma=(ab),\tau=(ac) & (\gamma)\\ (bc)[(ac)\Sigma]^a \text{ if } \sigma=(ab),\tau=(bc) & (\delta)\\ \tau(\sigma\Sigma)^a \text{ if } \sigma=(ab),\tau=(cd). & (\varepsilon)\end{cases}$$

As an example, let us calculate Π^a and Π^{ab} for $\Pi=(ab)(bc)(ad)(bd)(ab)(ac)$. We get successively:

$$\begin{array}{ll}\underline{(ab)(bc)}\,(ad)(bd)(ab)(ac) & \\ (bc)\,\underline{(ac)(ad)}\,(bd)(ab)(ac) & \\ (bc)(cd)\,\underline{(ac)(bd)}\,(ab)(ac) & \\ (bc)(cd)(bd)\,\underline{(ac)(ab)}\,(ac) & \\ (bc)(cd)(bd)(bc)\,\underline{(ac)(ac)} & \\ \underline{(bc)(cd)}\,(bd)(bc) & =\Pi^a\\ (cd)\,\underline{(bd)(bd)}\,(bc) & \\ (cd)(bc) & =\Pi^{ab}.\end{array}$$

We may describe the a-reduction of Π as a process in which a is pushed across Π until it disappears or remains in only the right-most factor. (We could similarly define another a-reduction, in which the product is scanned from right to left.)

Let Π be any product and let α be the permutation on Ω produced by Π. Then, as has been well known since Cauchy [**5**, pp. 98–100], α can be written as a product of disjoint cycles, this product being unique to within order of the factors provided we include all 1-cycles which may occur; we call this essentially unique product the *disjoint cycle decomposition* of α, respectively Π, and denote the number of factors in it by $c(\alpha)$, respectively $c(\Pi)$.

Our graph-theoretic terminology follows Bondy and Murty [**3**]. By a *trail* between the vertices u and v of an (undirected) graph, we mean either an edge uv or any finite sequence of distinct edges of the form $uu_1, u_1u_2, \ldots, u_nv$ (with $n \geqslant 1$). If $u = v$, the trail is said to be *closed.* Two vertices are *connected* if they are equal or if there exists a trail between them. Connectedness is an equivalence relation on the vertex set of the graph, and the corresponding equivalence classes with their induced graph structure are the *components* of the graph. A graph is *connected* if it has only one component and *acyclic* if it has no closed trails; a connected acyclic graph is a *tree.*

For any product Π, define the (undirected) *graph* of Π as follows. Its vertices are the n elements of Ω, its edges are the $|\Pi|$ factors of Π, and a vertex is incident with an edge if and only if it occurs in that edge. That the edges are undirected follows, of course, from the equality $(ab) = (ba)$. There will be no loops, but repeated occurrences of the same factor in Π give rise to multiple edges.

We shall need a result due essentially to Kirchhoff and Listing [**10**, pp. 51–54]: *a (finite) graph is acyclic if and only if the number of its edges plus the number of its components equals the number of its vertices.* For a brief proof, see §4.5 below.

3. Results and proofs.

Lemma 1. *The a-reduced form Π^a of a product Π has the following properties:*

(1) $\Pi^a \sim \Pi$;
(2) $|\Pi^a| \leqslant |\Pi|$;
(3) $|\Pi^a| \equiv |\Pi| \bmod 2$;
(4) *a can only occur in the right-most factor of Π^a;*
(5) *if $|\Pi^a| = |\Pi|$ and a occurs in Π, then a occurs in the right-most factor of Π^a;*
(6) *if Π does not move a, then a does not occur in Π^a;*
(7) *if Π moves a to b, then Π^a is of the form $\Sigma(ab)$ with a not occurring in Σ.*

Proof. (1) is evident from the five rules for a-reduction. (2) and (3) follow from the equation $|\Pi^a| = |\Pi| - 2\beta$ where β is the number of applications of rule (β) in the reduction of Π to Π^a. For (4) and (5) observe that a-reduction pushes all occurrences of a across the product until they disappear or arrive in the right-most factor; if a occurs in Π and rule (β) is never applied, then a cannot completely disappear. (6) and (7) follow at once from (1) and (4).

Corollary 1. *If $|\Pi^a| = |\Pi|$ and a occurs in Π, then a is moved by Π.*

This follows immediately from (5) and (6) of Lemma 1.

Theorem 1 (Cauchy). *A product of even length cannot be equivalent to a product of odd length.*

Proof. As usual (see [**9**, pp. 112–113] for example), it suffices to show that there cannot exist a product of odd length equivalent to I. Suppose on the contrary that such odd products do exist and, among them, let Π be one of minimal length. Let a be an element occurring in Π. Then by (1) and (3) of Lemma 1 and the minimality property of Π we have $|\Pi^a| \geqslant |\Pi|$, so that $|\Pi^a| = |\Pi|$ by (2) of Lemma 1. But then Corollary 1 implies that a is moved by Π, and therefore also by I, which is nonsense.

Notice that this proof does not depend upon disjoint cycle decomposition.

Theorem 2. *A minimal product moves all the elements occurring in it.*

Proof. For Π minimal we have $|\Pi^a| = |\Pi|$ so that, by Corollary 1, Π moves a if a occurs in Π.

LEMMA 2. *If a product* Π *is minimal and* a *does not occur in* Π, *then* $\Pi(ab)$ *is minimal.*

Proof. Let Σ be any product equivalent to $\Pi(ab)$. Since a does not occur in Π, a is moved to b by $\Pi(ab)$ and therefore by Σ also. By (7) of Lemma 1, Σ^a is thus of the form $\Pi_1(ab)$ with a not occurring in Π_1. Now $\Pi_1(ab)=\Sigma^a\sim\Sigma\sim\Pi(ab)$ implies $\Pi_1\sim\Pi$, so that $|\Pi_1|\geqslant|\Pi|$ since Π is minimal. Hence

$$|\Sigma|\geqslant|\Sigma^a|=|\Pi_1(ab)|=|\Pi_1|+1\geqslant|\Pi|+1=|\Pi(ab)|$$

as required.

LEMMA 3. *Let* Π *be any product and a any element of* Ω. *Then the graph of* Π *is acyclic if and only if the graph of* Π^a *is acyclic and* $|\Pi^a|=|\Pi|$.

Proof. Let us first study the effect of the five rules for a-reduction on the existence of closed trails in the graph of a product. Clearly, (α) and (ε) have no effect at all. The effect of (γ) and (δ) is to change a pair of adjacent edges into another pair of adjacent edges, namely two "sides" of a "triangle" into one of these two sides and the third side. This cannot affect the existence of closed trails, since the role of one side in closing a trail can also be played by the other two sides. The effect of (β) is to erase two edges linking the same two vertices, which will destroy at least one closed trail but cannot create any.

In other words, no rule creates closed trails, only (β) destroys them, and (β) cannot occur if Π is acyclic. Moreover, the only rule to affect the length of a product is (β), and (β) is used only if $|\Pi^a|<|\Pi|$. From these observations the result follows at once.

THEOREM 3. *A product* Π *is minimal if and only if its graph is acyclic.*

Proof. We use induction on $|\Pi|$. The result clearly holds for $|\Pi|\leqslant 1$. Assume it to be true (in each direction) for all products of length $k\geqslant 1$. Let Π be a product of length $k+1$ and let a occur in Π.

Suppose first that Π is minimal. Then, by Lemma 1, $|\Pi^a|=|\Pi|=k+1$ and Π^a is of the form $\Sigma(ab)$ with a not occurring in Σ. Clearly Σ is minimal and $|\Sigma|=k$. Hence the graph of Σ is acyclic by the induction hypothesis and, since a does not occur in Σ, the graph of Π^a is also acyclic. It now follows from Lemma 3 that the graph of Π is acyclic.

Conversely, suppose that the graph of Π is given to be acyclic. Then, by Lemma 3, the graph of Π^a is acyclic and $|\Pi^a|=|\Pi|=k+1$. By Lemma 1, Π^a is of the form $\Sigma(ab)$ with a not occurring in Σ. Clearly, the graph of Σ is acyclic and $|\Sigma|=k$. Hence Σ is minimal by the induction hypothesis. Since a does not occur in Σ, Lemma 2 implies that $\Pi^a=\Sigma(ab)$ is minimal. But Π and Π^a are equivalent and have the same length; thus Π is minimal.

We remark that, as a consequence of Theorem 3, if the factors of a minimal product are rearranged, the result is still minimal.

THEOREM 4. *A product* Π *is minimal if and only if* $|\Pi|=n-c(\Pi)$.

Proof. In the disjoint cycle decomposition of Π we may write each s-cycle $(a_1a_2\cdots a_s)$ as $(a_1a_2)(a_1a_3)\cdots(a_1a_s)$. Thus Π is equivalent to a product Π_1 of $n-c(\Pi)$ transpositions in which, when read from left to right, at least one new element is introduced with each factor. Therefore Π_1 is minimal by Lemma 2 and this proves the result.

(Note that the possibility of producing a given permutation by *some* product of transpositions does not require disjoint cycle decomposition but can be established directly by induction on the number of elements moved [**11**, pp. 13–14].)

We can combine and augment the above results on minimal products as follows.

THEOREM 5. *Let* Π *be a product whose graph has k components. Then the following conditions are equivalent*:

(a) Π *is minimal*;

(b) *the graph of* Π *is acyclic*;

(*c*) $|\Pi| = n - c(\Pi)$;
(*d*) $|\Pi| = n - k$.

Proof. The equivalence of (a) and (b) is Theorem 3 and that of (a) and (c) is Theorem 4; that (b) and (d) are equivalent is the Kirchhoff–Listing Theorem mentioned in §2.

COROLLARY 2. *Let* Π *be a product whose graph has* k *components. Then* $c(\Pi) \geqslant k$ *with equality if the equivalent conditions of Theorem 5 hold.*

Proof. Both "belonging to the same cycle in the disjoint cycle decomposition of Π" and "being connected in the graph of Π" are equivalence relations and the former is evidently a refinement of the latter; thus $c(\Pi) \geqslant k$. If the conditions of Theorem 5 hold, then (c) and (d) together give $c(\Pi) = k$.

COROLLARY 3 (Dénes & Berge). *The following conditions on a product* Π *are equivalent*:
(*e*) $|\Pi| = n-1$ *and* Π *produces an* n*-cycle*;
(*f*) *the graph of* Π *is a tree*;
(*g*) $|\Pi| = n-1$ *and the graph of* Π *is connected*;
(*h*) $|\Pi| = n-1$ *and the graph of* Π *is acyclic.*

Proof. The equivalence of (f), (g) and (h) is an immediate and well-known consequence of the Kirchhoff–Listing Theorem. That (e) is equivalent to them follows easily from Theorem 5; in fact, the equivalence of (e) and (h) follows from the equivalence of (c) and (b).

4. Remarks. 1. For a recent survey of proofs of Cauchy's Theorem (Theorem 1 above), see [**1**] and also Chrystal's elegant proof [**6**, pp. 28–30], Cartier's comparative study [**4**], the remarks in Herstein and Kaplansky [**9**, pp. 110–113, p. 116], and Miller's proof [**8**, pp. 48–49].

2. We have been unable to determine with certainty to whom Theorem 4 above is due; our oldest reference is Chrystal [**6**, pp. 28–30], but he does not claim the result. It is worth remarking that his argument yields a unified proof of Theorems 1 and 4 above.

3. The following lemma of Cauchy concerning disjoint cycle decomposition [**5**, pp. 100–103], which he used to prove Theorem 1 above [**5**, pp. 103–104], has been used by Dénes to establish Theorem 4 above [**7**, p. 65]; it is interesting that these proofs can also be combined into a single argument.

LEMMA (Cauchy). *If* α *is any permutation and* σ *is any transposition, then* $c(\alpha\sigma) = c(\alpha) \pm 1$.

Applying this result successively to $\alpha = I, \sigma_1, \sigma_1\sigma_2, \ldots, \Pi = \sigma_1\sigma_2 \cdots \sigma_l$, where $l = |\Pi|$, and noting that $c(I) = n$, we obtain $c(\Pi) = n + s - t$ where s and t are the numbers of times "$+1$" and "-1", respectively, occur. Since $s + t = |\Pi|$ we obtain

$$|\Pi| = n - c(\Pi) + 2s.$$

Hence $|\Pi| \equiv n - c(\Pi) \bmod 2$, which gives Cauchy's Theorem, and $|\Pi| \geqslant n - c(\Pi)$, which leads to Theorem 4 above.

Unfortunately, we have been unable to use Cauchy's Lemma to prove our results on the graphs of products of transpositions; here our reduction algorithm seems to be more powerful.

4. The essence of Corollary 3 above was first stated by Dénes [**7**, p. 64], but his proof that (e) is implied by the other conditions appears to suffer from a petitio principii. A more rigorous treatment has been given by Berge [**2**, Sect. 5 of Chap. 4]; it is based on a graph-theoretic version of Cauchy's Lemma and differs considerably from our proof.

5. The connection between finite loopless graphs and products of transpositions is sufficiently close to permit a proof of the Kirchhoff–Listing Theorem very similar to the proof of Theorem 4 described in §4.3 above. The role of Cauchy's Lemma is played by the following result. *If the graph* G' *is obtained from the graph* G *by adding an edge* ab, *then* G' *has the same number of components as* G *if* a *and* b *are connected in* G *and one less otherwise.* The proof thus obtained is essentially the same as Whitney's [**13**, p. 341].

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CLASSROOM NOTES

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ON THE USE OF A DIFFERENTIABLE HOMOTOPY IN THE PROOF OF THE CAUCHY THEOREM

R. VÝBORNÝ

1. Introduction. The homotopy version of the Cauchy Theorem has been well presented on several occasions (see, e.g., [1], [2]). The aim of this paper is to present a particularly simple proof using a differentiable homotopy. We offer two proofs, one for continuously differentiable f and one for the general case.

2. Notation. An open connected set in $\mathbb{C}$ will be denoted by G, its boundary by ∂G. The letter f will stand for a function which possesses a derivative f' everywhere in G. We shall integrate over paths. *A path* ϕ will be a continuous map $\phi:[0,1]\to G$ (or $\mathbb{R}^2$); a continuously differentiable path will be called *a road*. A path ϕ is said to be *closed* if $\phi(0)=\phi(1)$. In the sequel S will denote the square $[0,1]\times[0,1]$ and σ the path whose geometric image is ∂S.

3. Differentiable homotopy. DEFINITION. Two paths ϕ and ψ are said to be homotopic if there exists a function $H:S\to G$ such that (i) $H(0,s)=\phi(s), H(1,s)=\psi(s)$ for every $s\in[0,1]$; (ii) H is continuous on S; and either (iii) $H(t,0)=a, H(t,1)=b$ for all $t\in[0,1]$; or (iv) $H(t,0)=H(t,1)$ for all $t\in[0,1]$.

The function H is called a homotopy.

REMARK. If ϕ and ψ are homotopic and (iii) holds, one speaks of a homotopy with fixed ends; if (iv) holds, one says that ϕ and ψ are homotopic as closed paths. Since no confusion can arise in this paper, we shall use the abbreviated term homotopy (or homotopic).

DEFINITION. Two paths ϕ and ψ are said to be smoothly homotopic if they are homotopic and the function H satisfies

(v) H_t, H_s, H_{ts} exist and are continuous on S. (On ∂S the derivatives are understood to be "one-sided.")

THEOREM 1. *If ϕ and ψ are two smoothly homotopic roads then*

$$\int_\phi f = \int_\psi f. \tag{1}$$

The first proof uses the additional assumption that f' is continuous in G. Consider

$$I = \int_\sigma f dH = \int_\sigma f(H(t,s))(H_t(t,s)dt + H_s(t,s)ds). \tag{2}$$

By Green's Theorem

$$I = \int\int_S \left(\frac{\partial f H_s}{\partial t} - \frac{\partial f H_t}{\partial s} \right) dtds.$$

However,

$$\frac{\partial f H_s}{\partial t} = f' H_t H_s + f H_{st} = f' H_s H_t + f H_{ts} = \frac{\partial f H_t}{\partial s}.$$

Consequently, $I = 0$. Using either (iii) or (iv) to evaluate I directly, we have

$$I = \int_\phi f - \int_\psi f. \tag{3}$$

In the second proof, we also consider the integral (2). In view of (3), it is sufficient to show that $I = 0$. Assume, contrary to the desired result, that $I \neq 0$. Divide the square S into four equal squares S_1', S_2', S_3', S_4' as indicated in Figure 1, and denote their "boundary" paths by $\sigma_1', \sigma_2', \sigma_3', \sigma_4'$, respectively. Obviously,

$$I = \int_{\sigma_1'} f + \int_{\sigma_2'} f + \int_{\sigma_3'} f + \int_{\sigma_4'} f.$$

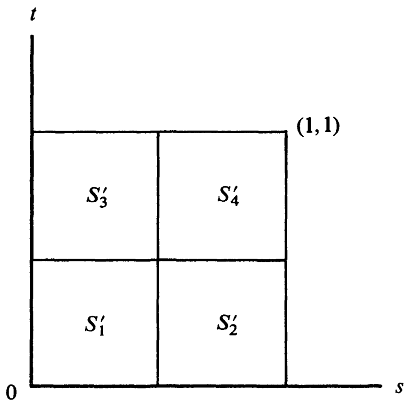

FIG. 1

It follows that there exists an integer $i, 1 \leqslant i \leqslant 4$ such that

$$\left|\int_{\sigma_i'} f\right| \geqslant \frac{|I|}{4}.$$

Denote $\sigma_1 = \sigma_i'$ and $S_1 = S_i'$. Now repeat this quartering process with S_1 and σ_1 and continue the process infinitely. We obtain a sequence of squares $S_1, S_2, \dots$ and corresponding paths $\sigma_i, i = 1, 2, \dots$ such that

$$\left|\int_{\sigma_i} f\right| \geqslant \frac{|I|}{4^i}, i = 1, 2, \dots; \tag{4}$$

and $\cap_{i=1}^{\infty} S_i = \{(s_0, t_0)\}$. Let M be the common bound for $|H|, |H_t|, |H_s|$ and ϵ a positive number satisfying $8M^2\epsilon < |I|$. By the differentiability of f at $H(s_0, t_0) = z_0$ there exist a positive number δ and a function η such that the equation

$$f(z) = f(z_0) + f'(z_0)(z - z_0) + \eta(z)(z - z_0),$$

with $|\eta(z)| < \epsilon$, holds for all $|z - z_0| < \delta$.

Let n be so large that $H(S_n) \subset \{z; |z - z_0| < \delta\}$; such an integer n certainly exists because H is continuous and the squares S_i shrink down to the point (s_0, t_0). Now the integral $\int_{\sigma_n} f(H_t dt + H_s ds)$ is the sum of three integrals

$$\int_{\sigma_n} f(z_0)\,dH, \quad \int_{\sigma_n} f'(z_0)(H - z_0)\,dH \quad \text{and} \quad \int_{\sigma_n} (H - z_0)\eta\,dH.$$

Direct evaluation shows that the first two integrals are zero. For the third one, we first observe that

$$|H(s,t) - H(s_0, t_0)| \leqslant M[|s - s_0| + |t - t_0|] \leqslant \frac{2M}{2^n}$$

and then we have the obvious estimate

$$\left|\int_{\sigma} \eta(H - z_0)\,dH\right| \leqslant \epsilon 2M \frac{1}{2^n} M \frac{4}{2^n} < \frac{|I|}{4^n}.$$

This contradicts (4). Hence $I = 0$ and the proof is complete.

4. Generalizations. There are several possible ways to generalize Theorem 1 to paths (not necessarily smooth). First, one can observe that the second proof remains valid if H_t and H_s exist with a possible exception of finitely many points, if they are bounded and if $H_t(\cdot, s)$ is piecewise continuous on $[0,1]$ for every $s \in [0,1]$ and $H_s(t, \cdot)$ is piecewise continuous on $[0,1]$ for every $t \in [0,1]$.

Using Theorem 1 and the homotopy H defined by $H(t,s) = t\psi(s) + (1-t)\phi(s)$ one can prove

THEOREM 2. *If ϕ and ψ are two roads such that the distance between ϕ and ψ is smaller than the distance of ϕ from ∂G and if either $\phi(0) = \psi(0)$, $\phi(1) = \psi(1)$ or ϕ and ψ are closed roads, then* (1) *holds.* (*Distance between ϕ and ψ is* $\sup\{|\phi(s) - \psi(s)|, s \in [0,1]\}$.)

The homotopy version of the Cauchy Theorem in full generality can be obtained from Theorem 2 using arguments similar to those presented in **[1]**.

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IS THERE LIFE AFTER MATHEMATICS?

LARRY COPES

Recently I was attending a seminar in which a graduate student was presenting Banach's Fixed Point Theorem. Something about the way David said and wrote what he did bothered me.

He said, "This is Banach's Fixed Point Theorem." He wrote, "*Fixed Point Theorem* (Banach, 1922)." Then he wrote and read aloud a statement of the theorem. Next he said, "The proof goes something like this"; he wrote "*Pf*:" and outlined a proof. Finally he said, "For example"; he wrote "*Ex*:" and gave several examples.

"Why should I be bothered by this?" I asked myself, even as I was. After all, it was what had taken place in every mathematics course I had taken since high school. Eventually the cause of my misgivings hit me: his presentation was backwards. Before reading on, you may wish to look back at the presentation above and see if you can figure out what I mean.

Death and life. David's presentation can be seen as having several pieces. They are: name of theorem, person to whom it is attributed, date, statement of theorem, proof, and examples or applications. What struck me so hard was that the way the mathematical idea being presented actually came about was probably almost in the reverse order. And the difference between the two orders is the difference between the life of the mind and the death that is in someone else's stale results.

What do I mean? Anyone who has done even limited mathematical investigation will agree that mathematical ideas usually arise from the explorer's intuition about something. We wish to define or verify logically our intuitive feelings about something, usually something generalized or abstracted from many examples. Perhaps we are given a problem. Perhaps we note that several specific mathematical objects or ideas have something in common. In any case, we get a hunch about a definition or theorem and want to verify it.

We then search for nonsuperficial relationships and come up with a conjecture of some sort: All such-and-suches have this property. Observing patterns in examples and applications is, therefore, the *first* step in the process, while the *second* step is making a hypothesis.

Now the second step, I admit, doesn't appear at all in the common form of slick presentations exemplified by David's. I believe this is because the initial conjecture probably is incorrect, and we tend to limit our class presentations to correct results. Only in the process of trying to test that conjecture—by proof or a search for counterexamples—do we gradually adjust it. Eventually we come up with a definition or theorem. So the proof and the theorem-statement stages of David's presentation evolve simultaneously. (Although I have used the pronoun "we" in this description, the evolutionary process often involves the work of many mathematicians over a long period of time.)

Eventually someone decides that the defined concept is useful or that the theorem has been proved, and sooner or later most of the mathematical community comes to agree. Consensus is reached—often very slowly—on a date, a person or persons to whom the idea may be attributed, and a name for the result—David's first three stages.

Thus the evolution of a mathematical idea—and "evolution" seems to be a more accurate description of the process than is either "discovery" or "creation"—takes a sequence of stages that are approximately the reverse of the steps in presentations of the result.

So what? So what if most presentations of mathematical ideas are backwards from the way they evolved? Teaching is, at least by some definitions, an efficient communication of a body of knowledge. We have learned ways of communicating those ideas not only efficiently but effectively; students don't need to reinvent the wheel in order to learn to use it or even to appreciate the greatness of the idea.

This is a good argument, but I believe it is less than perfect. It is based on the assumption that our goals are limited to student appreciation of, and skill at using, mathematical ideas. The assumption may be a fair one for engineers and business majors, but such narrow goals communicate to students a concept of mathematics that ignores most of what happens in mathematics—an inaccuracy in our teaching that surely is worse than omitting a condition in a definition or theorem! Don't we have an ethical obligation to provide undergraduate and graduate mathematics students with a proper perspective on mathematics, especially when that perspective is as different from the one we usually present as life is from death?

We *must* show our students how people come up with mathematical ideas and perhaps even develop their skill at generating good ideas for themselves. To do so, however, we must avoid the forms of presentation to which we are accustomed—those presentations that hide the evolving process of theorems and thus disguise the life of mathematics. But if we can't teach that way, what *do* we do?

An alternative. I've found that it is not hard to reverse the order of my presentations in courses from Calculus I on up. It is not too time-consuming to start with several well-chosen examples, point out some patterns, come up with a hypothesis or two, try to prove them, adjust them as needed, and derive a theorem.

For example, my standard way of presenting the Mean Value Theorem in Calculus I used to involve stating and proving Rolle's Theorem, giving some examples of how it applies to various functions, and assigning some homework exercises. The next day I would state and prove the Mean Value Theorem itself (using Rolle's Theorem), give some examples of how it applies, and assign some more homework exercises. In especially creative semesters I would show the MAA film "The Theorem of the Mean (Policeman)" for a humorous review. Total time: two and a half days.

My backwards alternative also takes two and a half days. I pose the problem: Is it trivial to prove the rather obvious fact that a traveler's journey must include some point at which his instantaneous speed equals his average speed for the whole journey? I point out how the statement might initially be "mathematized": For every function $f(x)$ defined on interval $[a,b]$, there is a point c in (a,b) such that

$$f'(c)=\frac{f(b)-f(a)}{b-a}.$$

Then I point out by counterexample that continuity must be an additional condition. Comparing the graph of our function to the graph of the linear function with the same endpoints, I reduce the conjecture to: For every continuous function $g(x)$ defined on $[a,b]$ with $g(a)=g(b)=0$, there is a point $c\in(a,b)$ such that $g'(c)=0$. In attempting to prove this statement, I point out, again by counterexample, that the additional condition of differentiability at c (and therefore everywhere in (a,b)) is required. Restatements and proofs of both theorems in final form need not be followed by examples, since we have been looking at special cases all along. The MAA film is still amusing and is shown if time remains.

Of course, if our goals include student awareness of the mathematical processes, we must include this in our testing. I find that careful assignment of out-of-class essays can be effective in communicating the importance to educated people of the ability to write effectively and carefully. The essay should describe one of the investigations pursued in class, including the false conjectures and dead ends as well as the ultimately successful paths. I ask students to submit the essay to count as the first question on an otherwise in-class exam, and make sure they

are warned early in the term that they will be responsible for all the work in the class sessions, not just for the theorems derived and proved.

To be honest I must point out some problems that I have encountered while teaching this way. The first is relatively minor: I find that I must have a much greater understanding of a mathematical result in order to teach this way rather than the traditional way. In particular, I must be aware of why certain proofs are clearer, if less efficient, than other proofs.

Second, students are thrown off slightly by an approach to a course that requires more than memorizing formulas and solving end-of-section problems. They are reasonably assured, however, when I tell them that it is only because they are advanced enough to be studying "higher mathematics" that they can think about it as they think about other subjects. After all, for years they have been doing in non-mathematics courses much of what I am asking of them in this course—taking careful notes, organizing them, and writing them up.

This comparison with other courses leads to the third problem, a more serious psychological one. It arises with students whose concept of knowledge in general is highly factual rather than relativistic. Students who still think that a poem (or a society) is either good or bad, and that it is the job of the English or politics teacher to tell them which it is, are likely to be upset with a consideration of mathematics as an evolving process. So are students who have sought out the sciences as the last remaining island of sanity in a sea of relativistic knowledge. My most successful responses to the trauma these students share with me is to let them know that I sympathize and that they too have the power to survive in such a world by imposing their own structures on the chaos.

Another alternative. The above alternative way of presenting material is aimed at the goal of familiarity with mathematical processes. It does little, however, to develop student skill at actually deriving mathematical ideas. Recently I feel I have had some success at carrying some courses one step further toward this latter goal.

Here again I start consideration of a topic with a problem or with several carefully chosen examples of a phenomenon. Instead of pointing out patterns to the students, I ask them to analyze the problem or search for similarities and differences among the examples. As they begin to see patterns and question their importance, the class sessions become alive, reflecting the life of mathematical investigation. It turns into a discussion, with me as moderator and resource person rather than judge. I steer the class to make and test conjectures by carefully summarizing what they say in such a way as to plant ideas. I also make sure that the last step in the process, after the summarizing statement and proof of a theorem, is generating a series of questions that arise out of the result. Indeed, I have come to see the courses I used to teach as neatly wrapped packages that had nothing to do with the sweat and joy of real mathematics.

As with the first alternative, I test on the achievement of the course goals. Here I assign students a problem or several examples of a mathematical phenomenon. I ask that they find or generate some patterns, prove or disprove some conjectures, establish a theorem, and pose some new problems.

The problem of success is intensified as this new teaching method is brought into play. In their mathematical careers the students in the class have achieved success, even distinction, by memorizing and spitting back definitions, theorems, and formulas. Earlier in my course they even mastered the ability to take notes on the development of mathematical ideas and to put them in the form of a paper. Now they are being asked to do something radically new—to do the equivalent of mathematical research, but on frontiers of their own knowledge. Can they possibly succeed?

In this case I have found that it's better not to let the class know too early what will be expected of them later in the course. As we begin a new section with this approach, I intentionally find something positive to say about anything anybody contributes, even if I have to pretend to misinterpret it somewhat. The result: more class contributions, more feeling of

success, and, yes, more thinking. When I *do* make homework assignments that ask them to engage in some fairly simple mathematical investigation, they know they can do it. Eventually most of them find that they can master these new skills and that they much prefer life over death.

Life and rebirth. I have argued here that, since the results of mathematical investigations are only a small part of all that is mathematics, a mathematics course must be built around the processes of mathematics in order to be accurate. These processes frequently take place in an order that is so backwards from the order of our usual presentations that the more common order appears as death contrasted to life. Hence any mathematics course that portrays mathematics honestly must be quite different—even backwards—from our usual teaching approaches.

But is any of this really new? I believe Socrates was saying the same thing when he compared a teacher to a midwife. The midwife's role is not to describe children already born, just as a mathematics teacher can't be satisfied with "presenting" results of previous investigations. The midwife and the teacher must both avoid the role of doing the job for the person they're trying to assist, and also the opposite extreme of merely observing passively. They both are involved with the producer, the product, and the vital relation between the two. The goal of both is to teach the art of using natural and trained skills in the life process.

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A DISCRETE APPROACH TO COMPUTER-ORIENTED CALCULUS

SHELDON P. GORDON

Introduction. During the past decade, computers have been introduced into a wide variety of undergraduate mathematics courses. In the calculus sequence, in particular, the computer is a marvelous pedagogical device. Numerous reports of such activities have been published, for example[**1**]-[**5**], [**7**]-[**11**], [**13**]-[**19**], [**21**], [**22**]. These calculus courses use a variety of approaches. At one extreme the computer is essentially relegated to the role of an auxiliary tool used merely to obtain print-outs and visual displays portraying limits and graphs of functions. The opposite extreme is a course which is essentially programming with applications to calculus; the calculus itself is secondary. Other courses graft numerical algorithms onto the main body of the calculus, either as a tool or to enhance understanding of the material, allowing the students different degrees of actual computer interaction.

This paper describes an approach to the calculus which incorporates the computer in a particularly natural way. The *discrete approach* uses results from the calculus of finite differences and finite sums, both for motivation and as significant tools leading to applications. This approach has many important and desirable benefits:

1. Students attain greater facility with, and understanding of, the basic concepts and methods of calculus.
2. Students come to appreciate the essential relationship between mathematics and computers, in the sense that most practical mathematical problems require computer solutions and appropriate mathematical formulations lead naturally to the use of the computer.
3. The approach provides an ideal context in which to develop several simple, yet useful, numerical algorithms for approximating functions and for actually finding where all those "given" functions come from.
4. It demonstrates the use of discrete as well as continuous models, especially in the nonphysical sciences which are now increasingly being presented mathematically and whose processes are inherently discrete rather than continuous.

This approach has been implemented on an experimental basis by the author and several

colleagues in the traditional calculus sequence at Suffolk County Community College and in an applied calculus course for business majors at C. W. Post College. In each instance the results were successful and student and instructor reactions were favorable. A detailed analysis of the results of these experiments will appear elsewhere [**8**]. Detailed amplifications and descriptions of the above list of advantages motivating the discrete approach also have been dealt with in a separate paper [**9**].

This paper will discuss some of the implications and advantages of the discrete approach and will illustrate some of its specific applications in calculus.

The Discrete Approach. The calculus of finite differences and finite sums [**12, 20**] contains discrete analogues of most of the major concepts and techniques of the infinitesimal calculus without using limits. For example, there are discrete parallels to such topics as maxima and minima, the chain rule, the fundamental theorem of integration, techniques of integration, Taylor series expansions, and partial derivatives. An indication of the parallelism can be seen from the sample formulas in Table 1.

TABLE 1

Discrete	Continuous
$\Delta[f(x)\pm g(x)]=\Delta f(x)\pm\Delta g(x)$	$D_x[f(x)\pm g(x)]=D_xf(x)\pm D_xg(x)$
$\Delta[f(x)\cdot g(x)]=f(x)\Delta g(x)+g(x)\Delta f(x)+\Delta f(x)\Delta g(x)$	$D_x[f(x)\cdot g(x)]=f(x)D_xg(x)+g(x)D_xf(x)$
$\Delta[f(x)/g(x)]=\dfrac{g(x)\Delta f(x)-f(x)\Delta g(x)}{g(x)g(x+h)}$	$D_x[f(x)/g(x)]=\dfrac{g(x)D_xf(x)-f(x)D_xg(x)}{g^2(x)}$
$\Delta^{-1}(x_n\pm y_n)=\Delta^{-1}x_n\pm\Delta^{-1}y_n$	$D^{-1}(f(x)\pm g(x))=D^{-1}f(x)\pm D^{-1}g(x)$
$\Delta^{-1}(cx_n)=c\Delta^{-1}x_n$	$D^{-1}(cf(x))=cD^{-1}f(x)$
$\Delta^{-1}(x_n\cdot\Delta y_n)=x_ny_n-\Delta^{-1}(y_{n+1}\Delta x_n)$ (Summation by parts)	$D^{-1}[f(x)D_xg(x)]=f(x)g(x)-D^{-1}[g(x)D_xf(x)]$
$\Sigma_{n=j}^{k}x_n=\Delta^{-1}x_n\big\vert_j^{k+1}$ (Fundamental Theorem of the Sum Calculus)	$\int_a^b f(x)dx=D^{-1}f(x)\big\vert_a^b$

The idea is to use the concepts from finite differences and sums to motivate the infinitesimal calculus and to provide the appropriate setting for solving "real" problems using discrete approximations. Specifically, most of the standard topics in calculus are taken up first in the discrete context, where the absence of limits reduces all work to arithmetic and algebraic manipulations. Most students obtain an easier and fuller understanding and greater facility with the techniques than with the purely continuous approach.

A course based on the discrete approach begins after the usual precalculus topics with the concept of sequence, which is fundamental to any computer usage and to much of the mathematical treatment that follows. The forward difference operator Δ is then introduced; it is defined by $\Delta x_n=x_{n+1}-x_n$. When this is extended to apply to an arbitrary function, using a step h, we have $\Delta f(x)=f(x+h)-f(x)$, which represents the change in f when x changes by h. Higher-order differences are defined by

$$\Delta^2 f(x)=\Delta[\Delta f(x)]=f(x+2h)-2f(x+h)+f(x)$$
$$\Delta^3 f(x)=\Delta\left[\Delta^2 f(x)\right]=f(x+3h)-3f(x+2h)+3f(x+h)-f(x)$$

and so on.

In this context it is clear to the student that a function experiences an overall increase when

$\Delta f(x)>0$ and an overall decrease when $\Delta f(x)<0$. Thus, after approximately one week of the course, the students can look for possible maxima and minima for a function f by seeking points where Δf is approximately zero or where Δf changes sign.

As an example we consider the function $f(x)=x^4-8x^3+22x^2-24x+10$ on the interval $[0,4]$. The appropriate values for f and its two first differences with a step $h=.5$ are shown in Table 2. From this we see that $f(x)$ experiences a net decrease between $x=0$ and $x=1$ and again between $x=2$ and $x=3$, since the differences are negative. Furthermore, the function has a net increase between $x=1$ and $x=2$ and again from $x=3$ to $x=4$. Thus, we can conclude that f possesses minima near $x=1$ and near $x=3$ and a maximum near $x=2$. This relatively coarse analysis can now be refined as much as desired by repeating it with smaller values for the step h. This process will almost certainly require the use of a computer, thus emphasizing the need for computers as tools in calculus.

An advantage of being able to do this kind of problem early is that it previews the notion of optimizing functions before the introduction of the limit, let alone the derivative. Simultaneously, this development introduces computer search operations for locating approximate values of critical points. It also provides an important and useful technique at an early stage of the course, thereby supplying practical motivation to the many students who are normally lost during the first month of purely mathematical development in the usual calculus course.

An example such as this one can be reconsidered at a later stage to illustrate the use of discrete analogues of the first and second derivative tests. We first note that information about the second difference at a point x is most accurately reflected by considering the value of $\Delta^2 f$ at the preceding point, $x-h$, since

$$\Delta^2 f(x-h)=f(x+h)-2f(x)+f(x-h)$$

is centered about x. Thus, we see from Table 2 that for $x=1$, $\Delta^2 f(.5)>0$, indicating that the

TABLE 2

$f(x)$	$\Delta f(x)$	$\Delta^2 f(x)$
$f(0)=10$	$\Delta f(0)=-7.4375$	$\Delta^2 f(0)=5.875$
$f(.5)=2.5625$	$\Delta f(.5)=-1.5625$	$\Delta^2 f(.5)=2.125$
$f(1)=1$	$\Delta f(1)=.5625$	$\Delta^2 f(1)=-.125$
$f(1.5)=1.5625$	$\Delta f(1.5)=.4375$	$\Delta^2 f(1.5)=-0.875$
$f(2)=2$	$\Delta f(2)=-.4375$	$\Delta^2 f(2)=-.125$
$f(2.5)=1.5625$	$\Delta f(2.5)=-.5625$	$\Delta^2 f(2.5)=2.125$
$f(3)=1$	$\Delta f(3)=1.5625$	$\Delta^2 f(3)=5.875$
$f(3.5)=2.5625$	$\Delta f(3.5)=7.4375$	
$f(4)=10$		

curve is concave up and hence that there is a minimum. Similarly, for $x=3$, $\Delta^2 f(2.5)>0$, also indicating the presence of a minimum. On the other hand, for $x=2$, $\Delta^2 f(1.5)<0$, indicating a maximum. It is clear that this technique can be easily implemented on the computer.

Another advantage of the discrete approach is that the introduction of sequences at the beginning of the course provides a technique for handling the important class of problems involving growth and decay. Most of these problems can be described by a relationship of the form $A_{n+1}=KA_n$, where A_n is the amount of something present at time n and K is a positive constant of proportionality. Then A_n is increasing if $K>1$ and decreasing if $K<1$. The solution is given by $A_n=A_0K^n$, where A_0 is the initial value for A.

Students seem to find this approach conceptually simpler and far easier than the usual solution based on integration and the properties of exponential and logarithmic functions. It provides additional practical motivation early in the course. In addition, by emphasizing the fact that the solutions to such problems, $A_n=A_0K^n$, satisfy $\Delta X_n=CX_n$, one can motivate the students

to seek a similar solution to the differential equation $y'=ky$ and lead them easily to the introduction of e via the limit. In many ways, beginning with e as a base is more natural for students than defining $\ln x$ as an integral.

Most processes involved in growth and decay problems are inherently discrete, so the procedure described above is more natural than the use of differential equations. The same comment applies to most of the processes studied in the biological sciences, operations research, business, and the social sciences, such as economics, sociology, and psychology (cf. **[6]**). Thus, the discrete approach provides an ideal context for developing a calculus course for the non-science major as well as for the science major.

As we have said, the idea behind the discrete approach is to introduce and treat various topics from calculus in their discrete formulations first. Once this has been done for a particular topic, the corresponding result from the infinitesimal calculus arises simply by passing to the limit as the step approaches zero. Thus, after the students have mastered a concept or technique from the finite calculus, it is a very simple matter to apply the limit process and derive the corresponding result in the continuous case. This natural transition has the additional advantage of presenting the students with a brief review of all concepts and methods as they are developed in the new context. Moreover, the transition can be made very quickly, so that in almost all cases the total time spent on both the continuous and discrete treatment of a topic is virtually the same as in the traditional course.

Approximate representations of functions. Still another advantage of the discrete approach is that it provides the opportunity to introduce discrete approximations into calculus courses in a simple, yet natural, manner. It thus supplies a suitable means for developing mathematics appropriate to applied mathematicians', scientists', or engineers' points of view, so that they will have an indication of how nonlinear or extremely complex situations can be treated. Although the phrase "given the function $f(x)$" is one of the most common expressions used in calculus, no attention is paid to where the function is coming from or who is giving it. In practice, such functions are usually obtained from relatively simple results on the approximation and interpolation of functions. Unfortunately, this is taken up only as part of a course in numerical analysis, a subject studied by few students. The majority of students taking calculus are never exposed to this critical area for applying calculus in practical situations. Even for those students who do take a course in numerical analysis, it remains an isolated area in mathematics with a philosophy different from that of most other mathematics courses. However, when we have the discrete formulations available in calculus, it is simple and natural to develop a few important approximation and interpolation methods, such as Lagrange's formula

$$P_n(x)=f(x_0)\cdot L_0(x)+f(x_1)\cdot L_1(x)+\cdots+f(x_n)\cdot L_n(x),$$

where

$$L_k(x)=\frac{(x-x_0)(x-x_1)\cdots(x-x_{k-1})(x-x_{k+1})\cdots(x-x_n)}{(x_k-x_0)\cdots(x_k-x_{k-1})(x_k-x_{k+1})\cdots(x_k-x_n)},$$

or Newton's formula

$$P_n(x)=f(x_0)+(x-x_0)\frac{\Delta f(x_0)}{h}+(x-x_0)(x-x_1)\frac{\Delta^2 f(x_0)}{2h^2}+\cdots+(x-x_0)\cdots(x-x_{n-1})\frac{\Delta^n f(x_0)}{n!h^n}.$$

Such developments give the student a feeling that this material constitutes an integral part of the subject; they also provide a marvelous opportunity to utilize the computer in solving problems which would otherwise have been far too complicated to consider. For example, it is easy to introduce the idea of fitting a polynomial graph to given discrete data.

Once the notion of interpolation is available, a great many other valuable techniques for enriching a calculus course become accessible. One of the most important involves the numerical approach to derivatives. From an ideal standpoint, the numerical calculation of derivatives

should be particularly natural for a computer-oriented calculus course, since all that needs to be done is to evaluate

$$\frac{\Delta f(x)}{h}=\frac{f(x+h)-f(x)}{h}$$

at a succession of ever smaller values of h to obtain an increasingly more accurate value of $f'(x)$. Unfortunately, theory and practice do not agree. Depending on the computer, there invariably comes a point where, for small h, the numerator becomes so close to 0 that the computer can no longer evaluate the quotient accurately and so the values calculated diverge sharply. This problem can often be neatly circumvented by utilizing simple interpolation techniques. For example, by approximating $f(x)$ by a fourth-degree polynomial centered at $x=a$, it is very simple to derive the formula

$$f'(a)\approx\frac{1}{12h}[f(a-2h)-2f(a-h)+2f(a+h)-f(a+2h)].$$

When h is taken reasonably small, this produces an amazingly accurate approximation to $f'(a)$ which is easy to calculate.

In a different direction, another "nice" result which can be included in this type of calculus course involves approximations to double integrals. In particular, using only the mathematics usually covered in calculus, it is possible quickly and easily to derive [7] a three-dimensional analogue of the trapezoid rule

$$\int_a^b f(x)\,dx\approx\frac{\Delta x}{2}[f(x_0)+2\{f(x_1)+f(x_2)+\cdots+f(x_{n-1})\}+f(x_n)].$$

If a function $f(x,y)$ is defined on a rectangular region R, one can easily show that

$$\iint_R f(x,y)\,dy\,dx\approx\frac{\Delta x\Delta y}{2}\left[\sum_{\text{boundary}} f(x_i,y_j)+2\sum_{\text{interior}} f(x_i,y_j)-f(x_0,y_0)-f(x_n,y_m)\right].$$

This special case of the Newton-Cotes formula can be obtained by approximations using solids surmounted by planes passing through three points determined by the partition of R.

Discussion. In order to incorporate the material on both finite differences and the computer into the first term of the calculus sequence, it was necessary to use approximately two weeks of class time. Normally this would constitute an inordinate portion of the term. However, the discrete approach involves developing the usual topics in the discrete case first and then carrying over in the limit to the continuous case. In each instance the author and the other instructors using the approach found that there is a considerable saving in time during the development of the continuous case, since the students were already familiar with the concepts from the discrete setting. In fact, this is true to such an extent that the net loss in time was virtually nil. At the same time, there was an overall gain in student understanding and facility, as measured by student performance on tests and by student comments (both oral and on course questionnaires).

In any realistic assessment of today's society and prospects for the future, it is clear that our lives are going to be increasingly affected by computers. This is especially true of the mathematics community. It is essential to emphasize applications in our courses. In particular, it is becoming increasingly important to incorporate computers into as many of our courses as possible. This must be done not merely as an educational aid but, more important, as a mathematical tool which is needed to solve the real-world problems which our students will one day face (as opposed to the relatively simple and artificial classroom problems usually presented). Not to do this is to do our students a disservice.

In order to use the computer to solve calculus problems, the problems must be transformed to finite difference analogues to enable the computer to handle them. Thus the discrete approach appears to be a natural way to incorporate computers into the calculus and show how they are actually used. Moreover, the discrete approach can be seen to encompass, and, more important, to supply a unifying context for, the efforts of the many people who have attempted to develop computer-oriented calculus courses to serve the needs of students in widely varying fields.

The discrete approach is an extremely valuable method which can be profitably incorporated in virtually any calculus course, especially if it is to have a computer orientation. However, it can also be used for motivational purposes or for introducing discrete models.

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PROBLEMS AND SOLUTIONS

The editors receive more problems than they can use, but they are always on the lookout for interesting and novel problems. The task of selection will be much easier if proposers will keep the following guidelines in mind:

Send all problems (both elementary and advanced) to A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131, in duplicate if possible. The editors urge proposers to include any solutions or information that will help the editors, including reasons why the problem is interesting. Problems in well-known textbooks and results that appear in generally accessible sources are not acceptable.

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution. If you submit a problem without a solution, you should tell the editors whether you know (or somebody else knows) how to solve the problem. If you are uncertain, the problem might be more suitable for the Research Problems section; see the instructions given there.

Proposers are asked to aim for the same audience as for the rest of the MONTHLY: *a rule of thumb is to think of people who have had at least a year of graduate work in Mathematics, very likely some years ago. Proposals that can be both understood and solved by high school students, or by those in the first two years of college, are usually more appropriate for the Mathematics Magazine or the Two-Year College Mathematics Journal. No precise line can be drawn between elementary and advanced problems, but in general an elementary problem will involve notions no more difficult than those encountered in the first year of graduate work and should be solvable by methods common at the same level. An advanced problem may deal with more advanced concepts or require more sophisticated techniques for its solution.*

A problem that takes half a page or more to state, or that involves several ad hoc definitions, is usually unsuitable. The statement of a problem should avoid specialized jargon and symbols whenever the English language is more readily understood: for example, "f is a continuous function" is preferable to "$f \in C$."

Before submitting a problem, be sure that it says exactly what you intend it to say. Solvers are not telepathic and will usually solve what was proposed, not what you had in mind.

PROBLEMS DEDICATED TO EMORY P. STARKE

Solutions of the problems in this issue dedicated to Professor Emory P. Starke should be mailed to Prof. A. P. Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, N.M. 87131 (USA), before September 30, 1979. To facilitate consideration, solutions should be typed (with double spacing).

S 11. *Proposed by R. C. Buck and E. F. Buck, University of Wisconsin, Madison*

A solid tetrahedron carries a continuous temperature distribution. What is the maximum number of points having the same temperature one can be sure of finding on the edges of the tetrahedron?

S 12. *Proposed by M. S. Klamkin, University of Alberta*

If a, a_1; b, b_1; c, c_1 denote the lengths of the three pairs of opposite sides of an arbitrary tetrahedron, prove that $a+a_1$, $b+b_1$, $c+c_1$ satisfy the triangle inequality.

S 13. *Proposed by H. Kestelman, University College, London*

A non-negative real matrix A with spectral radius 1 has the property that for some pair p, q,

the p, q element of A^j tends to 0 as $j \to \infty$. Show that, for some r, s, the r, s element of A^j is 0 for all positive integers j.

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA (USA) 94303. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (in duplicate, with double spacing) and should be mailed before September 30, 1979. Please enclose a self-addressed card or label (for acknowledgment).

E 2773. *Proposed by Michael W. Ecker, Pennsylvania State University, Worthington Scranton Campus*

Problem E 2704 [1978, 198] asked for the number of solutions to $x^2 = x$ in the ring $\mathbf{Z}_n$ of integers modulo n. Question: What is the number of solutions in $\mathbf{Z}_n$ of $x^3 = x$? (See p. 397.)

E 2774*. *Proposed by James Propp, Great Neck, N.Y.*

Prove or disprove that, given a convex two-dimensional figure S, six translates of S can fit inside a homothetic figure three times as large as S in linear dimensions.

E 2775. *Proposed by Ko-Wei Lih, Academia Sinica, Nankang, Taipei, Taiwan, Republic of China*

If we replace even integers by 0 and odd integers by 1 in the ordinary Pascal triangle, we get the following modulo 2 Pascal triangle:

$$\begin{array}{ccccccccc} & & & & 1 & & & & \\ & & & 1 & & 1 & & & \\ & & 1 & & 0 & & 1 & & \\ & 1 & & 1 & & 1 & & 1 & \\ 1 & & 0 & & 0 & & 0 & & 1 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Will 1101 or 1011 occur as a consecutive segment in any row of this modulo 2 Pascal triangle?

E 2776. *Proposed by Alan Wayne, Holiday, Florida*

(a) In the decimal system, find all twelve-digit positive integers n such that n^{102} ends at the right in the digits of n.

(b)* Is there a corresponding solution to the problem in numeration systems other than base ten?

E 2777. *Proposed by I. Borosh, H. Diamond, M. Gbur, & D. Hensley, University of Illinois, Urbana-Champaign.*

Let b/a be a reduced fraction greater than one. Let $r = r(a,b)$ denote the number of integers relatively prime to b in the sequence

$$[b/a], [2b/a], \ldots, [(a-1)b/a].$$

State and prove a rule for determining r as a function of a and b.

Here $[x]$ denotes the greatest integer in x. (Application: The number of primes $p \leqslant x$ lying in the sequence $\{[jb/a]\}_{j=1}^{\infty}$ is asymptotic to $rx/(\varphi(b) \log x)$, where φ denotes Euler's function. This application is problem 7.16 of *Les nombres premiers*, W. J. Ellison, in collaboration with M. Mendès-France, Hermann, Paris, 1975.)

E 2778. *Proposed by David J. Allwright, Cambridge University.*

Let k and r be integers with $r \geqslant 1$ and let z be a complex number with $|z| < 1$. Calculate the sum of $z^{\|N\|}$ as $N = (n_0, n_1, \ldots, n_r)$ ranges over all $(r+1)$-tuples of integers such that $n_0 + n_1 + \cdots + n_r = k$ and $\|N\| = |n_0| + |n_1| + \cdots + |n_r|$.

SOLUTIONS OF ELEMENTARY PROBLEMS

An Integer-Valued Function

E 2677 [1977, 738]. *Proposed by Erwin Just, Bronx Community College, CUNY.*

Let $n \geqslant 2$ be an integer. Show that there exists a function $f: \mathbf{R} \to \mathbf{R}$ such that $f(x)+f(2x)+\cdots+f(nx)=0$ for all x and $f(x)=0 \Leftrightarrow x=0$.

Solution by G. W. Peck, Massachusetts Institute of Technology. For $1 \leqslant x < n$ let $f(x)=1$. Define $f(x)$ for $nq^k \leqslant x < nq^{k+1} (k=0,1,2,\ldots, q=n/(n-1))$ by

$$f(x) = -\sum_{i=1}^{n-1} f\left(\frac{ix}{n}\right). \tag{1}$$

Define $f(x)$ for $2^{-k-1} \leqslant x < 2^{-k}$ $(k=0,1,2,\ldots)$ by

$$f(x) = -\sum_{i=2}^{n} f(ix). \tag{2}$$

Finally, set $f(x) = -f(-x)$ for $x<0$, and $f(0)=0$.

Then f satisfies the given equation and it is clear from (1) and (2) that $f(x) \equiv 1 \pmod n$ for all $x>0$. Thus $f(x)=0$ only for $x=0$.

Also solved by Columbia University Problem Group, Eric Grinberg, G. A. Heuer, L. E. Mattics, Anne Watkins & William Watkins, David Witte, and the proposer.

A Transcendental Function Satisfying a Duplication Formula

E 2692 [1978, 48]. *Proposed by Donald R. Woods, Stanford University.*

Show that the sequence of increasingly complex fractions

$$\frac{1}{2}, \left(\frac{1}{2}\right)\Big/\left(\frac{3}{4}\right), \frac{\left(\frac{1}{2}\right)\Big/\left(\frac{3}{4}\right)}{\left(\frac{5}{6}\right)\Big/\left(\frac{7}{8}\right)}, \left[\frac{\left(\frac{1}{2}\right)\Big/\left(\frac{3}{4}\right)}{\left(\frac{5}{6}\right)\Big/\left(\frac{7}{8}\right)}\right]\Big/\left[\frac{\left(\frac{9}{10}\right)\Big/\left(\frac{11}{12}\right)}{\left(\frac{13}{14}\right)\Big/\left(\frac{15}{16}\right)}\right], \ldots$$

approaches a limit, and find that limit.

What can be said about the more general sequence

$$\frac{x}{x+1}, \left(\frac{x}{x+1}\right)\Big/\left(\frac{x+2}{x+3}\right), \frac{\left(\frac{x}{x+1}\right)\Big/\left(\frac{x+2}{x+3}\right)}{\left(\frac{x+4}{x+5}\right)\Big/\left(\frac{x+6}{x+7}\right)}, \ldots ?$$

I. Solution by David Robbins, Phillips Exeter Academy (submitted by the proposer). The general sequence can be defined recursively by $f_0(x)=x$ and $f_{k+1}(x)=f_k(x)/f_k(x+2^k)$, $k \geqslant 0$. Now using induction on k it is easy to show that

$$f_k(x) = \prod_{r=0}^{2^k-1} (x+r)^{\theta(r)}$$

where $\theta(r) = +1$ or -1 depending on whether the number of ones in the binary expansion of r is even or odd.

Define $f(x)$ to be the infinite product $\Pi_{r \geqslant 0}(x+r)^{\theta(r)}$. This product is convergent because it

Volume of a Polytope

E 2701 [1978, 197]. *Proposed by Richard Stanley, M.I.T.*

Find the volume of the convex polytope determined by $x_i \geqslant 0$ $(1 \leqslant i \leqslant n)$, and $x_i + x_{i+1} \leqslant 1$ $(1 \leqslant i \leqslant n-1)$.

Solution by I. G. Macdonald, Queen Mary College, London, and Roger B. Nelsen, Lewis and Clark College (independently). The volume of the polytope is

$$V_n = \int_0^1 \int_0^{1-x_1} \cdots \int_0^{1-x_{n-1}} dx_n \cdots dx_2\, dx_1.$$

If we define polynomials $p_n(x)$ by $p_0(x) = 1$ and

$$p_n(x) = \int_0^{1-x} p_{n-1}(t)\, dt \qquad (n \geqslant 1) \tag{1}$$

then $V_n = p_n(0)$. It follows from (1) that the function

$$F(x,y) = \sum_{n \geqslant 0} p_n(x) y^n$$

satisfies the differential equation

$$\frac{\partial}{\partial x} F(x,y) + yF(1-x,y) = 0, \tag{2}$$

and the initial condition $F(1,y) = 1$. From (2) we derive

$$\frac{\partial^2}{\partial x^2} F(x,y) + y^2 F(x,y) = 0$$

and so

$$F(x,y) = A(y)\cos xy + B(y)\sin xy.$$

From (2) we have $\frac{\partial F}{\partial x}(0,y) = -y$ and so $B(y) = -1$. The initial condition $F(1,y) = 1$ now gives $A(y) = \sec y + \tan y$. Consequently $F(0,y) = A(y) = \sec y + \tan y$ and V_n is the coefficient of y^n in the Maclaurin expansion of $\sec y + \tan y$.

Explicitly one has

$$V_{2n} = (-1)^n E_{2n}/(2n)!,$$
$$V_{2n-1} = (-1)^{n-1} 2^{2n}(2^{2n}-1) B_{2n}/(2n)!,$$

where E_{2n} and B_{2n} are Euler and Bernoulli numbers, respectively.

Also solved by Victor Wei, Theodore Bolis, Murray Klamkin, Emeric Deutsch, August Sardinas, Peter Lindstrom, and the proposer.

Editor's Comments. Note that $\sec y + \tan y = \tan(\frac{y}{2} + \frac{\pi}{4})$. Several solvers observed that $n!\,V_n$ is the number of zig-zag permutations of $1, 2, \ldots, n$, i.e., permutations $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\alpha_1 < \alpha_2 > \alpha_3 < \cdots$.

Evaluation of a Determinant

E 2703 [1978, 198]. *Proposed by David Jackson, University of Waterloo.*

Let J be the $n \times n$ matrix whose entries are all ones and write $J = L + U$ where L (respectively U) is a lower (respectively upper) triangular matrix and the diagonal entries of L are zeros. Let $X = \operatorname{diag}(x_1, \ldots, x_n)$ where $x_1, \ldots, x_n$ are variables.

Prove that

$$\det\left(I - (XU)^{k-1} XL\right) = \sum_{s \geqslant 0} (-1)^s a_{sk} \qquad (k = 1, 2, 3, \ldots)$$

where a_j's are defined by

$$\prod_{i=1}^{n} \frac{1-(tx_i)^k}{1-tx_i} = \sum_{j\geqslant 0} a_j t^j$$

(t a new variable).

Solution by I. P. Goulden, University of Waterloo. If e is the column-vector of 1's then $J=ee'$ (where prime denotes the transposition). If a, b are column-vectors then $\det(I+ab')=1+b'a$. Using this we find that

$$\begin{aligned}\det\big(I-(XU)^{k-1}XL\big) &= \det\big(I+(XU)^k-(XU)^{k-1}XJ\big) \qquad (1)\\ &= \det\big(I+(XU)^k\big)\cdot\det\Big(I-\big(I+(XU)^k\big)^{-1}(XU)^{k-1}Xee'\Big)\\ &= \Big(\prod_{i=1}^{n}(1+x_i^k)\Big)\cdot\Big(1-e'\big(I+(XU)^k\big)^{-1}(XU)^{k-1}Xe\Big)\\ &= \Big(\prod_{i=1}^{n}(1+x_i^k)\cdot\Big(1+\sum_{m\geqslant 1}(-1)^m e'(XU)^{mk-1}Xe\Big),\end{aligned}$$

where the last equality should be interpreted in the sense of formal power series.

It is easy to verify that the (i,j)-entry of $(XU)^{mk-1}X$ is 0 if $i>j$, is x_i^{mk} if $i=j$, and is equal to

$$x_i x_j \sum_{a_r\geqslant 0} \prod_{r=i}^{j} x_r^{a_r}, \quad a_i+\cdots+a_j=mk-2,$$

if $i<j$. Consequently $e'(XU)^{mk-1}Xe$ is the coefficient of t^{mk} in

$$\prod_{i=1}^{n}\sum_{j\geqslant 0}(tx_i)^j=\prod_{i=1}^{n}(1-tx_i)^{-1}.$$

Let us write

$$\prod_{i=1}^{n}\big(1-(tx_i)^k\big)=\sum_{j\geqslant 0} b_j t^j, \qquad \prod_{i=1}^{n}(1-tx_i)^{-1}=\sum_{j\geqslant 0} c_j t^j.$$

Then it follows from (1) that

$$\begin{aligned}\det\big(I-(XU)^{k-1}XL\big) &= \Big(\sum_{r\geqslant 0}(-1)^r b_{rk}\Big)\cdot\Big(\sum_{m\geqslant 0}(-1)^m c_{mk}\Big)\\ &= \sum_{s\geqslant 0}(-1)^s \sum_{m+r=s} b_{rk}c_{mk}\\ &= \sum_{s\geqslant 0}(-1)^s \sum_{m+r=sk} b_r c_m\\ &= \sum_{s\geqslant 0}(-1)^s a_{sk}.\end{aligned}$$

Also solved by the proposer.

Idempotent Elements in $\mathbf{Z}/n\mathbf{Z}$

E 2704 [1978, 198]. *Proposed by S. Collins, S. M. Reddy, and N. J. A. Sloane, University of Iowa and Bell Telephone Laboratories, Murray Hill, N.J.*

Find the number of solutions of $x^2=x$ in the ring of integers modulo n.

Solution by O. P. Lossers, University of Technology, Eindhoven, The Netherlands. Let $n = P_1P_2\cdots P_k$, where the P_i are powers of distinct primes. Then $x^2 \equiv x (\text{mod } n)$ is equivalent to the set $\forall_i[x^2 \equiv x \text{ mod } P_i]$. Since $(x, x-1) = 1$, each congruence $x^2 \equiv x \text{ mod } P_i$ has exactly two solutions $x \equiv 0, 1 \ (\text{mod } P_i)$. Thus, by the Chinese remainder theorem, we get 2^k solutions to the proposed congruence. These solutions are clearly distinct.

Also solved by 91 other readers, and the proposers. (Stuart S.-S. Wang submitted five different solutions.)

Comment. Neal H. McCoy points out that Alfred L. Foster has studied idempotent elements in a commutative ring with unity and has shown that under ring multiplication and a suitable new addition, the set of idempotent elements is a Boolean ring. For details, he refers to Foster's paper in Duke Math. J., 12 (1945) 143–152.

Expected Number of Trials

E 2705 [1978, 198]. *Proposed by Clark Kimberling, University of Evansville, Indiana.*

For an experiment having m equally probable outcomes, find the expected number of independent trials for k consecutive occurrences of at least one of these outcomes.

Solution by Norman L. Johnson, University of North Carolina. Let E_k be this expected number. When the first sequence of $k-1$ consecutive identical outcomes occurs, then the conditional number of further trials needed is 1 with probability $1/m$, and E_k with probability $1-1/m$. Hence

$$E_k = E_{k-1} + 1/m + (1-1/m)E_k.$$

So $E_k = mE_{k-1} + 1$ and since $E_1 = 1$, we find that

$$E_k = 1 + m + m^2 + \cdots + m^{k-1} = \frac{m^k - 1}{m-1}.$$

Also solved by Barry Arnold, Stuart Black, D. M. Bloom, Theodore Bolis, Aage Bondesen (Denmark), Michael Cohen, Denny Culbertson, Daniel Gallin & James Finch, Irving Glick, Ellen Hertz, Joel Levy, Peter Lindstrom, O. P. Lossers (Netherlands), Paul Massell, Lennart Råde (Sweden), Michael Vowe (Switzerland), and Victor Wei.

Råde calls attention to his note, "Waiting for patterns in a sequence of random numbers," Zeitschrift für angewandte Mathematik und Mechanik, ZAMM, 56 (1976) 124–125, in which he solves the problem completely, and in addition finds the probability generating function, and some other expectations.

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to Professor Roger C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109. To facilitate their consideration, solutions of Advanced Problems in this issue should be typed (in duplicate with double spacing) and should be mailed before September 30, 1979.

6267. *Proposed by A. E. Fekete, Memorial University of Newfoundland.*

We say that two collineations of the real projective space PR^n are of the same type if their invariant configurations are projectively equivalent (i.e., there is a real projective collineation mapping one configuration into the other). Find an explicit formula determining the number of all different non-identity collineation types. For example, for $n=1$ there are 3 types: hyperbolic (two fixed points), parabolic (one fixed point) and elliptic (no fixed point). Also, define collineation types for the complex projective space PC^n and find their number.

6268. *Proposed by Gene Smith and Hugh M. Edgar, San Jose State University.*

Assume that the algebraic number field K possesses at least one proper intermediate field E, i.e., $Q \subset E \subset K$. Prove or disprove the following: K must have a strictly increasing chain

$$Q = K_0 \subset K_1 \subset K_2 \subset \cdots \subset K_{n-1} \subset K_n = K, \qquad n \geqslant 2,$$

of subfields such that K_i has a relative integral basis over K_{i-1} for $1 \leqslant i \leqslant n$. [Here Q is the rational number field and $S \subset T$ means that S is a proper subset of T.]

6269. *Proposed by Robert E. Shafer, Berkeley, California.*

Let $F(u) = u^{-u}\Gamma(u+\frac{1}{2})$ and

$$G(x,s,t) = \frac{1}{\left(x-s+\frac{1}{2}\right)\left(x-t+\frac{1}{2}\right)} - \frac{1}{\left(x+s+\frac{1}{2}\right)\left(x+t+\frac{1}{2}\right)}.$$

Prove that, for $0 \leqslant s < t \leqslant x$,

$$e^{(s-t)G(x,s,t)/24} < \frac{F\left(x-t+\frac{1}{2}\right)F\left(x+t+\frac{1}{2}\right)}{F\left(x-s+\frac{1}{2}\right)F\left(x+s+\frac{1}{2}\right)} < 1.$$

SOLUTIONS OF ADVANCED PROBLEMS

Noetherian Integral Domain

6177 [1977, 744]. *Proposed by Adrian R. Wadsworth, University of California–San Diego, La Jolla.*

Let K be a perfect field of prime characteristic. Prove that if R is a Noetherian integral domain with quotient field K then $R = K$.

Solution by Robert Gilmer, Florida State University. The following more general result is true: Assume that K is a field with the property that for each $x \in K$, there exist an integer $n(x) > 1$ and an element $y \in K$ such that $y^{n(x)} = x$. If R is a Noetherian integral domain with quotient field K, then $R = K$.

Proof. Assume that there exists a Noetherian domain R with quotient field K, where $R \neq K$. Let e be the identity element of K and let $R^* = R + Ze$ be the subring of K generated by R and e. The ring R^* is Noetherian, and we claim that $R^* \neq K$. If $e \in R$, then $R^* = R \neq K$. If $e \notin R$, let b be a nonzero nonunit of R. Since $R^* b \subseteq R$, it follows that $e \notin R^* b$, and hence $b^{-1} \notin R^*$ so that again $R^* \neq K$. Let D be the integral closure of R^*. Then $D \neq K$, and D is known to be a Krull domain [see Theorem 33.10 of M. Nagata, *Local Rings*, Interscience, 1962]; hence D admits a rank one discrete valuation overring V. Let v be a valuation on K associated with the valuation ring V. There is no loss of generality in assuming that Z is the value group of v. Assume that $x \in V$ has v-value 1. There exists an integer $n(x) > 1$ and an element $y \in K$ such that $y^{n(x)} = x$. Therefore $1 = v(x) = v(y^{n(x)}) = n(x)v(y)$, where $v(y)$ is an integer. The impossibility of such an equality yields a contradiction, thereby proving the result stated.

Solver's remark: Some authors assume as a part of the definition that a Noetherian integral domain contains an identity element. The solution above does not include this assumption.

Also solved by D. D. Anderson, J. T. Arnold, Bob Boyd, David E. Dobbs, Bruce Glastad, A. A. Jagers (Netherlands), Ernst Kani (Germany) and the proposer.

Tiling a Rectangle with Squares

6178* [1977, 744]. *Proposed by Robert Kowalski, Winona, Minnesota.*

Define the shape of a rectangle to be the ratio of the longer side to the shorter side. Suppose one has an unlimited number of congruent squares at one's disposal. Given shape α and an error ε, what is the least number of squares one needs to construct a rectangle whose shape differs from α by less than ε?

We give two solutions. The first is simple and direct, while the second gives an algorithm for computing the required number, together with a program implementing it.

I. *Solution by University of South Alabama Problem Group.* One square works if $\alpha=1$ so assume $\alpha>1$ and $\varepsilon>0$ is given. Let $A=\max\{1,\alpha-\varepsilon\}$ and take b to be the least positive integer such that the open interval $(bA, b(\alpha+\varepsilon))$ contains at least one integer and take a to be the least such integer; then $a\cdot b$ squares will suffice.

If $A<\frac{c}{d}<\alpha+\varepsilon$ and $cd<ab$, then by the definition of b we have $d\geqslant b$ and so $\frac{c}{d}\leqslant\frac{c}{b}<\frac{a}{d}\leqslant\frac{a}{b}$, which would mean that the choice of a wasn't minimal. Hence $a\cdot b$ is the least number of squares.

II. *Solution by Albert Nijenhuis, University of Pennsylvania and Dartmouth College.* The least number L of required squares is given by

$$L=\min\left\{mn|m,n>0,m,n\in Z,|\alpha-\frac{m}{n}|<\varepsilon\right\}$$

if a simple "formula" is wanted. As L depends on α and ε in a highly discontinuous fashion, no simple analytic expression can be expected. However, an algorithm is available, which is basically a continued fraction expansion of α. Here is a simple and elegant version. It assumes $0<\varepsilon<\alpha$.

(A) $p\leftarrow 0$; $q\leftarrow 1$; $p'\leftarrow 1$; $q'\leftarrow 0$;

(B) $m\leftarrow p+p'$; $n\leftarrow q+q'$; $r\leftarrow m/n$;

if $r\leqslant\alpha-\varepsilon$, then $p\leftarrow m$; $q\leftarrow n$; to (B)

if $r\geqslant\alpha+\varepsilon$, then $p'\leftarrow m$; $q'\leftarrow n$; to (B)

else $L\leftarrow mn$; Exit ∎

It is easy to see by induction that, throughout the iterations in step (B) we have $p'q-pq'=1$, $\frac{p}{q}\leqslant\alpha-\varepsilon$, $\frac{p'}{q'}\geqslant\alpha+\varepsilon, m=p+p', n=q+q'$, while at exit we also have $\alpha-\varepsilon<\frac{m}{n}<\alpha+\varepsilon$. Let now $\frac{a}{b}$ be any fraction in $(\alpha-\varepsilon,\alpha+\varepsilon)$, then (using exit values) we have $\frac{p}{q}<\frac{a}{b}<\frac{p'}{q'}$, and

$$\left(\frac{p'}{q'}-\frac{a}{b}\right)+\left(\frac{a}{b}-\frac{p}{q}\right)=\frac{p'}{q'}-\frac{p}{q}=\frac{1}{qq'}$$

or upon multiplying by $qq'b$

$$(p'b-aq')q+(aq-pb)q'=h.$$

As the parenthesized expressions are positive integers, we have $b\geqslant q+q'=n$. Similarly, the inequalities $\frac{q'}{p'}<\frac{b}{a}<\frac{q}{p}$ yield $a\geqslant p+p'=m$. Thus, $L=mn$ is minimal.

The following program for an HP25, one of the simplest programmable calculators, implements the algorithm. The user is to key in α, ENTER, then key in ε and start; L will be in the display on termination. Assignment of memories 0 through 7 to $m,n,p,q,p',q',\alpha-\varepsilon,\alpha+\varepsilon$, respectively; r remains in the memory stack.

01	fREG	09	STO 4	17	STO 1	25	GTO 35	33	STO 5
02	STO 7	10	RCL 2	18	÷	26	RCL 0	34	GTO 10
03	STO-6	11	RCL 4	19	RCL 7	27	RCL 1	35	RCL 0
04	R↓	12	+	20	$x\leftrightarrow y$	28	×	36	STO 2
05	STO+6	13	STO 0	21	$fx\geqslant y$	29	GTO 00	37	RCL 1
06	STO+7	14	RCL 3	22	GTO 30	30	RCL 0	38	STO 3
07	1	15	RCL 5	23	RCL 6	31	STO 4	39	GTO 10
08	STO 3	16	+	24	$fx\geqslant y$	32	RCL 1		

Description:
Step (B) starts at 10.
r is computed at 18, when m, n are in the memory stack.
r is compared with $\alpha+\varepsilon$ at 19-22; p', q' redefined at 30-34.
r is compared with $\alpha-\varepsilon$ at 23-25; p, q redefined at 35-39.
Exit at 26-29.

Also solved by Gerard J. Chang (Taiwan), C. V. Heuer, Frank Meyer, William Myers, and K. Roberts.

A Construction for Ideals

6180 [1977, 828]. *Proposed by L. C. Larson, Saint Olaf College, Northfield, Minnesota.*

Let A and B be ideals of a commutative ring R with unity. Show that $\{x\in R: xB\subseteq xA\}$ is an ideal if R is either an integral domain or a principal ideal ring, but that in general it need not be.

Solution by Edward T. Wong, Oberlin College, Oberlin, Ohio. This problem can be extended to R is either an integral domain or a *Bezout ring* (every finitely generated ideal is principal).

For any ideals A and B in R, let $T=\{x\in R: xA\subseteq xB\}$. If T contains a regular element, then $A\subseteq B$ and hence $T=R$. Thus T is always an ideal if R is an integral domain.

If $x\in T$ then $xR\subseteq T$. Therefore T is an ideal if and only if for all x,y in $T, xR+yR\subseteq T$. Suppose that for all x,y in T, $xR+yR=zR$ for some $z\in R$. Let $z=xt+ys$ and $x=zr_1, y=zr_2$. For any $a\in A, xa\in xB$ and $ya\in yB$. Hence $xat+yas=za\in xB+yB=zr_1B+zr_2B\subseteq zB$. This shows $z\in T, xR+yR=zR\subseteq T$, and T is an ideal.

However, it is not true in general. For example, let $F[x,y]$ be the polynomial ring in two variables over a field F and $R=F[x,y]/N$, where N is the ideal generated by xy. Let $A=\bar{x}R+\bar{y}R$ and $B=(\bar{x}+\bar{y})\mathrm{R}$, where $\bar{x}=x+N$ and $\bar{y}=y+N$. Since $\bar{x}\bar{y}=0, \bar{x}A=\bar{x}^2R=\bar{x}B$ and $\bar{y}A=\bar{y}^2R=\bar{y}B$. But $(\bar{x}+\bar{y})A=\bar{x}^2\mathrm{R}+\bar{y}^2R\not\subseteq(\bar{x}+\bar{y})B=(\bar{x}^2+\bar{y}^2)R$.

Also solved by J. T. Arnold & M. B. Boisen, I. M. Isaacs, A. A. Jagers (Netherlands), and by the proposer, who provides additional conditions for the set to be an ideal in the case the ring R is Noetherian.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR., AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf, Carleton, and Macalester Colleges
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We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Why the Professor Can't Teach: Mathematics and the Dilemma of University Education. By Morris Kline. St. Martin's Press, New York, 1977. 288 pp. \$10.00. (Telegraphic Review, April 1978.)

Editors' Note: Because of the special nature of this book and the attention it is getting from a great diversity of people interested in education, we have presented reviews from three points on the academic spectrum.

Mathematics education in the United States is ailing. This book presents Dr. Kline's diagnosis of those ills, his analysis of their causes, and his prescriptions for treatment. It is addressed to university administrators, parents, legislators, academic mathematicians, and concerned citizens, and it is dedicated to undergraduates in American colleges and universities. During most of the book the discourse leads one to believe that the patient is incurably afflicted —assailed by a host of malignancies that permeate mind, body, and spirit, debilitating rot that is the price of years of excess in pursuit of the damsel research. But on the penultimate page we find the first ray of hope: "Though reforms of various sorts have yet to be adopted, we need not be too despondent about our educational system." Following the damning accounts in the previous 270 pages, this straw of optimism will be clutched frantically by the reader who is going down for the third time.

Professor Kline carefully sets the tone of his work even before the Preface by quoting from Hamlet's soliloquy and by reproducing Dürer's *Melancholia*. In the Preface he warns, "We cannot in good conscience continue to vacillate, tread softly, and settle for bland statements." And he is true to his word. Throughout the book he says enough to offend nearly everyone who has taught mathematics at any level of our educational system during this century. But that is the role of the gadfly, one that he has willingly chosen in order to get our attention. He obviously *cares* about the quality of mathematical research, education, and cultural influence, and has elected hyperbole as a way of expressing the opinions he has developed during his long and distinguished career. One risk that an author assumes by adopting such a style is that the attention of the intended audience will be distracted from, rather than focused upon, the argument of the text. When generalizations are too sweeping, the mind of the reader sometimes sees counterexamples more clearly than overriding principles. Another risk is that the reader will weary of scolding and never bother to persevere to the proposals that the author offers as a resolution of the dilemma. In my opinion those proposals deserve to be considered by the mathematical community, and therefore in this review I shall abbreviate Professor Kline's lengthy description of the ills of mathematics in order to come more quickly to his suggestions for reform.

"Universities have many obligations and roles in our society, but the education of liberal arts students is undeniably the fundamental one, and any equivocation about the primacy of this function is an abdication of responsibility." Primarily addressing problems of undergraduate education in mathematics but suggesting that his observations have relevance to other segments of education, Kline contends that university policies and practices seriously shortchange the undergraduate. "In practice mathematics education has been a debacle," and "the prime culprit is the overemphasis on research." Universities have made research and publication the principal criterion for faculty rewards—retention, advancement, and prestige. "Research is the mink coat of the academic world." Excellence in teaching is sometimes given lip service, but only rarely is it regarded as essential. Conflict between good research and good teaching is inherent. Both demand much thought, time, and energy. But research requires deep specialized knowledge that usually is irrelevant to undergraduate teaching, whereas good teaching requires breadth of knowledge and the ability to motivate students whose interests span the entire spectrum of human thought. Graduate preparation to do research contributes very little to the ability of a person to teach undergraduates effectively, and since it is research output that is most highly esteemed, the system forces the young instructor to devote his energies toward journal publication rather than toward planning courses and working with students. Clearly the quality of undergraduate education suffers as a consequence. Undergraduates are shortchanged also by other current practices in education: large lecture classes, inexperienced and untrained teaching assistants, inappropriately designed courses, and inferior textbooks.

"In contrast to the universities," Kline asserts, "the four-year colleges do a far better educational job." Classes are smaller, and the faculty is mature and experienced. But facilities and resources are more limited, teaching loads are higher, the faculty is untrained in pedagogy, and administrators "seek to emulate universities and demand research as a condition for employment and advancement." In two-year colleges teaching problems are even more severe

because of the extreme variation of preparation and aspiration among the students.

Undergraduate teaching is debased not only by an overemphasis on research but by the very nature of mathematical research. Unlike most other disciplines, mathematics uses the term "research" exclusively to mean creative work, the production of new results, not the writing of expository or critical essays, nor papers that synthesize ideas or trace their historical development, nor distinctive textbooks. Kline strongly endorses the value of the latter type of writing and uses the term "scholarship" to distinguish it from "research."

Throughout the book the author indicts the trend of research in recent decades toward more and more abstraction and pointless generalization, a trend that is largely responsible for the isolation of mathematics from its sources of inspiration in science and the real world. Such research clogs journals with articles of dubious worth, and threatens to fragment mathematics into slivers of specialties whose practitioners are unable to communicate with one another about their work. He contends that mathematicians of today are totally ignorant of science and unable to relate their research to any question of interest to society. Neither are they able to use significant applications to demonstrate the relevance of the mathematics that they teach.

Separate chapters are devoted to criticism of mathematics education in the elementary schools, in the high schools, and in the colleges, with particular attention to so-called liberal arts courses, which the author describes as "probably the most important task of the professional mathematician." The themes in these chapters are similar: mathematics is best learned from experience with concrete examples rather than from axiomatic structures; to most students current curricula present mathematics as "abstract, dull, boring, and intrinsically meaningless." Improper training has made teachers at all levels incapable of presenting mathematics as a vital, significant part of our culture and heritage.

Chapter 2 is a useful summary of the historical development of mathematics education and research in America, and Chapter 10 is appropriately subtitled "A Tirade on Texts."

The strategy that Kline proposes for reforming mathematics education is to attack the evils at the top, within the universities, which have the ability to achieve balance between research and teaching. Specifically:

1. Universities should encourage and materially reward teaching and scholarship equally with research. Faculty appointments should be made separately for each of these three functions, and evaluation of faculty performance should be made against criteria that are appropriate for the principal activity for which the appointment was made. Research professors, scholars, and teachers should all have training and experience related to their respective roles in the university. Such a division of labor need not imply a separation of mathematics faculty into compartments isolated from each other, but it would assign primary responsibility and accountability for the quality of each of these essential functions of the university.

2. Graduate education in mathematics should be reorganized to provide appropriate training for research, for scholarship, or for teaching. The Ph.D. degree should train persons to do research, and the "Doctor of Arts" degree should be used to prepare scholars and teachers. The latter program, in particular, should include broad training in the art of teaching, with special attention being given to the needs of prospective teachers in two-year colleges. Graduate mathematics departments should respond also to the needs of engineers, scientists, and social scientists for advanced study that does not presume a later career in mathematical research.

3. University administration should be reorganized to create separate divisions for undergraduate and graduate study. This implies separate budgets, separate responsibility for faculty appointment, evaluation, and advancement, and for curricula. It does not imply that teachers in the undergraduate department cannot participate in graduate education, nor that research professors cannot teach undergraduates. Each division should draw on the strengths of the other.

4. New research institutes, similar to the Institute for Advanced Study, should be established to provide more opportunity for research and scholarship, free from the need for teaching to support these activities. Also some universities, like Rockefeller University, should be devoted wholly to graduate study and research.

5. Leadership in advancing these reforms should be assumed by the professional societies that are concerned with the quality of mathematical research, scholarship, and education: the National Academy of Sciences, the American Mathematical Society, the Mathematical Association of America, and the National Council of Teachers of Mathematics.

Professor Kline indicated in his Preface that he chose not to be wishy-washy. His language is acid, cynical, and scathing. His indictments are universal, and he ascribes to mathematicians the meanest of motives as well as gross defects of character and personality. His blanket condemnation of undergraduate mathematics never mentions the work of CUPM and other MAA programs of the past two decades. In his derogation of high school mathematics he fails to mention the spectacular success of the U.S. Olympiad teams. He derides a fictional (?) young colleague for not understanding what an Advanced Placement grade of 4.5 signifies! In urging elementary teachers to draw material from realistic experience he gives an example from baseball that will appear artificial to any Little Leaguer who has tried to steal home from third base.

These are a few illustrations of the irritations that most readers can expect to find. They should not overshadow the fact that the problems that Professor Kline has laid before us are cogent and urgent. It is appropriate to end this review with the words he used to end the book.

> It is now time for mathematicians to shed their narcissism, broaden their vision and interests, limit research to worthwhile papers, and thereby release time and energy to the many-sided needs and tasks of education. They must cater to the needs of all students, undergraduate and graduate alike. Only recognition of the interdependence of research, scholarship, and teaching can advance mathematics itself, improve teaching, and further the multitudinous valuable uses of mathematics in our society.

DANIEL T. FINKBEINER, Kenyon College

Dr. Kline is distinguished both as a scholar and as a historian of mathematics. But he is also an angry man. So it is important to read his book with enough care to separate truth from conjecture, and to disinter both from a deliberately vituperative language which will not win him many allies where he needs them most—among professional mathematicians. The title alone is disastrous to its main purpose. (On January 3, 1978, long before I was invited to review the book, Dr. Kline informed me that the subtitle was his original choice; the title used was insisted upon by the publisher.) If the reader happens to be a mathematician who loves to teach, like me, he is on the defensive as soon as he opens the book and launches his counterattack before he has read it all.

1. *Kline's indictment*; *the main charge*. Our undergraduate and high-school teaching of mathematics is a shambles. (Kline suggests that this generalizes to other fields, though the details are left to the reader.) The principal cause is the selfishness of university administrators who need lots of money. Money is best obtained through prestige. Prestige, in turn, comes from distinction in research [football and basketball also help], which administrators know to be "in direct conflict with teaching." Co-defendants are the professors who prosper by committing the offense of research to the neglect of teaching. They achieve higher salaries, promotion, and recognition if they concentrate on publication rather than on the classroom.

2. *Defense against the main charge*. Let me begin by agreeing with Kline that our teaching is deplorable, at least in the universities and high schools. But I quarrel with his simplistic analysis of the causes (and therefore with the futile remedies he proposes; see §5).

I believe the problem to be a social one. The prosecutor overlooks the fact that good teaching requires willing students, or at least nonresistant ones. The current mass production of vocation-oriented graduates who want degrees and not learning cannot be put at the doorstep of provosts, deans, and researchers. This is especially true in our state universities, which rely largely on public funds and which teach most of the students. It is *this* demand that creates calculus classes of 500 students, which *no one* can teach effectively. Doubling or even tripling teaching loads cannot solve the problem, because time, space, and money are not available.

And it is this demand that means his achieving a degree in a field no longer guarantees that the graduate has a thorough basic knowledge of that field; it only guarantees that he doesn't know anything else.

The society in which learning is admired and pursued for its own sake has disappeared. Industry has become a sport, and sport an industry. What is expected now is technological "progress" so that we can live longer to enjoy improved TV and to bolster up our defenses against communism. By some quirk which only mathematicians understand—and they are not about to "squeal"—it is generally supposed that research mathematics contributes to this progress. In fact 99.44 percent of it is as pure as Ivory soap and as useful to technology as a Cyrillic New Testament is to Billy Graham. (This includes so-called "applied" mathematics. Statisticians and computer scientists have long since declared their independence.)

But mathematicians are as human as politicians, and so have profited in salaries, in esteem, in travel grants, and in clinics for the microscopic examination of sets of measure zero. Given his choice, what sane young man would not choose to pursue "research," however inept he may be, rather than to explain the meaning of dy/dx, which, unlike Leibniz, he doesn't really understand, to students who don't want to. Even Ph.D.'s in Education are less concerned with teaching teachers than in statistical "research" in educational psychology, whatever that is.

These things are consequences of our technological bias, which has even forced some of the old-fashioned "ologies" to disguise themselves as social "science." In this form they find recognition and access to public research funds. So to blame administrators and professors for the sad state of teaching is like blaming the police for the rise of crime. If Kline had accused us of profiteering from the public misunderstanding of what we do I'd go along, but he doesn't.

3. *The case against research.* Since Kline claims that research interferes with teaching, the quality of that research seems irrelevant. Possibly his charge against the professors could be reduced from the first to the second degree if he could be convinced that their research is worthwhile; he doesn't say. In any case we are told that most current research is meaningless; it is largely abstract and has little contact with the real world, which gives rise to *genuine* problems. For him Marshall Stone represents the enemy, who is quoted as follows:

> Nevertheless the fact is that mathematics can equally well be treated as a game which has to be played with meaningless pieces according to purely formal and essentially arbitrary rules, but which become intrinsically interesting because there is such a great fascination in discovering and exploiting the complex patterns of play permitted by the rules. Mathematicians increasingly tend to approach their subject in a spirit which reflects this point of view concerning it . . . I wish to emphasize especially that it has become necessary to teach mathematics in a new spirit consonant with the spirit which inspires and infuses the work of the modern mathematician, whether he be concerned with mathematics in and for itself, or with mathematics as an instrument for understanding the world in which we live . . . In fact, the construction of mathematical models for various fragments of the real world, which is the most essential business of the applied mathematician, is nothing but an exercise in axiomatics . . . When an acceptable modern curriculum has been shaped in terms of its mathematical content, one must still be concerned with the spirit which animates the subject and the manner in which it is taught. It is here that it is highly appropriate to demand that, even in the earliest stages, an effort should be made to bring out both the unity and abstractness of mathematics.

This he counters by a quotation from Courant:

> A serious threat to the very life of science is implied in the assertion that mathematics is nothing but a system of conclusions drawn from the definitions and postulates that must be consistent but otherwise created by the free will of the mathematician. If this description were accurate, mathematics would not attract any intelligent person. It would be a game with definitions, rules and syllogisms without motive or goal. The notion that the intellect can create meaningful postulational systems at its whim is a deceptive half-truth. Only under the discipline of responsibility to the organic whole, only guided by intrinsic necessity, can the free mind achieve results of scientific value.

A plague on both their houses. On the one hand, to dismiss a large part of mathematics as "nothing but an exercise in axiomatics" is silly. For example, the restatement of mechanics in the language of modern differential geometry and topology clarifies concepts such as stability,

which have classically been somewhat vague; but it still requires hard concrete analysis to establish the stability of even the simplest nontrivial system. As to Courant on the scientific value of mathematics, let me remind the reader that much of the brilliant work of Poincaré, Birkhoff, and Wintner was suggested by problems of celestial mechanics, but what astronomer cares? And neither view mentions the ultimate test of great mathematics: great theorems, irrespective of the source. Gauss could find them everywhere, in number theory as well as in the theory of magnetism. (I urge that all participants in this debate read J. T. Schwartz's *The Pernicious Influence of Mathematics on Science*, Proc. Int. Conf. for Logic, Methodology and Philosophy, Berkeley, 1960.)

Just as in art and literature the question of what mathematics is important can be settled only by history, and not by pontification. And until history speaks I agree with Hilton that referees will have to make the judgments.

4. *Further charges*. The current textbooks, written by professors to make money, are pretty much of a kind, the parts are interchangeable, and students generally find the writing unreadable. Many of the writers pay lip service to applications, but don't really know enough to do them justice. Agreed. In fact, I fail to see why old Granville or Osgood went out of print, or why students of the current calculus books need training in weight-lifting to find them usable. (A publisher's representative recently asked in what way my proposed calculus book is "different." He busily recorded my claim that it would be shaped like a cube and give the answers only to the Fibonacci numbered problems.)

The training of high-school students gives them no idea why mathematics is either interesting or important. Again, I agree. My own experience in the training of high-school teachers convinces me that they can't teach what they don't know. But whose fault it is is hard to track down; I'm tempted to blame professional "educators," but I can't prove my case.

Dr. Kline deplores the indisputable fact that in the universities promotion depends mostly on publication; good teaching is seldom taken into account in judging. This coincides with my own experience on promotions committees where generally good teaching = no complaints in the dean's office. On the other hand, it is also my experience, and that of most of my colleagues, that with very rare exceptions the good teachers and the exciting lecturers who communicate their enthusiasm to students are also devoted to research and scholarship. But my experience is confined to universities; I cannot speak for the four-year colleges.

5. *Remedies*. Let me begin with a passage (pp. 235–6) from Kline's final chapter:

> The defects in our educational system cannot be eliminated by one measure. There is no one cure for all diseases. Yet, little by little, medicine has conquered some and alleviated the gravity of others. In the educational field the universities' insistence on research as the qualification for appointment and tenure of professors (despite the low quality of much of the research and its irrelevance to teaching), large lecture classes, the use of teaching assistants on a wide scale, and inadequate textbooks are all highly detrimental to the progress of mathematics and to the effectiveness of education. Some helpful steps are apparent, and we have to be willing to take them.
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> The first remedy lies in recognizing scholarship as well as research. Research in mathematics means the creation of new results or, at least, new methods of proof. Scholarship—which fundamentally implies breadth, knowledge in depth, and a critical attitude toward that knowledge—is currently deprecated. This distinction is not made in the social sciences, the arts, and the humanities. The person who digs up facts about an older civilization, who writes a detailed and perhaps critical biography of some major or minor historical or literary figure, or who puts together various theories of economics or government is considered creative, though there may be no single new fact in a given work. Of course, there are seminal thinkers in the nonmathematical fields. Some of their work is as novel, as creative, as anything produced in mathematics. But the distinction between old and new cannot be made as readily. In any case, in these fields new work is only a small part of what is accepted and even honored as research. The re-search of what has been done is accorded as much distinction as new work. In fact, a critical biography or evaluation of a man or an era is often lauded far more than the man or men whose work is being assessed. A lucid explanation or interpretation of mathematical research is worth far more than most research papers. Unfortunately, such presentations, even if of high quality, are held in low esteem. But it is scholars—people with a deep and broad knowledge of mathematics and an ability to communicate, whether or not they contribute new results—who can correct many evils and perform many vital tasks.

No one will quarrel with the importance of scholarship, but I think Kline makes two errors. First, scholarship in mathematics exists and *is* admired. There has been an abundance of good books, expository articles, and monographs, for example by Pólya, by Titchmarsh, by E. Artin, by J. Moser, by the writers of several volumes of Rota's new *Encyclopedia*. But is it only a coincidence that all of these scholars are also leaders in research? Second, it is not clear that the recognition of scholarship in the humanities and in the social sciences has done much for *their* teaching. For example, what do most history teachers know about the history of science, the major influence on mankind since the middle ages?

To support his ideas about scholarship Kline proposes a D.A. (Doctor of Arts) to replace the present Ph.D. for nonresearch scholars. In view of the shortage of jobs for almost any kind of degree during the next few years, judgment ought to be postponed.

As to Kline's proposal that graduate and undergraduate become independent institutions I have nothing to add to Hilton's commentary which appears in the *Mathematical Intelligencer*, Vol. 1, no. 2 (1978), pp. 78–80.

There is more, but room permits only a final quotation (p. 270):

> Reforms are needed not only to improve mathematics education. Both the survival of mathematics in the curriculum and of research itself are at stake. The concentration on pure, esoteric studies will ultimately mean less support from society and, as Richard Courant once predicted, all significant mathematics will be created by physicists, engineers, social scientists, and schools of business administration.

I believe the first statement, but none of the rest. Although I am an "applied" mathematician the Prime Number Theorem represents to me the ultimate in mathematical achievement. Like art and poetry such theorems will be created whatever support society chooses to grant.

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Excerpts from this book were published in the *Mathematical Intelligencer*, Vol. 1, No. 1 (1978), 5–14. In Vol. 1, no. 2 (1978), 76–80, of the same journal I wrote a response to Kline's argument, at the invitation of the editors. In reviewing the book now it would be gratuitous for me to repeat what I said there. The question I asked myself, in reading the book in its entirety, was whether, in my response, I had been fair to the argument of the book, taking into account the fact that I had based myself merely on a small part of the text (though a part selected by Kline himself). The conclusion I have reached is that I had been entirely fair, and that the strictures which I made against the book are even more valid when one reads the entire text. Moreover there is a further serious demerit of the book which comes to light on a more comprehensive and detailed reading, and this is that it is full of internal contradiction. I will develop this point later.

On the positive side, it should be admitted, the book does have certain distinct merits. It contains a valuable mini-history of mathematics in the United States; it draws attention—if attention is still required—to the fact that too much original work is published, and to many unfortunate consequences of that plethora of new mathematics; it advocates relating mathematics education to science and to science education, an admirable prescription and one to which a recent conference in Bielefeld devoted itself; and it provides an excellent recipe for good teaching. I say all this not in order to ingratiate myself with those who are sympathetic to Kline's point of view, but in order to insist that it is such a great pity that these valuable services are embedded in a text which is, in its essence, a farrago of prejudice. Almost every page contains some vituperative attack on Kline's fellow mathematicians; a detailed list of the unfairnesses and injustices which he perpetrates on his colleagues would require a review even longer than the book itself. Let me therefore try to give something of the flavor of the book, giving quotations where necessary.

Kline's own attitude toward the role he is playing is well illustrated by the quotation from *Hamlet* which appears before the Preface.

Whether 'tis nobler in the mind to suffer
The slings and arrows of outrageous fortune;
Or to take arms against a sea of troubles;
And by opposing end them.

Kline sees himself as battling against the forces of evil in education. But it is not plain in what respect Kline himself may be suffering or may have suffered, nor in what respect his own fortune has been outrageous. What risk is Kline now running by making this attack? Hamlet's choice was of the stuff of tragedy; Kline's choice, to publish this tirade or not, was made in a far less spiritual context.

Kline's method is that of vast generalization and oversimplification. Indeed it is an irony that, whereas he attacks research mathematicians for what he perceives to be their myopic passion for generalization in mathematics, he indulges in such immoderate generalization himself about the *entire* community of research mathematics. Indeed, generalization does not even stop there; for the blurb on the cover of Kline's book invites the reader to generalize further, from mathematics to the entire academic community. Such further generalization is often self-evidently absurd, since the generalized statement would not be simply wrong but actually meaningless. How do we generalize to the whole of academic research the statement that all good mathematical research has taken place in response to the attempt to solve nonmathematical problems? Amusingly, Kline is most ambivalent on this generalization beyond mathematics. On page 41 he states, "In mathematics, research has a very special meaning." However the previous sentence to the one quoted carries an asterisk referring to a footnote which says, "Though we shall discuss mathematical research, many of its features, as earlier noted, apply to other academic disciplines as well." Let us leave to Kline the reconciliation of those two statements!

The flavor of the book is conveyed by Chapter 1 in which a mythical Peter Landers takes a position at "Admirable University" (having gained his Ph.D. from Prestidigious (*sic*) University) and endeavors to do a good job of teaching in the face of insuperable odds. The chapter makes lively reading, as one would expect in view of the author's fluent style. This style, employed, as in his earlier works, to spread knowledge, may be characterized as limpid and felicitous; but here Kline uses his artistry to paint a picture that is the purest travesty.

And as the book begins, so it continues. We are treated to immoderate attacks on mathematical research, on the quality of the teaching done by young Ph.D.'s, on the publication and refereeing policies of the mathematical research community, and on present-day mathematics itself. In order not to make this review unduly long let me concentrate here on Kline's attack on the policies of mathematics journals publishing research. We learn on page 62 that most good mathematicians do not serve as referees. We are also told: "The narrowness of mathematicians also renders them unfit to discriminate between what is fundamental and what is trivial, between basic insights and mere technical byplay . . . Personal factors also intervene. Individuals favor friends and discriminate against rivals." On page 83 we find a further rich vein of calumnies: "Evaluation of research boils down to quantity of publication; the contents are irrelevant. As long as his peers accept it, a researcher can publish almost anything—and his peers do accept it, because they wish the same treatment." Kline returns to the same theme on page 248, where he writes: "The criterion of publication in a respectable journal also ensures nothing, because almost anything can be published these days." There are many other statements in the book of a similar tone; but let me complete this lugubrious list of quotations by citing finally a very explicit and particularly scurrilous attack. Kline writes, on page 268: "Most editors of the American Mathematical Monthly . . . also reject articles critical of the existing conduct of education."

I believe that everybody reading this review will know that Kline's charges against the editorial policy of mathematics journals are absolutely ridiculous and unfair. The distressing fact is, however, that most readers of Kline's book will not be able so surely to dismiss Kline's canards; and indeed many may well find that his ill-tempered remarks strike a sympathetic chord within their own thinking. For Kline is feeding a very popular prejudice—the self-serving

and dishonest modus operandi of the scientist and the academic. Kline is betraying his profession to the general public.

For what impression is Kline giving of the community of research mathematicians? The picture he paints is one of total lack of standards; total philistinism; total ignorance; total unconcern for teaching (typical here is Kline's remark on page 146, ". . . all of teaching is a chore to be disposed of as quickly as possible"); and, to top it all, a narrow and venal self-interest. I was particularly offended, in this last matter, by Kline's remarks on page 225, where he writes, "Professors do learn remarkably fast—what the market wants." I feel bound to add that Morris Kline is in a peculiarly weak position to make this particular charge against his colleagues.

If we are to attempt to view objectively the validity of Kline's argument—and I readily confess that this is a very difficult undertaking—then we should be concerned with its internal consistency. I therefore regard it as highly relevant to point out that the argument contains many explicit contradictions. I have referred already to Kline's ambivalence, to call it nothing worse, as to the special nature of mathematical research. Let me give a few further examples. On page 152 we read: "However, the lack of clear standards of teaching, the professor's ignorance of pedagogy, and the obligation . . . to cover ground prescribed in syllabi produce the same effect as incompetence and dishonesty." Yet, on page 153, we read, "Nevertheless, myopic professors impose their own interests on the students, with the result that their courses are largely useless to most students and to society."

On page 166 we are told (in my view, utterly erroneously), "Even the fact that the sum of the angles of a triangle is 180° is hardly attractive." This is immediately followed by "Moreover, beauty is a matter of taste . . . "! Again, on page 200 Kline quotes approvingly from A. N. Whitehead: "It is a profoundly erroneous truism . . . that we should cultivate the habit of thinking of what we are doing. The precise opposite is the case." Yet, on page 217, Kline is capable of writing: "But the symbolism is invented by human beings to express their thoughts. The symbols cannot transcend the thoughts."

A final example must suffice. In advocating the role of the scholar within the academic community (I dealt at length with this question in my article in the *Mathematical Intelligencer*), Kline remarks (page 237), "Scholars can elucidate the inscrutable results contained in research papers." Yet on the very next page he can write, "Without scholarship the currently vast number of proliferating disciplines steadily gain in quantity as they lose in quality, vision, and effective use of the little in them that is worthwhile." Kline seems able to dispense himself with the services of the scholar in forming his own judgments! Indeed he expresses himself even more strongly on page 251, where he writes, "The research has sunk to fruitless specialization and our classrooms on all levels are staffed with poorly trained or mistrained teachers." Evidently Kline does not find modern mathematical research inscrutable!

There is also much special pleading in Kline's argument. Again we must allow a few examples to illustrate the point of our charge.

As a first example consider the statement on page 84: "This recourse to peers is hardly likely to produce a fair judgment. We have already noted (Chapter 3) the defects of the refereeing process, and all of these apply as well to the judgment of published research by peers." The attack on the refereeing process in Chapter 3 begins, as we have remarked earlier, on page 62. It consists first of the statement, already quoted, that most good mathematicians do not serve as referees. It continues with a comment on blind refereeing, which is plainly irrelevant to peer evaluation. There follows a complaint about the number of papers published and not refereed. There is then an anecdote about Lord Rayleigh and the comment, "On the whole the refereeing of earlier times was competent and critical." This Kline explains by the fact that in those days the editors took a pride in the quality of the work published in their journals—with a very plain implication about the standards of contemporary editors. Thus I find nothing in Chapter 3 that refers in any way whatsoever to the case for or against the efficacy of peer evaluation.

As a second example of special pleading consider the content of page 201. Kline, arguing for the maintenance of certain topics in the high school curriculum, observes that students in college

"will be asked to perform calculations such as $\frac{1}{a}+\frac{1}{b}$, $a^5 \cdot a^3$, $(a+b)^2$." Devoting his argument to the first of these calculations, and concerned to justify the continued prominence of the addition of fractions in the high school curriculum, Kline argues that the addition $\frac{1}{a}+\frac{1}{b}$ is carried out in precisely the same manner as $\frac{1}{2}+\frac{1}{3}$. He concludes, "Hence, one must learn the skills of arithmetic to do algebra." Here Kline is advocating the presence in the pre-college curriculum of a topic whose only justification will come at a later stage of the mathematical education of the student, a stage which many students will not, in fact, reach. This runs counter to the very sensible principles he has himself enunciated for justifying the presence of some given material in the curriculum. Further, we do not add $\frac{1}{a}+\frac{1}{b}$ as for example we add $\frac{1}{2}+\frac{1}{2}$—surely about the most important example of the addition of fractions! Thus the connection between the adding of fractions and the adding of rational functions is by no means as simple as Kline makes out. Moreover, had he taken the example $(a+b)^2$, then the argument would have gone entirely the other way. For it is surely common ground that if called upon to compute $(6+4)^2$ we would be ill-advised to first expand the expression to $36+48+16$. The entire argument here appears specious but it is presented to the lay reader as being incontrovertible and obvious.

As a final example of false reasoning consider the extraordinary statement on page 126, "But number and geometric description are insignificant properties of real objects." The mind boggles at such an assertion. Offer one child one apple and another child two apples and see which feature of the difference between the two offerings is most readily apprehended by the child. If a footballer were asked to describe a football field would he be expected to omit the fact that it is rectangular? Kline bolsters his absurd statement by asking the question, "The rectangle may indeed be the shape of a piece of land or the frame of a painting, but who would accept the rectangle for the land or the painting?" Kline appears not to understand the nature of abstraction and thus not to be aware that the question is total nonsense.

It is surprising, in a great expositor and a great teacher, to find evidence of strange attitudes to teaching itself. Let us leave on one side Kline's naive faith in the efficacy of formal instruction in pedagogy; this at least is a point of view which many others adopt and a legitimate topic for argument. However I find myself appalled by the extreme vocationalism that Kline appears to be recommending in the design of the curriculum. On page 71 Kline, arguing that the typical researcher's knowledge is useless to 95 percent of the students who take mathematics, concludes that that knowledge should not have a place in what the student is taught. A little later he repeats that training students in the professor's speciality "is useless to almost all of his students."

I must simply set on record that I do not believe that one should determine the details of a curriculum on the basis of the techniques which one believes a student is likely to be using in the first few years of his subsequent employment. Our function is to educate; and we are at our best as educators when we are vitally interested in the subject matter of what we are teaching. This is, of course, not to say that syllabi should be *determined* by the tastes of the instructors. It is to say, however, that applications and illustrative problems and examples may be expected to arouse the enthusiasm of students if presented by enthusiastic (and, of course, skillful) teachers. Kline appears to reject this proposition. Indeed, he indulges in a peculiarly unappetizing sarcasm in stigmatizing the presentation of the Königsberg bridge problem, and Euler's solution, in a liberal arts course, asserting (page 123), "But mathematicians will not let the dead rest in peace, and they revive the problem as though it were the most momentous one facing our civilization." Of course, it is not revived in those absurd colors; it is revived as a beautiful, though elementary, example of combinatorial reasoning.

But even here there may be scope for argument. However, Kline appears to be bordering on the irresponsible when he advocates (page 189) that "positional notation in bases other than ten . . . should be taught to prospective teachers," though not to elementary school students, on the grounds that "a teacher must know more than he teaches." Where are Kline's principles now if he recommends something for the prospective teacher simply on the grounds that it will be something that his students will never have to know? On page 201 (a page from which we have

already quoted liberally!) in this same chapter on elementary education, one finds Kline arguing against the proposition that the prospect of the availability of hand calculators should influence the choice of curriculum. He says that "estimation as to the reasonableness of an answer calls for knowing a good many arithmetical skills." I might venture the criticism that one does not "know" a skill; my principal point here, however, is that the arithmetical skills required in estimation are not the ones which figure traditionally in the curriculum.

As Kline continues his prescription for elementary education—and let me repeat that he says many very valuable things here—he appears again to go off the rails in a remarkable sequence on page 203. A salesman selling a customer three pairs of shoes at twenty dollars a pair asks the customer for sixty dollars. Kline continues, "But the customer instead replies that three pairs of shoes at twenty dollars a pair is not sixty dollars, but sixty pairs of shoes, and he asks the salesman for the sixty pairs. Is the customer right? As right as the salesman." Once again the mind boggles. The salesman said correctly that the three pairs of shoes would *cost* sixty dollars. He never said that three pairs at twenty dollars a pair *is* sixty dollars. But what is all this extraordinary confusion in aid of? We find to our astonishment that Kline is describing a simple situation which "can be used to make the point . . . that numbers are abstractions"!

In this last excerpt from Kline's fantasy world we seem to come close to one of his real difficulties. He is not happy with the concept of abstraction, and this affects his attitude toward mathematics. We were already troubled on page 133, where Kline wrote: "But the creation of non-euclidean geometry shattered centuries of confidence in man's intellectual potential. Mathematics was revealed to be not a body of truths but a man-made approximate account of natural phenonema . . ." Surely we all know that what was shattered was man's belief in the categorical nature of euclidean geometry, not his confidence in his intellectual potential. Moreover it is precisely science and not mathematics which is an approximate account of natural phenonema. Mathematics certainly is a body of truths in the sense of being a body of deductive truths.

Kline's uncertainty in the presence of abstraction is nowhere more clearly revealed than on page 174, where he writes, "Negative numbers are not just inverses to positive integers under addition; they are the number of degrees below zero on a thermometer." This is plain nonsense. Negative numbers may be used to measure temperature, but they are not necessary for this purpose. Moreover, the number of degrees below zero in a temperature of minus sixteen degrees is sixteen degrees! When I first read Kline's remarkable statement I mentally added "and fractions are not parts of pies." To my astonishment I found on page 191 that Kline actually asserts that they are!

As I argued in the *Mathematical Intelligencer*, and as I have hinted more than once in what I have already written, Kline's thinking is constrained and distorted by his view of the relationship of mathematics to science; and I would like to devote the remainder of this review to an attempt to elucidate that view and to indicate its erroneous nature.

To Kline there are only "alleged" examples of mathematics being developed independently of science and proving subsequently useful to science. Kline claims that all that is good and healthy in mathematics has been directly inspired by science. In asserting and exploiting this proposition he has in mind the totality of mathematical knowledge. However, it is striking that, in endeavoring to persuade the reader of the correctness of his proposition, he writes, "Practically all of the major branches of mathematics were developed to solve scientific problems. . . ." Leaving aside the uncharacteristic caution that Kline exhibits in using the word "practically," what is striking here is that Kline is talking of *branches* of mathematics. Now it is not in dispute that major branches of mathematics have arisen in response to the desire to solve scientific problems. What is in dispute is whether all the significant results in any given branch have themselves been developed to solve scientific problems and whether their ultimate validity depends on their having been so developed. I would have thought that the situation was already sufficiently clear, but it may be valuable to point to two recent and very different examples to show that mathematics may well be developed purely for its intrinsic interest and power, and may then prove to be of great value outside mathematics.

The first example is simply stated. A recent paper in this MONTHLY (84 (1977) 82–107) is entitled "Error-Correcting Codes and Invariant Theory: New Applications of a Nineteenth-Century Technique." The author, Dr. N.J.A. Sloane of Bell Telephone Laboratories, received the Lester Ford Award at the recent Providence meeting for this paper. Kline characterizes invariant theory (page 56) as a nineteenth-century fad. It here proves invaluable to solve a real twentieth-century problem.

A second example is taken from the field that Kline seems to hold most dear, physics. Today's theoretical physicists are writing of gauge fields. It turns out that the mathematics is that of fiber bundle theory, a part of algebraic topology which has been developed very extensively in recent years as a purely mathematical discipline. A gauge type is nothing but a principal fiber bundle. A gauge potential is a connection on that bundle. Dirac's monopole quantization is the first Chern class of a bundle for the unitary group U(1). C. N. Yang has written (1977): "That non-abelian gauge fields are conceptually identical to ideas in the beautiful theory of fiber bundles, developed by mathematicians without reference to the physical world, was a great marvel to me. In 1975 I mentioned this to Chern, and said, 'This is both thrilling and puzzling, since you mathematicians dreamed up these concepts out of nowhere.' He immediately protested. 'No, no, these concepts were not dreamed up. They were natural and real.'" (Cf. p. 348, this issue.)

Chern is here taking a view of the influence of "reality" on mathematics far more subtle and sophisticated than that which Kline advocates. In our perceptions of geometric relationships it is natural for us to formulate the notion of a fiber bundle. This proves to be of enormous importance in many parts of mathematics. It is therefore no surprise that it is now proving important in physics.

Unfortunately for Kline, it seems that topology is a branch of mathematics to which he is peculiarly averse. His view of the nature of topology is well illustrated by what he writes on page 42: "One move was to enter the newer fields, such as the branch of geometry now called topology. The advantage of a new field for tyros in research is that very little background is needed and the best concepts and methodologies are only dimly perceived. Hence, because criteria for value are lacking, almost any contribution has potential significance. Publication is almost assured." Moreover, topology is listed on page 151 as one of the mathematical disciplines "devoid of applications." What Kline notices is that advances in topology are rarely achieved in response to physical problems; what his simplified view of the relation of mathematics to science has caused him, quite fallaciously, to infer is that topology must therefore be sterile and narrow, if not actually trivial.

Is it really necessary to continue to insist that mathematics is not the same sort of pursuit as science? I owe to Professor Rheinboldt the pertinent remark that applied mathematics is not just a part of mathematics but is mathematics plus something else, the art of applying it to "real" situations. I would have expected a celebrated practitioner of this art like Kline to understand this. Yet we find him making the analogy (page 175) between the problem for the mathematics student of applying mathematics, and the problem for the French youth of translating his thoughts into English. I can find no way of making this analogy valid.

We should share with Kline the view that mathematicians should be far more sensitive to the relationship of mathematics to science than they have been in recent years. This does not mean, however, that they should devote themselves exclusively, either in their research or in their teaching, to problems coming exclusively from outside mathematics. Kline's philistinism is surely quite as sterile as the vacuous generalization which he fairly castigates (while unfairly and unjustifiably regarding it as characteristic of the whole of modern mathematics).

To sum up we must conclude that Kline's book is an intemperate emotional outburst, and that the author has not only missed an opportunity to devote his great talents to the improvement of the quality of mathematical education but has also placed in the hands of the enemies of education in general and mathematics education in particular a potent if unreliable weapon.

PETER HILTON, Case Western Reserve University and Battelle Institute

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S, L. *Overcoming Math Anxiety*. Sheila Tobias. Norton, 1978, 278 pp, $10.95. [ISBN: 0-393-06439-5] A well-written commentary on the causes and consequences of avoiding mathematics, summarizing in a popular style evidence for the importance of mathematics, for the effects of socialization on female underachievement in mathematics, and on the psychology of learning mathematics. Extensive end note references help the reader assess Tobias's evidence for the "math anxiety" fad. The latter half, of uncertain value, rambles through word problems, fractions, and calculus (!) in an attempt to help motivated readers reduce their math anxiety. LAS

GENERAL, S(13). *Project-A-Puzzle, Second Edition*. Richard D. Porter. NCTM, 1978, ii + 62 pp, $4 (P). [ISBN: 0-87353-132-9] Puzzles (single page transparency masters), with follow-up activities, emphasizing logical and perceptional skills. LCL

GENERAL, L. *Using the Mathematical Literature, A Practical Guide*. Barbara Kirsch Schaefer. Dekker, 1979, ix + 141 pp, 45 sFr. [ISBN: 0-8247-6675-X] A fairly general survey of types of mathematical publications (journals, indexes, abstracts, books, bibliographies) with examples (rather than comprehensive lists) of each type. Except for some of the historical notes, little in this volume will be terribly informative to the experienced mathematician, although it would provide valuable information for a beginning graduate student. LAS

GENERAL, P, L. *Index of Mathematical Papers, Volume 9: An Annual Index to Mathematical Reviews, Volumes 53 and 54 (1977)*. AMS, 1979 [ISBN: 0-8218-4012-6]. *Part 1: Author/Key Index*, iii + 661 pp; *V. 2: Subject Index*, 528 pp, $120 set (P).

GENERAL, S*(13-18), L*. *The Ambidextrous Universe: Mirror Asymmetry and Time-Reversed Worlds, Second Revised, Updated Edition*. Martin Gardner. Scribner's, 1979, 293 pp, $9.95. [ISBN: 0-684-15789-6] This new edition of a classic book adds five chapters on time reversal to complement and refresh its deservedly popular discussion of symmetry in art and science. Its *raison d'être* is to explain in nontecnnical terms the significance for physics and cosmology of the fall of parity and of time-invariance symmetry. "Brims over with that sense of pleasure that the love of science can induce."--Jeremy Bernstein. LAS

GENERAL, S, P, L*. *On Aesthetics in Science*. Ed: Judith Wechsler. MIT Pr, 1978, 180 pp, $12.50. [ISBN: 0-262-23088-7] A series of essays developed from a course on aesthetics in science taught at MIT by art historian Wechsler, focusing on the process of modelling rather than on the artifacts of science. Includes essays by Cyril Smith, Philip Morrison ("On Broken Symmetries"), Arthur Miller, Seymour Papert ("The Mathematical Unconscious"), Howard Gruber and Geoffrey Vickers. LAS

GENERAL, S(15-17), L*. *The Encyclopaedia of Ignorance: Everything You Ever Wanted to Know About the Unknown*. Ed: Ronald Duncan, Miranda Weston-Smith. Pergamon Pr, 1978, x + 443 pp, $15 (P). [ISBN: 0-08-022426-1] Dozens of brief nontechnical essays by distinguished scientists stating "what it is they would most like to know." (Say the editors: "The more eminent they are, the more ready to run to us with their ignorance.") Mathematics influences many of the essays, especially those in physics (curvature of space, arrow of time, twistors and complex numbers), but is featured in only three: on computational complexity (by H.J. Bremermann), on applications in social science (by C.W. Kilmister), and on number theory (by H. Halberstam). LAS

BASIC, T(13: 1). *A Survey of Basic Mathematics, Fourth Edition*. Fred W. Sparks, Charles Sparks Rees. McGraw, 1979, xii + 543 pp, $13.95. [ISBN: 0-07-059902-5] Suitable for students with little or no high school mathematics. Contains arithmetic review and elements of a first course in algebra, as well as a numerical treatment of logarithms and trigonometry. Many exercises, review and test for each chapter and a test for the entire book. MW

BASIC, T(13: 1), L. *College Algebra and Trigonometry*. Dennis G. Zill, Jacqueline M. Dewar, Warren S. Wright. Wadsworth, 1979, xii + 492 pp, $14.95. [ISBN: 0-534-00612-4] Traditional topics and treatment of college algebra and trigonometry (beginning with angles and including right triangles) together with some discussion of matrices and systems of equations. Includes exercises, answers, chapter tests, index. JS

BASIC, T(13: 1). *Technical Mathematics*. Philip M. Jaffe, Rodolfo Maglio. Scott F, 1979, 563 pp, $13.95. [ISBN: 0-673-15111-5] Straight-forward text covering topics in arithmetic, algebra, geometry, and trigonometry which are immediately applicable to technical and shop work. Most concepts presented as rules with little explanation. Generous supply of word problems in general, technical and vocational categories provide applications in many fields. MW

PRECALCULUS, S(13), L. *Problem Book: Algebra and Elementary Functions*. A. Kutepov, A. Rubanov. Trans: Leonid Levant. MIR (US Rep: Imported Pub, 320 W. Ohio St., Chicago, IL 60610), 1978, 439 pp, $5.30. Translation of second Russian edition. No text, 1968 exercises, some oral, most routine, few challenging, arranged in order of increasing complexity. Excellent coverage of topics every calculus

teacher wishes his students would know, plus chapters on vectors, limits and derivatives. Answers provided where appropriate. Recommended investment at the price. JK

PRECALCULUS, T(13: 1). *College Algebra and Trigonometry as Socrates Might Have Taught Them.* Robert D. Hackworth, Joseph W. Howland. H & H Pub (1117 Webb Drive, Clearwater, FL 33515), 1978, 349 pp, $11.25 (P). Contains standard precalculus topics but crowded format makes reading uncomfortable. "As Socrates might have taught" is merely an elegant way of describing a semi-programmed approach where a student is urged to conceal answers and imagine a dialogue with a master teacher. MW

HISTORY, S*, P, L**. *Sofya Kovalevskaya: A Russian Childhood.* Sofya Kovalevskaya. Trans: Beatrice Stillman. Springer-Verlag, 1978, xiii + 250 pp, $14.80. [ISBN: 0-387-90348-8; 3-540-90348-8] A new translation of *From Russian Life: The Rajevski Sisters,* Sofya Kovalevskaya vivid portrait of her childhood in mid-nineteenth century Russia. A celebrated belletrist as well as the first pre-eminent "mathematical lady," Kovalevskaya recounts her earliest memories in extraordinary detail, including her and her older sister Anyuta's dalliance with Fyodor Dostoevsky. The volume includes an extensive biographical introduction by Stillman, editor's footnotes explicating names and events, Kovalevskaya own *Autobiographical Sketch,* and a brief article by Academician P.Y. Polubarinova-Kochina discussing the significance of Kovalevskaya mathematical work. LAS

HISTORY, L. *Norbert Wiener: Collected Work With Commentaries, V. I.* Ed: P. Masani. MIT Pr, 1976, xi + 761 pp, $35. [ISBN: 0-262-23070-4] First of four volumes, containing papers growing out of Wiener's early work: philosophy, potential theory, Brownian motion and ergodic theory. Each paper (or group of related papers) is followed by a brief commentary by a contemporary scholar, placing Wiener's ideas in the context of modern research. A comprehensive classified Wiener bibliography enables the reader to locate papers easily. LAS

FOUNDATIONS, P, L. *Remarks on the Foundations of Mathematics, Revised Edition.* Ludwig Wittgenstein. Trans: G.E.M. Anscombe. MIT Pr, 1978, 444 pp, $27.50. [ISBN: 0-262-23080-1] Rearranged and augmented selection from Wittgenstein's mathematical notebooks and manuscripts of the years 1937-1944. Includes previously unpublished material on mathematical proofs as "concept forming" and on the notion of "following a rule." Helpful index and descriptive table of contents. GHM

COMBINATORICS, T(14-18: 1), L. *Basic Techniques of Combinatorial Theory.* Daniel I.A. Cohen. Wiley, 1978, x + 297 pp, $17.95. [ISBN: 0-471-03535-1] Low key, very pleasurable introduction to elementary counting principles, including (generalized) binomial coefficients, generating functions, pigeon-hole principle, inclusion-exclusion, Pólya's theorem and much more. Attempts to "keep all notation transparent" and eschews formal proofs by induction (preferring the intuitive "and so on"). Over four hundred tantalizing exercises. Students ought to find it very absorbing. GHM

COMBINATORICS, T*(15-17: 1), S, P, L**. *Computers and Intractability: A Guide to the Theory of NP-Completeness.* Michael R. Garey, David S. Johnson. Freeman, 1979, x + 338 pp, $10 (P); $18.50. [ISBN: 0-7167-1045-5; 0-7167-1044-7] NP-complete problems are the hardest problems that permit verification of possible solutions by polynomial-time algorithms. Whether these problems can be solved in polynomial time (i.e., whether a solution can be found as well as verified) is one of the most important unsolved problems in mathematical science. Garey and Johnson's long-awaited monograph introduces examples and theory of NP-complete problems, and concludes with an extensive catalogue (100 pp.) of problems known to be NP-complete and a comprehensive bibliography cross-referenced to the text. An indispensible reference for anyone who cares about algorithms, as well as an attractive text. LAS

NUMBER THEORY, S, P. *Die Ersten 50 Millionen Primzahlen.* D. Zagier. Birkhäuser, 1977, 24 pp, sFr. 12.80 (P). [ISBN: 3-7643-0967-9] A concise exposition of the distribution and asymptotic properties of prime numbers, with tables, graphs and extensive footnotes. LAS

LINEAR ALGEBRA, T(14-15: 1), L. *Elementary Linear Algebra with Applications.* Francis G. Florey. P-H, 1979, xiii + 363 pp, $14.95. [ISBN: 0-13-258251-1] Elementary development of linear algebra, aimed at the sophomore level. Theory is standard, going as far as (but not proving) diagonalization of a real symmetric matrix and the Hamilton-Cayley theorem. Includes applications to economics, differential equations, quadratic forms, Fourier series; exercises; answers; index. Printed in blue. JS

LINEAR ALGEBRA, T(14: 1), S, L. *Applied Linear Algebra.* R.J. Goult. Ellis Horwood (Distr: Wiley), 1978, 196 pp, $25. [ISBN: 0-85312-076-5] Unpretentious and highly readable. Attractive non-encyclopedic package, less than 200 pages, in six chapters alternating between theory and applications. Fascinating mix of topics, unusual in books at this level. Problems at chapter ends mostly computational, with solutions and hints for almost every problem. Price is high. JK

LINEAR ALGEBRA, T(14: 1). *Introduction to Linear Algebra: Theory and Applications.* Peter V. O'Neil. Wadsworth, 1979, vi + 250 pp, $16.95. [ISBN: 0-534-00606-X] Standard semester course with no applications of the type one is entitled to expect from the subtitle. (Is one section on matrix multiplication and random walks enough?) Rarely goes outside linear algebra to motivate concepts. Distinguished by its treatment of real vector spaces and linear transformations before matrices, with claimed pedagogical advantages. The tone is generally honest, with an occasional nod to numerical methods. GHM

CALCULUS, T(13-15), S, L*. *Infinitesimal Calculus.* James M. Henle, Eugene M. Kleinberg. MIT Pr, 1979, ix + 135 pp, $12.50. [ISBN: 0-262-08097-4] A concise, elementary exposition of basic calculus theory based on Robinson's theory of infinitesimals, presented in an attractive two column counterpoint pitting historical notes, refreshing quotations, mathematical asides and exercises against the inexorable flow of basic theory. Contains none of the normal applications, so it is not (as Keisler's "nonstandard" *Elementary Calculus* is) a suitable text for ordinary freshman calculus; it is, however, an attractive option for a seminar, or as a supplement for an honors course. LAS

DIFFERENTIAL EQUATIONS, T(16: 1), S, L. *Ordinary Differential Equations.* V.I. Arnold. Trans: Richard A. Silverman. MIT Pr, 1978, ix + 280 pp, $8.95 (P). [ISBN: 0-262-51018-9] This textbook treats the subject of ordinary differential equations in a new way. The notions of vector field, phase space, phase flow and one parameter groups of transformations dominate the presentation. Many excellent drawings add to this geometrical approach. The exercises are thought provoking, but a few are routine. CEC

NUMERICAL ANALYSIS, P. *Recent Advances in Numerical Analysis.* Ed: Carl de Boor, Gene H. Golub. Acad Pr, 1978, ix + 270 pp, $12. [ISBN: 0-12-208360-1] 12 papers from a symposium held in Madison during May, 1978. RWN

NUMERICAL ANALYSIS, P. *Lineare Räume und Approximation.* Ed: P.L. Butzer, B. Szökefalvi-Nagy. Birkhäuser, 1978, 685 pp, sFr. 96. [ISBN: 3-7643-0979-2] Contains 47 of 48 lectures presented at the August 20-27, 1977 conference at Oberwolfach together with five papers submitted later and a section devoted to new and unsolved problems. JAS

NUMERICAL ANALYSIS, T(14-17: 1, 2), S, L. *A First Course in Numerical Analysis, Second Edition.* Anthony Ralston, Philip Rabinowitz. McGraw, 1978, xix + 556 pp, $21. [ISBN: 0-07-051158-6] Changes in this *Second Edition* reflect progress and developments in numerical analysis that are mainly due to increased use of digital computers in this area, especially use of recently developed mathematical software packages. Mathematically rigorous, yet oriented toward problem solving on a computer. Gives consideration to error calculation, computational efficiency, and comparisions of alternate algorithms for solving a problem. New edition includes spline interpolation, adaptive integration, the fast Fourier transform, the simplex method, simple and double QR algorithms. Chapter problems. Chapter bibliographies and bibliographic notes sections. Index. There is a separately published "Hints and Answers" to problems. RJA

OPTIMIZATION, P. *Nonlinear Programming 3.* Ed: Olvi L. Mangasarian, Robert R. Meyer, Stephen M. Robinson. Acad Pr, 1978, ix + 475 pp, $29.50. [ISBN: 0-12-468660-3] Seventeen of the papers from the third nonlinear programming symposium held at Madison on July 11-13, 1977. JAS

ANALYSIS, P. *Lecture Notes in Mathematics-694: Séminaire Pierre Lelong-Henri Skoda (Analyse) Année 1976/77.* Ed: Pierre Lelong, Henri Skoda. Springer-Verlag, 1978, 334 pp, $17.60 (P). [ISBN: 0-387-09101-7; 3-540-09101-7]. 14 *exposés* on complex analysis in finite and infinite dimensions. LAS

ALGEBRAIC GEOMETRY, T(18: 2), S, P, L. *Principles of Algebraic Geometry.* Phillip Griffiths, Joseph Harris. Wiley, 1978, xii + 813 pp, $42. [ISBN: 0-471-32792-1] A major expository presentation of much of the core of modern algebraic geometry. Background in sheaf and manifold theory with results from topology are presented, albeit quickly, to make this a self-contained monograph. For example, sheaves are motivated on one page, defined on the next, and their cohomology theory developed three pages later. A very solid introduction for advanced students and professional mathematicians, with good references and index but no exercises. The jacket claim that it will provide physicists as well as mathematicians with greater accessibility to the field seems a bit idealistic. JAS

GEOMETRY, T(16: 1), S, P, L. *Linear Geometry in Euclidean 4-space.* Yung-Chow Wong. SEAMS (Southeast Asian Mathematician Society, c/o Dept. of Math., Nanyang U., Singapore), 1977, ix + 217 pp, $10 (P). A concisely-written monograph containing an algebraic treatment of geometry of 2-planes in 4-dimensional real Euclidean space; presumes a good understanding of basic linear algebra. JNC

TOPOLOGY, P. *Algebraic and Geometric Topology.* Ed: R. James Milgram. Proc. of Symp. in Pure Math., V. XXXII. AMS, 1978. *Part 1,* vii + 412 pp [ISBN: 0-8218-1432-X]; *Part 2,* vi + 322 pp, $46.40 set. [ISBN: 0-8218-1433-8] These Proceedings of the 24th American Mathematical Society Summer Research Institute, which was held at Stanford University on August 2-21, 1976, contain write-ups of most of the main lectures as well as selected seminar talks and most of the problem sessions. JAS

TOPOLOGY, T*(18: 2), P. *Elements of Homotopy Theory.* George W. Whitehead. Grad. Texts in Math., V. 61. Springer-Verlag, 1978, xxi + 744 pp, $32. [ISBN: 0-387-90336-4; 3-540-90336-4] A presentation threading a course between the Scylla of categorical generalities and the Charybdis of state-of-the-art geometry. The reader who is acquainted with the fundamental group and (singular) homology theory will be introduced to the "elementary portion" of homotopy theory--i.e., no higher order cohomology operations, no localization and such. The approach concerns itself with the fundamental problems of extension, homotopy, and classification, with sufficiently elementary machinery to make the book appropriate for students or professionals who want to know "what's it (homotopy) good for." Each chapter ends with a set of exercises. JAS

PROBABILITY, P. *Transactions of the Twenty-Fourth Conference of Army Mathematicians.* US Army Research, Durham, NC, 1978, xv + 496 pp, (P). Invited and contributed papers, mostly on applications of stochastic processes, from a June 1978 conference at Charlottesville, Va. LAS

STATISTICS, T(13: 1). *A First Course in Statistics.* Marshall Gordon, Norman Schaumberger. Macmillan, 1978, ix + 214 pp, $7.95 (P). [ISBN: 0-02-406680-X] A compact introduction with surprisingly complete coverage: descriptive statistics; probability; binomial, normal, χ^2-distributions; nonparametrics. LCL

STATISTICS, S(16-18), P. *Minimizing the Sum of Absolute Deviations.* Vincent Sposito, Wendell Smith, Gary McCormick. Vandenhoeck & Ruprecht, 1978, 60 pp, DM 22 (P). [ISBN: 3-525-11243-2] A unified approach to regression under L_1, minimizing the sum of absolute deviations. Concepts, theorems and proofs, computational techniques concerning these estimators. LCL

STATISTICS, T(13: 1). *Developing Skills in Statistics.* Neal R. Townsend, Grayson H. Wheatley. Allyn, 1978, viii + 281 pp, $13.95. [ISBN: 0-205-05994-5] Usual topics of descriptive and inferential statistics with minimal mathematical demands. Attractive format, good exercise sets. LCL

COMPUTER PROGRAMMING, S(13-18). *The C Programming Language.* Brian W. Kernighan, Dennis M. Ritchie. P-H, 1978, x + 228 pp, $10.95 (P). [ISBN: 0-13-110163-3] C is a general-purpose programming language developed at Bell Laboratories. The Unix operating system and its software are written in C. The present text contains a tutorial introduction to C, chapters on each major feature of C, and a reference manual. Organized around the development of real programming examples. Index. RJA

COMPUTER PROGRAMMING, T(13-18: 1), S. *Structured Programming and Problem-Solving with Pascal.* Richard B. Kieburtz. P-H, 1978, xv + 365 pp, $10.95 (P). [ISBN: 0-13-854869-2] Contains the syntax of Pascal, generalities on problem solving using a programming language, special emphasis throughout on the use of structuring techniques in designing programmed solutions to problems. Includes applications in a variety of areas. Of particular interest is the discussion of computational techniques used in programmed creation of pictures, translation of languages, medical diagnoses, proofs of mathematical theorems. Chapter exercises. Appendix. Indices. RJA

COMPUTER PROGRAMMING, T(13: 1), S, L. *Programming for Poets: A Gentle Introduction Using FORTRAN with WATFIV.* Richard Conway, James Archer. Winthrop Pub, 1978, xiv + 332 pp, $10.95 (P). [ISBN: 0-87626-722-3] A nontechnical, nonmathematical approach. Explains programming by supervising the reading of selected programs (e.g., a word-replacement translator of natural language, coin tossing experiments, poetry composition, conversational program). Discussions of the nature and limits of programming. LCL

COMPUTER PROGRAMMING, S(13). *Some Common Basic Programs, Second Edition.* Lon Poole, Mary Borchers. Adam Osborne, 1978, xii + 193 pp, $8.50 (P). [ISBN: 0-931988-06-3] Not a book on how to program, but a collection of elementary programs on loans and investments, calculus, linear algebra, statistics and taxes. LLK

COMPUTER SCIENCE, T(17: 1), S, P, L. *Syntactic Pattern Recognition: An Introduction.* Rafael C. Gonzalez, Michael G. Thomason. Appl. Math. and Comp., No. 14. A-W, 1978, xix + 283 pp, $17.50 (P); $29.50. [ISBN: 0-201-02931-6] An introduction, with emphasis on practical aspects, to the concepts and techniques of applying formal language and automata theory to modeling structural relations in pattern classes. An interdisciplinary subject, touching on engineering, computer science, information science, physics, chemistry, biology, medicine, and applied mathematics. LCL

COMPUTER SCIENCE, S, L. *Home Computers: 2^{10} Questions & Answers.* Rich Didday. Dilithium Pr, 1977. *Volume 1: Hardware,* xi + 225 pp, $7.95 (P) [ISBN: 0-918398-00-2]; *Volume 2: Software,* xii + 175 pp, $6.95 (P). [ISBN: 0-918398-01-0] Dialog presentation of nuts and bolts of microprocessors with a good index. *Volume 1* starts with an introduction to jargon, then binary arithmetic and logic and a quick introduction to kits and microprocessors. *Volume 2* goes through beginning machine language (for the Imsai 8080) to an introduction to Basic followed by some "where do I go from here" generalities. A very informal presentation that would be useful for the (mathematician) parent whose teenage offspring has a chip on the shoulder, or for anybody who has got the bug himself. JAS

COMPUTER SCIENCE, P, L. *Advances in Computers, Volume 17.* Ed: Marshall C. Yovits. Acad Pr, 1978, xiv + 333 pp, $31. [ISBN: 0-12-012117-4] Five papers on natural language, networks, bubble memory, and right-of-access to government information. A well-edited volume, including a single name and subject index and a list of contents of all previous volumes in the series. LAS

COMPUTER SCIENCE, S(16-18), P, L. *Current Trends in Programming Methodology, V. IV: Data Structuring.* Ed: Raymond T. Yeh. P-H, 1978, xii + 321 pp, $19.95. [ISBN: 0-13-195735-X] The ten articles in this volume can be grouped into four areas: (1) a methodology for structured design and description of data; (2) a formal basis for specification, design, verification, and implementation of abstract data types; (3) programming language design; (4) formal models of data structures. Extensive bibliography. RJA

COMPUTER SCIENCE, T(15-18: 1, 2), S, L. *The Structure of Computers and Computations, V. 1.* David J. Kuck. Wiley, 1978, xxi + 611 pp, $23.95. [ISBN: 0-471-02716-2] Begins with an overview of computer systems and then presents the theoretical background needed for the remainder of the text. Chapters on processors, control units, main memories, interconnection networks, and memory hierarchies. The entire text attempts to illuminate the interplay between the structure of a computer and the structure of those computations that the computer must perform. Chapter problems. References. Index. RJA

APPLICATIONS S(11-13). *Applications of Secondary School Mathematics.* Bernice Kastner. NCTM, 1978, vi + 106 pp, $7 (P). [ISBN: 0-87353-127-2] An attractive compendium of brief, independent, classroom-sized enrichment units suggesting answers to "What's it good for?": perception, growth, wave motion, music, economics, molecules, relativity, all treated with at most a little bit of calculus. Each section includes references for further reading. LAS

APPLICATIONS, P. *Multichannel Time Series Analysis with Digital Computer Programs, Revised Edition.* Enders A. Robinson. Holden-Day, 1978, xxvii + 298 pp, $9.95 (P). [ISBN: 0-8162-7254-1] Paperback reprint of 1967 original edition: Fortran programs for wave form manipulations (deconvolutions, filtering, spectral analysis), with full explanations of their theory and use. Opens with a nonmathematical introduction, and concludes with an appendix on relations to operator theory. LAS

APPLICATIONS (CRYSTALLOGRAPHY), P, L. *Crystallographic Groups of Four-Dimensional Space.* Harold Brown, *et al.* Wiley, 1978, xiv + 443 pp, $38.50. [ISBN: 0-471-03095-3] This book describes the two-, three-, and (for the first time) four-dimensional crystallographic groups, primarily by the use of tables. LCL

APPLICATIONS (ECONOMICS), T(17-18: 1), S. *Introduction to Mathematical Economics.* Murray C. Kemp, Yoshio Kimura. Springer-Verlag, 1978, 249 pp, $19.80. [ISBN: 0-387-90304-6; 3-540-90304-6] Rigorous presentation of mathematical topics used in advanced economic analysis. Many topics not

covered in other places. "Introduction" may be misleading, since it would apply to upper-level graduate students. Readable, concise. Good bibliography, name index, subject index. WC

APPLICATIONS (MECHANICS), P. *Modern Problems in Elastic Wave Propagation.* Ed: Julius Miklowitz, Jan D. Achenbach. Wiley, 1978, x + 561 pp, $32.50. [ISBN: 0-471-04696-5] This book contains the text of the 25 main papers presented at the International Union of Theoretical and Applied Mechanics Symposium on the title topic; also included are abstracts of a number of shorter papers which were presented at this conference held at Northwestern University, September 12-15, 1977. JAS

APPLICATIONS, (PHYSICS), S**, P, L*, *Einstein's Universe.* Nigel Calder. Viking Pr, 1979, 154 pp, $10. [ISBN: 0-670-29076-9] A brilliantly written popular exposition of Einstein's theories in the context of modern cosmology, reinforced by homely examples (e.g., an ambitious engineer squeezing the earth to a black hole in order to extract every drop of hydroelectric energy from a waterfall) and vivid expression ("This incest among the gravitons produces the curvature of space."). Calder, a superb science writer, uses Einstein's universe as a vehicle for a comprehensive tour of contemporary physics, providing numerous distinctive views of the "great divide" between relativity and quantum theory. The book accompanies the recently broadcast BBC-PBS television program of the same name, but, apart from a few illustrations in common, is totally independent of it. LAS

APPLICATIONS (PHYSICS), P. *Solitons and Condensed Matter Physics.* Ed: A.R. Bishop, T. Schneider. Springer-Verlag, 1978, xi + 341 pp, $29.50. [ISBN: 0-387-09138-6; 3-540-09138-6] Proceedings of the symposium held at Oxford, England, June 27-29, 1978 wherein mathematicians and physicists got together to talk about the development and speculate about the future of non-linear physics. JAS

APPLICATIONS (PHYSICS), P. *Mathematical Foundations of Quantum Scattering Theory for Multiparticle Systems.* I.M. Sigal. Memoirs No. 209. AMS, 1978, iii + 147 pp, $8.40 (P). [ISBN: 0-8218-2209-8] "A proof of the completeness of the scattering eigenfunctions for systems of an arbitrary but fixed number of particles." JAS

APPLICATIONS (PHYSICS), S*(16-18), P, L. *Differential Forms in Mathematical Physics.* C. von Westenholz. Stud. in Math. and its Appl., V. 3. North-Holland, 1978, xv + 487 pp, $61.25. [ISBN: 0-7204-0537-8] A relatively straightforward presentation of the basics of modern differential geometry very neatly interspersed with physics applications in which the author explains the gains in intuition resulting from the geometer's tools. Of special interest to advanced undergraduate and beginning graduate physics students with mathematical interests, it is nonetheless hard to recommend it due to a price of noticeably more than 10 cents a page. JAS

APPLICATIONS (PHYSICS), T(18: 1, 2), S. *Analysis, Manifolds and Physics.* Yvonne Choquet-Bruhat, Cécile Dewitt-Morette, Margaret Dillard-Bleick. North-Holland, 1977, xvii + 544 pp, $19.50. [ISBN: 0-7204-0494-0] A substantial self-contained exposition of the basics of algebra, topology, analysis, including distributions, and various tools of the differential geometer and topologist. Extensive indices of both terminology and symbols and substantial problem sets should help the reader benefit from the rather formidable mass of material presented. But beware, the presentation may be too abstract for physicists and the *raison d'être* too physical for mathematicians. Needed: a sympathetic and expert instructor. JAS

APPLICATIONS (SOCIAL SCIENCE), S(17), P, L? *Graph Theory and Its Applications to Problems of Society.* Fred S. Roberts. CBMS Reg. Conf. in Appl. Math., No. 29. SIAM, 1978, v + 122 pp, $11.50 (P). This monograph is based on a series of ten lectures delivered at the CBMS-NSF Regional Conference on Graph Theory held at Colby College in 1977. The number and variety of applications is striking. An excellent bibliography is included. CEC

APPLICATIONS (SOCIAL SCIENCE), T(15-17: 1), S. *Mathematical Models in the Social Sciences.* John G. Kemeny, J. Laurie Snell. MIT Pr, 1978, vii + 145 pp, $5.95 (P). [ISBN: 0-262-61030-2] Paperback reprint of 1972 MIT Press hardcover edition (TR, February 1973) of the 1962 Ginn original. Remains a valuable source of interesting applications of sophisticated mathematical techniques. LAS

APPLICATIONS (SOFTWARE ENGINEERING), S(15-18), P. *Structured Analysis and Design.* Infotech, 1978. *V. 1: Analysis and Bibliography,* vi + 273 pp; *V. 2: Invited Papers,* ii + 373 pp, $260 set. [ISBN: 8553-9430-7] *Volume 1* begins with basic issues in the area of software design and then proceeds to design tools, requirements analysis, specification, and finally to a discussion of current design methodologies. Extensive annotated bibliography; references for the volume, contributor and subject indexes. *Volume 2* contains the papers and a subject reference. RJA

APPLICATIONS (VECTOR ANALYSIS), S(13). *Elementary Vectors, Third Edition (SI Units).* E.O.E. Wolstenholme. Pergamon Pr, 1978, vi + 121 pp, $3.50 (P). [ISBN: 0-08-021654-4] A short introduction to vectors. Includes scalar and vector products, differentiation and integration of vectors, applications involving motion of a particle, equations of lines, planes, curves and curved surfaces. *Second Edition,* TR, August-September 1971. LLK

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; William Carlson, St. Olaf; Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; Lorraine L. Keller, St. Olaf; Joseph Konhauser, Macalester; Loren C. Larson, St. Olaf; George H. Mills, St. Olaf; R.W. Nau, Carleton; John Schue, Macalester; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Martha Wallace, St. Olaf.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036

PERSONAL ITEMS

University of Akron: Dr. Judith A. Palagallo, formerly at Hartwick College, Oneonta, New York, and Dr. Harold L. Putt, formerly at Bowling Green State University, have been appointed Assistant Professors. Assistant Professors William W. Hokman and Phillip H. Schmidt have been promoted to Associate Professors.

University of California, Berkeley: Professor Shoshichi Kobayashi has accepted the position of Chairperson of the Department of Mathematics for 1978-79. Professor Murray H. Protter has accepted the position of Director of the Center for Pure and Applied Mathematics for 1978-79. Professor Hale Trotter, Princeton University, was a Visiting Professor for the Winter Quarter, 1979. Dr. Sy Friedman, M.I.T., and Dr. Bruce Reznick, Stanford University, are NSF Postdoctoral Fellows for 1978-1979. Professor Abraham H. Taub has retired; a Symposium on Relativity was held in his honor in August, 1978. Professor Elwyn R. Berlekamp has been elected to the National Academy of Engineering.

Bethel College, St. Paul, Minnesota: Dr. James T. Holmes, Westmont College, has been appointed Associate Professor. Mr. Eric J. Gossett, University of Arizona, has been appointed Instructor.

California State College, San Bernardino: Professor John Hafstrom will retire in June, 1979. Professor René Dennemeyer retired in February, 1979.

Colorado School of Mathematics, Golden: Drs. Robert S. Fisk, Pacific Lutheran University, Robert Underwood, University of South Carolina, Thomas E. Kelley, Princeton University, and Dave Schmitz, Iowa State University, have been appointed Assistant Professors.

Southern Methodist University: Instructor Carolyn Shull has been promoted to Assistant Professor. Associate Professor Richard K. Williams has been promoted to Professor.

Michigan Technological University, Houghton: Dr. Beverly Gimmestad, Metropolitan State (Denver), has been appointed Assistant Professor. Assistant Professor William P. Francis has been promoted to Associate Professor. Associate Professor Arthur Boggs has retired with the title of Associate Professor Emeritus.

University of Mississippi: Drs. Bruce Glastad, Florida State University, William A. Staton, University of Houston, and Edward C. Waymire, University of Arizona, have been appointed Assistant Professors.

Wellesley College: Associate Professor Howard Wilcox has been promoted to Professor. Assistant Professor Frederic Shultz has been promoted to Associate Professor.

Southern Connecticut State College: Dr. Ruth Silverman, New Jersey Institute of Technology, has been appointed Associate Professor. Associate Professors Helen G. Bass and Kerry E. Grant have been promoted to Professors.

Associate Professor Marvin C. Papenfuss, Loras College, Dubuque, Iowa, has accepted the position of Chairman of the Mathematics Department.

Associate Professor Gary L. Sherman, Rose-Hulman Institute of Technology, Terre Haute, Indiana, has been promoted to Professor.

Associate Professor Herbert Koller, University of Bridgeport (Connecticut) is spending the academic year 1978-79 on a leave of absence at the Hong Kong Polytechnic Institute.

Professor Robert H. Sorgenfrey, UCLA, has retired with the title of Professor Emeritus.

Professor Donald Greenspan, formerly at the University of Wisconsin, at Madison, has been appointed Professor at the University of Texas at Arlington.

Professor Yudell L. Luke, University of Missouri-Kansas City, has been elected Curator's Professor.

Dr. Peter Zito Daffer, University of South Carolina, has been appointed Assistant Professor at Louisiana Tech University, Ruston.

Instructor Sr. M. Kathleen Scannell, Gwynedd-Mercy College, Gwynedd Valley, Pennsylvania, has been promoted to Assistant Professor.

Professor Theodore Laetsch, University of Arizona, has been named Head of the Department of Mathematics.

Professor Walter W. Leighton, University of Missouri-Columbia, has retired with the title of Distinguished Professor Emeritus.

Dr. Dale W. Lick, formerly Dean of the School of Sciences and Health Professions, Old Dominion University, Norfolk, Virginia, became President of Georgia Southern College, Statesboro, Georgia, on July 1, 1978.

Associate Professor Dana Sudborough, Southern Oregon State College, has retired with the title of Associate Professor Emeritus.

Associate Professor Robert A. Shive, Jr., Millsaps College, is on a one-year appointment as an intern in the American Council on Education Fellows Program in Academic Administration.

Assistant Professor Keith Harrow, Brooklyn College (CUNY), has been promoted to Associate Professor.

Assistant Professor F. P. Glick, University of British Columbia, has been promoted to Associate Professor.

Dr. Ronald Shepler has been appointed Assistant Professor at Ferris State College, Big Rapids, Michigan.

Dr. Maryam Hastings, formerly Instructor at St. Peter's College, has been appointed Assistant Professor at William Paterson College of New Jersey.

Associate Professor Ronald E. Loser, Adams State College, Alamosa, Colorado, has been promoted to Professor.

Assistant Professor William Fisher, California State University, Chico, has been promoted to Associate Professor.

Professor Michel L. Balinski, CUNY New York, has been appointed Professor at Yale University.

Dr. Robert Wagner, Vanderbilt University has been appointed Associate Professor at Duke University.

Dr. Werner Rheinboldt, University of Maryland, has been appointed Andrew Mellon Professor at the University of Pittsburgh.

Assistant Professor George Ball, Alfred University, Alfred, New York, has been promoted to Associate Professor.

Associate Professor Paul Froeschl, Saint Mary's College, Winona, Minnesota, has been appointed Division Coordinator, Division of Mathematical and Natural Sciences.

Assistant Professor Cornelius P. (Neil) Aboff, Rose-Hulman Institute of Technology, Terre Haute, Indiana, died in July, 1978.

Association Professor Lawrence A. Dalton, Georgian Court College, Point Pleasant, New Jersey, died on June 10, 1978, at the age of 46. He was a member of the Association for ten years.

Professor Karel de Leeuw, Stanford University, died on August 18, 1978, at the age of 58.

Dr. Lewis Parker Siceloff, Monroe, North Carolina, died on December 7, 1978. He was a member of the Association for sixty-three years!

Professor Williston C. Hobbs, Prince George's Community College, died in 1978.

Associate Professor Francis A. Varrichio, St. Peter's College, River Edge, New Jersey, died in 1978 at the age of 56. He was a member of the Association for twenty three years.

Professor Zeev Nehari, Carnegie Mellon University, died on September 1, 1978. He was a member of the Association for twenty seven years.

Byron Cosby, Professional Lecturer, Washington, D.C., died on April 30, 1978.

Professor W.H. Gage, University of British Columbia, died on October 3, 1978.

WORKSHOP ON MATHEMATICS APPLICATIONS IN MEDICINE AND BIOLOGY

The Northeast Section of the M.A.A. will sponsor a summer workshop at the University of Maine, Orono 04469, June 18-22, 1979, concerning *Applications of Mathematics in Medicine and Biology.* Topics to be covered include discrete population dynamics, harvesting problems in discrete population models, deterministic models for communicable diseases, Monte Carlo models for common source epidemics. The principal lecturer will be Maynard Thompson, Professor of Mathematics at Indiana University, and co-author (with Daniel P. Maki) of *Mathematical Models and Applications.* Two lectures will be given each day. Other lectures on special topics will be arranged. The cost of the workshop, including room and board, is $110 for M.A.A. members, $120 for non-members.

ICME 80 FIRST ANNOUNCEMENT

The first announcement concerning the 1980 International Congress on Mathematical Education is scheduled to be distributed early this spring. (The Congress will take place at the University of California, Berkeley, August 10-16, 1980.) If you wish to receive a copy of the first announcement, please make your request to: Office of Mathematical Sciences, National Research Council, 2101 Constitution Avenue, NW, Washington, D.C. 20418.

COMPUTATIONAL PROBABILITY LECTURE SERIES, BALTIMORE, MARYLAND

The Department of Mathematical Sciences of the Johns Hopkins University and the Johns Hopkins University Press are sponsoring a week of lectures on computational probability, to be held 16-20, July, 1979. Principal speaker will be Professor M.F. Neuts of the University of Delaware; there will be opportunity for contributed talks as well. For information, contact Alan F. Karr, Department of Mathematical Sciences, The Johns Hopkins University, Baltimore, MD 21218.

OPERATIONS RESEARCH: MATHEMATICS AND MODELS

The American Mathematical Society, in conjunction with its eighty-third summer meeting in Duluth, Minnesota, will continue its short course series with a course entitled "Operations Research: Mathematics and Models." The one and one-half day course will be held on Sunday and Monday, August 19 and 20, 1979, in Bohannan Hall, room 90, on the University of Minnesota, Duluth campus.

The program is under the direction of Saul I. Gass, Chairman of the Faculty in Management Science and Statistics, College of Business and Management, University of Maryland, and Ralph Disney, Department of Industrial Engineering and Operations Research, Virginia Polytechnic Institute and State

University. It will consist of six seventy-five minute lectures. Each lecture will be self-contained and be devoted to a different area of application; health care delivery systems (William P. Pierskalla, University of Pennsylvania); fire department allocation and deployment (Warren E. Walker, The Rand Corporation); queueing networks (Ralph L. Disney, VPI & SU); fishery management (Frederick C. Johnson, National Bureau of Standards); military (Seth Bonder, Vector Research and University of Michigan); and agriculture (Robert B. Rovinsky, U.S. Department of Agriculture).

Time schedules and further information about registration and accommodations will be published in the April, June and August 1979 issues of the *NOTICES*, or may be obtained by writing or calling the American Mathematical Society's Meeting Arrangements Department, P.O. Box 6887, Providence, RI 02940; Telephone (401) 272-9500, Ext. 239.

UNDERGRADUATE MATHEMATICS APPLICATIONS PROJECT

The Modules and Monographs in Undergraduate Mathematics and Its Applications Project (UMAP) has been operating under a grant from the National Science Foundation to Education Development Center, Inc., of Newton, Massachusetts, since 1976.

One goal of UMAP is to develop, through a community of users and developers, 300 instructional modules and 40 monographs for use in mathematics classes and classes where mathematics is applied. To date the Project has some 95 modules in print and another 120 manuscripts in various states of development. These modules are written for undergraduate students and enable them to learn professional applications of mathematics in such fields as biomedical sciences, economics, American politics, harvesting, international relation, numerical methods, computer science, earth science, and navigation. All modules are peer and student reviewed, field-tested and revised before being made generally available.

Another goal of UMAP is to organize a Consortium of users and developers within the mathematics community to carry on the work of the Project after the funding period. Toward this goal the Project has already enlisted the active participation of more than 750 authors, peer-reviewers, student review advisors, field-testers and module users throughout the country.

The UMAP Catalog and further information about the Project may be obtained by writing to EDC/UMAP, 55 Chapel Street, Newton, MA 02160.

NEW JOURNAL FROM ADELPHI UNIVERSITY

The mathematics department at Adelphi University intends to launch a biannual journal devoted to reviews of non-print media in mathematics. The journal, to appear April 1980, is to be called *Adelphi University Reviews; Mathematics Media*. We would be glad to hear from anyone willing to review for us. The journal will also consider articles bearing upon the use or production of non-print media for use in undergraduate mathematics.

Please address all correspondence to: *Adelphi University Reviews; Mathematics Media*
Department of Mathematics
Adelphi University
Garden City, New York 11530

CONFERENCE NUMERICAL ANALYSIS OF SEMICONDUCTOR DEVICES

This conference will be held at Trinity College, University of Dublin, Ireland, on June 27-29, 1979, under the auspices of the Numerical Analysis Group of Trinity College. Topics include: *Finite Difference and Finite Element Techniques for Semiconductor Devices, Numerical Simulation of Bipolar and Field Effect Transitors, Numerical Simulation of Power Semiconductor Devices, e.g. Thyristors, Numerical Simulation of Impact and Transfer Electron Devices, Numerical Simulation of Energy Conversion Devices, e.g. Solar Cells,* and *Numerical Methods used in Integrated Circuit Design.* There will be invited speakers from France, Germany, Japan, The Netherlands, Northern Ireland, and the U.S.A. For further information contact: Secretary, NASECODE I, 39 Trinity College, Dublin 2, Ireland.

DYNAMICS AND MODELLING OF REACTIVE SYSTEMS

The Mathematics Research Center at the University of Wisconsin-Madison will hold an Advanced Seminar on Dynamics and Modelling of Reactive Systems, October 22-24, 1979. The program will consist of approximately fourteen invited lectures on recent advances in modelling and mathematical analysis of chemically reactive systems, including applications to chemical, biological and atmospheric processes. The lecturers will include N.R. Amundson, R. Aris, D.G. Aronson, G.F. Carrier, M. Feinberg, E.D. Gilles, P.S. Gough, L.N. Howard, J.B. Keller, D. Luss, J. Rinzel, R.A. Schmitz, J.H. Seinfeld and F.A. Williams. The program committee consists of W.E. Stewart (Chairman), J.R. Bowen, C. Conley and W.H. Ray. A detailed program will be available in August. Further information may be obtained from Mrs. Gladys Moran, Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

MEETING OF THE LOUISIANA-MISSISSIPPI SECTION

Professor Jackie B. Garner, Section Chairman, called the Fifty-Sixth Annual Business Section to order on January 27, 1979 at noon; about seventy persons attended the meeting. Since the meeting was held concurrently with the joint AMS-MAA meeting, no technical papers were presented separately. The minutes of the 1978 Section Meeting were read and approved. Professor J.R. Foote gave the Treasurer's Report.

Professor Thomas A. Atchison, Section Governor, reported on certain items which were discussed at the meeting of the Board of Governors. The revised Section By Laws have been approved; the Board recommends against national collection of Section dues. The Committee on Placement Examinations, chaired by Section member R.D. Anderson, is working on tests for about 200 subscribers nationally, and produces a newsletter for subscribers. There is an annual fee of $75; further information may be obtained from A.D. Wilcox at MAA Headquarters. The purchase of a building to house MAA in Washington has been completed; extra space has been leased to other groups at a rate which assists in making mortgage payments, cutting previous rental costs being paid by MAA to about one half.

Professor Ben Mitchell reported on conduct of and the financial state of giving the Annual High School Mathematics Examination in Louisiana. Professor David Cook reported for the State of Mississippi.

The site of meeting committee recommended acceptance of the bid made by Mississippi State University for the February 1981 meeting; this bid was approved. Thanks were given to the Mississippi University for Women for its bid, with the hope that the offer would be made again for the 1983 meeting in Mississippi. The 1980 meeting is scheduled for February 15-16, 1980, at Louisiana Tech University.

Professor Garner announced and reviewed various items of interest to the Section. A Section Newsletter was initiated over a year ago by Robert A. Shive, Jr., and Joe R. Foote with Professor Foote serving as Editor. Three issues have appeared. The idea was approved at the 1978 Business Session. Professor Foote has been appointed to the subcommittee on Public Information, of the Committee on Exchange of Information on Mathematics. Professor Foote has asked the Executive Committee to replace him as Secretary-Treasurer and Newsletter Editor. It was announced that Professor John L. Tilley of Mississippi State University will assume these duties for the period 1979-1981. Professor Garner commended Professor Foote for his service to the Section over the past four-five years, and those present expressed their approval with applause.

Professor Garner asked that members respond to a call for Papers to be issued later this year, and to send all newsworthy items to Professor Tilley.

J. R. FOOTE, *Secretary*

FEBRUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The Annual Meeting of the Northern California Section was held on February 24, 1979, jointly with the Northern California Section of SIAM, at Sonoma State University, Rohnert Park.

At the business meeting, with William Chinn, Section Chairman, presiding, Stephen T. Tschantz and Daniel Knierim were honored for their performance on the 1977 Putnam Competition. Professor Bacon of Stanford delivered a eulogy for Professor Karel de Leeuw who was killed in August, 1978.

The membership elected the following section officer: Vice Chairman—Sister Madeleine Rose Ashton, College of Holy Names. Succeeding William Chinn as Chairman is Carroll Wilde and succeeding Jane Day as Program Chairman is William Chinn.

The following invited papers were presented:

The Unreasonable Effectiveness of Mathematics, Richard Hamming, Naval Postgraduate School
Properties of the Symmetric Group, Derrick Lehmer, U. C. Berkeley
Isoperimetric Inequalities, Pure and Applied, Robert Osserman, Stanford University
From Tiling to Algebra and Combinatorics, Sherman Steen, U. C. Davis

A luncheon was held at the college dining hall and featured a talk by George Polya on *Some Mathematicians I Have Known.*

WE NEED YOUR HELP...

The Mathematical Association of America's newly formed Committee on Improving Remediation in the Colleges is seeking information from universities and colleges about effective efforts in the area of mathematics remediation. The Committee especially solicits information about programs that have been shown to prepare students deficient in mathematics for successful performance in the mathematics courses required for degree programs. The Committee also welcomes information about experimental programs that show promise even though documentation of their success is not yet available. Course content, organization, and emphasis as well as descriptions of curricular materials used are of particular interest. In addition, the Committee wishes to identify programs that have been developed to correct directly the causes of inadequate student preparation in mathematics. Persons who can inform the Committee of such programs are asked to communicate with Professor Joan Leitzel, Department of Mathematics, The Ohio State University, 231 West Eighteenth Avenue, Columbus, Ohio 43210.

MATHEMATICAL ASSOCIATION OF AMERICA

THE SIXTY-SECOND ANNUAL MEETING OF THE ASSOCIATION

The Sixty-Second Annual Meeting of the Association was held at the Convention Center in Biloxi, Mississippi, from Friday to Sunday, January 26-28, 1979, in conjunction with meetings of the American Mathematical Society, the Association for Women in Mathematics, the Mathematicians Action Group, and the National Council of Teachers of Mathematics.

Sessions of the MAA were held on Friday morning at 9:00 A. M., Saturday at 9:00 A. M., and on Sunday at 9:00 A. M. and 1:30 P. M. At 10:30 A. M. on Friday and on Sunday at 10:00 A. M., the program was divided into parallel sessions, one in the Convention Center Coliseum and the other in Rooms 4-8 of the Convention Center. Presiding at the lecture by Professor John O. Riedl Jr. was Professor David E. Cook; at the lecture by Professor Raymond Smullyan, Professor Virginia Carlton; at the lecture by Professor Spanier, Professor Wanda W. Helm; at the lecture by Professors Fey and Kilpatrick, Professor James E. Keisler; at the lecture by Professor Fred S. Roberts, Professor K. B. Reid; at the lecture by Professor Diaconis, Professor Eleanor B. Walters; at the lecture by Professor Andrew M. Gleason, Professor Wallace C. Pye; at the lecture by Professor Buck, Professor Billy R. Sneed; at the film presentation by Professor Banchoff, Professor David I. Schneider.

The Program Committee consisted of Brooks Reid, Chairman; Richard D. Anderson, Virginia Carlton, David E. Cook, James E. Keisler, Wallace C. Pye, Billy R. Sneed, Eleanor B. Walters.

FIRST SESSION OF THE ASSOCIATION

Panel Discussion: ''Innovations in the Teaching of Calculus.''

A panel discussion featuring persons affiliated with the Open University Program: Dr. M. Crampin, Academic Course Team Chairman, Mathematics Foundation Course, Ms. Jean Nunn, Producer BBC/OUP, Professor Louis C. Leithold, Author and Consultant, Professor R. M. Pengelly, Dean of Mathematics Faculty, Open University, and John Richmond, Senior Producer Mathematics, BBC/OUP.

An innovatory teaching package on the Calculus forms part of the new Foundation Course in Mathematics offered by the British Open University. Print, television, radio, audio-tapes, calculator activities, assignments and tutorials are integrated into a high quality package taking the student from first ideas on differentiation to an understanding of Taylor Series.

The presentation explained the education partnership between the Open University and the BBC and a team-based approach to the production of teaching materials. The importance of this is emphasized by the fact that this presentation was a team effort.

Panel Discussion: ''The Advanced Mathematics Test of the Graduate Records Examination.''

A panel discussion with Professor Richard D. Anderson, Louisiana State University, Professor Israel N. Herstein, University of Chicago, Professor Gloria C. Hewitt, University of Montana, and Mr. J. R. Jefferson Wadkins, Educational Testing Service.

As an introduction, four queries were discussed: (1) What is the GRE Advanced Mathematics Test? (2) What kinds of questions appear on the Test? (What topics are covered? What is the level of difficulty?) (3) What are the purposes and uses of the Test? (4) How can a score by student S on edition E be compared with a score by student T on edition F where $E \neq F$ and where even $E \cap F = \phi$? A description is then given of the test construction process with emphasis on the role played by the Committee of Examiners, a group representing the U. S. graduate mathematics community.

''Calculus with Programmable Calculators'' by Professor John O. Riedl, Jr., Ohio State University.

SECOND SESSION OF THE ASSOCIATION

Panel Discussion: ''National Science Foundation Programs in Mathematics Education and Research.''

A panel discussion with Dr. William H. Pell, Head, Mathematical Sciences Section, NSF, Dr. Alphonse Buccino, NSF Directorate for Science Education, Dr. Thomas A. Keenan, NSF Program Director for Software Systems Science, and Dr. Alvin I. Thaler, NSF Program Director for Algebra and Number Theory.

Business Meeting of the Association: Announcement of the recipients of the Award for Distinguished Service to Mathematics and the Chauvenet Prize for 1979.

Panel Discussion: ''Prospects in Mathematics Education in the 1980's: A Report on a Conference.''

A panel discussion with Professor Gloria F. Gilmer, Milwaukee Area Technical College, Professor Donald L. Kreider, Dartmouth College, and Professor James W. Vick, University of Texas at Austin.

THIRD SESSION OF THE ASSOCIATION
(Joint with the National Council of Teachers of Mathematics)

''Godelian Puzzles'' by Professor Raymond M. Smullyan, Herbert H. Lehman College and the Graduate Center, CUNY.

Godel's Island consists entirely of ''knights'' who always tell the truth and ''knaves'' who always lie. Certain knights are called ''established'' knights. Each inhabitant has a unique club named after him. A is called a ''friend'' of B if and only if A claims that B belongs to B's club. Every member of Godel's Club has a friend who is not an established knight, and every nonmember has a friend who is an established knight.

Theorem: At least one knight is unestablished.

This, which combines several problems in ''What Is The Name Of This Book?,'' is closely related to Godel's Incompleteness Theorem.

''The Claremont Program in Applied Mathematics'' by Professor Jerome Spanier, Claremont Graduate School.

A description was given of a relatively new, career-oriented MA program in the Mathematical Sciences, risen out of the ashes of a small, traditional Ph. D. program in 1972. With considerable emphasis on courses which stress mathematical modeling and on the unique Mathematics Clinic in which real-world problems are attacked, this program has altered the character of graduate training in Claremont and has had considerable influence on the undergraduate curricula as well. The Clinic was described in detail and examples of projects undertaken in the Clinic were presented.

''The NSF Status Surveys on School Mathematics'' by Professor James T. Fey, University of Maryland, and Professor Jeremy Kilpatrick, University of Georgia.

The National Science Foundation recently completed three different studies of current curricula and instructional practices in school mathematics, science, and social science--grades K-12. The findings related to mathematics have been synthesized and summarized by a committee of the National Council of Teachers of Mathematics and the results were reported at the 1979 meetings of NCTM and MAA.

''Intersection Graphs, Food Webs, Ecological Phase Space, and the Boxicity Conjecture'' by Professor Fred S. Roberts, Rutgers University.

The intersection graph of a family of sets has the sets in the family as vertices and an edge between two sets if they intersect. Intersection graphs of families of real intervals and of boxes in n-space are studied. Applications of intersection graphs from genetics, archaeology, transportation science, etc. were mentioned. This talk investigated the smallest number of ecological dimensions needed to account for competition; equivalently, the smallest n such that a graph whose edges represent competition is the intersection graph of boxes in n-space. The construction of competition graphs from acyclic digraphs representing food webs was discussed, leading to a serious ecological/mathematical conjecture.

Panel Discussion: ''Basic Skills: Placement and Pedagogy.''

A panel discussion with Professor Thomas A. Carnevale, Essex County College, Professor Bernard L. Madison, Louisiana State University, and Professor Amber Steinmetz, Three Rivers Community College.

Problems and methods of placement of students at the basic skills level were discussed, especially as these relate to the Placement Testing Program of the MAA. Two methods of teaching basic skills were presented. One is a variation of the traditional lecture method and the other is the laboratory or learning center approach.

FOURTH SESSION OF THE ASSOCIATION

''Mathematical Problems in ESP Research'' by Professor Persi W. Diaconis, Stanford University and Bell Telephone Laboratories.

Modern ESP experiments offer new problems in combinatorics and mathematical statistics. For example, standard card guessing experiments use a 25 card deck consisting of 5 types of cards repeated 5 times each. A guesser tries to guess at the cards one at a time. In many experiments the guesser is given ''feedback''--told if each guess is correct or not. Dr. Ronald L. Graham and Professor Diaconis have analyzed the guesser's optimal strategy and ways of scoring such experiments which do not depend on what strategy the guesser uses.

''Curvature and Arc Length'' by Professor Andrew M. Gleason, Harvard University.

''Sherlock Holmes in Babylon'' by Professor R. Creighton Buck, University of Wisconsin.

The speaker presented a resume of the deductive steps that had led Neugebauer and Sachs to their interpretations of the scientific meaning of the Babylonian mathematical tablet known as Plimpton 322, and mentioned alternate viewpoints and some related questions that remain unanswered.

SPECIAL SESSIONS OF THE ASSOCIATION

Film showings were held in the Coliseum of the Convention Center on Thursday and Saturday at 7:00 P. M. The following films were shown on Thursday:

7:00 - 7:22 P.M.	Adventures in Perception (Maurits Escher)
7:25 - 7:45 P.M.	Cycloidal Curves or Tales from the Wanklenberg Woods (A film by the late Carl B. Allendoerfer.)
7:50 P.M.	Films of the Topology Films Project
7:50 - 7:55 P.M.	Zooms
7:57 - 8:00 P.M.	Sierpinski's Curve Drawn as a Function of Time
8:01 - 8:20 P.M.	Regular Homotopies in the Plane, Part II
8:25 - 9:30 P.M.	Computer Animation and the Geometry of Surfaces in 3- and 4-space. A presentation by PROFESSOR THOMAS F. BANCHOFF, Brown University, of films made in collaboration with DR. CHARLES STRAUSS.

The following films were shown on Saturday:

7:00 - 8:01 P.M.	Let Us Teach Guessing (George Polya)
8:05 - 8:12 P.M.	Newton's Equal Areas
8:15 - 8:24 P.M.	Linear Programming
8:27 - 8:42 P.M.	The Theorem of the Mean-a film of the MAA Calculus Films Project
8:45 - 8:58 P.M.	Dihedral Kaleidoscopes-a film of the College Geometry Project
9:00 P.M.	BBC Broadcast as Part of the Calculus Block of the Open University's Foundation Courses in Mathematics
9:00 - 9:25 P.M.	Behaviour of Functions
9:30 - 9:55 P.M.	Taylor Polynomials

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors met on Thursday, January 25, 1979, at 9:00 A. M. in the Coronet Room of the Broadwater Beach Hotel with 38 members present. Among the items of business transacted were the following:

The Board elected Professor Jacqueline C. Moss, Paducah Community College, as Second Vice-President for the two-year term 1979-80.

The Board elected Professors Gloria F. Gilmer, Milwaukee Area Technical College, and Peter L. Duren, University of Michigan, as Governors-at-Large for the three-year term 1979-81.

Professor Doris W. Schattschneider, Moravian College, was elected Editor of MATHEMATICS MAGAZINE for the five-year term beginning on January 1, 1980. Professor Schattschneider will succeed Professors J. Arthur Seebach, Jr. and Lynn A. Steen of St. Olaf College current Co-Editors.

It was voted to approve the recommendation of the Committee on Earle Raymond Hedrick Lectures that Professor Mary Ellen Rudin, University of Wisconsin, be invited to deliver the twenty-seventh set of Earle Raymond Hedrick Lectures at the Association's meeting at the University of Minnesota, Duluth, on August 21-23, 1979.

It was voted to approve the recommendation of the Nominating Committee for a Secretary for the Period 1980-84 that David P. Roselle, Virginia Polytechnic Institute and State University, be reelected as Secretary.

The Board approved establishment of a Joint MAA-SIAM Committee on Applications of Mathematics in the College Curriculum. The Board passed the following resolution: ''Resolved that MAA joins SIAM in the establishment of a Joint MAA-SIAM Committee on Applications of Mathematics in the College Curriculum. The Committee may study any questions relating to the applications of mathematics in the undergraduate or graduate college curriculum, but it shall not speak for any one of the two societies unless specifically authorized to do so by the Presidents of both organizations. Any recommendations of the Committee shall be presented to the SIAM Council and the MAA Board of Governors. The Committee shall consist of seven members, three appointed by the President of SIAM, three appointed by the President of MAA, and a Chairman jointly appointed by the Presidents of both societies. One of the members appointed by MAA shall be a member of CUPM. Members of the Committee shall serve for staggered terms. The Committee shall continue in existence until discharged by the SIAM Council and the MAA Board of Governors.''

President Alder requested election of the following Nominating Committee for 1979: David P. Roselle, Chairman; Joan P. Leitzel, Howard E. Zink. The Board elected this Committee whose charge it is to present five nominees for each of the offices of President and First Vice-President.

The operating budget for 1979 was approved and the Board discussed the budget for 1980. In particular, it was reported that a dues increase will be necessary to balance the budget for 1980. Accordingly, the Board approved the schedule of dues given below. It was noted that this is the first dues increase in two years and that the dues for Students, Unemployed, and Emeritus members are not being increased.

M=MONTHLY, G=MATHEMATICS MAGAZINE, T=TWO-YEAR COLLEGE MATHEMATICS JOURNAL

	M	M+G	M+T	T	T+G	M+G+T
Regular	$30	$38	$38	$24	$30	$44
Regular (1st 2 yrs)	25	33	33	19	25	39
Student or Unemployed	15	21	21	10	15	26
Emeritus	15	21	21	10	15	26
Contributing	60	68	68	54	60	74
Sponsor	90	98	98	84	90	104
Patron	150	158	158	144	150	164
Family	$10 - No Journals					
Life	Regular $500; Patron $1,000 or more					

The Board heard a report that the MAA offices had been relocated to the new Washington headquarters complex on August 26, 1978. The MAA now occupies the first three floors of 1529 Eighteenth Street, N. W., and has the remaining two floors of this building and all of 1527 Eighteenth Street under long-term lease. On account of this income from rentals and the donations totalling approximately $360,000 to the Building Fund, the Association now has lower net cost for its headquarters space than it did earlier. There is being planned an official opening of the headquarters in 1979.

The Board heard the report of the Committee on Placement Examinations in which it was reported that there are currently about 200 institutional subscribers to these multiple-choice placement examinations. The existing tests are basic skills, basic algebra, more advanced algebra, and trigononetry and elementary functions. A different version of the trigonometry and elementary functions test and a new calculus-readiness test are in preparation. Additional information about this program can be obtained by writing MAA, 1529 Eighteenth St., N. W., Washington, D. C. 20036.

Upon recommendation of the Committee on Sections, the Board approved changing the name of the Philadelphia Section to the Eastern Pennsylvania and Delaware Section. Also approved were by-law revisions by the Louisiana-Mississippi, North Central, Northeastern and Seaway Sections.

The Board approved creation of a standing Committee on Short Courses. This Committee hopes to organize the first mini-course for presentation at the August, 1979 meeting in Duluth.

The Board accepted a grant of $2250 from the Prudential Foundation. This grant was used to finance a conference, ''Math Opens Doors Everywhere,'' for guidance counselors in New Jersey. This conference was conceived, organized, and conducted by Professor Eileen L. Poiani, Governor of the New Jersey Section and Director of of the MAA lectureship program, ''Women and Mathematics.''

The Board approved the University of Michigan as the site of the August, 1980 meeting. With this addition, the meetings now scheduled are the following:

Duluth, Minnesota	August 21-23, 1979
San Antonio, Texas	January 5-7, 1980
Ann Arbor, Michigan	August 18-20, 1980
San Francisco, California	January 9-11, 1981

It was announced to the Board that MAA President Henry L. Alder has been elected Vice-Chairman of the Council of Scientific Society Presidents for 1979. According to CSSP custom, President Alder will thus serve as CSSP Chairman during 1980.

ANNUAL BUSINESS MEETING OF THE ASSOCIATION

The Annual Business Meeting of the Association was held on Saturday, January 27, 1979, in the Coliseum of the Convention Center, with President Alder presiding.

The Association's eighteenth Award for Distinguished Service to Mathematics was made to Professor Otto Neugebauer of Brown University. The citation (which appears on pages 77-78 of the February issue of the MONTHLY) was prepared by Professor Ralph P. Boas and read by MAA President-Elect Dorothy L. Bernstein. Professor Neugebauer was unable to attend the Biloxi meeting but sent his greetings to those present.

The Chauvenet Prize for 1979 was presented to Dr. Neil J. A. Sloane of Bell Telephone Laboratories for his paper, ''Error Correcting Codes and Invariant Theory: New Applications of a Nineteenth Century Technique,'' which appeared in the AMERICAN MATHEMATICAL MONTHLY (84), 82-107. Further details concerning this Prize and its recipient appear in the February, 1979, issue of the MONTHLY, page 79.

The Secretary called upon First Vice-President Peter J. Hilton who delivered a resolution of appreciation for retiring President Henry L. Alder's hard work on behalf of the Association. Professor Hilton called attention to Professor Alder's devoted service to the Association both as Secretary and as President. He said that the organization was indeed fortunate to have been the benefactor of such dedicated and excellent service.

The Secretary presented his report announcing first that Professor Jacqueline C. Moss had been

elected Second Vice-President of the Association for the two-year term 1979-80.

On behalf of all members of the Association, the Secretary thanked Professor Howard E. Zink of Lane Community College for his service as Second Vice-President during the past two years. The Secretary said that Professor Zink's counsel with regard to the purchase of a headquarters building had been particularly helpful and had been appreciated by all members of the Executive and Finance Committees.

The Secretary called attention to the excellent program for the Biloxi meeting. He thanked all members of the Program Committee, especially Professor K. Brooks Reid, Chairman. The Secretary also thanked Professor John O. Riedl Jr. of Ohio State University for having substituted on the program for his colleague Professor Harry P. Allen when it had been necessary for the latter to leave the meeting in order to attend to a family illness.

The Secretary thanked the Committee on Local Arrangements for their careful attention to the many details that make a meeting such as this one a success. He noted that this is the first Annual Meeting not held at the site of a college. Consequently, the Committee on Local Arrangements had had to add a certain amount of travel and other inconvenience to its usual duties. The Secretary particularly thanked Professor Thomas A. Atchison, Chairman of the Committee, for his hard work on behalf of the Association.

President Alder next introduced President-Elect Dorothy L. Bernstein of Goucher College. President-Elect Bernstein described the challenges now facing collegiate mathematics educators and solicited help in addressing those problems.

POSTER SESSION

Posters were on display in the Exhibit Area of the Biloxi Convention Center after 9:00 A. M. on Friday, and authors were available between noon and 1:00 P. M. on Saturday to discuss their displays. The following abstracts were submitted:

ELIZABETH BERMAN, University of Missouri--Kansas City, Kansas City, MO 64110

''Reading Graphs''

Most textbooks in basic mathematics explain how to produce a graph, given data. But few cover the useful skill of producing data, given a graph. I present a few examples of reading graphs, preparing the student for algebra, statistics, and calculus. I ask the student for the following procedures, not necessarily in technical terms:

Given the value of the independent variable, find the corresponding value of the dependent variable.

Given a value of the dependent variable, find the corresponding value of the independent variable (equation solving).

Extrapolate.

Find the intersection of two lines, corresponding to the solution of two simultaneous linear equations. Given a scatter diagram, draw a line to fit the points.

Decide from a graph where a function is increasing, where decreasing, and what are the extrema.

E. P. MILES, JR., Florida State University, Tallahassee, FL 32306

''Colorful Mathematics or Functional Design''

The author's abstract by this title in 1955 was illustrated by hand-drawn color block graphs. Recently developed computer programs now apply his methods to create multicolor designs. Ten representative patterns generated on an Intecolor 8051 were be displayed and discussed. Character mode block graphs are generated by evaluating functions of x and y limited to the range 0,1,2,3,4,5,6, and 7 by such devices as the greatest integer function, modular arithmetic, and the absolute value function. The eight numbers in the range of f(x,y) are identified respectively with the eight colors of the additive (light) system of colors: black, red, green, yellow, blue, magenta,

cyan, and white. Thus, all characters with coordinates such that $1 \leq f(x,y) \leq 2$ will be assigned the color red. Actually under interactive use any possible subset of the eight colors can be assigned to the eight or fewer numbers generated by a particular function. Titles for patterns displayed are: 1. Expanding Hearts, 2. Pastel Parabolas, 3. Collision Course, 4. A Close Encounter of the Infinite Kind, 5. Fishy, 6. Sinusoidal Shapes, 7. Tangential Flags, 8. Hyperbolic Cross Currents, 9. Sunset at Magic Mountain and 10. Random Reflections. Polaroid color photographs of sixty additional interesting patterns were available in album form for examination during the presentation hour.

JANE I. ROBERTSON, Schoolcraft College, Livonia, MI 48152

''Casting Out Nines: A Check for Whole Number Arithmetic''

Schoolcraft College, with its open-door policy, offers a course in remedial arithmetic. The regular textbook material on whole number arithmetic can be expanded to include the check, ''Casting Out Nines''. This check, once learned with pencil and eraser, can later be used with machine calculations.

The poster displayed worksheets stressing the format of the calculations and their check, An accompanying text gives background, rules, explanation, discussion, and students' reaction.

MEETINGS OF OTHER ORGANIZATIONS

The AMS held sessions from Wednesday through Saturday, January 24-27. Invited addresses were delivered in the Coliseum according to the following schedule:

Wednesday

9:00 A.M. ''Some Applications of Algebraic Geometry to Ring Theory''
Michael Artin, Massachusetts Institute of Technology

10:30 A.M. ''Manifolds and Submanifolds''
Julius L. Shaneson, Rutgers University

1:00 P.M. COLLOQUIUM LECTURE I:
''Complex Analysis and Algebraic Geometry''
Phillip A. Griffiths, Harvard University

3:30 P.M. ''Proper Holomorphic Maps''
John E. Fornaess, Princeton University

8:30 P.M. JOSIAH WILLARD GIBBS LECTURE
''What are Solitons and Inverse Scattering Anyway, and Why Should I Care?''
Martin D. Kruskal, Princeton University

Thursday

9:00 A.M. ''Problems With Different Time Scales and Their Numerical Solution''
Heinz-Otto Kreiss, University of Uppsala

10:30 A.M. ''Entropy, Isomorphism, and Reparametrization for Finite Measure-preserving Actions of Certain Continuous Groups''
Jacob Feldman, University of California, Berkeley

1:00 P.M. COLLOQUIUM LECTURE II

2:10 P.M. ''Space-times with Distribution Valued Curvature Tensors''
Abraham H. Taub, University of California, Berkeley

5:00 P.M. AMS Business Meeting

Friday

1:00 P.M. COLLOQUIUM LECTURE III

2:15 P.M. ''The Heart Valve Problem of Cardiac Fluid Dynamics and Its Numerical Solution''
Charles Peskin, New York University

3:30 P.M. ''Representations of Classical Groups''
Bhama Srinivasan, Clark University

Saturday
1:00 P.M. COLLOQUIUM LECTURE IV

The Association for Women in Mathematics sponsored a panel discussion, ''ERA and Bakke,'' at 4:00 P.M. on Saturday. Professor Judith Roitman served as moderator.

The Mathematics Action Group held a panel discussion, ''How Can We All Have Jobs,'' at 8:00 P.M. on Thursday.

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements consisted of Thomas A. Atchison, Chairman; Frank T. Birtel, ex-officio; Wendell Deer, Stephen A. Doblin, Roosevelt Gentry, James E. Keisler, William J. LeVeque, ex-officio; Eldon L. Miller, Carol B. Ottinger, Charles S. Rees, David P. Roselle, ex-officio; Robert A. Shive, Jr., and Billy R. Sneed.

Beginning at 9:00 P. M. on Friday, January 26, there was a social at the Broadwater Beach Hotel. Food and drinks were available as was dancing to a New Orleans style jazz band.

In recognition of the increased interest in running among mathematicians, Addison-Wesley sponsored footraces on Saturday, January 27. Races of approximately two and six miles were featured and approximately 200 persons entered these races.

David P. Roselle, Secretary

ACADEMIC MEMBERS ELECTED INTO THE ASSOCIATION

In accordance with the amendment adopted at the business meeting of the Association at Stillwater on August 30, 1961, the Board of Governors at its meeting in Biloxi, Mississippi, on January 25, 1979, elected to membership the thirty-third set of applicants for academic membership (for the previous lists of applicants see the March, 1979, issue of this MONTHLY). Approval of election was given to the following applicants for academic membership:

Albion College, Albion, Michigan
Bethany College, Bethany, West Virginia
Capilano College, North Vancouver, British Columbia
C. W. Post College, Greenvale, New York
Essex County College, Newark, New Jersey
Georgia State University, Atlanta, Georgia
McNeese State University, Lake Charles, Louisiana
Monmouth College, West Long Branch, New Jersey
Mount Royal College, Calgary, Alberta
Northeastern University, Boston, Massachusetts
Northwestern College, Orange City, Iowa
University of Georgia, Athens, Georgia

David P. Roselle, Secretary

CALENDAR OF FUTURE MEETINGS

Fifty-ninth Summer Meeting, University of Minnesota, Duluth, August 21–23, 1979.

Sixty-third Annual Meeting, San Antonio, Texas, January 5–7, 1980.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, last weekend in April or first weekend in May. Deadline for papers 6 weeks before meeting.

FLORIDA, early March. Deadline for paper titles 2 weeks before meeting.

ILLINOIS, first Friday/Saturday in May.

INDIANA

INTERMOUNTAIN

IOWA, third weekend in April. Deadline for papers February 1.

KANSAS, March or April. Deadline for papers January 1.

KENTUCKY, early April. Deadline for papers 6 weeks before meeting.

LOUISIANA–MISSISSIPPI, Friday-Saturday before February 20. Deadline for papers 3 months before meeting.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK, Adelphi University, May 5, 1979.

MICHIGAN, University of Detroit, May 4–5, 1979.

MISSOURI, late March/early April. Deadline for papers January 31.

NEBRASKA, April.

NEW JERSEY, early November and early May.

NORTH CENTRAL, end of October and April. Deadline for papers October 1 and April 1.

NORTHEASTERN, University of Maine, Orono, June 22–23, 1979.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO, College of Wooster, Wooster, fall 1979.

OKLAHOMA–ARKANSAS, (approx.) Friday and Saturday of first weekend in April. Deadline for papers 3 weeks before meeting.

PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 15–16, 1979.

PHILADELPHIA, Drexel University, Philadelphia, November 17, 1979.

ROCKY MOUNTAIN, last weekend in April or first in May. Deadline for papers 8 weeks before meeting.

SEAWAY, SUNY, College at Oneonta, May 4–5, 1979. Deadline for papers 6 weeks before meeting.

SOUTHEASTERN

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, usually in April. Deadline for papers 2 weeks before meeting.

TEXAS, Friday and Saturday in early April. Deadline for papers March 1.

WISCONSIN, Friday and Saturday between mid-April and first week in May. Deadline for papers 6 weeks before meeting.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, San Francisco, January 3–8, 1980.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES

AMERICAN MATHEMATICAL SOCIETY, University of Minnesota, Duluth, August 22–25, 1979.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Louisiana State University, Baton Rouge, June 25–28, 1979.

ASSOCIATION FOR COMPUTING MACHINERY, Plaza Hotel, Detroit, Michigan, October 29–31, 1979.

ASSOCIATION FOR SYMBOLIC LOGIC, New York City, December 28–29, 1979.

ASSOCIATION FOR WOMEN IN MATHEMATICS

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES, University of Saskatchewan, Saskatoon, June 4–5, 1979.

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS, Washington, D.C., August 13–16, 1979.

MU ALPHA THETA, Athens State College, Athens, Alabama, August 5–8, 1979.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Seattle, Washington, April 16–19, 1980.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Marc Plaza Hotel, Milwaukee, Wisconsin, October 15–17, 1979.

PI MU EPSILON, University of Minnesota, Duluth, August 21–23, 1979.

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Radisson Muehlbach, Kansas City, Missouri, November 1–3, 1979.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Royal York Hotel, Toronto, Canada, June 11–13, 1979.

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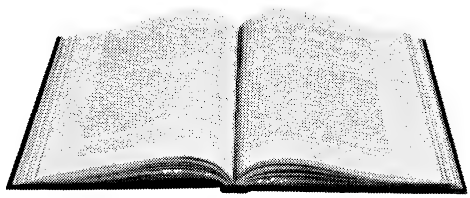

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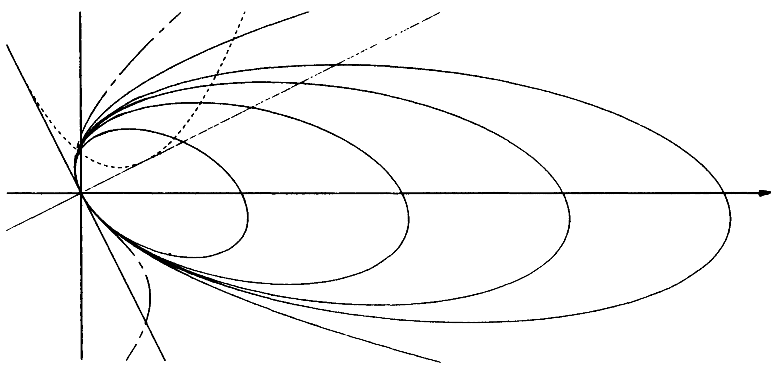

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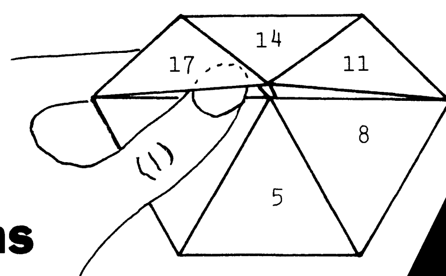

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1979

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Research papers per se are unsuitable; see statement of policy (Vol. 85, p. 1).

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HOW TO DO ARITHMETIC

JANE I. ROBERTSON

Introduction. A generation ago arithmetic was very simple—dull, but simple. By sixth grade a child could add, subtract, multiply, and divide and thought no more about it. The algorithms for getting the right answer were intended to be used just that way.

Then two groups revised those algorithms. Starting in the late 1940's the elite fraternity of computer designers, who built the machines to make hand computation obsolete, examined the rules every schoolchild knew and invented entirely new algorithms for machines. Starting in the 1950's the "new math" group, who brought the curriculum into the twentieth century, modified the old single-purpose algorithms (which were designed solely to get the right answer) to dual-purpose algorithms (to get the right answer and understand why).

Both groups made significant contributions to mathematics but at the expense of competency in arithmetic. Computing machines make it easy to think that mastery of a basic skill can be replaced by pushing a few buttons. The dual-purpose algorithms are difficult to use because the old algorithms had been refined to such simplicity that they could not be modified without affecting their usefulness. It is now harder to be competent and seductively easy not to try.

This paper sets forth the single-purpose algorithms and some of those devised by both groups. Since arithmetic is taught only to little children, elementary school teachers, remedials, and graduate electrical engineers, the developments are quite different. Parts of this paper will be familiar to some and entirely new to others, but that is the present fragmented state of arithmetic and, to my mind, another cause for the decline in competency in arithmetic.

Numeral Systems. The four arithmetic operators, $+ - \times \div$, combine two real numbers (except division by zero) to produce a unique result. The result is independent of the numeral system used to represent the real numbers; however, the choice of algorithm to perform the operation depends heavily on the numeral system used. These algorithms specify not just the list of rules but also the format in which the calculation is laid out.

There are many place-value numeral systems of the form $\Sigma_{i=m}^{n} a_i r^i$. While it is common practice to choose $r = 2, 3, \ldots$, and $\{a_i\} = \{0, 1, \ldots, r-1\}$, it is not required, and departing from common practice can yield strange systems. For a mild example, choose $r = 1$, $m = 0$, and $\{a_i\} = 1$ for all i; then numerals are:

$$\begin{aligned} 1 &= 1\times 1^0 \\ 11 &= 1\times 1^1 + 1\times 1^0 \\ 111 &= 1\times 1^2 + 1\times 1^1 + 1\times 1^0 \\ &\vdots \end{aligned}$$

Since all digits within a numeral are interchangeable, all places have equal value, and there you are in Roman numerals with no place value, no carries, and no decimal point. This degenerate system represents only the natural numbers, and the characters V, X, etc., are shorthand notation for strings of ones. These characters are arranged within a numeral in a sophisticated order, but without place value. This paper discusses how to do arithmetic when $r = 1, 2, 10$—or proceeding from most to least familiar: decimal, binary, Roman.

The decimal system ($r = 10$) has a base big enough and small enough for humans to use. It is big enough to represent large numbers with short numerals, and small enough that the digits all

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Jane Ingersoll Robertson studied mathematics at the University of Michigan and has been at various times a numerical analyst, an operations analyst, and a teacher of mathematics. She wishes to acknowledge that her interest in arithmetic was inspired by James E. Robertson (now of the University of Illinois) and fostered by observing her children's homework.—*Editors*

have distinctive shapes. The associated addition and multiplication tables are big enough to contain the commonly needed information, and small enough to be memorized and used for mental calculation. The decimal system, so easily read and memorized, permitted development of algorithms for all four operations that are simple enough to be used reliably by average people.

The decimal system is like our 26-letter alphabet. The alphabet brought reading and writing within the capability of virtually everyone. The decimal system made arithmetic literacy available to all. Both depend on easy access to writing materials.

The binary system ($r=2$) is functionally related to electronic computers: 1 represents a closed circuit and 0 an open circuit. It is much easier and cheaper to build a reliable device to detect presence (1) or absence (0) of electricity than to build one to detect ten different $(0,1,\ldots,9)$ degrees of presence of current. The addition table

+	0	1
0	0	1
1	1	10

and the multiplication table

×	0	1
0	0	0
1	0	1

contain only $0,1,10$ as elements; so they are easy to wire into electronic computers. Simple reliability is the basic reason why computers are designed to do arithmetic in the binary system.

The 2-digit base that is reliable for machines is notoriously unreliable for humans. It is hard to read the binary numerals 1110110011 and 1101110011, but easy to discriminate between their decimal equivalents, 947 and 883. It is easy to memorize 863–2413, but how does one memorize its binary equivalent, much less tell it to a telephone operator with any assurance of being understood? The 2-digit base is clearly better suited to the on-off mentality (?) of a computer.

Roman numerals ($r=1$) are superb for counting. It is simple for a person to make piles of 5 or 10 objects, then combine them into piles of L, C, or D objects to get a grand total of M objects. This method is reliable and error detection reasonably easy. A person can keep track of numbers with counters such as beads.

The numeral system with radix 1 is so limited that addition and subtraction within the natural numbers are very easy, but multiplication and division are very hard. Enlarging the radix to 2 or 10 introduces zero, place value, carries, and the decimal point. The place-value algorithms for addition and subtraction are more complex, but they can be incorporated into algorithms for multiplication and division that are relatively easy to use.

Addition. The single-purpose decimal algorithm is a very simple list of rules.

1. Write the addends in column form with corresponding digits in the same column. Draw a horizontal line under the bottom addend.
2. Add the least significant digits, i.e., the far right column. This produces a sum digit and a carry digit. Popular usage has it that "no carry" means the carry is zero.
3. Record the sum digit in its column under the horizontal line.
4. The carry rule: Enter the carry digit in the next-left column.
5. Repeat steps 2, 3, 4, proceeding from least to most significant until all the sum digits have been generated.

The sum digits are generated sequentially, proceeding from least to most significant, as a consequence of the carry rule #4. For example:

111 1	← carries	
9666030		
+6688776		(1)
16354806		

To get the millions digit it looks as if we add $1+9+6$ to get 6-carry-1. Actually we say $1+9=10$ then $10+6$ produces 6-carry-1. We have done two successive additions in our head, then written down the sum digit and (optionally) its carry in the adjacent column. There are 11 additions done sequentially: 7 for the given digit pairs and 4 to assimilate carries. Most people don't need to write down carries in simple problems; the format with columns lined up is a sufficiently powerful visual aid for getting the sum with speed and accuracy. This algorithm is a time-tested balance between mental arithmetic and the written record.

This balance is ruined in the dual-purpose algorithm because the format has been changed to sentence form. For example:

$$1776+1978+10025+5280=19059$$

The eye must hop about to find and add corresponding digits, then record the sum digit on the far side of the equals sign, then cross the equals sign again with the carry to start searching for the next group of digits to add. This extra burden on the eye distracts the memory, eliminates the flexibility in adding digits, destroys the natural place to mark down carries, and impedes the simple visual check. The dual-purpose algorithm invites error because it upsets the delicate balance between mental calculation and the written record.

The new math group originally introduced sentence form into the elementary school curriculum for two very good reasons. Many children had developed an almost superstitious belief that they could only add numbers that were stacked on top of each other, and experience with other formats helped break down superstition. Use of sentence form anticipates equations that are the natural language of algebra.

Schoolchildren now have drills in both algorithms. Since the time—and patience—allotted to drills is necessarily limited, the time spent with sentence form means just that much less time devoted to training the eye to use column form. The children's attention is spread too thin and competency declines.

The single-purpose algorithm is valid but clumsy for adding two binary numerals. They are more than three times as long as decimals and adding generally produces many more carries because the base is so small. For example, (1) above is:

	11	1	11			1		
100	100	110	111	110	111	101	110	
11	001	100	001	000	000	001	000	(2)
111	110	011	000	110	111	110	110	

This is $24+6$ sequential additions, 24 for digit pairs and 6 for carry assimilation.

One can compare incidence of carries generated by adding two numbers in decimal and binary. In (1) above there is one string that is 3 carries long and one string that is 1 carry long. In (2) there are three strings, one each of length 1, 2, and 3. Usually strings of carries are of higher frequency and longer in binary than in decimal.

For more than two addends the next-left carry rule (#4) does not hold because the sum of a column of digits (#2) is a sum digit and more than one carry digit which are added to successive columns according to place value. See the example in (5b) below.

Exceptions to the next-left carry rule are so rare in decimal that there are few who mention that the next-left rule is usable but not accurate.

The 24+6 sequential additions of (2) are too many for man or machine. Humans simply don't add in binary. Computers are wired to cut down addition time by adding each digit pair simultaneously—a technique not available to humans. Each pair produces a sum digit and a carry digit that is sent to the next-left adder. The process is repeated until the carries have rippled down the line and been assimilated. Schematically, (2) becomes:

100	100	110	111	110	111	101	110		
11	001	100	001	000	000	001	000		
111	101	010	110	110	111	100	110	pseudosum 1	(3)
	1		1			1		carry 1	
111	100	010	100	110	111	110	110	pseudosum 2	
	1		1					carry 2	
111	110	010	000	110	111	110	110	pseudosum 3	
		1						carry 3	
111	110	011	000	110	111	110	110	sum	

There are two obvious consequences of simultaneous addition. The sum digits are not produced sequentially but according to when the last carry ripples down and is assimilated. The sum is formed in four sequential additions: one to add the given digits and three more to assimilate carries.

The algorithm (3) has been improved by sending any carry to the next pair of like digits and assimilating it there. The sum digit for the pairs passed over is automatically set to 0. (3) then becomes:

100	100	110	111	110	111	101	110		
11	001	100	001	000	000	001	000		
111	100	010	000	110	111	100	110	pseudosum 1	(4)
	1	1				1		carry 1	
111	110	011	000	110	111	110	110	sum.	

The sum is formed in two sequential steps.

The major challenge in the design of arithmetic units has been to speed up carry assimilation. The algorithm (4) is only one such scheme, but it illustrates some differences between humans and machines. In computers the carry rules need not be simple—only a few engineers must use them. Sum digits need not be generated sequentially. Consequently, there are a variety of rules for special-purpose machines. This is a far cry from one simple next-left carry rule that serves a person a lifetime.

To add with Roman numerals, collect characters and then do some consolidation. For example:

$$\begin{array}{r} \text{XXXVIII} \\ +\quad \text{XVII} \\ \hline \text{XXXXVVIIIII} = \text{LV} \end{array}$$

It is much like playing with poker chips, occasionally trading in 5 whites for a red, or 2 reds for a blue. Addition started this way. The place-value algorithm evolved slowly and it requires good penmanship to line up rows and columns plus memorization of a large addition table, neither being required by Roman numerals.

It is critically important to understand that a schoolchild must transfer his number representation from $r=1$ (collect and consolidate) to $r=10$ (place value and carries) at this point so that he can master the single-purpose decimal addition algorithm that will enable him to

multiply and divide. Not to make the transfer means that he is effectively living in the Roman Empire where all but the simplest multiplication and division was done by professionals. In today's world, failure to use place value means that a person had better keep his calculator (and a spare battery) handy because he can't multiply and divide without it.

Subtraction. The single-purpose algorithm for subtraction of signed decimal numbers involves decisions about sign and magnitude. First, scan the minuend and subtrahend to see if the signs are alike or opposite. For like signs: use column form; "put the big number on top"; subtract corresponding digits from right to left; assimilate borrows in the next-left column, either by subtracting 1 from the minuend digit or by adding 1 to the subtrahend digit; do not mark down borrows. The difference digits are then generated sequentially from least to most significant. The difference takes the sign of the minuend if it is larger in absolute value, and its opposite if smaller. For minuend and subtrahend with opposite signs, change the sign of the subtrahend and add. The difference takes the sign of the minuend. Subtraction is in two steps: one for the difference digits and another for the difference sign.

As a bonus feature the single purpose algorithm contains the simple visual check of subtracting down and adding up. For example:

$$\begin{array}{r} 7615982 \\ -6688776 \\ \hline 927206 \end{array}$$

The check reflects the operation; subtraction is the inverse of addition. This may be second nature to some people, but the check is neither known nor taught in many elementary classrooms today.

The new math ruined this algorithm by requiring that borrows be assimilated in the minuend and marked down. For example:

$$\begin{array}{r} 6507 \\ \cancel{7}\cancel{6}\cancel{1}59\cancel{8}2 \\ -6688776 \\ \hline 927206 \end{array}$$

This obscures the given data, impedes the simple visual check, and is difficult to use in the long-division algorithm. Arithmetic competency has got to decline when the algorithm encourages defacing given data and discourages the habit of checking.

It gets worse in sentence form.

Marking borrows was introduced originally as a learning device, but actually it turns out to be a crutch that most can't throw away. It appears that a child can learn to assimilate borrows written or mental, but not both.

As in addition, the decimal algorithm is valid but clumsy for subtracting in binary. Humans don't. Computers can't use the algorithm because they can't "put the big number on top."

"Putting the big number on top" is peculiar to subtraction. Humans decide which is the big number by doing a trial subtraction, visually comparing as many corresponding digits from most to least significant as necessary. Computers must actually perform the subtraction to compare numbers.

Computers eliminate the trial subtraction by using the characteristic that their number system is finite, the maximum number of digits in each numeral being set by design specification. The finite number system is familiar to us all in the odometer of a car. In a new car it reads 00000. As the car is driven, the mileage is recorded and can get all the way to 99999, and the next one mile brings it full circle to 00000. There is no place to store the lead digit of 100000. Similarly, if one were to go -2 miles in a new car, the odometer would go backward to 99998; then go forward $+5$ miles and it would read 00003. Thus the finite number system enables us to write

positive and negative numbers with both sign and magnitude contained in the numeral. This is called complement representation. If there are 5 digits in each numeral, then the integers, which are $-50000 \leqslant \mathcal{I} \leqslant 49999$ in signed magnitude form, become $50000 \leqslant \mathcal{I} \leqslant 49999$ in complement form on the odometer. Positive numbers have lead digits 0 through 4, and negative numbers have lead digits 5 through 9. That's how you know you've gone 2 miles backward when the hypothetical odometer reads 99998, the complement form of -2.

The complement form of numbers makes possible a one-step algorithm for subtraction, which holds regardless of signs and magnitudes. The difference is the minuend plus the complement of the subtrahend.

The fundamental dissimilarity between computers and humans is that computers use finite number systems, and humans use infinite number systems. The distinctly different subtraction algorithms are a consequence of this dissimilarity. Speculative question: As calculators infiltrate the elementary and secondary schools, how will students' perception of infinity be altered?

Subtraction with Roman numerals is very easy. It is "take-away" with a little conversion between characters. Just stay in the natural numbers.

Multiplication. Multiplication is defined independently of addition, but the familiar algorithm for generating the product involves prerequisite knowledge of the $\times$-table for finding the partial products, place value for positioning them, and addition for summing them. An example in both decimal and binary shows how the size of the base affects the algorithm.

```
For a 1-digit multiplier:     947     1110110011
                               ×3             ×1
                             ----    -----------
                             2841     1110110011
```

The decimal product is formed sequentially from least to most significant with next-left carry assimilation. This is easy for humans who have memorized the $\times$-table and mentally assimilate carries. The binary product is always the multiplicand or 0, because 0 and 1 are the only binary digits.

For a multidigit multiplier do a problem in:

(5a) decimal

```
   947
  ×883
  ----
  2841
 7576
7576
------
836201
```

(5b) binary

```
                       1110110011
                      ×1101110011
                      -----------
                       1110110011
                      1110110011
                    1110110011
                   1110110011
                  1110110011
                 1110110011
                1110110011
              -------------------
pseudosum 1   1000111000001000101
carry 1       1111 11111 11 1 1
              -------------------
pseudosum 2   0110101100101100001
carry 2       1  1 11 1  1     1
              -------------------
product       1100110000100110101001
```

(5c) showing cumulative sums, S_i, of partial products, p_i.

```
 p_i   S_i               1110110011
                        ×1101110011
                        -----------
 p_1   S_1               1110110011
 p_2                    1110110011
                        -----------
       S_2             101100011001
 p_3                  1110110011
                     --------------
       S_3           10001100100 1001
 p_4                 1110110011
                    ---------------
       S_4          1011110010101001
 p_5                1110110011
                   ----------------
       S_5         11010100101101001
 p_6              1110110011
                 ------------------
       S_6       1010101110001101001
 p_7             1110110011
                 ------------------
product = S_7 →  1100110000100110101001
```

The decimal algorithm (5a) is good for humans. The partial products are easy to position with least significant digit below its multiplier digit. It is possible to sum them without writing down carries. The final product is compact.

It is virtually impossible for a human to multiply in binary (5b) without graph paper; the base is too small. True, each partial product is the multiplicand weighted by its position, but the myriad of 0's and 1's dance before the eyes. Carries go over more than one column; so they must be marked down and assimilated in separate steps.

Computers multiply in binary for simplicity and speed using algorithms like (5c). The computer forms a series of cumulative sums, successively shifting then adding in each partial product until the product is formed. Adding only two numerals at a time ensures that within each addition the carries go next-left.

Multiplication by repeated addition is straightforward for human and computer once place value has been mastered. The human decimal algorithm relies on relatively large memorized addition and multiplication tables together with a low incidence of carries to give the right balance of mental calculation and the written record. The binary computer algorithm relies on the small addition and multiplication tables to utilize a very fast shift and add.

The new math version of (5a) has come to require that children write in all the positional zeros associated with a partial product—including rows of zeros for zero multiplier digits. Many children develop an almost superstitious belief that they must write those zeros. It is supposed to yield understanding. Mostly it yields writer's cramp.

Multiplying in Roman numerals is repeated addition without place value.

Division. Division is unlike other operations. The Romans just left it in the hands of professionals and were thankful it was seldom used. The simple problems could always be managed by counting: XXXXV ÷ IX = V is easy if you count out 45 beans into piles of 5 each. The hard problems require a place-value algorithm. An example in decimal and binary will show how the size of the base affects the usefulness of the algorithm. The remainder and decimal point present no special difficulty and discussion of them is omitted.

```
   947             1110110011
 ______       ______________
3)2841       11)101100011001
  27            11
  --            --
   14           101
   12            11
   --           ---
    21          100
    21           11
    --          ----
                  100
                   11
                  ---
                    11
                    11
                   ---
                     100
                      11
                     ---
                       11
                       11
                       --
```

The important points are that:

1. There is no direct method for getting the quotient digits: they are found by guessing. In order to guess efficiently a person must be able to multiply and subtract mentally, otherwise the writing and erasing get out of hand. Guessing in decimal takes practice because there are 10

different digits in the base. Guessing in binary is easy, but it is impossible not to make mistakes in the written work without using graph paper to line up rows and columns.

2. The algorithm is iterative. The quotient digits are generated from most to least significant. Even computers must find quotient digits sequentially.

It is little wonder that the decimal algorithm took so many centuries to be refined. The balance between mental and recorded calculations is highly sophisticated. It does not translate into binary.

Computers can speed up division by using an algorithm based on a redundant number system. Just as the multiplication algorithm uses addition, so the division algorithm uses subtraction. In computers subtraction leads to complement representation. This form was further modified so that numerals contain both positive and negative digits. For example:

$$\begin{array}{ll} & 375 = 4\bar{3}5 \\ \text{where} & 375 = 300 + 70 + 5 \\ \text{and} & 4\bar{3}5 = 400 - 30 + 5 \end{array}$$

The bar over the digit means it is negative. In this system $r = 10, \{a_i\} = \{0, \pm 1, \pm 2, \ldots, \pm 9\}$. There are more than ten (the radix) elements in the base; so number representation is not unique. A radix 10 example with numbers in signed-magnitude form will suggest how computer designers came to use a redundant base. (6a) uses the familiar base $\{0, 1, \ldots, 9\}$, and (6b) uses the redundant base.

(6a)

$$\begin{array}{r} 375 \\ 4\,)\overline{\,1500} \end{array}$$

$$\left.\begin{array}{r} -\ 4 \\ \hline +\ 11 \\ -\ 4 \\ \hline +\ 7 \\ -\ 4 \end{array}\right\} \begin{array}{c} 3 \\ \text{subtractions} \end{array}$$

$$\begin{array}{r} \hline +\ 30 \end{array}$$

$$\left.\begin{array}{r} -\ 4 \\ \hline +\ 26 \\ \vdots \\ +\ 6 \\ -\ 4 \end{array}\right\} \begin{array}{c} 7 \\ \text{subtractions} \end{array}$$

$$\left.\begin{array}{r} \hline +\ 20 \\ -4 \\ \hline +\ 16 \\ \vdots \\ +4 \\ -4 \end{array}\right\} \begin{array}{c} 5 \\ \text{subtractions} \end{array}$$

$$\begin{array}{r} \hline 0 \end{array}$$

(6b)

$$\begin{array}{r} 4\bar{3}5 \\ 4\,)\overline{\,1500} \end{array}$$

$$\left.\begin{array}{r} -\ 4 \\ \hline +\ 11 \\ -\ 4 \\ \hline +\ 7 \\ -\ 4 \\ \hline +\ 3 \\ -\ 4 \end{array}\right\} \begin{array}{c} 4 \\ \text{subtractions} \end{array}$$

$$\begin{array}{r} \hline -\ 10 \end{array}$$

$$\left.\begin{array}{r} +\ 4 \\ \hline -\ 6 \\ +\ 4 \\ \hline -\ 2 \\ +\ 4 \end{array}\right\} \begin{array}{c} 3 \\ \text{additions} \end{array}$$

$$\left.\begin{array}{r} \hline +\ 20 \\ -4 \\ \hline +\ 16 \\ \vdots \\ +4 \\ -4 \end{array}\right\} \begin{array}{c} 5 \\ \text{subtractions} \end{array}$$

$$\begin{array}{r} \hline 0 \end{array}$$

In both cases guessing quotient digits is forbidden. Computers can't guess, but they can try sucessive values for each quotient digit. In both cases the divisor, 4, is subtracted 1, 2, 3 times from 15. This is where the similarity between (6a) and (6b) ends.

The human (6a) successively subtracts until the remainder first becomes smaller than the divisor, then records the number of subtractions, 3, as the lead quotient digit and uses the remainder, $+3$, to find the next quotient digit. The procedure is repeated until the problem ends.

The computer cannot tell which is the larger of two positive numbers without actually subtracting, but it can easily distinguish positive and negative numbers in complement form by the lead digit. So the computer successively subtracts until the remainder first changes sign from $+3$ to -1. In our automobile the odometer changes from 00003 to 99999.

There are now two choices. The computer could back up one step to restore the smallest positive remainder, $+3$, and continue calculating like a human. The calculation would look like (6a) with two extra steps, overshoot and restore, inserted.

Alternately the computer can do nonrestoring division. It can use the first negative remainder as the basis for the second quotient digit. The second quotient digit will correspondingly be negative. It is found by successively trying $0, -1, -2, \ldots$ until it (in this case $-3=\bar{3}$) produces a change in sign in the remainder (from -2 to $+2$). With a positive remainder, the next quotient digit is positive. And so the calculation proceeds, with the quotient digits being alternately positive and negative, as the remainders alternate from positive to negative.

The answers, 375 and $4\bar{3}5$, are equivalent. Humans prefer 375 because we are accustomed to scan numerals from most to least significant digit. Computers use $4\bar{3}5$ because it arises naturally out of the division algorithm.

The division algorithm (6b) was worked out about 20 years ago. Actually, it isn't much better than (6a), but it led to significantly better algorithms. The key is the use of positive and negative digits in a numeral. This is a good example of an early step in a productive development in arithmetic. It is a long way from putting 45 beans in 9 piles.

Conclusion. The algorithms for whole number arithmetic that are designed for specific purposes are most useful. Notable examples are the single-purpose algorithms which comprise the basic skill and those developed for arithmetic units in computers. The dual-purpose algorithms are partially responsible for the decline in competency because the algorithms require a student to do two things at the same time: be right and understand why.

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HIGHER MATHEMATICAL EDUCATION IN THE PEOPLE'S REPUBLIC OF CHINA

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Introduction. In October 1977, a delegation from the National Council of Teachers of Mathematics visited the People's Republic of China. In the almost one and a half years between our visit and that of the Pure and Applied Mathematics Delegation significant events had occurred. Chairman Mao had died. Teng Hsiao-ping, twice removed from leadership, had been reinstated as vice-chairman of the party. The "Gang of Four" had been disgraced and arrested. All of these events had important consequences for educational policy in China. In his article [2] on the trip of the Pure and Applied Mathematics Delegation in this MONTHLY, Victor Klee, writing after some of these events, asked whether they foretold "another change in the Chinese educational system." The answer is yes. This article, in part, will describe some of the changes.

Our delegation consisted of twenty-three mathematics educators from the United States and Canada. There was a variety of educational levels and types of institutions represented. The greatest part of our time in China was spent visiting educational institutions and talking to educators. A full report [1] containing our observations on all levels of mathematical education in China will be available. This article will concern itself only with post-middle-school education. It is based on visits to Peking University and Futan University (Shanghai), conversations with two Chinese mathematicians from Peking University who traveled with us during our trip, a visit to Peking Teachers University, a visit to a Worker's Training School at a machine-tool factory, and a discussion there about a Worker's College run by the factory. Although this is a limited experience on which to draw conclusions, the changes in education occurring in China have so much interest that it seems worthwhile to share what we learned.

The Organization of Peking and Futan Universities. Peking University, founded in 1898, is now organized into 20 departments. There are 10 science departments, 7 liberal arts departments, and 3 foreign language departments. There are 2,700 teachers and researchers and 7,000 students (38 percent are women), including 130 foreign students from 37 countries. There are 140 teachers and researchers in the mathematics department, and 150 students. One-third of the faculty is at or above the rank of lecturer (which corresponds to our assistant professor rank). Enrollment at Peking University was 10,000 students prior to the Great Proletarian Cultural Revolution, and there seem to be plans to increase to that number again. Before the Cultural Revolution, programs in the liberal arts schools required five years of study, and programs in the sciences required six years. Now both are set at three years. There are plans to lengthen the course of study for some specialties. In particular, we were told that mathematics programs were to be extended to four years in the near future.

The university admits students from all over the country, and admission is highly sought after. There is no tuition at Peking University. In fact, students receive a stipend while attending, as well as free food, lodging, and medical care. Students who have been workers for at least five years receive their worker's salaries in addition to the other benefits.

There were no degrees or graduate programs at Peking University at the time of our visit, although there were plans for starting graduate programs again. At present, students who wish to continue their studies past the undergraduate level must obtain a position at a university or research institute and participate in seminars. This may account for there being almost as many teachers as students in the mathematics department.

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Futan University, founded in 1905, seems to be quite similar in structure to Peking University. It has developed very rapidly since liberation in 1949. In pre-liberation days there were only 400 teachers, but now there are 2,200 teachers and researchers. (Approximately one-half of the faculty do research only. The rest teach—usually six hours per week—and participate in research seminars.) There are 14 departments at Futan University: 7 in the liberal arts (Chinese literature, history, philosophy, political economy, journalism, international politics, foreign languages) and 7 in the sciences (biology, physics, chemistry, mathematics, computer science, electronics, and optics). There are 4,000 students. In addition Futan University has a mathematics research institute, a genetics institute, a historical geography research institute, a linguistics laboratory, and a Chinese language research institute.

We had the opportunity to visit the mathematics library at Peking University. They have 30,000 mathematics books and 400 periodicals, including 50 periodicals from the United States. The library seems to be well stocked although somewhat dated. During a cursory glance at some of the stacks, the following books were noticed by one observer: *Lie Algebras* by Jacobson, *Multilinear Algebra* by Marcus, *Non-commutative Rings* by Herstein, *Category Theory* by Mac Lane, *Number Theory* by Weil, *The Four-Color Problem* by Ore, *Analysis* by Dieudonné, *Studies in Topology* by Stavrakas, *Algebraic Topology* by Switzer, *Topics in K Theory* by Hodgkin and Snaith.

The function of the mathematics department at Peking University is to train scientific workers and mathematics researchers. The department is not specifically involved with training high school or university teachers, although some of their students will teach at this level. They are concerned about elementary and high school education, since it affects the quality of university students and the general scientific development of the country. Before the Cultural Revolution the university had a middle school attached to it which served to prepare workers, peasants, and soldiers for university studies. This school was closed when all middle schools began serving this function, and the university started enrolling students directly from the ranks of workers, soldiers, and peasants. Members of the mathematics faculty are presently assisting in the development of new teaching materials upon the invitation of the Ministry of Education. Also, curriculum writers come to Peking University to consult with the faculty. They have been discussing the modernization of the mathematics curriculum but said they will undertake a careful study before making any decisions about introducing modern mathematics into the primary curriculum. One question that the Chinese are considering is whether set theory should be earnestly studied in the primary schools.

The academic calendars at the two universities are similar. There are two terms: The first runs from the beginning of September into January, and the second runs from the beginning of February into July. Each term consists of eighteen weeks of teaching, with classes meeting six days per week, and two weeks for review and examinations. Faculty devote about two weeks each term to productive labor and have a one-month summer vacation and a half-month winter vacation. Students spend about two months each year working in the factories, the communes, or the army. They have a three-week winter vacation and a two-week summer vacation.

The Curriculum. The mathematics department at Peking University has three specialties:

1. Computer mathematics
2. Mathematics—pure and applied
3. Information theory.

A typical program for students specializing in pure and applied mathematics would include, during the first year, analytic geometry, algebra, mathematics analysis (this is a calculus course), and physics. There would also be a review of middle-school mathematics, since the students would have been working for two or three years. In the second year students would continue with analysis and physics and add courses in differential equations, complex variables, and algebra (one-half year). During the third year they would study real variables, probability and statistics, numerical analysis, and computer programming. Electives taken by students who

continue beyond three years are chosen from courses in modern algebra, differential geometry, topology, and logic.

Unlike American students, Chinese students concentrate their course work in their major department as soon as they enter the university. Students of mathematics, for example, study mathematics almost exclusively. The only exceptions are physics, foreign languages, and politics. The faculty are concerned that their program overburdens the students and are trying to find a way to reduce the burden.

The program at Futan University is similar. Students in their first year study mathematical analysis (calculus) and solid analytic geometry, higher algebra, general physics, foreign languages, politics, and physical education. In the second year students continue with mathematical analysis and higher algebra during the first term and with ordinary differential equations and complex variables during the second term. Partial differential equations, probability and statistics, functions of real and complex variables, and differential geometry are studied in the third year. For students studying beyond three years, electives are chosen from topology, modern abstract algebra, theoretical physics, mechanics, and computer analysis. Operations research is available as a special elective course.

A professor at Peking University described the first-year algebra course he teaches. The curriculum is as follows: polynomials, including factorization and proof of unique factorization; real and complex polynomials; the fundamental theorem of algebra (no proof); symmetric polynomials (the fundamental theorem of symmetric polynomials might be proved); linear equations; the Gaussian elimination process; concepts of linear independence and dependence; criteria for solvability of equations; structural solutions for nonhomogeneous systems; determinants; Cramer's rule; operations on matrices; abstract linear spaces over subfields of complex numbers; the theory of linear transformations; similarity transformations; characteristic roots and vectors; introduction to Jordan canonical form (no proofs); and the concepts of groups, rings, and fields (definitions and simple properties). All students, including students in numerical analysis and computing, take this algebra course.

The course entitled Introduction to Real Analysis, taken in the third year, includes the following subjects: point set topology of the real line and the plane, inner and outer measure, the measurability of sets, Lebesgue integral, Fatou's lemma, the dominated convergence theorem, Levi's theorem, and L^2 as an example of concrete Hilbert space. The class meets four hours a week.

There is an advanced algebra course available as an elective. In this course students study groups, rings, and fields in more detail than in the first-year course. For example, in group theory they study Lagrange's theorem, the homomorphism theorems, permutation groups, and sometimes Sylow's theorems or group representations. In rings they study ideals and quotient rings, polynomial rings, rings with ascending chain condition, and algebraic numbers. Galois theory is a major part of the course—they prove the fundamental theorem of Galois and derive the conditions for the solution of equations by radicals.

Although it is difficult to make any definitive conclusion in such a short visit, we had the impression that the mathematical preparation of a mathematical major at Peking and Futan universities is comparable to the preparation available at most American universities.

Peking Teachers' University. Peking Teachers' University, founded in 1902, trains middle-school teachers. It has 15 departments with over 1,000 teachers and researchers and over 3,000 students. In addition there are about 15,000 students enrolled in correspondence and/or short-term courses. There is a kindergarten, a primary school, and a middle school connected to the university.

There are three courses required of all full-time students: history of the Chinese Communist Party, politics, and philosophy. Since the Cultural Revolution, the university has been emphasizing the combination of theory and practice. With this goal in mind the university runs several small factories where teaching, research, and productive work are combined. Also, all students

spend some time each year in factories, communes, or the army. Practice teaching in the middle school is required of all senior students.

Since our visit was brief and our delegation was scheduled to deliver some talks about U.S. education, we were not able to obtain detailed information about the curriculum at Peking Teachers' University. However, our general impression is that there is more emphasis on subject matter and less on professional educational courses than there is in the United States.

Education of Workers. There is an impressive dedication to ongoing education in China. Workers are frequently seen reading and studying during their breaks. Factory administrators talk about workers becoming technicians and technicians becoming engineers and designers. The commune we visited had a spare-time college, and it seems that most of the factories run spare-time colleges of various types.

A factory we visited (the Shanghai Machine Tool Plant) ran a Worker's College and a Worker's Training School. Although the college was not in session when we were there, we obtained information from administrators. The teachers at this Worker's College are graduates from Teachers' Universities and Mechanical Colleges. There are five mathematics teachers who also work in the plant. In order to qualify for the Worker's College, students must have three years of work experience and a cultural level equivalent to a middle-school education. Nevertheless, since the teachers found that the preparation of the workers was very uneven, they start with a review of elementary mathematics. They then go on to higher mathematics, including differential equations and other theoretical developments. We were told that in the past there was no examination for this college but that they were now giving a cultural examination in order to select the best students. The examination includes Chinese mathematics, politics, and mechanics. It is also required that the prospective students be ideologically advanced, that they be healthy, and that they be under 30 years old. The college offered only one course—design and manufacture of machine tools—for the past three groups of students. Now there are three specialties: (1) design and manufacture of machine tools, as above, (2) automation of machine tools, and (3) hydraulic devices of machine tools.

We had the opportunity to visit the Worker's Training School run by the faculty. This is a two-year school for male and female middle-school graduates selected by the state from throughout China. During the first year the students study politics, mathematics, physical culture, mechanical design (mechanics, heat treatment, and metal materials management). There are 34 periods each week for these subjects, distributed as follows: politics 4, math 8, physical culture 2, mechanical design 20. The second year consists of a week of apprentice training in a workshop alternated with a week of classroom learning. We saw students, both girls and boys, in the factory doing heavy metal work as well as in classroom situations. Students may be sent to different parts of China after successful completion of their training.

Professors at Work. Professors spend six to eight hours per week in class and do the usual preparation and committee work outside of class. Although this is quite similar to what American professors do, there are important differences. For one, it is quite rare for teachers to move from one institution to another. They usually stay at one institution during their entire career. A more striking difference is that most professors have had work experience in the countryside. One mathematician talked about this. He said that the first time he went to the countryside (by this he meant any rural setting outside of a city) was in 1958, when he was 30 years old—eight years before the start of the Cultural Revolution. He has since been there several times. Sometimes he went to the suburbs of Peking, which are made up of farms and communes, and other times he went farther away—usually to small towns near Peking. During one visit he did physical labor with commune members, and on another he taught math and science in elementary school. He also gave lectures on political theory and technical innovation in agriculture. His general observation was that it was a "good experience." The length of his stay varied from six months to a year. Up until the time of his first visit he had had no contact

with farmers, since he had grown up in the city. He thinks the policy of having professors and intellectuals spend time away from the university will continue, but there will be some changes. The duration of the stay, for example, may be shortened, or perhaps mathematicians will go to factories instead of the countryside and do work that is more closely connected with their research.

When asked whether a very talented young person should go to the countryside, this mathematician replied, "Yes, everyone should spend some time in the countryside." When asked how he changed as a result of the experience, his reply was, "I now understand more about the country and how to construct socialism." The experience also increased his awareness concerning the value of applications of mathematics. Since he now knows more about the peasants and the way they live, he thinks more about what their needs are as he goes about his research: "I must think about the society and about the revolution." He thinks it is important that professors and peasants become friends. Many professors have had no contact with peasants and vice versa. "We should work on mathematics, but we should also know how workers and peasants live." He said he plans to continue to work in pure mathematics (algebra) as well as in applications, perhaps in statistical problems or optimization theory. Concerning the latter, he mentioned that in 1971–1973 he spent some time in a Peking steel plant helping workers learn the Fibonacci search method. He lectured on the method and helped workers devise an experiment to determine the best temperature for a thermal treatment of steel. He first learned of the Fibonacci method in 1953 from a book by an American mathematician. In recent years, the mathematician Hua Lo-keng has been popularizing it in China and attempting to promote its use by workers in order to improve production.

Chinese mathematicians are proud of their achievements in making mathematics accessible to factory workers. Factory workers have been taught various types of experimental design that can be used to improve production. At present there is a group at the Institute of Mathematics in Peking developing ways of teaching linear programming to factory workers. The plan is to encourage technicians and engineers to popularize linear programming after learning the methods from mathematicians.

I questioned our two Chinese companions on their parents' attitudes toward their going to the countryside. One said that, since the first time he went was nine years after the revolution, his parents had already adapted and did not raise any fuss. The other's parents also thought it was a good idea, but they were quite anxious about his health. In talking about this he said, "Members of the older generation have difficulty, but most of them have changed because they are impressed by the achievements of the revolution." Because he was not in good physical health, he once worked as an accountant in a commune, and another time raised pigs. He also helped improve the mathematical training of rural teachers. The first time he went to the farm he found it difficult for a few months. He went with six or seven colleagues, and the peasants thought that these professors were "different." He was able to get the peasants to feel better about him by eating with them, helping them in organizing their accounting system, and buying food for them. He felt that in general he was warmly received by the peasants. He also mentioned that older professors do not have to go to the countryside, although some do volunteer.

Chinese Interest in American Education. The Chinese expressed a strong interest in American education. Some indication of their specific interests about college-level mathematics is given by the questions they asked following a talk on undergraduate mathematics education in the United States. Some of their questions were:

1. How are media being used in teaching?
2. When do we use epsilon-delta arguments and when do we avoid them?
3. What were the reasons for the changes in mathematical education over the past ten years?
4. How have modern mathematical developments influenced undergraduate mathematics courses?

5. How are we organized for teaching classical mathematics and how are we organized for using modern methods?
6. How do we provide general mathematical background knowledge and also knowledge in special fields?
7. How do we teach logical deduction?
8. How is theory combined with practice in the USA?
9. How do we handle the teaching of analysis and calculus in high school and college?
10. What does "application" mean?
11. Why have the analytical geometry texts been weakened?
12. What organizations or institutions are responsible for the training of elementary and secondary teachers of mathematics?
13. What is the curriculum for training elementary and secondary school teachers?
14. What are the different kinds of colleges and universities in the USA?

Higher Education and the "Gang of Four." During the briefings we received at the institutions we visited, the "Gang of Four" were under continual attack. Their activities and policies were described in the following terms: The "Gang of Four" had gone too far in discrediting the value of education and had said that it is better to have a non-educated worker than an educated elite. They tried to convince students to be their followers. At Peking University one of their followers said that there is only one specialty at the university—the struggle with the capitalist-roaders—and that is the only qualification for graduation. The "Gang of Four" made people look foolish, particularly faculty, and tried to discredit academic and scientific knowledge because they felt it would make people reactionary. They discouraged students from studying and teachers from teaching and doing scientific research, especially research in basic theory. The "Gang of Four" tried to stir up bad feelings between teachers and students and treated teachers and intellectuals as enemies. One of the followers of the "Four" said that teachers should not feel comfortable, because then they would restore capitalism. As a result of this, the teaching quality has been lowered.

An often quoted example used to illustrate the educational policies of the "Gang" was that they praised a student for turning in an unanswered examination paper. They did not mention to us, as was reported at the time, that the student had written a letter on the back of the paper pointing out that the examination was a surprise and that, since he had been working in the countryside, he had little time to study. One author reports that the student wrote that if he had abandoned his work to study he "would have been guilty of being unworthy of the revolutionary cause which concerns both the poor and lower-middle peasants and myself" [3].

It seems to this author that the "Gang of Four" stood for continuing and strengthening the reforms of the Cultural Revolution. Their priority was to "put politics in command" and to continue the struggle against elitism and class privilege in the educational system. They sought an educational system which was egalitarian and free from foreign influence.

While not dismissing the function of education in providing the knowledge and training to modernize China, they would not sacrifice socialist values to achieve this goal. The defeat of their position is manifested by the changes described in the following section.

Changes in Higher Education. It is difficult to give an accurate or complete assessment of the changes taking place in Chinese higher education. Since the Cultural Revolution began there has been a struggle over educational policy in China, and the controversy continues. In simple terms, the debate is over the importance of building socialism versus the importance of increasing production and achieving technological equality with more developed countries. At present, the side advocating the importance of increasing production and achieving modernization has won. One indication of this is the difference in briefings received by the Pure and Applied Mathematics Delegation in May 1976 and those received by our delegation in November 1977. Klee reports in [1] that the 1976 briefings included a description of the struggle against right deviationism and an emphasis on the importance of "combining theory with

practice," avoiding the three divorces (from practice, from the everyday concerns of the workers and peasants, and from proletarian politics), and the "open door policy" (the policy which encourages workers and peasants to attend the university and education institutions to be concerned with production). In contrast, our briefings included a description of how the "Gang of Four" had interfered with educational programs, and the emphasis was on the university's role in producing trained scientific workers so that China could achieve modernization. To be sure, the importance of "combining theory and practice" was still mentioned, but their statements about the value of basic theoretical research clearly indicated a shift in policy.

One specific change occurred a few days before our visit to Peking University. The Central Committee of the Communist Party issued a document on the enrollment of students which changed the admissions process. Whereas, from the time of the Cultural Revolution, middle-school graduates had to spend two or three years in the countryside, the factories, or the army before attending the university, now 20 to 30 percent of the students will go directly to the university. We were told that this percentage will probably increase. The applicants must be middle-school graduates or must have attained an equivalent educational level, and will be given an entrance examination. Whereas in the past students have been selected to go to the university by local work groups and their selection had been based on service to the community and ideological strength, now the entrance examination and academic achievement will play a more important role. The faculty who discussed the policy change with us said that the advantage of having students participate in labor before attending the university was that they had a raised consciousness about the needs of the people. A disadvantage was that after two or three years away from school they had forgotten a lot and had difficulty resuming their studies.

Another specific change in university education is the reinstitution of titles. At the time of the Cultural Revolution titles of faculty at universities were abolished. Now they are being restored. One professor was concerned about this. He thought that titles and ranks could denote an actual division between different kinds of work, but he cautioned that people must not take the rank as the goal. He said that the goal still is to serve the people and to educate the students.

There has also been a change in the relative value being assigned to applications and to theory. There is a greater stress given to basic research and theoretical work. We were told that theoretical work had been impeded by the "Gang of Four" and that now Chairman Hua and Vice-Chairman Teng have stated that basic theory will be stressed because of its long term significance. However, we were also told that they must be concerned with the application of the basic theory, because this directly concerns the country's development.

We had the opportunity to discuss the issue of pure versus applied mathematics with the mathematics faculty at Peking University. Although they have a group of pure mathematicians and a group of applied mathematicians, they stated that there is an emphasis on applications in all the mathematics that they study. The deputy chief of the mathematics department reported, in response to a question about combining theory and practice, that although it is more difficult to apply pure mathematics one mathematician, Liao Shan-tao, who had been specializing in topology, is now studying global analysis and structural stability. This approach combines theory and practice.

Our discussion indicated a definite trend toward being more willing to do pure research in mathematics while maintaining concern about the applications of theory because of the state of the country's development. They stated that the study of each—pure and applied—should promote the other and gave, as examples, applications of group theory and finite fields.

Thus far we have mentioned changes within higher education. One indication that mathematics and science education will be given a high priority within the general society is the reinstatement of the Scientific and Technological Commission in the state council. This, commission, eliminated at the time of the Cultural Revolution, has the status of a governmental ministry. It will coordinate scientific and technological development throughout China. There have already been provincial conferences aimed at increasing the production of scientists and

technicians. Within eight years complete new curricula for mathematics, physics, chemistry, astronomy, geology, and biology will be established.

Summary. The Great Cultural Revolution had a profound effect on higher education in China. Some of the policies instituted at that time are being changed: the goal is to achieve modernization, and the educational system is seen as playing a key role in achieving that goal. The priorities of developing social consciousness and eliminating the elitism in educational institutions have given way to priorities of increasing production and developing a modern technological state. Entrance examinations and faculty titles have been reinstated. There are plans to increase the time of undergraduate study and to re-establish graduate programs. There was no discussion during our briefings of the need to change traditional teaching methods. There is increasing encouragement for professors to do basic research.

However, not all of the reforms of the Cultural Revolution are being abandoned. There is still emphasis on everyone's knowing the significance of physical labor, and students will be spending time working in the countryside and in the factories. School factories will continue to exist. The stress on applications of mathematical theory continues to be strong, as does the emphasis on making mathematical applications accessible to workers and peasants. The stated goals of the policy changes are to help the country as a whole, with the aim of education being to serve the people rather than to benefit the individual.

It is likely that the debate about the relative importance of building a socialist consciousness and achieving modernization will continue and that the educational system will be a key area in the controversy. The Chinese seem to be philosophical about changes in policy, saying that each change is an attempt to find a correct policy. As one of their old proverbs states, "The tree may prefer calm, but the wind will not subside."

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A GLIMPSE OF MATHEMATICS STANDARDS IN THE PEOPLE'S REPUBLIC OF CHINA

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An eight-year plan to improve the development of the natural sciences in the People's Republic of China was announced at the March 1978 meeting of the National Science Conference in Peking. Under the provisions of the plan, an army of scientists and technicians will be trained to help China obtain the "four modernizations" (modernization of agriculture,

industry, national defense, and science and technology) and secure its place as an industrialized world power. All agencies of the state, especially education, have been mobilized to contribute to this effort. New educational priorities are focusing on improved teaching of science and mathematics at all levels, from nursery school to the university. School mathematics is undergoing the most profound reforms ever experienced in the PRC. A unified ten-year curriculum has been devised and a supporting textbook series published. Mathematics texts are modern in content and reflect contemporary methods of instruction. "Key schools" have been established to attract and nurture mathematically and scientifically talented students. A resurrected Olympiad examination scheme will be used to locate budding academic talent. The first series of such national examinations has already been given in mathematics. Two hundred thousand students entered the competition; of these, 350 regional finalists participated in the final round of tests held on May 21, 1978, in Peking. The 57 winners were given national acclaim.

While all of these innovations have had an impact on the Chinese educational scene, perhaps none has provided so dramatic a testimony to the resumption of academic normalcy as the reinstitution of university entrance examinations. University entrance procedures were radically altered by the reform of the Great Cultural Revolution, and entrance examinations were eliminated in 1973. The resumption of university entrance examinations was welcomed by China's young; now, once again, they had a definite set of academic standards to strive for. Over five million candidates sat for the first round of these examinations given in the fall of 1977. Each province, autonomous region, and municipality set its own examination questions. While subject selection on the examinations varied according to proposed major, all candidates had to offer a paper in mathematics. Peking's examination was given from December 10 to 12 at 200 sites throughout the city. Peking has always been noted for academic excellence; its examination questions represent a high level of academic achievement by contemporary Chinese standards. The mathematics questions from the Peking examination are given below.

Mathematics Problems of the 1977 Higher Education Entrance Examination of Peking

(1) Solve the equation $\sqrt{x-1}=3-x$.

(2) Evaluate $2^{-\frac{1}{2}}+\dfrac{2^0}{\sqrt{2}}-\dfrac{1}{\sqrt{2}-1}$.

(3) Given $\lg 2=0.3010$, $\lg 3=0.4771$, find $\lg\sqrt{45}$.

(4) Prove $(1+\tan\alpha)^2=\dfrac{1+\sin 2\alpha}{\cos^2\alpha}$.

(5) Find an equation for the line passing through the intersection of the lines $x+y-7=0$ and $3x-y-1=0$ and the point $(1,1)$.

(6) This year, the value of production of a factory for July is one million dollars. If, for each month, the value is increased by 20% over the previous month, what is the total value from July to October?

(7) Given the quadratic function $y=x^2-6x+5$,
1. Find the coordinates of the vertex of its graph and the axis of symmetry.
2. Sketch its graph.
3. Find the x-intercept and y-intercept of the graph.

(8) A ship is sailing eastward at 20 miles/hour. At point A, a lighthouse B is seen from the ship in the direction N 45° S. One hour later, the ship is at point C and the lighthouse is seen in the direction N 15° S. Find the distance CB between the ship and the lighthouse.

(9) The bisector of $\angle A$ in $\triangle ABC$ meets BC at D and circumscribed circle at E. Prove that $AD\cdot AE=AC\cdot AB$.

(10) What are values of m for which the straight line $y=x+m$ meets the ellipse $\frac{x^2}{16}+\frac{y^2}{9}=1$ at (a) one point, (b) two points, (c) no points? Sketch the graphs when the two curves meet at one point.

University entrance examinations have since been unified and now follow a set of national criteria established by the Ministry of Education.

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GEOMETRICAL PROPERTIES OF THE PENOSCULATING CONICS OF A PLANE CURVE

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1. Introduction. The penosculating conics originally arose in the study of approximations to a plane curve and the related concept of the aberrancy of a curve. The concept of aberrancy itself —although little known—should be of interest to every calculus student, since it provides a geometrical interpretation of the *third* derivative of the function representing a curve [7]. Specifically, the *aberrancy* of a curve at an arbitrary point P is defined as the tangent of the angle δ formed between the normal at P and the limiting position of a line drawn from P to the midpoint of a chord parallel to the tangent line at P as the chord approaches P. The angle δ itself is called the *angle of aberrancy*, and the line through P which makes the angle δ with the normal at P is called the *axis of aberrancy*. An expression for the aberrancy of a C^3 curve given by $y=f(x)$ based on this definition was first derived correctly by Transon [8] and is

$$\tan\delta = y' - \frac{1+(y')^2}{3(y'')^2}y''', \tag{1}$$

where prime denotes differentiation with respect to x. The aberrancy is zero at every point of a circle and a non-zero constant at every point of a logarithmic spiral. The aberrancy thus depends on the third derivative and is an easily visualized geometrical quantity which, loosely speaking, may be said to measure the asymmetry of a curve with respect to the normal to the curve at a point. In much the same way that the slope and the curvature of a curve at a point are geometrical representations of the first and second derivatives, respectively, the aberrancy serves as a geometrical interpretation of the third derivative.

Now it is well known from the calculus that the slope and the curvature may be defined in terms of the first- and second-order osculants to a curve at a point, respectively. In particular, the curvature is (except for a sign) the reciprocal of the radius of the so-called *osculating circle* (also called the *circle of curvature*), i.e., the unique circle whose derivatives up to and including order two agree with those of the curve at the point (second-order contact). Transon [8] also showed how in a similar way the aberrancy may be defined alternatively in terms of the third- and the fourth-order osculant to a curve at a point. Thus he showed that the angle of aberrancy δ in (1) is the angle between the normal to the curve at P and the axis of the *osculating parabola* at P, i.e., the unique parabola whose derivatives up to and including order three agree with those of the curve at P (third-order contact). On the other hand, δ is also the angle between the

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normal to the curve at P and the diameter of the *osculating conic* at P, i.e., the unique conic which makes fourth-order contact with the curve at P. In this context, the center of the osculating conic is also known as the *center of aberrancy* of the given curve at P, and its distance from the point of contact P with the given curve is called the *radius of aberrancy*. (At a point of the given curve where the osculating conic is a parabola, the center of aberrancy recedes to infinity and the radius of aberrancy becomes infinite.) It is of interest that the radius of aberrancy depends on the *fourth* derivative of the given curve at P as well as on those of lower order. Hence the radius of aberrancy serves as a geometrical interpretation of the fourth derivative (for details see [7]).

Transon [8] went on to consider the case where only derivatives up to and including order *three* are prescribed in fitting the general conic to a given curve at a point P. In this case the resulting osculant to the given curve at P is not a unique curve, but forms a pencil of conics. Clearly, the additional prescription of the fourth derivative at P would reduce this pencil to the unique osculating conic; i.e., the pencil falls short of being this unique conic by assignment of a single parameter. Hence the pencil of conics which make third-order contact with a given plane C^3 curve at a point P has been termed the pencil of *penosculating conics* [10]. Alternatively, since it may be shown that each member of this pencil has four coincident points in common with the given curve at P (four-point contact), this pencil has also been called the pencil of *four-point conics* [6]. (In general, a *pencil of conics* is defined as the set of conics which pass through four points in the plane. In the special case considered here, the four points coincide.) Since the members of the pencil of penosculating conics all have the same first three derivatives as the given curve at the point of contact P, they all have the same slope, curvature, and aberrancy as the given curve at P. In particular, they all have the same angle of aberrancy δ at P, and hence all of their centers lie on a straight line through P which makes the angle δ with the normal to the given curve at P. This is the aforementioned line which is known as the *axis of aberrancy* in analysis [2], [9], and as the *affine normal* in differential geometry [1], [5]. A few basic geometrical properties of the pencil of penosculating conics were obtained by Transon [8] and Wilczynski [10]; here we prove additional properties of this pencil.

2. The pencil of penosculating conics and the enveloping parabola. To construct the pencil of penosculating conics, consider an arbitrary plane C^3 curve and let P be a point on this curve which is not a point of inflection and at which the pencil is to make third-order contact. For convenience, let the origin of a Cartesian coordinate system be located at P, let the y-axis be aligned with the axis of aberrancy of the given curve at P, and let the x-axis be normal thereto. Assume that the given curve can be represented explicitly by a single-valued function $y=f(x)$ in this coordinate system and that $f''(0)\neq 0$. By this choice of axes, the angle of aberrancy δ at P, i.e., the angle between the y-axis and the normal to the curve at P, is made equal to the angle between the x-axis and the tangent to the curve at P, i.e., to the slope angle $\theta=\arctan f'(0)$ (see Fig. 1). From (1) this implies that by this choice of axes the third derivative of the function representing the curve has been made to vanish at P, i.e., $f'''(0)=r=0$, and only the first two derivatives, say, $f'(0)=p$ and $f''(0)=q\neq 0$, need to be prescribed at P to determine the pencil of penosculating conics. The pencil must contain a singly infinite parameter, say k, and this parameter is chosen here so as to represent the ordinate of the center $(0,k)$ of an arbitrary conic of the pencil on the axis of aberrancy. The equation of the pencil is now derived by implicitly differentiating the equation of the general conic through the origin, namely, $Ax^2+2Bxy+Cy^2+Dx+Ey=0$, three times, imposing the prescribed values of the derivatives at the origin, fixing the center of the conic to be at the designated point, and then solving for the coefficients of the resulting system of equations. The desired pencil of penosculating conics thus becomes

$$(p^2+kq)x^2-2pxy+y^2+2kpx-2ky=0, \tag{2}$$

with the ordinate k of the center of any conic on the axis of aberrancy serving as the parameter.

The members of the pencil are ellipses (hyperbolas) when the discriminant $\Delta = kq$ is positive (negative). For $k=0$, the conics degenerate into the double tangent at P, namely, $(px-y)^2=0$. For $k=\pm\infty$, (2) becomes the so-called *osculating parabola*

$$qx^2+2px-2y=0,$$

with focus at $(-p/q,(1-p^2)/2q)$ and directrix $y=-(1/2q)(p^2+1)$, i.e., perpendicular to the axis of aberrancy. As noted above, there is precisely one member of the pencil which makes at least fourth-order contact with the given curve at P. This unique conic is called the *osculating conic* of the given curve at P. For its determination, the fourth derivative of the given curve at P is required in addition to the first three derivatives. Thus, letting $f^{\text{iv}}(0)=s$, this condition when imposed on (2) determines the value of the parameter to be $k=3q^2/s$ and thereby selects the unique osculating conic of the pencil. The center $(0,3q^2/s)$ of this osculating conic is the center of aberrancy of the given curve at P and its distance from P is the radius of aberrancy.

Transon [8] observed that both sets of axes of the penosculating conics envelop a certain fixed parabola and gave a geometrical construction for it. Analytically, this *enveloping parabola* may be derived by eliminating the parameter k simultaneously from the equation of the family of axes $\mathfrak{F}(x,y,k)=0$ of the conics (2) and from $\partial\mathfrak{F}/\partial k\ (x,y,k)=0$. The result is

$$q^2y^2+2q(p^2-1)y+4pqx+(p^2+1)^2=0. \tag{3}$$

This fixed parabola has $x=0$ as its directrix and $(-2p/q,(1-p^2)/q)$ for its focus. The following properties of the enveloping parabola are easily proved:

1. The directrix of the enveloping parabola is the axis of aberrancy.
2. The focus of the enveloping parabola is the projection of the center of curvature of the given curve at P onto the axis of aberrancy reflected in the normal to the given curve at P.
3. The enveloping parabola is tangent to the normal at the center of curvature $(-p(p^2+1)/q,\ (p^2+1)/q)$ and to the tangent at $(-(p^2+1)/pq,-(p^2+1)/q)$. The focus of the enveloping parabola lies on the line segment connecting these points.
4. Any two orthogonal tangents to the enveloping parabola drawn from a point on the axis of aberrancy (i.e., the axes of some penosculating conic) touch the enveloping parabola at two endpoints of a chord which contains the focus. Conversely, any two tangents to the enveloping parabola drawn from the endpoints of a chord which contains the focus are orthogonal and meet on the axis of aberrancy (*orthoptic* property of the parabola, see [11]).
5. The focus of the osculating parabola is the midpoint of the line segment from the point of contact to the focus of the enveloping parabola.

To illustrate the foregoing, the enveloping parabola and representative members of the family of penosculating conics on the concave side of a given curve are drawn in Figure 1.

Knowledge of the enveloping parabola facilitates the construction of any member of the pencil of penosculating conics. For once the center of the desired conic on the axis of aberrancy is designated, its *axes* may be drawn by constructing orthogonal tangents from this center to the enveloping parabola. One of these two axes intersects the normal to the given curve at a point N of the line segment which connects the point of contact P with the center of curvature O of the given curve at P. This is the *focal axis* (principal axis).

To locate the foci of the penosculating conic under consideration, construct first the *auxiliary circle* with PO as diameter. Then draw the chord DD' of this circle which is perpendicular to PO and which intersects PO at N. The two points where the two rays drawn from P and going through D and D' intersect the focal axis are the *foci* F and F' of the conic (see Fig. 2). This determination of the foci is based on the converse of Keill's construction. In its direct form [11], Keill's construction yields the center of curvature corresponding to a point P on a general conic from knowledge of its foci and the normal to the conic at P.

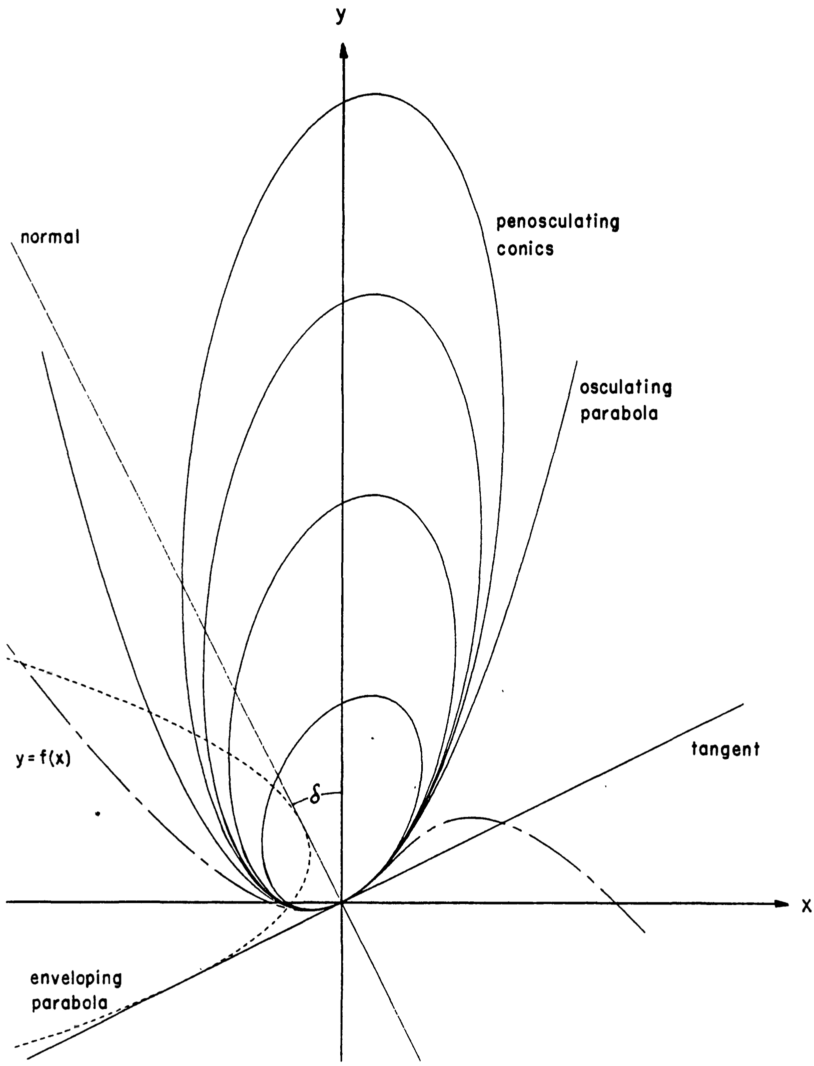

FIG. 1

3. The locus of foci of the penosculating conics. Having shown how to construct the foci of any individual penosculating conic in the last section, we will now obtain the locus of foci of the entire pencil of penosculating conics. The derivation of this locus is greatly facilitated by the following remarkably simple relationship between the locus of foci and the previously determined enveloping parabola:

The locus of foci of the pencil of penosculating conics (four-point conics) is the pedal curve with respect to the point of contact of the enveloping parabola.

The *pedal curve* of any given curve with respect to a given fixed point (called the *pedal point*) is the locus of the foot of the perpendicular dropped from the fixed point to a tangent which moves along the given curve (cf. [11]).

To prove this relationship between the locus of foci and the enveloping parabola, let P be the point of contact, let n be the normal to the given curve at P, and let $\mathcal{Q}$ be the axis of aberrancy through P (see Fig. 2). Let O be the center of curvature corresponding to the point P of the

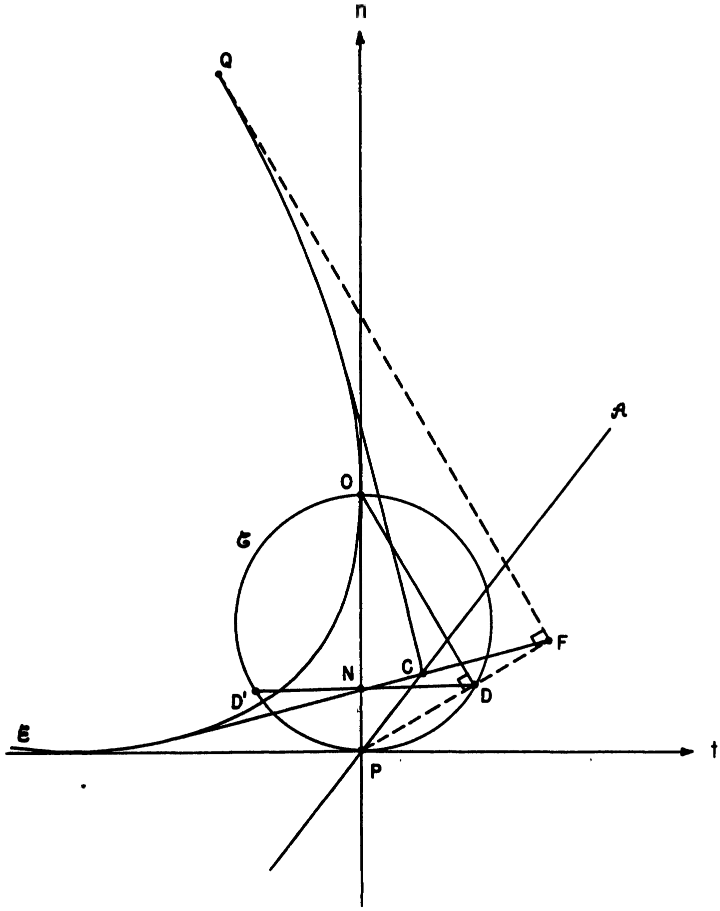

FIG. 2

given curve, let $\mathcal{C}$ be the auxiliary circle with PO as diameter, and let $\mathfrak{E}$ be the enveloping parabola of the axes of the penosculating conics. Let PFQ be any right angle with the side FQ tangent to the enveloping parabola at Q and the side FP passing through the point P. It remains to show that F is the focus of a penosculating conic. Now, one side of the right angle PFQ intersects the auxiliary circle $\mathcal{C}$ at a point D. Draw the chord DD' of the auxiliary circle perpendicular to the normal n, and let the point where DD' intersects n be denoted by N. The line through F and N intersects the axis of aberrancy at a point C, and this point is the center of some penosculating conic. Now the angle PDO is inscribed in the auxiliary circle $\mathcal{C}$ with PO as diameter, and hence a right angle. Thus by the converse of Keill's construction, outlined in the last paragraph of section 2, the point F is a focus of the penosculating conic through P with C as center. The locus of these foci is thus the pedal curve of $\mathfrak{E}$ with P as pedal point.

The desired locus of foci is now easily derived on using this result. Let $\tilde{y}-y=(dy/dx)(\tilde{x}-x)$ be an arbitrary tangent line to the enveloping parabola at (x,y), with $(\tilde{x},\tilde{y})$ as a running point on that line, and let $\tilde{y}(dy/dx)+\tilde{x}=0$ be a line through the pedal point $(0,0)$ and orthogonal to the given tangent line. Then the common point of intersection $(\tilde{x},\tilde{y})$ of these two lines is a point on the pedal curve, and solving for it from the above equations yields the *pedal transformation*

$$\tilde{x}=-\frac{\left(y-x\dfrac{dy}{dx}\right)\dfrac{dy}{dx}}{1+\left(\dfrac{dy}{dx}\right)^2},\qquad \tilde{y}=\frac{y-x\dfrac{dy}{dx}}{1+\left(\dfrac{dy}{dx}\right)^2}.$$

By applying this transformation to the enveloping parabola (3) one obtains the locus of foci of the penosculating conics

$$(qx-p)y^2+(p^2-1)xy+(qx+p)x^2=0,$$

where the tildes have been dropped. On solving for y, this curve may be written in the explicit form

$$y=\frac{(1-p^2)x \pm x\sqrt{(p^2+1)^2-4q^2x^2}}{2(qx-p)}. \tag{4}$$

This crunodal cubic curve is an *oblique* strophoid [3] with its double point at the point of contact P. The slopes of the tangent at the double point are $-1/p$ and p; i.e., the locus is tangential to the normal and to the tangent of the given curve at P. The curve has the asymptote $x=p/q$ and intersects the x-axis at the origin and at $x=-p/q$. The domain of (4) is confined to $|x| \leqslant (p^2+1)/2q$, with the extrema in x occurring at $(-(p^2+1)/2q, -(p^2+1)(p-1)/(2q)(p+1))$, and $((p^2+1)/2q, (p^2+1)(p+1)/(2q)(p-1))$. A graph of this locus of foci, showing its relationship to the enveloping parabola, is presented in Figure 3. A typical penosculating ellipse and a representative penosculating hyperbola are also entered in this figure, together with their foci and centers.

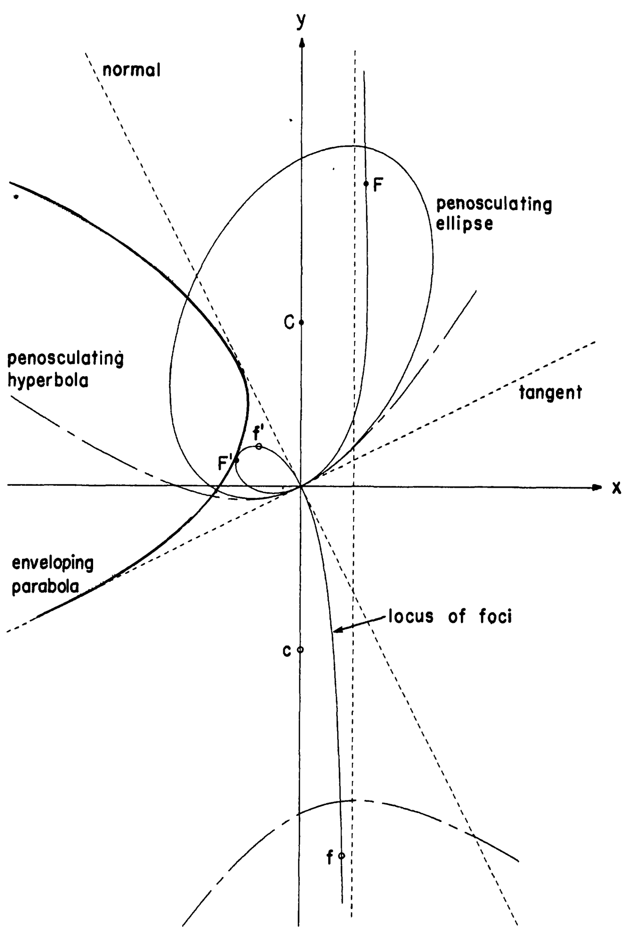

FIG. 3

In the special case where $p=1$, equation (4) reduces to that of the *right* strophoid

$$y^2=x^2\frac{1+qx}{1-qx}.$$

In this case the extrema in x become $(-1/q,0)$ and $(1/q,\infty)$; hence the domain is confined between the x-intercept and the asymptote. Moreover, the curve becomes symmetrical with respect to the x-axis. (The case $p=-1$ yields the same result with the x-axis reversed.)

The locus of foci helps to elucidate the transition of the penosculating conics from ellipses to hyperbolas (without going through the usual intermediate case of the parabola!) as their centers traverse the point of contact P on the axis of aberrancy. To see this, let us return to the general case and consider a penosculating conic corresponding to a given curve which is concave upward ($q>0$) at P. If the center of this conic lies on the positive half of the y-axis, it will be an ellipse ($k>0$). Its foci lie at the two points of intersection of its focal axis with a branch of the strophoid, namely, that branch which starts at the point $(-p/q,(1-p^2)/2q)$, touches the tangent line at P, and approaches the upper half ($y>0$) of the asymptote. As the center of the conic approaches P, the foci move toward each other, and in the limit of the degenerate (double tangent line) conic ($k=0$), become coincident at P. As the center C moves to a point on the negative half of the y-axis, the conic transforms into a hyperbola ($k<0$) and the foci shift to the other branch of the strophoid, i.e., the branch which starts at the point $(-p/q,(1-p^2)/2q)$, touches the normal at P, and approaches the lower half ($y<0$) of the asymptote. As the center moves to infinity (along either the positive or negative half of the y-axis) one of the foci moves to infinity along the asymptotic part of the curve, while the other focus approaches the point $(-p/q,(1-p^2)/2q)$ on the strophoid. The conic thus becomes the osculating parabola ($k=\pm\infty$) with its focus at this point.

4. The penosculating conics of minimum and maximum eccentricity. Wilczynski **[10]** pointed out that the pencil of penosculating conics in general contains no circle and hence must contain an ellipse of minimum eccentricity, although he did not actually obtain this ellipse. The equation of this ellipse will now be derived, and in addition it will be shown that the pencil also contains an equally important hyperbola of maximum eccentricity.

The square of the eccentricity of the general conic $Ax^2+2Bxy+Cy^2+Dx+Ey+F=0$ can be expressed [4] in terms of its first two invariants, $I_1=A+C$ and $I_2=AC-B^2$, as follows: $e^4/(1-e^2)=(I_1^2-4I_2)/I_2$. Applying this to the pencil (2) enables the relationship between the eccentricity e and the ordinate k of the center of the corresponding penosculating conic to be expressed as

$$\frac{(e^2-2)^2}{1-e^2}=\frac{(p^2+1+kq)^2}{kq}. \tag{5}$$

Figure 4 shows a graph of this (e,k)-relation for fixed values of p and q. It is evident from this graph that the range of eccentricities which actually occurs for members of the pencil of penosculating conics is confined to a strip between a certain minimum value e_m and a maximum value e_M. The actual minimum (m) and maximum (M) of the curve are seen from (5) to occur at the points whose coordinates are given by $k_m=(p^2+1)/q$, $e_m^2=2p(\sqrt{p^2+1}\,-p)$ and by $k_M=-(p^2+1)/q$, $e_M^2=2$, respectively.

The value $k_m=(p^2+1)/q$ corresponds to a unique *minimum eccentricity ellipse* whose equation, from (2), is

$$(2p^2+1)x^2-2pxy+y^2+\left[2(p^2+1)/q\right](px-y)=0.$$

The center $(0,(p^2+1)/q)$ of this ellipse is the projection of the center of curvature on the axis of aberrancy, and its axes are

$$y-\left(p\pm\sqrt{p^2+1}\,\right)x-(p^2+1)/q=0,$$

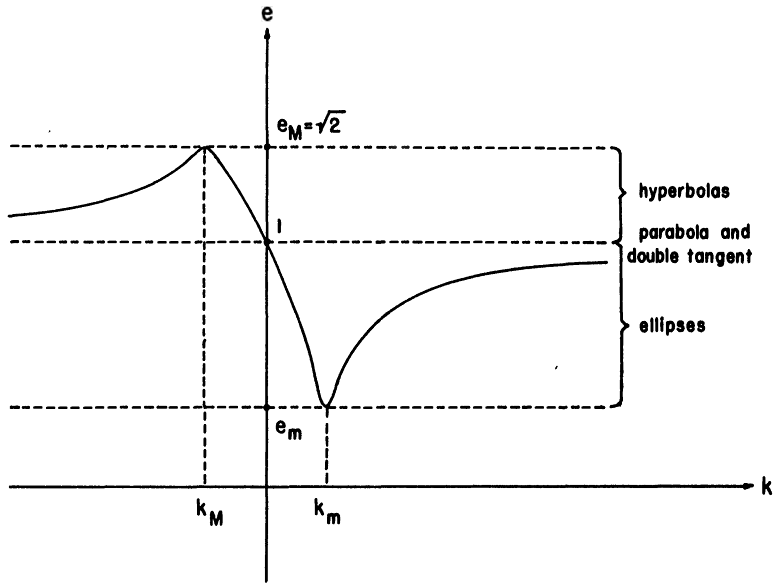

FIG. 4

i.e., parallel to the bisectors of the angle between the axis of aberrancy and the tangent at P.

The value $k_M = -(p^2+1)/q$ corresponds to a unique *maximum eccentricity hyperbola* with eccentricity $e_M = \sqrt{2}$. Its equation is

$$x^2+2pxy-y^2+\left[2(p^2+1)/q\right](px-y)=0,$$

and it is evident from its eccentricity and also from the form of its equation that it is an *equilateral* hyperbola; in fact it is the only equilateral hyperbola contained in the pencil. The axes of this hyperbola are

$$y \pm \left[\left(\sqrt{p^2+1} \pm 1\right)/p\right]x+(p^2+1)/q=0.$$

They are parallel to the bisectors of the angle between the axis of aberrancy and the normal at P. Its asymptotes are

$$y-\left(p \pm \sqrt{p^2+1}\right)x+(p^2+1)/q=0.$$

From the foregoing and Figure 4 it is evident that *for any attainable eccentricity e such that $e_m < e < e_M$ there exist precisely two members of the pencil of penosculating conics which have this eccentricity*. In the special case $e=1$, the unique osculating parabola may be thought of as being paired with the degenerate (double tangent line) conic.

From (5) it is easily seen that if $(0,k_1)$ and $(0,k_2)$ are the centers of two penosculating conics with the same eccentricity, then $k_m^2 = k_M^2 = k_1k_2$. Hence it follows that *the centers of any two penosculating conics having the same eccentricity are inverse points with respect to a circle whose diameter is the line segment connecting the centers of the minimum eccentricity ellipse and the maximum eccentricity hyperbola.*

5. The pencil of penosculating parabolas. Finally, it should be mentioned that geometrical properties analogous to those presented in this paper may also be obtained for the pencil of *penosculating parabolas*, i.e., the parabolas which make second-order contact (or equivalently, three-point contact) with a given curve at a point. These properties may be derived by methods similar to those employed in the previous sections and are merely summarized here. The envelope of the axes of the pencil of penosculating parabolas is a deltoid (three-cusped

hypocycloid), and the locus of foci of the pencil is the inscribed circle of this deltoid. Hence the locus of foci in this case is the orthoptic curve of the envelope of the axes of the pencil of penosculating parabolas. (The *orthoptic curve* of a given curve is the locus of intersection of two mutually orthogonal tangents to the given curve [11].) Moreover, the directrices of the penosculating parabolas all intersect at a single point—thus forming a pencil of lines—and the locus of vertices of the pencil of penosculating parabolas forms a bifolium with a tacnode at the point of contact with the given curve.

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V-FLEXING THE HEXAHEXAFLEXAGON

T. BRUCE McLEAN

1. Introduction. In 1939, A. H. Stone discovered a flexible hexagon with six different faces and B. Tuckerman provided a traverse that illustrated the various ways that you could pinch flex your way to each of the faces. Martin Gardner [1] gives an interesting history of their origin and relates that R. P. Feynman and J. W. Tukey joined Stone and Tuckerman on the first flexagon committee. Finally, in 1957, Oakley and Wisner [5] published their definitive work on flexagons and set to rest all flexagon questions if we restrict ourselves to the fascinating pinch flex.

Many amateurs found trouble in their first attempt at pinch flexing and messed the flexagon up so badly that four triangles of one face ended up with two from another. This produces a face that was never counted by the committee. Stone has indicated that he would correct any individual that handled the flexagon in such a precarious manner, but was pleased that order could result from this new flex. We name this modification a V-flex in honor of two students, Bob Verrey and Alan Moluf, who found several hundred of the new faces in 1965 while

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attending Findlay High School in Ohio. Their most amazing contribution to the final traverse was that they completely rewound the flexagon using only the V-flex. This is equivalent to turning the strip of triangles upside-down before winding it into a hexagon. Those that have painted each of the six faces a different color will note that two alternating colors will make up a face after it has been rewound.

It is the purpose of this paper to take Stone's hexahexaflexagon, count the number of faces that appear, and provide a traverse that shows how to pinch flex and V-flex your way through the 3420 faces that are now possible.

2. Construction. Stone folded and rolled a strip of eighteen equilateral triangles as described by each of the authors in the bibliography. Throughout this paper, we will assume that it has been constructed as shown in Figure 1. Carefully mark off eighteen equilateral triangles on a roll of adding tape, numbering them consecutively. It is important to number the back side of each triangle the same as the front. Roll the strip into one that is half as long in the direction shown in Figure 1. Fixing triangles 1 through 5, fold the remainder under. Fixing the first eleven triangles, fold the remainder under, placing 17 on top of triangle 1. Finally, fold triangle 18 back under 1 and tape the two segments labeled a–b with one piece of tape.

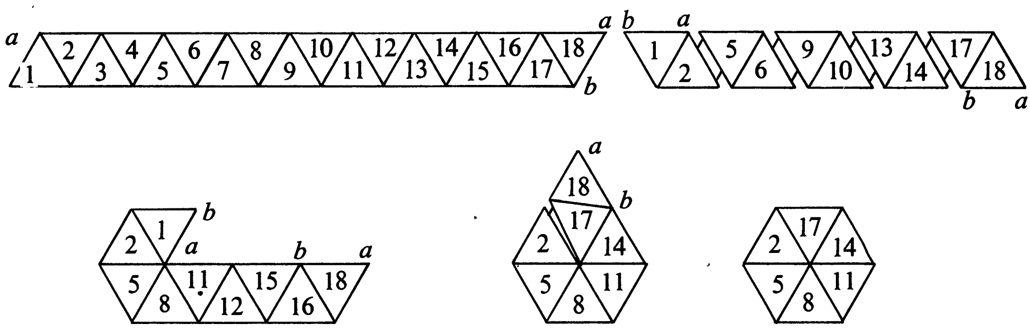

FIG. 1

Please note that it was not necessary to number all eighteen triangles in the past, because the pinch flex produced a triplication of what happened in the first six triangles. Since the V-flex destroys this symmetry, it is mandatory that we label each of the triangles.

3. Notation. To record this initial face of the hexahexaflexagon, we will use a notation that is similar to the one by Oakley and Wisner [5] and label it

$$(p_1\text{-}p_2\text{-}p_3\text{-}p_4\text{-}p_5\text{-}p_6)$$
$$=(2\ 3-5\ 4\ 7\ 6-8\ 9-11\ 10\ 13\ 12-14\ 15-17\ 16\ 1\ 18).$$

This shows the six triangular regions that form the hexagon, separated by dashes, with the six uppermost triangles to the left. The hidden triangles then follow in the actual order in which they occur. Oakley and Wisner [5] refer to these triangular regions as regular pats and, since all of ours are regular, we will just call them *pats*. The number of triangles that make up a pat p will be called the *degree of the pat* and denoted by $D(p)$. Finally, we will say that pat $p_1=2\ 3$ is in *position one* and consecutively number the positions from left to right. This corresponds to numbering them counterclockwise in Figure 1 with $p_6=17\ 16\ 1\ 18$ being in position six at the top.

4. V-flex. Throughout this paper we assume that the flexagon is never turned over except for the purpose of this demonstration. The reason that we go through this flex twice is to make sure that one can master the V-flex. A lot of practice is needed before continuing the reading. In

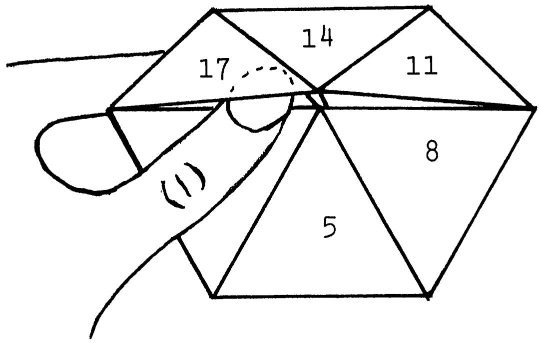

FIG. 2

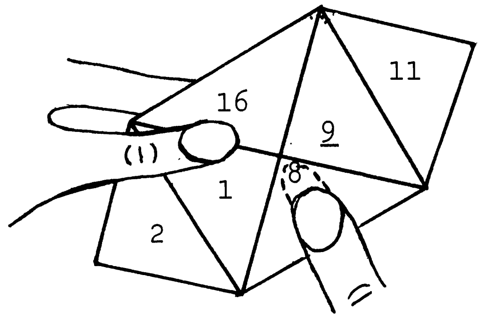

FIG. 3

Figure 2, we rotate pat 5 4 7 6 to position three at the bottom and fold the upper half of the flexagon away from us until it forms a right angle with the lower half. Insert the left thumb under the half-pat 17 16, separating it from the lower half-pat 1 18. In Figure 3, we continue the separation until we can see that triangle 16 is perpendicular to triangle 1. Simultaneously, pat 8 9 is splitting with some help from the right hand if needed. As shown in Figure 3, we pinch triangle 8 with the right thumb on top and the right index finger underneath. We pull triangle 8 towards us as far as we can, and it will look like Figure 4. Finally, we finish the V-flex by pushing the lower right corner of triangle 8 under the next half-pat 6 7 4 5, producing the pat 6 7 4 5,8 as shown in Figure 5. Thus we have V-flexed the face

$$(17\ 16\ 1\ 18-2\ 3-5\ 4\ 7\ 6-8\ 9-11\ 10\ 13\ 12-14\ 15)$$

into

$$(16\ 17-1\ 18\ 3\ 2-6\ 7\ 4\ 5,8-9-10\ 11-13\ 12\ 15\ 14).$$

For those familiar with this notation, you will notice that the first two and the last two triangular pats behaved like a pinch flex, while the center two were backwards and destroyed the symmetry.

To return to the starting position, we turn the flexagon over and rotate pat 9 into position three and V-flex again. Now when we turn it over again and rotate pat 5 4 7 6 into position three, we can start over again from Figure 2. Without any further explanation, we provide the return to the initial face in the following figures.

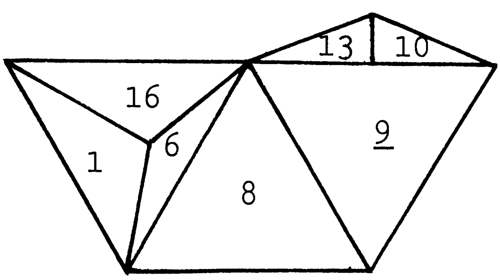

FIG. 4

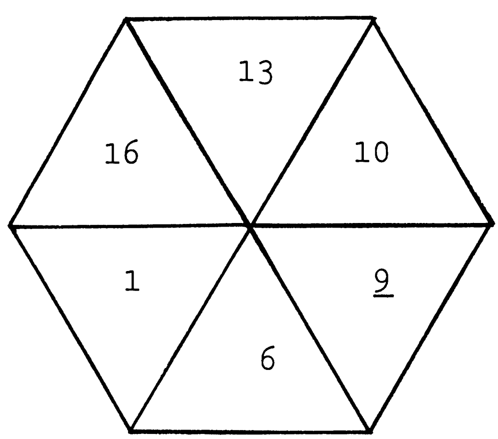

FIG. 5

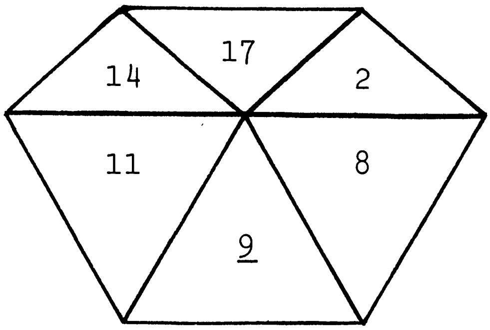

FIG. 6

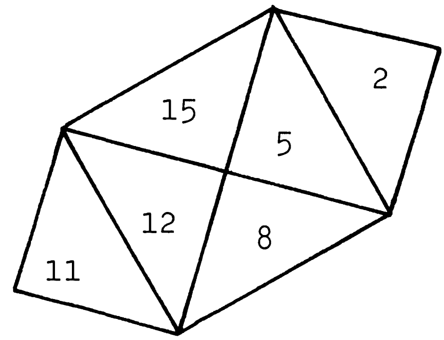

FIG. 7

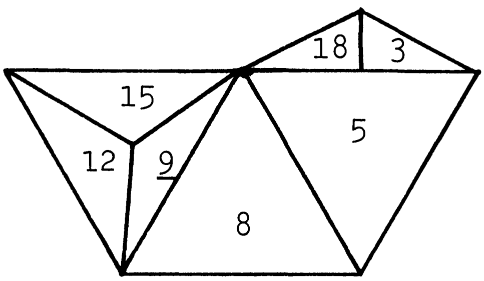

FIG. 8

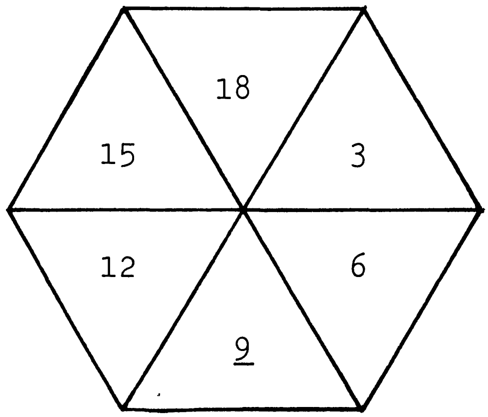

FIG. 9

5. Pinch Flex. The pinch flex is well known and all of the authors in the bibliography describe it. For our purpose, we rotate pat 5 4 7 6 into position three. Fold down and together the pats in positions three and four with the right hand. Then, with the left hand, push in the corner of the flexagon between the pats in position six and one. Finally, open into the new face without rotating. Thus we have started with

$$(17\ \ 16\ \ 1\ \ 18-2\ \ 3-5\ \ 4\ \ 7\ \ 6-8\ \ 9-11\ \ 10\ \ 13\ \ 12-14\ \ 15)$$

and pinch flexed into

$$(16\ \ 17-1\ \ 18\ \ 3\ \ 2-4\ \ 5-7\ \ 6\ \ 9\ \ 8-10\ \ 11-13\ \ 12\ \ 15\ \ 14).$$

We can continue to pinch flex until we have alternating singleton pats. At this time we need to rotate 60° and continue. This procedure will take us through the Tuckerman traverse.

6. Pats. All of the numbers referred to will be positive integers less than or equal to eighteen. Because we need to think of 16, 17, 18, 1, 2 as being consecutive, we state the following. Numbers $a_1,\dots,a_n$ are *consecutive* if they are consecutive in the usual sense or if there is a k less than n for which $a_1,\dots,a_k$ and $a_{k+1},\dots,a_n$ are consecutive in the usual sense and $a_k=18$ and $a_{k+1}=1$.

We describe what is an acceptable pat similar to the inductive definition of Oakley and Wisner [5]. Each number p is a *pat of degree one*. Recall that in Section 4, we took a pat $p=5\ 4\ 7\ 6$ and a pat $q=8$, reversed them both, and formed a new pat $\bar{p}\bar{q}=6\ 7\ 4\ 5,8$. To do this in general, we take two pats $p=a_1\dots a_{3k+c}$ and $q=b_1\dots b_{3n+c}$ where c is either 1 or 2 and permit $\bar{p}\bar{q}=a_{3k+c}\cdots a_1, b_{3n+c}\cdots b_1$ to be a *pat of degree* $3(k+n)+2c$. We also require $a_1,\dots,a_{3k+c}$, $b_1,\dots,b_{3n+c}$ to be a permutation of $3(k+n)+2c$ consecutive numbers. From this definition, it is impossible to have a pat of degree zero (mod 3); and for our purposes, we will not need pats of

degree larger than eleven. Oakley and Wisner [5] refer to the comma in the definition of $\bar{p}\bar{q}$ as a *thumbhole*. For each pat that is not a singleton, it is where you can insert your thumb to separate the upper half of the pat from the lower half. Because the thumbhole always occurs in the middle of a pat of degree two or four, we will never show it. For ease of recognition, we will indicate the location of the thumbhole for pats of higher degree.

If $p = a_1 \cdots a_{3k+c}$ and $q = b_1 \cdots b_{3k+c}$ are pats, then *q is a translation of p* if there exists an integer t, so that for every number i, $1 \leqslant i \leqslant 3k+c$, $a_i \equiv b_i + t$ (mod 18). If two pats are each of degree four or less, then they are translations of each other. However 3 4 1 2,5 is not a translation of 1,4 5 2 3. Naturally, *p will equal q* if q is a translation of p where $t=0$. It is easy to see that translation is an equivalence relation. We are interested in the number of equivalence classes of pats to assist us in counting faces. Fortunately, Table I from Oakley and Wisner [5] provides the data and we provide a brief summary below.

TABLE I

Number of Equivalent Pats of Degree $D(p)$ and Examples

$D(p)$	Partitions of $D(p)$	Number of Pats	Examples
1	1	1	1 2 3 4
2	1+1	1	1 2 2 3 3 4 4 5
4	2+2	1	2 1 4 3 3 2 5 4
5	4+1=1+4	2=1+1	3 4 1 2, 5 1, 4 5 2 3
7	5+2=2+5	4=2+2	5 2 1 4 3, 7 6
8	7+1=4+4=1+7	9=4+1+4	3 4 1 2, 7 8 5 6
10	8+2=5+5=2+8	22=9+4+9	6 5 8 7 2 1 4 3, 10 9
11	10+1=7+4=4+7=1+10	52=22+4+4+22	9 10 3 4 1 2 7 8 5 6, 11

The table above points out that there is 1 translation class of degree 2 and 52 classes of degree 11. Since there are 18 pats in each class, we note that there are a total of 18 different pats of degree 2 and 936 different pats of degree 11.

7. Faces. We start with a face $(p_1\text{-}p_2\text{-}p_3\text{-}p_4\text{-}p_5\text{-}p_6)$ that is composed of 6 pats and review the rules that governed a face before we had a V-flex. We will use $\oplus$ to denote addition modulo 6.

(a) p_i is a pat for each i,

(b) $p_i - p_{i\oplus 1}$ is a permutation of six consecutive numbers,

(c) $\sum_{i=1}^{6} D(p_i) = 18.$

From these rules, it is easy to prove that for each i, $D(p_i) = D(p_{i\oplus 2})$. When the V-flex is added to the system, rule (b) becomes

(b) $p_i - p_{i\oplus 1}$ is a permutation of $3k$ consecutive numbers.

Instead of having alternating pats of the same degree, we now have that $D(p_i) \equiv D(p_{i\oplus 2})$ (mod 3). Thus by softening rule (b), many more faces are now possible.

Before counting the faces, we now have enough notation to mathematically define the pinch flex and the V-flex. If P is a face that has been rotated so that neither p_1,p_3, nor p_5 is a singleton (pat of degree one), then a function F defined by

$$F(P) = F(\bar{a}_1\bar{b}_1 - p_2 - \bar{a}_3\bar{b}_3 - p_4 - \bar{a}_5\bar{b}_5 - p_6)$$
$$= (a_1 - \bar{b}_1\bar{p}_2 - a_3 - \bar{b}_3\bar{p}_4 - a_5 - \bar{b}_5\bar{p}_6)$$

is called a *pinch flex*. A pinch flex is possible from every face, since the triangles that are singletons can be rotated into positions 2, 4, and/or 6. For the V-flex, we need a face P that can be rotated so that neither p_1, p_4, nor p_5 is a singleton. Then a function V defined by

TABLE 2								
Number of Translation Classes and Number of Faces per Class								
Number of Singletons	$D(p_1)$	$D(p_2)$	$D(p_3)$	$D(p_4)$	$D(p_5)$	$D(p_6)$	Number of Classes	Faces per Class
0	4	2	4	2	4	2	1	6
1	1	5	4	2	4	2	2	18
	1	2	7	2	4	2	4	18
	1	2	4	5	4	2	2	18
	1	2	4	2	7	2	4	18
	1	2	4	2	4	5	2	18
2	1	8	1	2	4	2	9	18
	1	5	1	5	4	2	4	18
	1	5	1	2	7	2	8	18
	1	5	1	2	4	5	4	18
	1	2	1	8	4	2	9	18
	1	2	1	5	7	2	8	18
	1	2	1	5	4	5	4	18
	1	2	1	2	10	2	22	18
	1	2	1	2	7	5	8	18
	1	2	1	2	4	8	9	18
3	1	11	1	2	1	2	52	18
	1	8	1	5	1	2	18	18
	1	8	1	2	1	5	18	18
	1	5	1	5	1	5	2	18
	1	5	1	5	1	5	2	6
							189×18	=3402
							3× 6=	18
							192 classes	3420 faces

$$V(P) = V(\bar{a}_1\bar{b}_1 - p_2 - p_3 - \bar{a}_4\bar{b}_4 - \bar{a}_5\bar{b}_5 - p_6)$$
$$= (a_1 - \bar{b}_1\bar{p}_2 - \bar{p}_3\bar{a}_4 - b_4 - a_5 - \bar{b}_5\bar{p}_6)$$

is called a **V-flex**. If a face has three singletons, then no V-flex is possible. One with two singletons can be rotated so they end up in positions 2 and 6 so that a V-flex is possible. Also, if there is one singleton in the face, three different V-flexes are possible. Finally, if there are no singletons, then there are six different ones. Three will be placed into one translation class and the other three resulting faces will also be translations of each other.

We will not count a rotation of a face as being different, but we do wish to count a translation if the uppermost triangles are different. To do this, we define faces P and Q to be *equal* if there exists an integer r so that for each i, $p_i = q_{i\oplus r}$. Even though we are going to count translations, it will be easier if we first of all count translation classes. Thus, faces P and Q are in the same translation class if there exist integers r and t such that, for each i, pat $q_{i\oplus r}$ is a translation of p_i by t. As an example,

$$(2\ 1\ 4\ 3-5\ 6-8\ 7\ 10\ 9-11\ \ 12-14\ 13\ 16\ 15-17\ 18)$$

is in the same class as

$$(5\ 4\ 7\ 6-8\ 9-11\ \ 10\ 13\ 12-14\ \ 15-17\ 16\ 1\ 18-2\ 3)$$

where $r=0$ and $t=3$. However, these are two different faces, because for any r, p_i will not equal $q_{i\oplus r}$ for each i.

We now write down all of the translation classes, being careful to follow the rules of this section. In most cases, the number of classes that we give for each row is just the product of the number of pats that are possible from Table 1. (See Table 2.) The last two rows are symmetrical, and this doesn't apply. The classes and faces are listed by the number of singletons that they contain.

All of the above classes, with the exception of the first and the last two, permit multiplication by 18 because there is no symmetry within their windings; and we will show later how to translate a face by one triangle. By observing the initial winding at the beginning of the paper, it will be self-evident why a translation by 6 is the same as a rotation of $-60°$. Thus, we only multiply the first class by 6. For the last two classes, we list a representative of each, so that one can see that the same is true for them as well.

$$(1\text{-}2,5\ \ 6\ \ 3\ \ 4\text{-}7\text{-}8,11\ \ 12\ \ 9\ \ 10\text{-}13\text{-}14,17\ \ 18\ \ 15\ \ 16)$$

$$(1\text{-}4\ \ 5\ \ 2\ \ 3,6\text{-}7\text{-}10\ \ 11\ \ 8\ \ 9,12\text{-}13\text{-}16\ \ 17\ \ 14\ \ 15,18)$$

Even though the next to the last two classes are wound 1-5-1-5-1-5, these classes can be multiplied by eighteen because they have one pat of degree five that is not translation equivalent to the other two and symmetry is destroyed. Thus, we have 189 classes with 18 faces each and another 3 classes of 6 faces each, for a total of 3420 different faces according to our definition. We point out that some of these faces are counted as being different because their pats are wound differently even though their uppermost triangles may be identical.

8. Traverse. We begin this section by drawing the traverse for the first 100 equivalence classes that have fewer than 3 singletons. All of these classes can be visited using only the V-flex. See Figure 10.

While we are at any of the above classes, pinch flexing three times will return us to the point of departure. This traverse is given below in Table 3. Since we return to the class from which we started, only the first two pinch flexes are shown. For example, the first line should be 1-101-102-1.

Going back to Figure 10 and starting at class 1, rotate the initally wound flexagon so that pat 5 4 7 6 is in position three. We now move counterclockwise on the main cardioid by V-flexing and then rotating the lone singleton to position three before each successive V-flex. As the traverse shows, 15 of these V-flexes with the singleton at the bottom on the last 14 will return us to class one with no translation taking place. All of the remaining paths are traveled in a clockwise direction and may be initiated at any time from the cardioid by holding the singleton in position two or six before the next V-flex. The letter A on the traverse will indicate leaving the cardioid with the singleton in position two, while a B is for leaving from position six. We will refer to these departures as VA and VB, respectively. Only one letter is placed at most of the points on the cardioid because space prohibited placing the other one on. Please remember that the cardioid is traveled consecutively and counterclockwise while the remaining paths are traveled clockwise and consecutively until returning to the cardioid. Once the departure from the cardioid is made, the two singletons must be placed in positions two and six simultaneously before the next V-flex. This is easy to remember since a proper VA or VB will leave the singletons in the proper position.

We translate a face by one triangle as follows. We pick a representative from each of the first fifteen classes by traversing the entire cardioid. Starting around the cardioid a second time, we stop temporarily at class 5. By applying a VA to the representative of class 5 and taking the short cut to class 9, the singleton of this face of class 9 is five larger (mod 18) than the singleton of the representative that we noticed the first time we passed through class 9. This is recorded on the traverse in Figure 10 with the symbol 5 shown on the path entering class 9 from class 5. Since 5 and 18 are relatively prime, we will eventually be able to translate any face by one

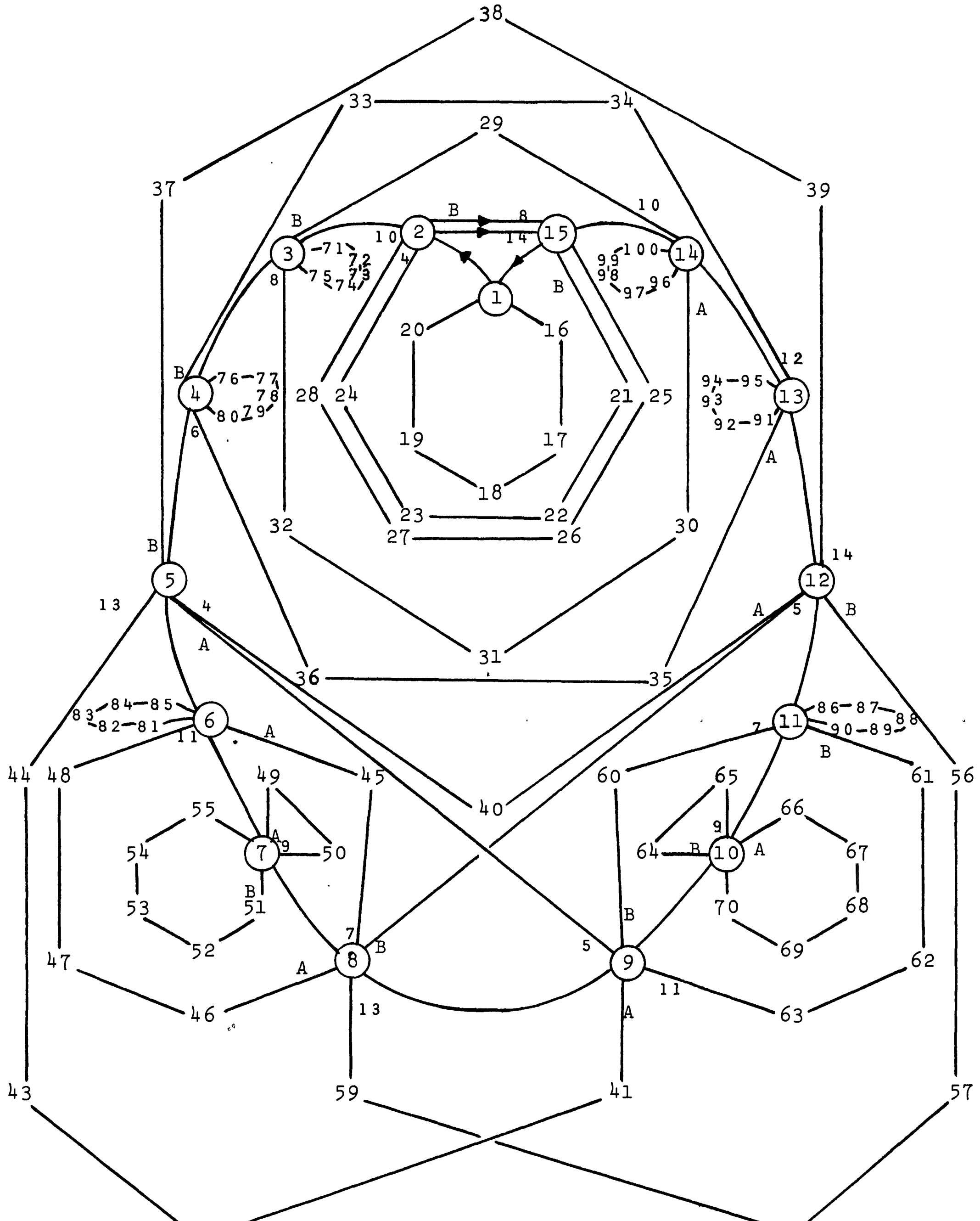

FIG. 10

triangle. Obviously, there are easier ways to accomplish this. Thus, given enough time, we can travel to each of the faces.

TABLE 3

Traverse for the Pinch Flex

1	101	102	26	94	134	51	151	48	76	138	32
2	16	103	27	135	77	52	153	154	77	27	135
3	104	105	28	71	136	53	155	156	78	173	174
4	106	107	29	21	24	54	145	42	79	175	176
5	64	108	30	137	95	55	46	149	80	37	141
6	109	110	31	59	41	56	157	90	81	148	44
7	111	112	32	76	138	57	158	159	82	177	178
8	35	113	33	139	75	58	160	67	83	179	180
9	114	36	34	96	140	59	41	31	84	150	47
10	115	116	35	113	8	60	40	45	85	49	152
11	117	118	36	9	114	61	161	70	86	164	65
12	119	50	37	141	80	62	162	87	87	62	162
13	120	121	38	142	143	63	66	163	88	181	182
14	122	123	39	91	144	64	108	5	89	183	184
15	124	20	40	45	60	65	86	164	90	56	157
16	103	2	41	31	59	66	163	63	91	144	39
17	99	125	42	54	145	67	58	160	92	185	186
18	126	127	43	146	147	68	165	166	93	187	188
19	128	72	44	81	148	69	167	168	94	134	26
20	15	124	45	60	40	70	61	161	95	30	137
21	24	29	46	149	55	71	136	28	96	140	34
22	129	130	47	84	150	72	19	128	97	189	190
23	131	132	48	51	151	73	169	170	98	191	192
24	29	21	49	152	85	74	171	172	99	125	17
25	133	100	50	12	119	75	33	139	100	25	133

9. Conclusion. The next obvious step in this adventure is to count how many of the faces actually appear different on the surface. It is our belief that when the individual pats that make up a face get rotated you have a different appearing face. This would mean that the hexahexaflexagon with just the pinch flex would have 9 faces since 3 of the faces appear a second time in the traverse with their pats rotated. We simply state that when the V-flex is added and rotations of pats are counted, then our figures show that 1314 of the previously defined 3420 faces actually appear to be different on the surface. We had to modify our numbering system by labeling each of the six corners of each of the triangles to obtain this count of 1314.

Other projects that haven't even begun would be to discover the traverse for larger hexaflexagons and for some of the tetraflexagons. Alan Moluf was very successful in building and V-flexing many different types of flexagons, including square ones. Martin Gardner [2] provides us with some information on square ones, but probably not V-flexible as they are described.

Upon request, the author will furnish a computer printout that shows (1) one representative from each of the 192 equivalence classes, (2) the degree of each pat that makes up the face, and (3) the location of the thumbholes. With this information, it is easy to observe that none of the classes is a translation of any of the others.

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UNIVERSAL ALGEBRA AND EULER'S OFFICER PROBLEM

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Introduction. At a first glance it seems unlikely that there is any useful connection between universal algebra and combinatorics. Both are areas of mathematics hard to describe precisely, and it might seem as if this is the only characteristic they have in common. Universal algebra falls somewhere between algebra and logic and is concerned with problems and concepts which have some "universal" flavor, i.e., are common to different branches of algebra. On the other hand, combinatorics is usually concerned with specific problems—for example, the study of finite patterns satisfying certain conditions and the existence and enumeration of such patterns.

Second thoughts suggest, however, that since combinatorics has made frequent use of "classical" algebraic structures, such as finite fields, permutation groups, matrices, etc., perhaps the universal-algebraic approach and the more exotic algebraic structures of recent years may also be useful. This is indeed the case, and a persuasive argument can be made that universal algebra rather than classical algebra often provides the appropriate point of view for applying algebra to combinatorics. We wish to illustrate here the fruitful interplay between universal algebra and combinatorics by applying universal algebra to the famous Euler conjecture on the existence of mutually orthogonal latin squares. The universal algebraic approach not only gives very simple counterexamples with a minimum of computation but it puts the conjecture in a context where it becomes, not just a special problem, but a part of a general theory (which is, after all, the main purpose of universal algebra).

We show here just one application of the rapidly growing use of universal algebraic methods in combinatorics. We give a brief survey of other work in the same vein at the end of the paper.

1. Euler's Officer Problem. Suppose we have thirty-six officers, six from each of six different regiments and, in each regiment, one from each of six different ranks. We wish to arrange these officers in a 6×6 square so that (i) each regiment is represented exactly once in each row and column, (ii) each rank is represented exactly once in each row and column. Can this be done? The problem was put to Euler in 1779. In trying to solve it, he generalized it and made a far-reaching conjecture which was laid to rest only in 1959–60 by the combined work of E. T. Parker, R. C. Bose, and S. S. Shrikhande [2], [3], [4].

To put the problem and its generalization in a more formal context, we need several

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definitions. By a *latin square of order n*, based on $a_1, a_2, \ldots, a_n$, we mean an $n \times n$ array of n^2 cells, each of which is occupied by one of $a_1, a_2, \ldots, a_n$, and such that in each row and each column of the array each a_i occurs exactly once. (Throughout the paper, we will use $1,2,\ldots,n$ for the entries, rather than $a_1, a_2, \ldots, a_n$.) For example, Figure 1 shows latin squares of order 3 and 4. Two latin squares $\mathcal{A}$, $\mathcal{B}$ of order n are said to be *orthogonal* if when we superimpose $\mathcal{B}$ on $\mathcal{A}$ so that the cell in the ith row, jth column contains the ordered pair (a_{ij}, b_{ij}), a_{ij} from $\mathcal{A}$, b_{ij} from $\mathcal{B}$, then every ordered pair from $\{1,2,\ldots,n\}$ occurs in some cell (and hence occurs exactly once). In other words, the mapping $(i,j) \rightarrow (a_{ij}, b_{ij})$ is a bijection. In Figure 2, (a) shows two orthogonal 3×3 latin squares, (b) shows two 3×3 squares which are not orthogonal.

1	3	2
3	2	1
2	1	3

1	2	3	4
2	1	4	3
3	4	1	2
4	3	2	1

FIG. 1

(a)

1	2	3
2	3	1
3	1	2

1	2	3
3	1	2
2	3	1

→

1,1	2,2	3,3
2,3	3,1	1,2
3,2	1,3	2,1

(b)

1	2	3
2	3	1
3	1	2

1	3	2
3	2	1
2	1	3

→

1,1	2,3	3,2
2,3	3,2	1,1
3,2	1,1	2,3

FIG. 2

The *Officer Problem*, in this terminology, asks whether there are two orthogonal latin squares of order six. Numbering the regiments and ranks from 1 to 6, each officer becomes an ordered pair (i, j) where i denotes his regiment and j his rank. The requirements of the problem are equivalent to arranging the officers so that the regiments and the ranks are orthogonal latin squares of order six.

There are over 800 million different latin squares of order six, yet no two of them are orthogonal. Euler was not able to prove this, but he generalized the problem and conjectured that, if $n \equiv 2 \pmod 4$, then there do not exist two orthogonal latin squares of order n. It is not clear what led to this, the famous *Euler conjecture*—perhaps he was able to construct two orthogonal latin squares of every other order. This is not hard to do by constructing two of order four, two of order eight, and two of any odd order. A direct product construction then gives a pair of orthogonal latin squares of all orders except twice an odd number, that is, all numbers not congruent to 2, modulo 4.

The early results about the conjecture tended to support it. Tarry, 1900, proved that there was no pair of 6×6 orthogonal latin squares, thus solving the original Officer Problem. No group table of order $n \equiv 2 \pmod 4$ can have a latin square orthogonal to it, Mann [**15**]. Furthermore, Macneish described a simple method, using finite fields and direct products, which gives orthogonal squares of all orders $n \not\equiv 2 \pmod 4$ but fails for orders $n \equiv 2 \pmod 4$. This led him to the conjecture, a generalization of Euler's conjecture, that if $n = p_1^{s_1} p_2^{s_2} \cdots p_t^{s_t}$, where the p_i are primes, then the maximum number of pairwise orthogonal latin squares of order n is $\min_i (p_i^{s_i} - 1)$. His construction showed that there were at least this many.

The combined work of Parker, Bose, and Shrikhande settled the conjecture completely by showing that Macneish's conjecture was false and that Euler was almost totally wrong. Except for $n = 2, 6$, there exists a pair of orthogonal latin squares for all $n \equiv 2 \pmod 4$.

The original counterexamples required difficult techniques and the constructions are quite complicated. We will show that universal algebra insight (or perhaps it should be hindsight) not only enables us to give counterexamples with a minimal amount of computation but provides a pleasing background of general theory into which the problem fits quite naturally. Sections 2 and 3 give some elementary ideas from universal algebra; Section 4 is the core of the paper; Section 5 weaves together the algebra with a major branch of combinatorics, the theory of block designs; and finally, in Section 6, we construct the counterexamples.

2. Algebras. By an *algebra* $\mathcal{A}$ we will mean a non-empty set A, the *elements* of the algebra, and a finite set Ω of mappings $A \to A, A^2 \to A, A^3 \to A, \ldots$, the *operations* of the algebra. We write $\mathcal{A} = (A, \Omega)$. A mapping $f: A \to A$ in Ω is called a unary operation, a mapping $g: A^2 \to A$ is called a binary operation, and so on. To denote the value an operation takes, we will either use ordinary functional notation, $f(a)$, $g(a_1, a_2)$, etc., or, for binary operations (with which we will be mainly concerned), the usual notation $a_1 \cdot a_2$, $a_1 + a_2$, $a_1 \times a_2$, $a_1 \circ a_2$, etc.

The reader will already be familiar with many different types of algebras, groups, rings, fields, lattices, etc. The usual ideas of subalgebra, homomorphism, direct product, from groups and rings, can be extended in the obvious way to apply to the general notion of an algebra we have given. Only the direct product construction will be needed, however.

We will be working mainly with algebras called *quasigroups*, usually described in terms of one binary operation *multiplication*, denoted by $x \cdot y$, which satisfies the conditions that for any a, b in the quasigroup, there exist unique solutions x, y in the quasigroup to the equations

$$a \cdot x = b, \qquad y \cdot a = b.$$

·	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

(a)

·	1	2	3	4
1	4	3	1	2
2	3	4	2	1
3	2	1	3	4
4	1	2	4	3

(b)

·	1	2	3	4	5
1	1	3	4	5	2
2	5	2	1	3	4
3	4	5	3	2	1
4	2	1	5	4	3
5	3	4	2	1	5

(c)

FIG. 3

In Figure 3, we show the multiplication tables for quasigroups of orders 3, 4, 5. The quasigroups shown in tables (a), (c) have the extra property that multiplication is *idempotent*, $x^2 = x$ for every element x.

Our interest in quasigroups is due to the fact that the multiplication table of a quasigroup is a latin square. Conversely, by bordering a latin square with $1,2,\ldots,n$ as side-line and head-line, we obtain the multiplication table of a quasigroup. A latin square is idempotent if its main diagonal is $1,2,\ldots,n$ in that order.

We can also make a pair of orthogonal latin squares into an algebra. We require a set $A=\{1,2,\ldots,n\}$ and two binary operations $x\cdot y$, $x\circ y$ such that A is a quasigroup under each operation. This gives us two latin squares of order n. The definition of orthogonality corresponds exactly to the requirement that, for any a, b in A, the equations $x\cdot y=a$, $x\circ y=b$ have unique solutions for x and y.

THEOREM 1. *The quasigroups* $(A,\ \cdot)$, $(A,\ \circ)$ *have orthogonal tables if and only if, for any* a, b *in* A, *the equations*

$$x\cdot y=a$$
$$x\circ y=b$$

have unique solutions in A.

3. Varieties. Universal algebra deals with classes of algebras as much as with single algebras. One of the most natural classes to consider is called a *variety* (or *equational class*). Let $\mathcal{A}=(A,\ \Omega)$ be an algebra. If $u(x_1,x_2,x_3,\ldots)$, $v(x_1,x_2,x_3,\ldots)$ are two "well-formed" expressions built up from variables $x_1,x_2,x_3,\ldots$ and the operation symbols of Ω, such that $u(a_1,a_2,a_3,\ldots)=v(a_1,a_2,a_3,\ldots)$ is true in $\mathcal{A}$ for all elements $a_1,a_2,a_3,\ldots$ in A, then we say that $\mathcal{A}$ satisfies the *identity*

$$u(x_1,x_2,x_3,\ldots)=v(x_1,x_2,x_3,\ldots).$$

By a *variety* V we mean a class of operationally similar algebras $\mathcal{A}=(A,\ \Omega)$, i.e., with the "same" set Ω of operations, consisting of all algebras which satisfy some given set of identities involving variables and the operations Ω. In other words, a variety is a class of algebras which can be characterized by axioms, each of which has the form of an identity. Groups, rings, lattices, are familiar examples of varieties. However, neither the class of integral domains nor that of fields is a variety.

One of the main virtues of a variety as a class of algebras is that it is closed under the most important ways in which we construct new algebras from old, taking subalgebras, quotient algebras, and direct products. (A famous theorem of Birkhoff [1] states that the converse of this is also true.)

From the way we have defined quasigroups, it is not clear that this class of algebras is a variety. However, a simple trick, replacing the unique solutions x, y of the equations $a\cdot x=b$, $y\cdot a=b$ by new operations (see Evans, [7]) enables us to redefine the class of quasigroups as a variety.

A *quasigroup* $\mathcal{A}=(A;\ \cdot,\ \diagdown,\ /)$ is an algebra consisting of a non-empty set A and three binary operations, *multiplication* $(\cdot)$, *left-division* $(\diagdown)$, *right-division* $(/)$, such that the identities

$$\begin{aligned} x\cdot(x\diagdown y)&=y, & (x/y)\cdot y&=x,\\ x\diagdown(x\cdot y)&=y, & (x\cdot y)/y&=x, \end{aligned} \tag{3.1}$$

hold for all x, y in A. These identities are equivalent to the requirement that the equations $a\cdot x=b$, $y\cdot a=b$ have unique solutions $x=a\diagdown b$, $y=b/a$.

4. Varieties of orthogonal latin squares. A similar procedure to that used above to define quasigroups in terms of identities enables us to turn the class of pairs of orthogonal quasigroups into a variety. We begin with an algebra $\mathcal{A}=(A;\cdot,\diagdown,/,\circ,\circ\!\!\!\diagdown,\circ\!\!\!/,*,+)$ having eight binary operations. We assume that the operations $\cdot$, $\diagdown$, $/$ satisfy the quasigroup identities (3.1) and that A under the operations $\circ$, $\circ\!\!\!\diagdown$, $\circ\!\!\!/$ also satisfies quasigroup identities $(3.1)^\circ$ similar to (3.1) with $\circ$, $\circ\!\!\!\diagdown$, $\circ\!\!\!/$ replacing $\cdot$, $\diagdown$, $/$. Hence, the tables for $\cdot$ and $\circ$ are latin squares on A. We further

assume that $\mathcal{Q}$ satisfies the identities

$$\begin{aligned}(x*y)\cdot(x+y)&=x\\(x\cdot y)*(x\circ y)&=x\\(x*y)\circ(x+y)&=y\\(x\cdot y)+(x\circ y)&=y\end{aligned}\tag{3.2}$$

for all x, y in A. These identities (3.2) guarantee that the equations $x\cdot y=a$, $x\circ y=b$ have unique solutions $x=a*b$, $y=a+b$.

THEOREM 2. *In each algebra of the variety $V^{(2)}$ defined by* (3.1), (3.1)°, (3.2), *the operations* $\cdot$, $\circ$ *have orthogonal latin squares as tables.*

Furthermore, any pair of orthogonal latin squares can be regarded as an algebra in this variety. We define $\cdot$, $\circ$ by the latin squares, then add the appropriate division operations. We define $x*y$, $x+y$ to be the unique elements u, v satisfying $u\cdot v=x$, $u\circ v=y$.

We will need some generalizations of this result. By a *k-set of mutually orthogonal latin squares* (m.o.l.s.) of order n, we mean k latin squares on $\{1,2,\ldots,n\}$ such that each pair is orthogonal.

THEOREM 3. *For each $k\geq 2$, there exists a variety $V^{(k)}$ of algebras such that k of its binary operations form a k-set of* m.o.l.s. *on each finite algebra in $V^{(k)}$.*

Proof. We can write down the list of operations and identities for $V^{(k)}$. We need $3k$ binary operations to give k quasigroups and $4k$ identities similar to (3.1) to define them as quasigroups. Then $\dfrac{k(k-1)}{2}$ quadruples of identities like (3.2) will give the orthogonality of each pair of the quasigroup multiplication tables.

Let $W^{(k)}$ denote the variety obtained from $V^{(k)}$ by adding the *idempotent law* $x\cdot x=x$, $x\circ x=x$, etc., for each binary operation in $V^{(k)}$. Then $W^{(k)}$ is a *subvariety* of $V^{(k)}$, by which we mean that every $W^{(k)}$-algebra is a $V^{(k)}$-algebra.

Since every k-set of m.o.l.s. can be turned into a $V^{(k)}$-algebra and every k-set of idempotent m.o.l.s. can be turned into a $W^{(k)}$-algebra, we may justifiably claim that

(i) $V^{(k)}$ is the variety of all k-sets of m.o.l.s.

(ii) $W^{(k)}$ is the variety of all k-sets of idempotent m.o.l.s.

The next well-known theorem describes some algebras in $V^{(k)}$ and $W^{(k)}$.

THEOREM 4.

(i) *There exists an algebra in $V^{(k)}$ of order q for every prime power $q\geq k+1$.*

(ii) *There exists an algebra in $W^{(k)}$ of order q for every prime power $q\geq k+2$.*

Proof. Let $(F, +, \cdot)$ be the finite field of order q. To prove (i) let $m_i(x,y)=f_ix+y, i=1,2,\ldots,k$ for any $k\leq q-1$ non-zero elements $f_1,f_2,\ldots,f_k$ of F and define k quasigroup multiplications m_i on F by these equations. By Theorem 1, the corresponding tables are orthogonal latin squares and so we have an algebra in $V^{(k)}$. To prove (ii) let $m_i(x,y)=f_ix+(1-f_i)y, i=1,2,\ldots,k$ for any $k\leq q-2$ elements of F, $f_1,f_2,\ldots,f_k$ not equal to 0 or 1. Define k quasigroup multiplications m_i on F by these equations. The corresponding tables are idempotent latin squares, each pair of which is orthogonal, and so we have an algebra in $W^{(k)}$.

Since a variety is closed under direct products, we have a slight generalization of a theorem of Macneish **[14]**. A higher dimensional analogue of this is given in Evans **[9]**.

THEOREM 5. *If there is a k-set of* (*idempotent*) m.o.l.s. *of each of the orders n_1, n_2, then there is a k-set of* (*idempotent*) m.o.l.s. *of order n_1n_2.*

Another theorem of Macneish **[14]** has a similar mild generalization.

THEOREM 6. *If* $n = p_1^{s_1} p_2^{s_2} \cdots p_t^{s_t}$, *where the* p_i*'s are primes, then*
(i) *there are at least* $\min_i(p_i^{s_i} - 1)$ m.o.l.s. *of order* n
(ii) *there are at least* $\min_i(p_i^{s_i} - 2)$ *idempotent* m.o.l.s. *of order* n.

Proof. This follows immediately from Theorems 4 and 5.
We will need one more piece of information connecting the varieties $V^{(k)}$ and $W^{(k)}$.

THEOREM 7. *Let* $L_1, L_2, \ldots, L_k$ *be a k-set of* m.o.l.s. *of order n. Then there exists a* $(k-1)$*-set of idempotent* m.o.l.s. *of order n.*

Proof. Let the symbol 1 occur in the cells $(1, i_1)$, $(2, i_2), \ldots, (n, i_n)$ in the table L_k. Then these cells in each of $L_1, L_2, \ldots, L_{k-1}$ must be occupied by $1, 2, \ldots, n$ in some order. Apply the appropriate permutation to the symbols in each of $L_1, L_2, \ldots, L_{k-1}$ to obtain $L_1', L_2', \ldots, L_{k-1}'$ where in each, the cells $(1, i_1), (2, i_2), \ldots, (n, i_n)$ are occupied by $1, 2, \ldots, n$ in their natural order, i.e., j in cell (j, i_j). Note that $L_1', L_2', \ldots, L_{k-1}', L_k$ are still m.o.l.s. Now permute simultaneously the columns in $L_1', L_2', \ldots, L_{k-1}'$ so that the cells $(1, i_1), (2, i_2), \ldots, (n, i_n)$ are moved to the main diagonal. We obtain idempotent latin squares $L_1'', L_2'', \ldots, L_{k-1}''$ which are still mutually orthogonal.

THEOREM 8. *If* $V^{(k)}$ $(k \geqslant 3)$ *contains an algebra of order n, then* $W^{(k-1)}$ *contains an algebra of order n.*

We can give a more algebraic flavor to the above proof by describing directly the construction of the $W^{(k-1)}$-algebra from a $V^{(k)}$-algebra. The $k-1$ quasigroup multiplications of the algebra in $W^{(k-1)}$ are *isotopes* of $k-1$ quasigroup multiplications of the k multiplications of an algebra in $V^{(k)}$. In other words, we choose $k-1$ of the k quasigroup multiplications in the $V^{(k)}$-algebra, say $m_1, m_2, \ldots, m_{k-1}$, then define new quasigroup multiplications $m_i'(x,y) = \{m_i(x, y\alpha)\}\beta, i = 1, 2, \ldots, k-1$, where α, β are permutations depending on the kth quasigroup multiplication m_k in the original $V^{(k)}$-algebra.

5. Block designs. We need some simple ideas about certain kinds of patterns called *block designs*. By a *2-design* we mean a finite set A and a finite collection $B_1, B_2, B_3, \ldots$ of subsets of A, called *blocks* such that
(i) $|B_i| \geqslant 2, i = 1, 2, 3, \ldots$
(ii) each pair of distinct elements x, y in A belong to exactly one block.
We write

$$D = \{A; B_1, B_2, B_3, \ldots\}$$

for such a system. We remark that we have adopted, for simplicity, slightly nonstandard terminology in calling such a system a 2-design. In general, by a t-(v, K, λ) design is meant a block design in which $|A| = v, |B_i| \in K$ and every t-element subset of A occurs in exactly λ blocks.
Examples of 2-designs are abundant—e.g., see M. Hall [13]. We will need only one.

Example. Let π be a finite plane (affine or projective). Then the set of points in π forms a 2-design if the blocks are taken to be the lines in π.
We leave the proof of the next theorem to the reader.

THEOREM 9. *If* $D = \{A; B_1, B_2, B_3, \ldots\}$ *is a 2-design and S is a subset of A such that* $|B_i - S| \geqslant 2$, *then* $\{A - S; B_1 - S, B_2 - S, \ldots\}$ *is a 2-design.*

The next theorem has been quoted and rediscovered so often that it is now almost a piece of folklore of universal combinatorial algebra although it first appeared, at least implicitly, in Stein [21]. For certain kinds of varieties, it allows us to use 2-designs to construct new algebras in the variety from known algebras.
We will call a variety V a b.i.-variety (for *binary idempotent*) if it satisfies the following:

(i) every operation is binary;
(ii) every operation satisfies an idempotent law, $x \cdot x = x$, etc.;
(iii) the defining identities involve at most 2 variables.

THEOREM 10. *Let $D = \{A; B_1, B_2, B_3, \ldots\}$ be a 2-design and let V be a* b.i.*-variety. If V contains algebras of order $|B_i|$, $i = 1,2,3,\ldots$ then V contains an algebra of order $|A|$.*

Proof. This is almost obvious. Construct a V-algebra on each B_i. Since each pair of elements of A occurs in exactly one B_i, each operation is defined unambiguously on A for each pair of elements, $x \neq y$. Idempotence takes care of the other products. The defining identities of V involve only two variables and so hold universally in A because they hold in each B_i.

The next result may be thought of as perhaps the main theorem in the paper. It follows immediately from Theorems 8, 10 and gives the results used by Parker **[18]**, in disposing of the Macneish conjecture, and Bose, Shrikhande **[2]**, in their first counterexample to the Euler conjecture.

THEOREM 11. *Let $D = \{A; B_1, B_2, B_3, \ldots\}$ be a 2-design such that on each B_i we can construct a k_i-set of* m.o.l.s. *Then, on A, we can construct a k-set of idempotent* m.o.l.s. *where $k = \min_i k_i - 1$.*

Proof. On each B_i, we can construct a $V^{(k)}$-algebra (Th. 4). Hence, on each B_i we can construct a $W^{(k-1)}$-algebra (Th. 8). Hence, by Theorem 10, we can construct a $W^{(k-1)}$-algebra on A.

Parker's result is the special case where all the $|B_i| = k$, the order of some projective plane. Here, $k_i = k - 1$, and so we obtain $k - 2$ m.o.l.s. on A.

6. Euler's Conjecture. Our strategy should now be clear. We know that we can produce a k-set of idempotent m.o.l.s. and hence an algebra in $W^{(k)}$ of any prime power order $q \geqslant k + 2$. Furthermore, using a 2-design $D = \{A; B_1, B_2, B_3, \ldots\}$ we can construct a $W^{(k)}$-algebra on A if we can construct one on each B_i. All we have to do now is produce the appropriate 2-design with $|A| \equiv 2 \pmod 4$ and each $|B_i|$ a prime power $\geqslant k + 2$, in order to obtain a k-set of m.o.l.s. of order $|A|$, contradicting the Euler conjecture.

Let π be the projective plane of order 7. This has $57 = 7^2 + 7 + 1$ points and each line contains $8 = 7 + 1$ points. Choose three points on one line and remove these from π to obtain a block design (see Theorem 9 and example in Section 5) having 54 elements with each block containing 8, 7, or 5 points. Since there are $W^{(3)}$-algebras of orders 5, 7, 8, there is one of order 54. That is, there is a set of three mutually orthogonal idempotent latin squares of order 54.

As another example, consider the projective plane of order 8 having $73 = 8^2 + 8 + 1$ points and remove three non-collinear points. The resulting block design has 70 elements and the blocks have 7, 8, or 9 elements. Hence, there is a set of five mutually orthogonal idempotent latin squares of order 70.

THEOREM 12. *There are an infinite number of values of $n \equiv 2 \pmod 4$ for which there exist a pair of* m.o.l.s.

Proof. The variety $V^{(2)}$ contains an algebra of every odd order $\geqslant 3$ and also one of order 54. Since the product of an odd positive integer and 54 is congruent to 2 modulo 4, the theorem follows.

We can also obtain (a weaker form of) Parker's counterexample to the Macneish conjecture, which he found just before Bose and Shrikhande found the first counterexample to the Euler conjecture. Parker produced four m.o.l.s. of order 21, whereas the Macneish conjecture predicts only two. We produce three. Let π be the projective plane of order 4. As a block design, this contains 21 points with each block containing 5 points. Hence, since there is a $W^{(3)}$-algebra of order 5 there is one of order 21. That is, there are three idempotent m.o.l.s. of order 21.

7. Some references. As we remarked in the introduction, the classical algebraic structures, such as finite fields and permutation groups, have long been used in the theory and construction of combinatorial designs. However, the not-so-familiar Veblen-Wedderburn systems and near-fields also have a long history (in the foundations of geometry) going back to the beginning of this century. A discussion of these algebras is given in Hall [12]. They can be used, as fields were in Theorem 4, to obtain sets of m.o.l.s. Schroeder, toward the end of the nineteenth century, was probably the first to study the algebraic structure of quasigroups, although this area of algebra has only begun to flourish since the 1930's. The recent book [6] by Denes and Keedwell is a comprehensive treatment of latin squares and contains a detailed account both of the history of the Euler conjecture and of its disposal. Bruck [5] describes how to associate an algebra with a block design—in the special case of Steiner triple systems where all the blocks have size three. Recently Ganter and Werner [11] used universal algebraic techniques to study algebras which are associated with 2-designs where all the blocks have the same prime-power size, and Quackenbush [19] has extended this to the case where the block sizes lie in some set of prime powers. Stein [20], [21] and Mendelsohn [17] were among the first to view combinatorial problems from a universal algebraic point of view. A detailed study of algebras associated with orthogonal latin squares is in preparation (Evans [10]), and a brief survey of these results may be found in [8] which also contains a survey paper by Mendelsohn [17] on universal algebra and combinatorics. The above list is far from comprehensive, but the papers described do give a representative flavor of the approach I have tried to illustrate in this paper.

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PROGRESS REPORTS

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It is easy to be too busy to pay attention to what anyone else is doing, but not good. All of us should know, and want to know, what has been discovered since our formal education ended, but new words, and relations between them, are growing too fast to keep up. It is possible for a person to learn of the title of a recent work and of the key words used in it and still not have the faintest idea of what the subject is.

Progress Reports is to be an almost periodic column intended to increase everyone's mathematical information about what others have been up to. Each column will report one step forward in the mathematics of our time. The purpose is to inform, more than to instruct: what is the name of the subject, what are some of the words it uses, what is a typical question, what is the answer, who found it. The emphasis will be on concrete questions and answers (theorems), and not on general contexts and techniques (theories). References will be kept minimal: usually they will include only one of the earliest papers in which the answer appears and a more recent exposition of the discovery, whenever one is easily available.

Everyone is invited to nominate subjects to be reported on and authors to prepare the reports. The ground rules are that the principal theorem should be old enough to have been published in the usual sense of that word (and not just circulated by word of mouth or in preprints); it should be of interest to more than just a few specialists; and it should be new enough to have an effect on the mathematical life of the present and near future. In practice most reports will probably be on progress achieved somewhere between 5 and 15 years ago.

SEMISIMPLE ARTINIAN RINGS OF QUOTIENTS

JONATHAN S. GOLAN

If R is a commutative ring and A is the set of all regular elements of R (namely those elements of R that are not zero-divisors), then a standard construction, analogous to the construction of the rational numbers from the integers, allows us to obtain the ring of quotients of R by "inverting" all of the elements of A. Moreover, we have reason to expect that the ring thus obtained will be "nicer" than the ring we started out with. Thus, for example, the rational numbers form a field, while the integers are only an integral domain.

Can we do the same thing for noncommutative rings? In other words, if R is a noncommutative ring and if A is the set of all regular elements of R, does there exist a ring S containing R such that (i) every element of A is invertible in S, and (ii) every element of S is of the form $a^{-1}r$, where $a \in A$ and $r \in R$? If such a ring S does exist, we say that S is *a classical left ring of quotients* of R. A necessary and sufficient condition for a noncommutative ring R to have a classical left ring of quotients was given by Jacobson (1943) and Asano (1949), based on the work of Ore (1931): it is that for each r in R and each a in A there must exist elements r' in R and a' in A such that $a'r = r'a$. If R is a domain (i.e., if $A = R \backslash \{0\}$), then this condition is equivalent to the condition that any two nonzero left ideals of R have a nonzero intersection. It is satisfied, for example, by domains satisfying the ascending chain condition on left ideals.

Having found a necessary and sufficient condition for the existence of a classical left ring of quotients, we now look for conditions that guarantee that this ring is "nice" in some prescribed sense. One class of rings that are surely nice consists of the ones isomorphic to full matrix rings over division rings—such rings are called *simple artinian*. Another class of "nice" rings consists of the ones isomorphic to finite direct products of simple artinian rings—such rings are called *semisimple artinian*. The first big breakthrough in solving our problem was made by Goldie (1958, 1960) and by Lesieur and Croisot (1960), who completely characterized the noncommutative rings R whose classical left ring of quotients S is "nice" in the above senses. Indeed, S is simple artinian if and only if R satisfies the following three conditions:

(G1) R contains no infinite direct sum of nonzero left ideals;

(G2) R satisfies the ascending chain condition on left annihilators;

(P) Any product of nonzero two-sided ideals of R is nonzero.

Moreover, S is semisimple artinian if and only if condition (P) is replaced by the weaker condition.

(SP) The square of a nonzero two-sided ideal of R is nonzero.

Rings satisfying conditions (G1) and (G2) have since become known as *left Goldie rings*. Rings satisfying the ascending chain condition on left ideals are surely left Goldie rings.

We could, in a manner analogous to the above, define the notion of a classical *right* ring of quotients of a noncommutative ring and the right-handed version of Goldie's theorems would then tell us when this ring of quotients is "nice". In general, there is no reason to suppose that the right and left rings of quotients so defined would coincide, and it is therefore surprising to find out that there are important classes of rings for which that is indeed so. In 1960, Posner proved that if R is a ring satisfying condition (P), and satisfying a polynomial identity with coefficients in its center, then R is both a left and a right Goldie ring and the left and right classical rings of quotients of R coincide and satisfy the same identity as R. Posner's Theorem was generalized by Small (1966) to the case of rings satisfying the weaker condition (SP). In particular, Small's results imply that if R is a subring of the full ring of matrices over a commutative ring C and if (i) R satisfies (SP) and (ii) C satisfies the ascending chain condition on annihilators, then R is both a left and a right Goldie ring and the left and right classical rings of quotients of R coincide.

The initial results obtained by Goldie triggered a surge of interest in rings of quotients of noncommutative rings and their properties. Since the classical left ring of quotients of a ring R is but one of a large family of "rings of quotients" which can be defined for R, it is natural to ask when these rings can be "nice" in the above senses. This problem was solved by Sandomierski (1967) for the case of Utumi's "maximal left ring of quotients," and in the more general case by Page (1972), who showed that, if S is a left quotient ring of R (in any of the accepted senses) that contains R and is semisimple artinian, then S must equal the maximal left ring of quotients of R. Zelmanowitz (1967) has shown that if a ring R has a classical left ring of quotients which is semisimple (or simple) artinian, then the same is true for the classical left ring of quotients of the ring of endomorphisms of any finite-dimensional torsionless left R-module. This result has recently been extended to other rings of quotients by Leu and Hutchinson (1977).

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MISCELLANEA

24. [People believe] that beating the market is in the short run largely accidental and in the long run impossible. Success is explained by the same law of averages that says if you can flip a half dozen heads in a row, the next half dozen tosses will come up tails.

—Harvey D. Shapiro, *New York Magazine*, August 14, 1978, p. 49.
(Suggested by W. F. Eberlein)

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ON ISOPERIMETRIC INEQUALITIES RELATED TO A PROBLEM OF MOSER

E. LUTWAK

The setting for this note will be the Euclidean plane. The letter K will be used to denote compact convex sets, exclusively. The area of K will be denoted by $|K|$ and the length of its boundary will be denoted by $L(K)$. If a fixed direction in the plane is taken to be our reference, then every angle θ will determine a direction (with respect to the reference direction). We shall use $b_K(\theta)$ to denote the width of K in the direction (determined by) θ; that is, the distance between the supporting lines of K which are perpendicular to the direction θ. K is said to have constant width if it has the same width in all directions.

For a given θ, let $R(K,\theta)$ denote the rectangle circumscribed about K that has one of its sides perpendicular to the direction determined by θ. Clearly,

$$|R(K,\theta)| = b_K(\theta)b_K\left(\theta+\frac{\pi}{2}\right).$$

If we take the minimum over all θ, we obtain

$$\left[\min_\theta |R(K,\theta)|\right]^{\frac{1}{2}} \leqslant \frac{1}{2\pi}\int_0^{2\pi} b_K(\theta)^{\frac{1}{2}} b_K\left(\theta+\frac{\pi}{2}\right)^{\frac{1}{2}} d\theta, \tag{1}$$

with equality if and only if $b_K(\theta)b_K(\theta+\frac{\pi}{2})$ is constant for all θ. From the Cauchy-Schwarz inequality, we have

$$\left[\int_0^{2\pi} b_K(\theta)^{\frac{1}{2}} b_K\left(\theta+\frac{\pi}{2}\right)^{\frac{1}{2}} d\theta\right]^2 \leqslant \left[\int_0^{2\pi} b_K(\theta)\,d\theta\right]\left[\int_0^{2\pi} b_K\left(\theta+\frac{\pi}{2}\right)d\theta\right], \tag{2}$$

with equality if and only if $b_K(\theta)/b_K(\theta+\frac{\pi}{2})$ is constant for all θ. Since both integrals on the right side of (2) are equal, we can combine (1) and (2) to obtain

$$\min_\theta |R(K,\theta)| \leqslant \frac{1}{4\pi^2}\left[\int_0^{2\pi} b_K(\theta)\,d\theta\right]^2, \tag{3}$$

with equality if and only if K is of constant width. If we combine (3) with the fact (see, for example, [**1**, p. 65]) that

$$L(K) = \frac{1}{2}\int_0^{2\pi} b_K(\theta)\,d\theta,$$

we obtain:

THEOREM.

$$\min_\theta\ |R(K,\theta)| \leqslant \frac{1}{\pi^2}L(K)^2,$$

with equality if and only if K is of constant width.

L. Moser (see, for example, [**2**, p.244]) posed the question of whether every closed curve of length 2π can be accommodated in a rectangle of area 4. Since any curve of length 2π has a convex hull (smallest convex set containing the curve) whose boundary's length is less than or

equal to 2π, the theorem can be used to give an affirmative answer to Moser's question.

The theorem can easily be generalized. We can consider, for example, the problem of circumscribing a convex set with parallelograms with a fixed inner angle ϕ.

Let ϕ be fixed such that $0<\phi\leqslant \frac{\pi}{2}$. For a given θ, let $P_\phi(K,\theta)$ denote the parallelogram circumscribed about K whose sides are perpendicular to the directions determined by θ and $\theta+\phi$. Clearly, $P_{\pi/2}(K,\theta)=R(K,\theta)$. Since

$$|P_\phi(K,\theta)|=\csc\phi\, b_K(\theta)b_K(\theta+\phi),$$

the exact same procedure can be used to establish:

THEOREM.

$$\min_\theta |P_\phi(K,\theta)|\leqslant \frac{\csc\phi}{\pi^2}L(K)^2 \qquad \left[0<\phi\leqslant\frac{\pi}{2}\right],$$

with equality if and only if K is of constant width.

Both theorems have easily proven extensions to higher dimensional spaces.

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A NOTE ON PARTITIONS AND TRIANGLES WITH INTEGER SIDES

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In a recent paper [2], Jordan, Walch, and Wisner characterize the number $T(n)$ of incongruent triangles with integer sides that have perimeter n. They determine $T(n)$ by first noting that $T(4)=0$, $T(6)=T(8)=1$, $T(10)=2$, $T(12)=3$, $T(14)=4$, and then proving two theorems equivalent to the assertions: (1) $T(2n+12)=T(2n)+n+3$; (2) $T(2n)=T(2n-3)$. In this note we remark that $T(n)$ may be simply handled by relating it to $p_3(n)$ and $p_2(n)$, the number of partitions of n into 3 and 2 parts, respectively. In the following $[x]$ denotes the greatest integer in x and $\{x\}$ is the nearest integer to x.

THEOREM. $T(n)=p_3(n)-\sum\limits_{1\leqslant j\leqslant\left[\frac{1}{2}n\right]} p_2(j)$.

Proof. Each partition of n into three parts yields a unique triangle of the desired type and conversely, except when the sum of the smallest two parts does not exceed the largest part. This happens for each partition of j into two parts c and d with $1\leqslant j\leqslant\frac{1}{2}n$, for then $c+d+(n-j)$ is the related partition of n and $c+d\leqslant n-j$. Hence $T(n)=p_3(n)-\Sigma_{1\leqslant j\leqslant[\frac{1}{2}n]}p_2(j)$.

COROLLARY. $T(n)=\left\{\frac{n^2}{12}\right\}-\left[\frac{n}{4}\right]\left[\frac{n+2}{4}\right]$.

Proof. Since $p_2(n)=[\frac{1}{2}n]$ (see [1, p. 81, Ex. 1]), it is a simple problem in mathematical induction to prove that

$$\sum_{1\leqslant j\leqslant\frac{1}{2}n} p_2(j)=\left[\frac{n}{4}\right]\left[\frac{n+2}{4}\right].$$

The formula $p_3(n)=p(\text{“}\{1,2,3\}\text{”},n-3)=\{n^2/12\}$ for $n\geqslant 0$ is given in Example 2 of [1, p. 81].

We note that this corollary gives us an operational formula through which we may easily compute $T(n)$; furthermore, all the assertions for $T(n)$ described in the first paragraph are easily deduced from it.

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ON THE COEFFICIENTS OF A PARTIAL FRACTION DECOMPOSITION

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Let $R(z)$ be the rational function $P(z)/Q(z)$, where the polynomials P and Q have no common factors, P has simple zeros, and the degree of P is greater than the degree of Q. If m is a positive integer, we can form the partial fraction decomposition

$$\left\{\frac{1}{R(z)}\right\}^m=\left\{\frac{Q(z)}{P(z)}\right\}^m=\sum_{j=1}^{n}\sum_{r=1}^{m}A_{j,r}^{(m)}(z-z_j)^{-r} \tag{1}$$

where $z_j, j=1,2,\dots,n$ are the simple zeros of P and the $A_{j,r}^{(m)}$ are complex constants in general.

We will now establish several formulae for the $A_{j,r}^{(m)}$ which permit their recursive evaluation. Let

$$\phi(z)\equiv\frac{z-z_j}{R(z)}=\left\{a_1+a_2(z-z_j)+a_3(z-z_j)^2+\cdots\right\}^{-1}$$

where $a_j=R^{(j)}(z_j)/j!$. From (1), it follows that

$$(m-r)!\,A_{j,r}^{(m)}=\lim_{z\to z_j}D^{m-r}\left\{\frac{z-z_j}{R(z)}\right\}^m=D^{m-r}\{\phi(z)^m\}\Big]_{z=z_j}.$$

It also follows from the Lagrange reversion of a power series [1] that if $w=R(z)=a_1(z-z_j)+a_2(z-z_j)^2+\cdots$, then

$$\psi(z)\equiv\int_{z_j}^{z}[\phi(t)]^\alpha\,dt=\sum_{k=1}^{\infty}\frac{[R(z)]^k}{k!}\left\{D^{k-1}(\phi(z))^{\alpha+k}\right\}\Big]_{z=z_j}.$$

By direct differentiation of the latter expression, we obtain

$$\frac{d^n\psi(z)}{dR(z)^n}\Big]_{z=z_j}=\left\{D^{n-1}\phi(z)^{\alpha+n}\right\}_{z=z_j}.$$

Consequently, for $\alpha=r-1$,

$$(m-r)!\,A_{j,r}^{(m)}=D^{m-r}\phi(z)^m\Big]_{z=z_j}=\frac{d^{m-r+1}\psi(z)}{dR(z)^{m-r+1}}\Big]_{z=z_j}.$$

If we define

$$G_{m,r}(z) \equiv \frac{d^{m-r+1}\psi(z)}{dR(z)^{m-r+1}}$$

or, equivalently,

$$G_{m,r}(z) = \left\{ \frac{1}{R'} \frac{d}{dz} \right\}^{m-r} \frac{\phi}{R'}^{r-1},$$

it then follows that

$$G_{m+1,r} = \frac{1}{R'} G'_{m,r} \quad \text{and} \quad G_{m,m} = \frac{\phi^{m-1}}{R'}. \tag{2}$$

Thus,

$$(m-r)!\,A_{j,r} = G_{m,r}(z_j) \quad \text{and} \quad G_{m,m} = \left\{ \frac{1}{R'(z_j)} \right\}^m = \left\{ \frac{Q(z_j)}{P'(z_j)} \right\}^m. \tag{3}$$

The coefficients $A_{j,r}^{(m)}$ can be calculated successively by use of equations (2) and (3). For example,

$$A_{j,2}^{(3)} = G_{3,2} = \frac{G'_{2,2}}{R'} = \frac{1}{R'}\left(\frac{\phi}{R'}\right)' = \frac{1}{R'}\left\{ \frac{\phi'}{R'} - \frac{\phi R''}{(R')^2} \right\}\Bigg]_{z=z_j}.$$

When r is close to m, it is easier to use the formula

$$(m-r)!\,A_{j,r}^{(m)} = D^{m-r}\phi(z)^m\big]_{z=z_j}.$$

We find that $A_{j,m}^{(m)} = \phi(z_j)^m = R'(z_j)^{-m}$ and

$$A_{j,m-1}^{(m)} = D\phi(z)^m\big]_{z=z_j} = -\frac{m}{2} \cdot \frac{R''(z_j)}{R'(z_j)^{m+1}}.$$

It is also possible to establish formulae connecting the $A_{j,r}^{(m)}$ with $A_{j,r}^{(m-1)}$. Let D_j be a closed disk about z_j as center containing no other zeros of P. Then

$$A_{j,r}^{(m)} = \frac{1}{2\pi i} \oint_{C_j} (z-z_j)^{r-1} R(z)^{-m}\, dz \tag{4}$$

where C_j is the boundary circle of D_j. Using the partial fraction decomposition of $1/R(z)$, we have

$$\begin{aligned} A_{j,r}^{(m)} &= \frac{1}{2\pi i} \oint_{C_j} \frac{(z-z_j)^{r-1}}{R(z)^{m-1}} \left\{ \sum_{k=1}^{n} \frac{1/R'(z_k)}{z-z_k} \right\} dz \\ &= \sum_{k=1}^{n} \frac{1}{R'(z_k)} \frac{1}{2\pi i} \oint_{C_j} \frac{(z-z_j)^{r-1}}{z-z_k} \left[\sum_{p=1}^{n} \sum_{q=1}^{m-1} \frac{A_{p,q}^{(m-1)}}{(z-z_p)^q} \right] dz. \end{aligned}$$

To evaluate these integrals we need to consider the cases $k \neq j$ and $k = j$ and use Cauchy's integral formula for derivatives. We find that

$$A_{j,r}^{(m)} = \begin{cases} \displaystyle\sum_{\substack{k=1 \\ k\neq j}}^{n} \sum_{q=1}^{m-1} \frac{1}{(z_j-z_k)^q} \left\{ \frac{A_{k,q}^{(m-1)}}{R'(z_j)} + \frac{(-1)^{q-1} A_{j,q}^{(m-1)}}{R'(z_k)} \right\}, & r=1, \\ \displaystyle\frac{A_{j,r-1}^{(m-1)}}{R'(z_j)} + \sum_{\substack{k=1 \\ k\neq j}}^{n} \sum_{q=r}^{m-1} \frac{(-1)^{q-r} A_{j,q}^{(m-1)}}{(z_j-z_k)^{q-r+1} R'(z_k)}, & 1<r<m, \\ \displaystyle\frac{A_{j,m-1}^{(m-1)}}{R'(z_j)}, & r=m. \end{cases}$$

For another recurrence formula, consider

$$A_{j,r}^{(m)}=\frac{1}{2\pi i}\oint\frac{(z-z_j)^{r-1}}{R(z)^m}dz=\frac{1}{2\pi i}\oint\frac{(z-z_j)^{r-1}}{R(z)^{m+1}}\cdot R(z)dz$$

$$=\frac{1}{2\pi i}\oint\frac{(z-z_j)^{r-1}}{R(z)^{m+1}}\cdot\left\{\sum_{k=1}^{\infty}\frac{R^{(k)}(z_j)}{k!}(z-z_j)^k\right\}dz$$

$$=\sum_{k=1}^{m-r+1}\frac{R^{(k)}(z_j)}{k!}A_{j,r+k}^{(m+1)}.$$

There are also relationships which exist between the $A_{j,r}^{(m)}$ for different values of j. If we expand both sides of (1) in powers of $1/z$, the left-hand side starts with a term of degree $m(n-k)$, where k is the degree of Q. However, the right-hand side is

$$\sum_{j=1}^{n}\frac{A_{j,1}^{m}}{z}\left\{1+\frac{z_j}{z}+\left(\frac{z_j}{z}\right)^2+\cdots\right\}+\sum_{j=1}^{n}\frac{A_{j,2}^{(m)}}{z^2}\left\{1+\frac{2z_j}{z}+3\left(\frac{z_j}{z}\right)^2+\cdots\right\}+\cdots.$$

Consequently, when $m(n-k)$ is not a small positive interger, we have

$$\sum_{j=1}^{n}A_{j,1}^{(m)}=0,\qquad\sum_{j=1}^{n}\left(A_{j,1}^{(m)}z_j+A_{j,2}^{(m)}\right)=0,$$

$$\sum_{j=1}^{n}\left(A_{j,1}^{(m)}z_j^2+2A_{j,2}^{(m)}z_j+A_{j,3}^{(m)}\right)=0,\qquad\text{etc.}\tag{5}$$

As an application of (2) and (5) we see that if $R=P/Q$, where the degree $P\geqslant$ degree $Q+2$, then with $g_1=1/R'$ and $g_{k+1}=g_k'/R'$, we have for all positive integers m that

$$\sum_{j=1}^{n}g_m(z_j)=\sum_{j=1}^{n}G_{m,1}(z_j)=(m-1)!\sum_{j=1}^{n}A_{j,1}^{(m)}=0.$$

This is a generalization of a result of Lott [2].

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FURTHER COMMENTS ON THE VARIATION FUNCTION

A. M. RUSSELL

In [1] Dr. F. N. Huggins presents certain properties of the variation function, with particular interest being centered upon the properties that the variation function inherits from its parent function.

The purpose of this note is to present an additional example, and also to provide a stronger version of Theorem 7 in [1]. Both contributions deal with the relationship between the derivatives of a function and of its variation.

Accordingly, let f be a function of bounded variation on $[a,b]$. For each $x\in[a,b]$, let V_f denote the function defined by $V_f(x)=V_a^x(f)$, where $V_a^x(f)$ is the total variation of f on $[a,x]$. As pointed out in [1], if a function f is of bounded variation on $[a,b]$, then both f and V_f are differentiable on $[a,b]$, except possibly on a set of Lebesgue measure zero. Furthermore, the set

of points in $[a,b]$ at which $f'(x)$ exists is not necessarily the same as the set of points in $[a,b]$ at which $V_f'(x)$ exists. To substantiate this claim Huggins provides an example in which V_f has a derivative, when f does not. It is perhaps more surprising to note that even when f has a derivative, V_f may not. The following example illustrates this case:

$$f(x)=x^2\cos(x^{-\beta}),\qquad 0<x\leqslant 1,\qquad 1<\beta<2,$$
$$f(0)=0.$$

We now present a stronger version of Theorem 7 in [1]. It is shown in [1] that if f is differentiable on $[a,b]$ and has a bounded derivative, then $V_f'(x)=|f'(x)|$ almost everywhere (a.e.). This result can be extended to arbitrary functions of bounded variation, giving rise to the following theorem. The proof is straightforward and quite accessible to the advanced undergraduate, so in keeping with [1], it is presented for convenience.

THEOREM. *If f is of bounded variation on $[a,b]$, then*

$$V_f'(x)=|f'(x)|\ \text{ a.e.}$$

Proof. A subdivision of $[a,b]$ by points $x_0,x_1,\dots,x_m$, where $a=x_0<x_1<\cdots<x_m=b$ will be denoted by Δ. We now consider a sequence $\{\Delta_n\}$ of subdivisions of $[a,b]$ with the property that the approximating sums

$$\sum_{k=1}^{m}|f(x_k^{(n)})-f(x_{k-1}^{(n)})|$$

corresponding to the subdivision $\Delta_n=\{a=x_0^{(n)},x_1^{(n)},\dots,x_m^{(n)}=b\}$ are within 2^{-n} of the total variation $V=V_a^b(f)$. For simplicity we suppress the superscript "n" associated with the points of Δ_n. No confusion will arise.

Corresponding to each Δ_n we define the following function f_n. In each subinterval $[x_{k-1},x_k]$ of Δ_n, let

$$f_n(x)=\begin{cases} f(x)+c_k, & f(x_k)-f(x_{k-1})\geqslant 0\\ -f(x)+d_k, & f(x_k)-f(x_{k-1})<0,\end{cases}$$

where the constants c_k and d_k are determined so that $f_n(a)=0$, and the values taken at the subdivision points x_k agree. Consequently,

$$f_n(x_k)-f_n(x_{k-1})=|f(x_k)-f(x_{k-1})|,$$

and so,

$$0\leqslant V_f(b)-f_n(b)=V_f(b)-\sum_{k=1}^{m}[f_n(x_k)-f_n(x_{k-1})]\leqslant 2^{-n}.$$

We now show that $V_f(x)-f_n(x)$ is an increasing function on $[a,b]$. Let $x<y$, and suppose that x and y belong to the same subinterval $[x_{k-1},x_k]$ of Δ_n. Then,

$$V_f(y)-V_f(x)\geqslant|f(y)-f(x)|\geqslant f_n(y)-f_n(x),\tag{1}$$

which on rearrangement gives

$$V_f(y)-f_n(y)\geqslant V_f(x)-f_n(x).$$

When x and y do not belong to the same subinterval, we apply the inequality (1) to each of the subintervals $[x,x_i],[x_i,x_{i+1}],\dots,[x_j,y]$, and add.

Since the positive term series $\sum_1^\infty[V_f(x)-f_n(x)]$ is dominated by $\sum_1^\infty 2^{-n}$, it is convergent for all x in $[a,b]$. By Fubini's theorem on term-by-term differentiation of series (see [2]), the differentiated series converges a.e., and hence

$$V_f'(x)-f_n'(x)\to 0\ \text{ as } n\to\infty\ \text{ a.e.}$$

The required result now follows by observing that $f_n'(x)=\pm f'(x)$, and $V_f'(x)\geqslant 0$ since $V_f(x)$ is nondecreasing.

REMARKS. A more comprehensive account of the relationship between V_f' and f' can be found in Chapter 4 of [3]. The theorem presented in this paper is in fact Theorem 7.9 of [3], although the proof given here is perhaps more elementary. To be more specific, a more complete result concerning the relationship between V_f' and f' is contained in de La Vallée Poussin's Decomposition Theorem (Theorem 9.6 of [3]). For convenience we cite it here; but for consistency, adopt the notation of [1]. In the statement of the theorem, if f is a nonnegative, additive function of intervals, then for any set X, $f^*(X)$ will denote the infimum of the sums $\Sigma_k f(I_k)$, where $\{I_k\}$ is an arbitrary sequence of intervals such that $X \subset \cup_k I_k^0$.

DE LA VALLÉE POUSSIN'S DECOMPOSITION THEOREM. *If f is a function of bounded variation, and if $E_{+\infty}$ and $E_{-\infty}$ denote the sets in which f has a derivative equal to $+\infty$ and to $-\infty$, respectively, then*

(i) *for any bounded Borel measurable set X, at each point of which f is continuous, we have the relations*

$$f^*(X) = f^*(X \cap E_{+\infty}) + f^*(X \cap E_{-\infty}) + \int_X f'(x)\,dx,$$

$$V_f^*(X) = f^*(X \cap E_{+\infty}) + |f^*(X \cap E_{-\infty})| + \int_X |f'(x)|\,dx;$$

(ii) *the two derivatives f' and V_f' exist and satisfy the relationship $V_f'(x) = |f'(x)|$ at any point x of continuity of f, except at most at the points of a set N such that $V_f^*(N) = |N| = 0$ (that is, a set which is at the same time of measure (L) zero, and of measure (V_f) zero).*

Saks then goes on to point out an interesting immediate consequence of de La Vallée Poussin's Theorem. In order that a continuous function f of bounded variation be absolutely continuous, it is necessary and sufficient that the set function f^* should vanish identically on the set of points at which f has an infinite derivative. In particular, therefore, any continuous function of bounded variation that is not absolutely continuous has an infinite derivative on a nonenumerable set.

Finally, Saks points out, in particular, that the results above cannot be extended directly to additive functions of an interval in the plane.

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MISCELLANEA

25.

THE BERNOULLIS

Remarkable truly
The house of Bernoulli.

Two Jacobs, three Johanns in bold explorations,
Also a Daniel and two Nicolauses.
Eight of them, covering three generations.
(Nobody ever refers to the spouses.)

Katharine O'Brien

RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.

THE ELUSIVE EXPANSIVE PROPERTY

RICHARD K. WILLIAMS

If h is a homeomorphism of a metric space (X,d) onto itself, then h is said to be **expansive** if there exists $\delta>0$ such that $x,y\in(X,d)$, $x\neq y$ implies $d(h^n(x),h^n(y))>\delta$ for some integer n. As usual, h^n denotes the nth iterate of h.

Expansive homeomorphisms have been studied for almost 30 years. For instance, the definition as well as some basic results appear in [2]. In some of the earlier writings, the word "unstable" is used instead of the word "expansive." In 1950, Utz [6] wrote what is probably the first paper devoted entirely to expansive homeomorphisms.

In 1962, Bryant [1] proved that there does not exist an expansive homeomorphism on either an open or a closed arc and, in 1960, Jakobsen and Utz [3] showed the non-existence of an expansive homeomorphism on the closed unit disk. However, in [7], this author gave an example of a homeomorphism which *is* expansive on the open unit disk.

In [6], Utz showed that if f is a homeomorphism of (X,d_1) onto (Y,d_2), if f^{-1} is uniformly continuous, and if h is expansive on (X,d_1), then fhf^{-1} is expansive on (Y,d_2). Thus, there exist no expansive homeomorphisms on closed 1-cells or closed 2-cells, but there is one on the open 2-cell. If the expansive homeomorphism on the open 2-cell is called $h(x)$, then $H(x_1,x_2,\ldots,x_n)\equiv(h(x_1), h(x_2),\ldots,h(x_n))$ defines an expansive homeomorphism on the open $2n$-cell.

In the past 16 to 18 years, there has been a conspicuous absence of results on the possibility of generalizing the non-existence theorem in [3], i.e., it has not been shown whether or not there exists an expansive homeomorphism on the closed unit ball in 3 or more dimensions, or equivalently on the closed n-cell, $n\geqslant 3$. Since the surface of the closed unit ball is invariant under such a homeomorphism, non-existence proofs would be achieved if non-existence proofs on S^n, $n\geqslant 2$, were constructed. These results are also missing.

In [5], Reddy gave an example of an expansive homeomorphism on the surface of a torus in 3 dimensions, and in [4], O'Brien and Reddy generalized this result by showing the existence of expansive homeomorphisms on each compact orientable surface with positive genus. While these results generate many elegant examples, they obviously do not handle the (seemingly) simple case of the surface of the unit ball in 3 dimensions, S^2.

It is also not known whether or not there exists an expansive homeomorphism on a compact infinite-dimensional space. By the theorem of O'Brien and Reddy in [4], however, there are expansive homeomorphisms on compact spaces of arbitrarily large dimension.

Thus, while some quite sophisticated higher dimensional results are known, there remain unanswered some very elementary questions about the existence of expansive homeomorphisms on S^2 or on the closed unit ball in 3 dimensions.

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CLASSROOM NOTES

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AN ULTIMATE PROOF OF ROLLE'S THEOREM

ALEXANDER ABIAN

The almost universal procedure of proving Rolle's theorem runs as follows. First it is shown [**1**, p. 16] that a real-valued function f continuous on $[a,b]$ attains a maximum and a minimum on $[a,b]$. Then from the hypotheses of Rolle's theorem it is derived that a maximum or a minimum of f is attained at an interior point c of $[a,b]$, and then it is shown [**1**, p. 36] that the derivative f' of f vanishes at c, i.e., $f'(c)=0$.

Below we prove Rolle's theorem by by-passing both the first step mentioned above and even the fact that f is bounded on $[a,b]$. Our proof is highly constructive in the sense that it can be readily computerized and c can be computed within any desired degree of accuracy.

In what follows, every function is real-valued and is defined on a real closed interval $[a_i,b_i]$ whose midpoint is denoted by m_i.

Let g be a function defined on $[a_i,b_i]$. Then $[a_i,b_i]$ is called *g-updown* if and only if

$$g(m_i)\geqslant g(a_i) \text{ and } g(m_i)\geqslant g(b_i) \text{ where } m_i \text{ is the midpoint of } [a_i,b_i]. \tag{1}$$

Let $[a_i,b_i]$ be a g-updown interval and let p_i,m_i,q_i, divide $[a_i,b_i]$ in fourths. We call the *successor* of the g-updown interval $[a_i,b_i]$ the *first* (reading from left to right) of the three closed intervals:

$$[p_i,q_i], \qquad [a_i,m_i], \qquad [m_i,b_i] \tag{2}$$

(given in the above specific order) *at whose midpoint the value of g is not exceeded by the value of g at the midpoint of either of the two other intervals.*

We denote by $[a_{i+1},b_{i+1}]$ the successor of the g-updown interval $[a_i,b_i]$. Clearly, $[a_{i+1},b_{i+1}]$ itself is a g-updown interval *whose length is the half of that of* $[a_i,b_i]$.

This work was partly supported by Iowa State University SHRI.

THEOREM (Rolle). *Let f be a function continuous on the closed interval $[a_0,b_0]$ with $a_0<b_0$ and let f be differentiable in the open interval (a_0,b_0). If $f(a_0)=f(b_0)=0$ then $f'(c)=0$ for some $c\in(a_0,b_0)$.*

Proof. Let g be a function on $[a_0,b_0]$ defined by:

$$g=f \quad \text{if} \quad f(m_0)\geqslant 0, \qquad \text{otherwise}, \quad g=-f \tag{3}$$

where, as expected, m_0 is the midpoint of $[a_0,b_0]$.

From (3), in view of the hypotheses of the Theorem, it follows that g is continuous on $[a_0,b_0]$ and differentiable in (a_0,b_0). Moreover,

$$g(a_0)=g(b_0)=0 \quad \text{and} \quad g(m_0)\geqslant 0 \tag{4}$$

which by (1) implies that $[a_0,b_0]$ *is g-updown.*

Let us consider the nested sequence of successive g-updown intervals

$$[a_0,b_0],[a_1,b_1],[a_2,b_2],\ldots,[a_i,b_i],\ldots. \tag{5}$$

From (4), (2) and (1), it readily follows:

$$0\leqslant g(m_0)\leqslant g(m_1)\leqslant g(m_2)\leqslant\cdots\leqslant g(m_i)\leqslant\cdots \tag{6}$$

and

$$g(m_i)\geqslant g(a_i) \quad \text{and} \quad g(m_i)\geqslant g(b_i) \quad \text{for every} \quad i\in\omega. \tag{7}$$

In view of our earlier remark, the lengths of the intervals in (5) tend to zero. Let c be the unique common point of $[a_i,b_i]$'s. Obviously, $c\in[a_0,b_0]$ and

$$\lim_{i\to\infty} a_i=\lim_{i\to\infty} b_i=\lim_{i\to\infty} m_i=c \quad \text{with} \quad a_i\leqslant c\leqslant b_i \quad \text{for every} \quad i\in\omega, \tag{8}$$

which by continuity of g implies

$$\lim_{i\to\infty} g(a_i)=\lim_{i\to\infty} g(b_i)=\lim_{i\to\infty} g(m_i)=g(c). \tag{9}$$

Moreover, from (6), (9) and (7) it follows that

$$g(c)\geqslant g(a_i) \quad \text{and} \quad g(c)\geqslant g(b_i) \quad \text{for every} \quad i\in\omega. \tag{10}$$

We claim that c is an interior point of $[a_0,b_0]$, i.e., $c\in(a_0,b_0)$.

Indeed, if $g(m_i)>0$ *for some i* then from (6) and (9) it follows that $g(c)>0$, which by (4) implies $a_0\neq c\neq b_0$ and therefore $c\in(a_0,b_0)$. On the other hand, if $g(m_i)=0$ *for every i*, then from (2) and (8) it follows that $c=m_0$, which again implies $c\in(a_0,b_0)$. Thus, $c\in(a_0,b_0)$ and $g'(c)$ exists. However, from (10) it follows that $g(c)-g(a_i)\geqslant 0$ and $g(c)-g(b_i)\geqslant 0$; whereas, as (8) shows, $c-a_i\geqslant 0$ and $c-b_i\leqslant 0$ for every $i\in\omega$. Thus, from (8) it follows that $g'(c)=0$, which by (3) implies $f'(c)=0$, as desired.

REMARK. It is also interesting to note that the fact that a function f continuous on $[a,b]$ attains a maximum and a minimum can be proved [2, p. 91] constructively, without invoking the boundedness of f on $[a,b]$ (which is usually invoked). Naturally, the Attainment Theorem implies that f is bounded on $[a,b]$.

The author thanks Professor James L. Cornette for useful discussions.

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ON ROLLE'S THEOREM

Hans Samelson

In many years of teaching calculus the writer has found Rolle's Theorem in Differential Calculus a peculiarly rigid object. It seems to require a completely fixed proof and almost a fixed set of words. It was a surprise therefore that in an honors calculus class last year a different proof developed. We have to show: Given f on $[a,b]$, differentiable on (a,b), continuous at a and b, $f(a)=f(b)$; then there exists x_0 in (a,b) with $f'(x_0)=0$. We use two principles:

(1) If f is continuous on $[c,d]$ and $f(c)=f(d)$, then there exist α,β in $[c,d]$ with $\beta-\alpha=\frac{1}{2}(c-d)$ and $f(\alpha)=f(\beta)$ (a special case of P. Lévy's theorem about horizontal chords). For the proof one considers the function $g(x)=f(x+\frac{1}{2}(d-c))-f(x)$, notices $g(\frac{1}{2}(c+d))=-g(c)$, and applies Bolzano's Intermediate Value Theorem.

(2) If f is differentiable at an interior point x, then for any two sequences α_n,β_n with $\alpha_n\leqslant x\leqslant\beta_n,\alpha_n<\beta_n,\alpha_n\to x,\beta_n\to x$, the difference quotient $(f(\beta_n)-f(\alpha_n))/(\beta_n-\alpha_n)$ converges to $f'(x)$. (Proof by observing geometrically or analytically that in the case $\alpha_n<x<\beta_n$ the above difference quotient lies between $(f(\beta_n)-f(x))/(\beta_n-x)$ and $(f(x)-f(\alpha_n))/(x-\alpha_n)$.)

Now to Rolle's theorem: Applying (1) repeatedly, we get a sequence $\{[\alpha_n,\beta_n]\}_0^\infty$ of subintervals of $[a,b]$ such that (i) $\beta_n-\alpha_n=(1/2^n)(b-a)$, (ii) $f(\alpha_n)=f(\beta_n)$, (iii) $[\alpha_{n+1},\beta_{n+1}]\subset[\alpha_n,\beta_n]$. Clearly, the sequences $\{\alpha_n\}$ and $\{\beta_n\}$ have the same limit x_0, and satisfy $\alpha_n\leqslant x_0\leqslant\beta_n$. It is easy to arrange matters so that $x_0\neq a,\neq b$; in the "dangerous" case, where $f((a+b)/2)$ and also one of $f((3a+b)/4),f((a+3b)/4)$ equals $f(a)$, one chooses the second or third quarter of $[a,b]$ as $[\alpha_2,\beta_2]$.

Applying (2) to the sequences $\{\alpha_n\},\{\beta_n\}$, we find $f'(x_0)=0$ since the difference quotients all vanish.

We note that this is really a different proof; the point x_0 found is not necessarily the absolute (or even a local) maximum or minimum.

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Added in proof. A proof of the Mean Value Theorem, pretty much along these lines, was given by D. Pompeiu [1]. Some comments on this appear in [2]. (Thanks to R. P. Boas, Jr., for the reference.)

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PRISMATOID, PRISMOID, GENERALIZED PRISMOID

A. Day Bradley

A Prismoid is a Solid, or Body somewhat resembling a Prism, but that its ends are any dissimilar Plane Figures of the same number of Sides, the upright Sides being Trapezoids. If the ends of the Prismoid be bounded by dissimilar Curves it is sometimes called a Cylindroid.

Charles Hutton, *Mathematical & Philosophical Dictionary,* Vol. 2, London, 1796, p. 284.

It may happen that figures seemingly quite different have identical expressions for their areas or volumes. This is true for that somewhat ungainly solid, the prismatoid, and for the

generalized prismoid. "The prismatoid is a polyhedron all of whose vertices lie in two parallel planes called bases. If the two bases have the same number of sides, the prismatoid is called a prismoid. A *generalized prismoid* is any solid having two parallel base planes and having the area of its sections parallel to the bases given by a quadratic, or cubic, function of their distances from one base" [1]. The formula for the volume of the generalized prismoid has an obvious relation and historical connection with Simpson's Rule for the area between a curve and the X axis.

Logically the first formula to have been developed would have been the well-known $\frac{1}{6}h(B_1+4M+B_2)$ for the volume of the prismatoid, where h is the altitude, B_1 and B_2 are the areas of the bases, and M the area of the midsection. However, this formula was proved by Jakob Steiner (1796–1863) in 1842 long after the proof of Simpson's Rule and the formula for the generalized prismoid. Steiner's proof is the same as that familiar to students when secondary school mathematics included solid geometry [2]. In Figure 1, P is any point in the midsection of the prismatoid, and E and F are the midpoints of AC and CB. The volume is found by adding the volumes of the two pyramids with vertex P and bases B_1 and B_2, and all pyramids like P-ABC with vertex at P and base a triangle in a lateral face.

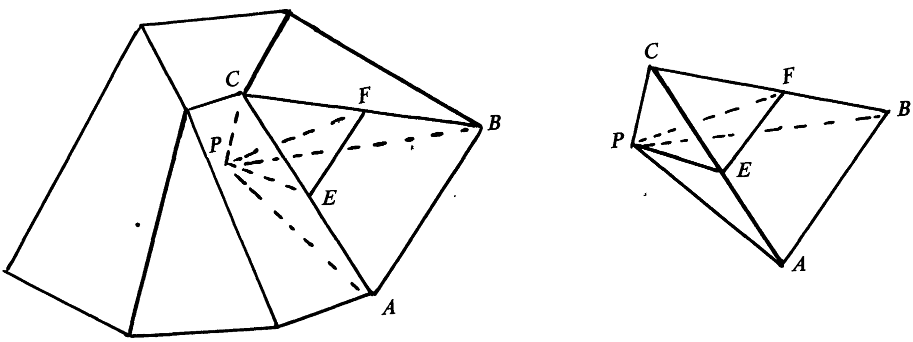


FIG. 1

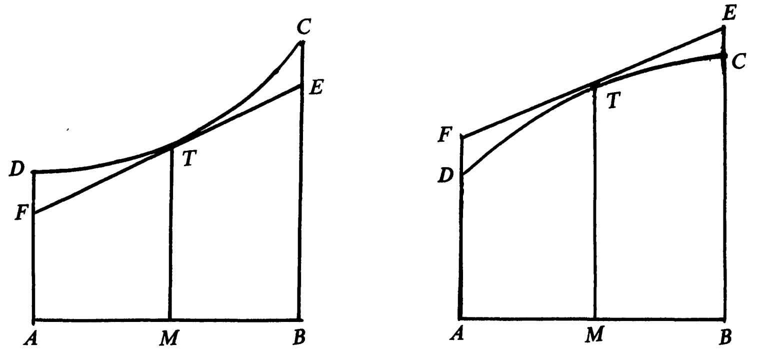
$$\text{The area of } ABC = 4 \text{ (area of } CEF).$$
$$\text{The volume of } P\text{-}ABC = 4 \text{ (volume of } P\text{-}CEF).$$
$$\text{The volume of } P\text{-}ABC = 4\cdot\frac{1}{3}\cdot\frac{h}{2} \text{ (area of } PEF)$$
$$= \frac{h}{6} \text{ (4 area } PEF).$$

Summing all such pyramids and adding the two pyramids with vertex P and bases B_1 and B_2, we have

$$\text{Volume of the prismatoid} = \frac{h}{6}(B_1+4M+B_2).$$

Simpson's Rule for the area between the X axis and the arc of the curve $y=ax^2+bx+c$ may be stated, $\frac{1}{6}h(y_1+4y_m+y_2)$ (Fig. 2).

Thomas Simpson (1710–1761) proves the rule which bears his name by applying two theorems on the parabola and the formula for the area of a trapezoid. Without specific statement Simpson employs the following theorems: if MT in Figure 2 bisects AB and therefore the chord CD of the parabola, then the tangent at T is parallel to DC; and the area of the parabolic segment DTC is two-thirds the area of the parallelogram $DCEF$.



(a) (b)

$$A(0,0), M\left(\frac{h}{2},0\right), B(h,0)$$

$$D(0,y_1), T\left(\frac{h}{2},y_m\right), C(h,y_2)$$

FIG. 2

$$\begin{aligned} \text{The required area} &= ABCD - \frac{2}{3}(ABCD - ABEF) && \text{Figure 2(a)} \\ &= ABCD + \frac{2}{3}(ABEF - ABCD) && \text{Figure 2(b)} \\ &= \frac{1}{3}(ABCD) + \frac{2}{3}(ABEF) \\ &= \frac{h}{3}\left[\frac{y_1+y_2}{2}\right] + \frac{2h}{3}[y_m] \\ &= \frac{h}{6}(y_1 + 4y_m + y_2). \end{aligned}$$

Simpson gives the extension to the well-known surveyor's rule and to the volume of "conic frustrums." If any conic be rotated about its axis the content of the frustrum is

$$(d_1^2 + 4d_m^2 + d_2^2)\frac{\pi}{4}\frac{h}{6}. \qquad \text{Figure 3}$$

(See [3].)

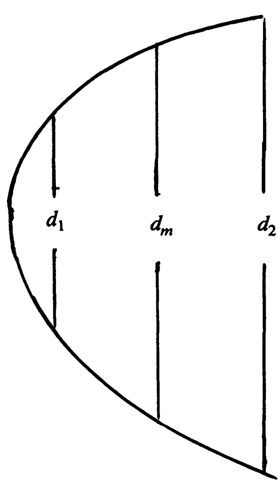


FIG. 3

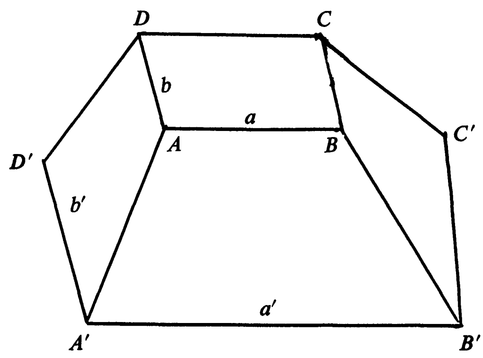


FIG. 4

This formula was welcomed by the authors of works on the "art or science" of gauging. Gauging is the "measuring of casks and other things falling under the cognizance of the officers of the excise and hath received its name from the gauge or rod used by the practioners of the art"[4]. Thomas Moss notes that the "general rule was first and very judiciously introduced into the present subject (gauging) by the late ingenious Mr. Robert Shirtcliffe"[5]. Shirtcliffe's proof in modern notation is that,

$$\int_0^h f(x)\,dx \equiv \frac{h}{6}[f(0)+4f(h/2)+f(h)],$$

where $f(x)=ax^2+bx+c$. His proof, like that of many calculus texts, does not give any hint of the method of discovery and he makes no claim for priority [6]. Did Shirtcliffe know of Simpson's work even though Simpson did not publish until three years after Shirtcliffe?

Other writers on gauging included the generalized prismoid formula, not always being careful to indicate its limitations. Thomas Moss [7] says,

> Or universally let the Solid be of what Form soever; add the two extreme Areas, and four times that in the Middle together; multiply the Sum by one-sixth of the Distance of the extreme Areas, and the Product will be the Measure of the Solid nearly.

Charles Baillairge [8] contended that the generalized prismoid formula should be universally applied because of its simplicity.

> The prismoidal formula does not bring out the true content of casks of all varieties of sizes, to within the tenth or twentieth and up to the half or thereabouts of one per cent. Notwithstanding which it is the only practical formula which can bring out anything like a reliable result. The true formulas for casks never can, nor will they ever be applied, they are too lengthy, too abstruse.

Curiously a special case of the prismoid has been a favorite with Heron of Alexandria (c. +75), with early writers on the calculus, and with school texts as late as 1916. This is a solid whose bases are nonsimilar rectangles, with their sides respectively parallel (Fig. 4). Heron divided the solid into a parallelepiped, two wedges, and a pyramid and found the equivalent of this formula [9]:

$$h\left[\frac{(a'+a)(b'+b)}{4}+\frac{(a'-a)(b'-b)}{12}\right].$$

Thomas Simpson in his *Fluxions* [10] finds the volume by integration, as does William Emerson, both obtaining $\frac{1}{6}h(2ab+2a'b'+ab'+a'b)$. Wells and Hart [11] find the volume of this prismoid by a simple dissection into two wedges (Fig. 5). The solid is divided into the two wedges by the plane through AB and $D'C'$.

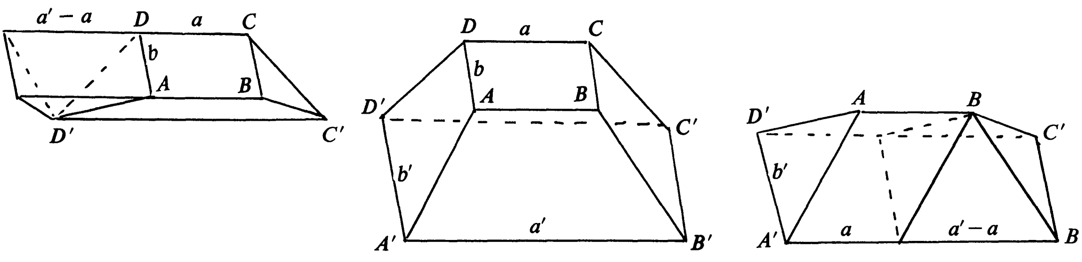


FIG. 5

The wedge with base $A'B'C'D'$ and edge AB

$$=\frac{1}{2}\text{ (parallelepiped)}+\text{pyramid}$$

$$= \frac{h}{2}ab' + \frac{h}{3}(a'-a)b'$$

$$= \frac{h}{6}(ab' + 2a'b').$$

The wedge with base $ABCD$ and edge $D'C'$

$$= \frac{1}{2}\ (\text{parallelepiped}) - \text{pyramid}$$

$$= \frac{h}{2}a'b - \frac{h}{3}(a'-a)b$$

$$= \frac{h}{6}(a'b + 2ab).$$

$$\text{The volume of the solid} = \frac{h}{6}(2ab + 2a'b' + ab' + a'b).$$

Strangely neither Simpson, nor Emerson, nor Wells and Hart have indicated that the solid is a special case of both the prismoid and the generalized prismoid.

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MORE ON GRADIENT CHARACTERIZATIONS OF ANALYTICITY

BERTRAM WALSH

In a recent article in this Monthly [1] H. S. Bear and G. N. Hile gave a proof of the following theorem of V. K. Dzyadyk [2] under the additional assumption that $u, v \in C^2(G)$.

THEOREM. *Let G be a region in the complex plane. Let $u, v \in C^1(G)$. Then $u+iv$ or $u-iv$ is analytic on G if and only if $\nabla u \cdot \nabla v = 0$ and $|\nabla u| = |\nabla v|$ on G.*

In giving their proof Bear and Hile asserted that the proof given in [2], under the assumption $u,v \in C^1$, was incomplete. For that reason the following demonstration of Dzyadyk's original assertion, using information absent from both [1] and [2], may be of interest. The missing information is the following celebrated theorem of T. Radó [4], which we state in a convenient form.

THEOREM. *Let g be continuous on an open set G in the complex plane, and suppose g is analytic on* $G \backslash g^{-1}[\{0\}]$. *Then in fact g is analytic on all of G.*

Proof of Dzyadyk's theorem. As in [1] or [2] one sees readily that G is the union of three disjoint sets: the set Z of points z where $\nabla u(z) = \nabla v(z) = 0$, the set $A^+ \subseteq G \backslash Z$ where u and v satisfy the Cauchy-Riemann equations, and the set $A^- \subseteq G \backslash Z$ where u and $-v$ satisfy the Cauchy-Riemann equations. Continuity of ∇u and ∇v implies that A^+ and A^- are open and Z closed. Since u is then the real part of the analytic function $u+iv$ on A^+ and the real part of the analytic function $u-iv$ on A^-, u is harmonic on $A = A^+ \cup A^-$ and thus C^∞ there. It follows that the function $g = u_x - iu_y$, which is continuous on G in any event, satisfies the Cauchy-Riemann equations on A and is thus analytic there. Since $A = G \backslash g^{-1}[\{0\}]$, Radó's theorem implies that g is analytic on G and thus that u_x and u_y are harmonic on G. By symmetry, v_x and v_y are also harmonic on G. If we assume that $A^+ \neq \varnothing$ then the harmonic functions $u_x - v_y$ and $v_x + u_y$ vanish on a nonempty open subset of G and hence vanish identically: u,v satisfy the Cauchy-Riemann equations in G and $u+iv$ is analytic. Similarly, $u-iv$ is analytic if $A^- \neq \varnothing$; if $G = Z$, then everything was constant anyway.

It may be worth reminding the reader that Radó's theorem is a consequence of the basic potential-theoretic fact [3, Cor. 7.8, p. 131] that if $s: G \to (-\infty, +\infty]$ is a superharmonic function on an open set $G \subseteq \mathbb{C}$, then any closed set $K \subseteq s^{-1}[\{+\infty\}]$—where necessarily $K^0 = \varnothing$—is a removable singularity set for locally bounded harmonic functions. For the hypotheses imply that the function $s = -\log|g|$ is a superharmonic function whose infinities are precisely $g^{-1}[\{0\}]$, so $\operatorname{Re} g$ and $\operatorname{Im} g$ have harmonic extensions to G; since $g^{-1}[\{0\}]$ is nowhere dense, the extensions agree with the original definitions of $\operatorname{Re} g$ and $\operatorname{Im} g$. Since $\operatorname{Re} g$ and $\operatorname{Im} g$ are each other's harmonic conjugates near some point of G they must be conjugates throughout G, so g is analytic on G.

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MATHEMATICAL EDUCATION

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THE BLIND STUDENT IN THE MATHEMATICS CLASSROOM

W. R. UTZ

1. Introduction. As visually handicapped students appear in more college classrooms, it is appropriate to share experiences in the teaching of blind students. There are colleges with

enrollments limited to the deaf, but I believe that there is no college that accepts only blind students; even on the high school level it is becoming more common to integrate blind students with sighted students.

My own campus has encouraged handicapped students for over a decade and has a sizable professional staff to give guidance and assistance to blind students. In particular, I frequently have blind students in my own classes. I like having them, and the sighted students also appreciate their presence, since the class sessions become especially well organized: I plan what I am going to say, I speak more slowly than usual, and I repeat certain sentences for emphasis. There is no reduction in course content.

2. Blind Students in Any Classroom. All blind or visually impaired students have support devices in the way of readers, books on tapes, calculators, tape recorders, etc. These students are trained to be independent before they come to college, and independence is encouraged by those directing their work. Thus if a student does not have a textbook or reader, it is probably because the student has not followed directions for securing one; the office for the handicapped will very likely not be sympathetic with you if you try to make arrangements for a book or a reader. The attention a teacher gives to a blind student out of class is no greater than that given to a sighted student.

Most of these students have adjusted to a long, white, collapsible walking probe in contrast to a seeing-eye dog. The rare student with a dog presents no special problem.

Textbooks are available on cassette tapes given a few weeks' notice. The American Printing House for the Blind provides such a service. In some states a copy of the text may be sent to the state prison, where certain inmates are prepared to respond swiftly with the tapes. However, I have never had a student who needed to have the textbook on tapes, since the visual textbook, the classroom, and a reader seem sufficient. Some textbooks, including mathematics texts, exist in braille but are not practical on the college level in classes of both blind and sighted students, since a braille edition is very large, requiring many volumes. These textbooks are useful in high schools for the blind, where they are kept in the classroom and used in a laboratory style. In such classrooms the students do their work on a braillewriter.

Recently the remarkable Kruzweil Reading Machine has begun to appear on campuses. It is able to read (i.e., vocalize) books in about 200 type styles and does it very well. Students can make tapes as it "speaks" if they wish; unfortunately, this machine does not read mathematics.

The tape recorder is used by many blind students, but others are able to record enough notes with a slate and stylus. Sometimes they have memories sufficiently acute to enable them to dispense with notes.

One should guard against the common misconception that all blind students compensate with superior memories and complementary senses. Multihandicaps are common among the blind, especially those who are congenitally blind. If blindness is due to rubella, there are almost always multiple handicaps, possibly impaired hearing. Congenital deafness is a greater handicap in adjusting to life than congenital blindness; it is exceedingly rare for a person who is born both blind and deaf to reach college. One should not assume that a blind student has superior hearing.

When a blind student appears in your classroom, give directions for finding a convenient seat. At the end of the first class period, have conversations with the blind students to discover their names, backgrounds, and any special problems they may have. This is best done by going to each seat and placing a hand on the chair by way of contact. Face the blind person when you speak, and keep eye contact as you would with a sighted student. Give more than the usual verbal feedback in a conversation to compensate for the absence of the visual exchange. If the class is long and, as a consequence, has an intermission, the student will appreciate directions to a rest room, water fountain, and soft drink machine. Classroom doors should be kept entirely open or entirely closed, since a partly opened door is a hazard to the blind.

No vocabulary adjustments are needed. Blind students also say, "I can't see that" (for "I don't understand that") and "It looks easy to me." The sentence "The vision which moves us now is nothing less than the image of world federation" occurs in a recent publication of the National Federation of the Blind.

3. The Mathematics Classroom. Frequently the first apprehension of a teacher confronted with a blind student is how to test the student. This is no trouble at all, for the student knows (even so, an understanding is appropriate) that prior to the first test or written assignment he is to have a person to read to him—a reader. He secures the reader himself from a list provided by the services for the handicapped. The reader is paid from funds available for this purpose. One of the nice things about teaching blind students is that money and services are available for any reasonable academic need. Tests may be carried away from the classroom by the student and taken in the company of the reader. The time limit for the test may be specified, as well as the way it is to be submitted. I will accept tests in the handwriting of the reader, but I caution the student to have the test read back to him, since I take exactly what is written.

Learning braille well enough to read a test is not too difficult (partly because you can also see it), but it is unnecessary to do this. A good command of braille by an adult suddenly blinded is not trivial to acquire. The braillewriter is slightly larger than a typewriter (it usually has six keys), while the slate and stylus are small enough to carry in one's pocket. Blind students do not expect sighted people to read braille. If the student sends you a note or letter, he will probably type it on an ordinary typewriter.

I include blind students in classroom questions and find that their performance is consistent with their homework and tests. This gives me some confidence that their written work is original. When a blind student is reciting in class and the problem is long, I write on the blackboard for him as he dictates. From time to time, as the problem develops, I review for him what is on the blackboard. Assignments, like everything else, must be vocalized; I do this in a part of the room close to the blind student, who may have activated a tape recorder to receive the assignment. The sighted students soon forget that there are blind students in the class and so have to be prompted to read everything they have written on the blackboard. Also, they have to be cautioned against using pronouns such as "this" and "that" in lieu of reading the entire line being cited.

Blind students do require a little more attention in problems involving drawings and curves. The most satisfactory way to deal with this aspect of a course is with a device called a raised line drawing board. This is an inexpensive, simple device easily mistaken for a portable clip board. The face of the board is rubber-coated. One places a thin sheet (about $8\frac{1}{2}\times 11$ inches) of acetate on the board; then a ball-point pen, for example, may be used to draw figures. The acetate rises where the stylus or pen marks the sheet and the student can feel the figure. All of this can be done rapidly. If there are several blind students in the class, the teacher can make a drawing on a sheet for one student who can then pass it on to the others.

In classes where drawings are not frequent, or lacking a raised line drawing board, there are a number of substitutes. For example, the shapes of simple curves, such as conics, may be communicated by the teacher's making an impression on the back of the student's hand with the blunt end of a pencil or pen. Complicated curves and figures may be traced by the teacher on heavy paper with Elmer's glue. The glue will stand up in a ridge above the paper when dry. Obviously, this requires some warning, as it takes about four hours for the glue to dry well.

Another device may be made by scoring a portable softwood board into vertical and horizontal lines. Fixed thumbtacks can be used to denote the origin and units on the coordinate axes of a rectangular coordinate system. Round-headed map tacks can be used to plot points on a curve.

Much more sophisticated devices exist. The intrinsic difficulty in those that use the braille principle is that braille is raised above the paper by making an indentation on the reverse side of the paper, and so a machine has to keep this transformation in mind.

Both talking calculators and braille calculators are available, but the hand calculators that I have seen are limited in their operations and cannot be programmed. Students can be taught to punch cards and use a computer, however.

4. Sources. The Library of Congress Talking Book Program is splendid for leisure experiences on tapes. The following two organizations are much more helpful for the college student and teacher of mathematics. The American Printing House for the Blind, Inc., 1839 Frankfort Ave., Louisville, KY 40206, is supported by the U.S. Government. It is responsive to requests for material in braille and in leading teachers to a knowledge of what may be obtained. The American Foundation for the Blind, Inc., 15 West Sixteenth St., New York, NY 10011, supplies relevant publications, including teaching hints, and sells teaching and learning aids to teachers and students. The raised line drawing board and the calculators mentioned above may be purchased, for example, from the American Foundation for the Blind.

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PREPARATION OF FOREIGN GRADUATE STUDENTS TO TEACH MATHEMATICS: AN EXPERIMENTAL COURSE

SUZANNE DAMARIN AND GREGORY WEST

Like mathematics departments in many universities, the Ohio State University department enrolls a number of non-American graduate students who are supported by teaching assistantships. The preparation of these students for their teaching responsibilities, which often begin only a few months after their arrival in this country, has generally been inadequate, leaving both the foreign teaching assistants (TAs) and the undergraduates assigned to the TAs' courses uncomfortable and dissatisfied.

Approximately 30 percent of the teaching assistants employed by the Mathematics Department have come to Ohio State from countries outside North America. These foreign TAs are assigned responsibility for conducting recitation sessions that support precalculus instruction in large lectures. Typically, the TA is assigned two 30-student sections each academic quarter and can expect to teach more than 600 students while at Ohio State.

In the past, foreign TAs at Ohio State have been required to take nondegree-credit English composition courses and to participate in the teaching seminars required of all new TAs. For the summer of 1977 a unique experimental course, combining English language training with guidance and practice in teaching mathematics, was initiated by the Departments of Mathematics and English. The University supported this effort with an instructional grant. The course addressed itself to issues outlined in the CUPM pamphlet on the teaching of college mathematics [1], the nature of the local teaching situation and its problems, and the use of the English language in the mathematics classroom.

Description of the course. The course was organized into four components, each with its own objectives. The first three weeks were designed to familiarize the TAs with the Ohio State student and with the precalculus mathematics curriculum. A variety of activities were used in this phase. Reading assignments ranged from Pólya's "On Learning, Teaching, and Learning Teaching" [2] to articles critical of foreign TAs that had appeared in the student newspaper [3]. Videotapes of mathematics classes in progress were shown and discussed as part of a global introduction to the issues facing the foreign TA. Problem sets and past examinations used in remedial and precalculus mathematics courses were given to the TAs, and each TA was required to summarize orally the content of several problem sets to the rest of the group. This approach provided the TAs with an overview of the content and objectives of the precalculus curriculum,

as well as with practice and feedback related to mathematical terminology and usage.

During the second component the TAs were involved in teaching a minicourse to volunteers who responded to advertisements for a "Free Algebra Review." This course was held daily, except Friday, for five weeks; topics included simultaneous linear equations, quadratic equations, graphing functions, laws of exponents, and absolute value. A "master teacher" introduced the course and taught periodically as needed; the TAs taught most sessions, with two TAs teaching one half-hour apiece for each session. TAs were required to consult the master teacher in the process of preparation. This phase was designed so that the TA took increasing responsibility for the planning of instruction. During the first weeks the master teacher prepared carefully structured materials from which the TA was to work; later the TAs drafted their own materials and revised them in consultation with the master teacher before teaching.

Selected segments of the videotapes made of all minicourse sessions were viewed and discussed by the entire group on Fridays. The segments analyzed in these sessions were organized around a skill area, such as interpreting student remarks and questions or involving students in problem solving. Language problems common to several TAs were discussed in these sessions, and alternate ways of reading mathematical expressions (orally) were presented. In addition, each TA was required to view the full videotape of his own teaching with the instructors.

The third component of the course lasted two weeks. It was devoted to correction of group problems identified during the minicourse and to preparation for teaching a precalculus course, which was the fall-quarter course assignment for all of the TAs in the experimental course. In this phase TAs studied the text materials they would be using; in class, they were called upon to explain problems or statements related to the text. They were also given advice concerning the relative difficulty of topics in the course and the nature of frequently occurring misunderstandings.

The fourth component of the course was weekly individual consultation between the TA and the English instructor. In these sessions the TA was videotaped for 15 mintues, either conversing with the instructor or tutoring a student. The segments with the instructor were natural-speed conversations on unrehearsed topics in which the instructor used a variety of native-speaker reductions and question forms. In the tutoring sessions the TA helped the student to solve a series of related problems. The instructor-TA sessions gave the TA more opportunity to work on comprehension of instructor-produced reductions and questions, while the student-TA conferences more naturally elicited TA-produced questions. After the conversation or tutoring session, the TA and the instructor replayed and discussed the videotape for the balance of the hour. Segments in which there was comprehension loss by the instructor, student, or TA were replayed until the TA isolated the problem. The instructor then orally drilled the TA on the particular reduction or question form and gave assignments from a pronunciation text [**4**].

Evaluation. The success of the experimental course must be examined from several points of view. Because the group was small and there is no possibility of comparison with a "control group," the evaluation must be somewhat guarded. However, some encouraging observations can be made; not the least of these is the decline in the number of complaints about new foreign TAs during fall 1977.

The *English Placement Test* was given as a final examination in the experimental course. This is a test on nontechnical topics of written English used by the Ohio State English Department to place foreign students in one of three English courses. Each composition is read quickly by three instructors from the English as a Second Language section, and each grader independently assigns a placement level to the composition by comparing it to model themes. All TAs in the experimental group advanced at least one placement level on this test.

The TAs evaluated their own improvement on each of 12 dimensions related to teaching mathematics at Ohio State. They were instructed to rate their improvement on the scale: 3 for much improvement, 2 for some improvement, 1 for little improvement, and 0 for no improve-

ment. A summary of their responses is presented in Table 1. In ten of the twelve categories the median rating was 2 or 3. The TAs assigned one of the lowest ratings to improvement of English grammar; however, the results of the placement test indicated that they had improved on this dimension as much as would have been expected had they enrolled in the appropriate English grammar and composition courses.

TABLE 1. TA Self-evaluations of Improvement

Dimension	Rating					
	3	2	1	0	blank	Median
English grammar	0	6	3	3		1
Comprehension of student remarks	1	8	3	0		2
Ability to paraphrase statements	1	5	3	0	1	2
Flexibility in classroom presentation	4	6	1	0	1	2
Feeling at ease when teaching	4	6	1	0	1	2
Technical vocabulary	5	4	2	0	1	2
Ability to restructure problems	0	5	5	1	1	1
Ability to judge appropriate quantity of material to be covered	4	5	3	0		2
Ability to judge appropriate level of material to be covered	6	4	2	0		3
Ability to involve students in problem solving	2	8	1	0	1	2
Understanding American students' backgrounds and needs	8	2	1	1		3
Ability to respond to student questions	0	10	1	0	1	2

The instructors identified four dimensions on which they evaluated each of the TAs. The *Teaching Style* dimension was defined by organization, ability to involve students in discussion and problem solving, forcefulness in presenting new material, and related abilities. *Language*: *Technical* referred to correctness of the TAs' grammar, pronunciation, sentence structure, question forms, and vocabulary, while *Language*: *Functional* referred to the ability to make oneself understood by students through clarity of exposition, ability to paraphrase, etc., and comprehension of student remarks and questions and related issues. The *Participation/Interest* dimension referred to the TAs' level of involvement in the various parts of the course. Students were graded using an 8-point scale on each dimension; an overall grade was also assigned. Table 2 presents means and standard deviations of these grades, while Table 3 presents correlations among them.

Examination of these tables reveals that "teaching style" was the dimension on which the TAs scored lowest. It is also the component most highly correlated to the overall grade. The comparatively low grades on this dimension were attributable to two problems. Some of the TAs were simply not forceful enough in their teaching; their problems included lack of clarity and an apparent inability to "think on their feet" (in English). Others tended to lecture, often above the heads of the audience, and to pay little attention to the needs and questions of the class.

The low correlation between *Language*: *Technical* and *Language*: *Functional* is of particular interest. The ability of a TA to communicate with freshman mathematics students is quite different from the technical quality of the TA's spoken English. Neither aural comprehension of students' questions and remarks nor the ability to use redundancy, paraphrase, diagrams, or other means of clarifying teacher statements is a direct outgrowth of technical competence in spoken (or written) English. One man whose presentations were quite clear (almost faultless) was not able to understand student questions *at all*. On the other hand, two TAs with obvious grammatical deficiencies were able to communicate quite well with students.

TABLE 2. Means and Standard Deviations

Dimension	Mean	S.D.
Teaching Style	4.50	3.03
Language: Technical	5.17	2.11
Language: Functional	4.67	2.27
Participation/Interest	5.75	2.36
Overall Grade	5.00	2.00

TABLE 3. Correlations Among Scores on Evaluation Dimensions

	Language Technical	Language Functional	Participation/ Interest	Overall Grade
Teaching Style	.409	.607	.754	.942
Language: Technical		.197	.109	.484
Language: Functional			.155	.776
Participation/Interest				.624

Observations and conclusions. Many cultural factors other than language are related to foreign TAs' interactions with students. The transition from a society with a caste system or with a highly selective educational tracking system to an open-admission university must be a difficult one. While all the TAs in the experimental course adjusted remarkably well to the Ohio State situation, the difficulty of the transition was apparent in some (but not all) cases in a variety of ways: (a) difficulty in selecting and adhering to an appropriate level, (b) unrealistic expectations with respect to students' background knowledge as evidenced by presenting skeleton derivations and expecting students to fill in gaps mentally, (c) derogatory remarks to the instructors and among themselves concerning students' levels of achievement and motivation, and (d) occasional obvious anger at students for not understanding.

In addition to this general cultural transition required of foreign TAs, for some of them there was a need to adapt to a new curricular sequence. There were several points in the algebra materials used where the "natural" order of presentation of topics or the "natural" explanation of the foreign TA diverged from the American treatment of topics. While the orderings and explanations natural to the TA were certainly mathematically valid, they did not always mesh well with the local curriculum.

The instructors of the experimental course were able to address these issues as well as other problems related to the teaching of mathematics to American students. On the basis of the success of the course, a slightly modified version was required of incoming foreign TAs in the summer of 1978.

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MATHEMATICS IN BIOMEDICINE

C. Y. WANG

I need not elaborate on the success of mathematics applied to astronomy, physics, chemistry, or engineering. But what has mathematics, an exact science, to do with inexact sciences, such as biology or medicine? What can it do? Is biomathematics just a fancy alias for statistics?

Table 1 shows some of the uses of mathematics in biomedicine.

TABLE 1
Mathematics Used in Biomedicine

(I) *Service Mathematics* (biometrics). Calculus, probability, statistics, operations research, numerical analysis, logic.
Used in: Computational tools, experimental design, data analysis, curve fitting, regression, inferences, correlation, data processing, hospital planning, and medical care.

(II) *Descriptive Mathematics* (shape and form). Geometry, sets, number theory, graphs, topology, probability, dimensional analysis, coding theory.
Used in: Taxonomy, phylogeny, diagnosis, nerve networks, kinematics of motion, structural principles.

(III) *Functional Mathematics* (dynamic models). Analysis, differential equations, integral equations, numerical analysis, mechanics.
Used in biomechanics: solid, fluid, nervous, transport phenomena.
compartmental problems: enzyme, tracers, diffusion, genetics, population, epidemiology, ecology.
control theory: feedback, regulation, systems theory.

These branches of mathematics can be considered applied mathematics, since they have been applied to real phenomena in biomedicine. We see that the same branch of mathematics can appear to be either "pure" or "applied." I would define applied mathematics as "useful mathematics," the mathematics which is directly related to physical problems.

Applied mathematics involves modeling, which may be statistical, geometric, or dynamic. The following are the three equally important steps in the modeling process:

1) Formulation of a mathematical model of the physical phenomena.
↓
2) Solution of the resulting mathematical problem.
↓
3) Evaluation of the model and physical interpretation of the results.

In many cases the easiest step is obtaining the mathematical solution. Notice also the feedback which is essential in revising the model if it does not describe the real situation. Only through this interaction can we produce viable models and develop new mathematical tools.

Let me also differentiate "applicable mathematics" from "applied mathematics." I believe all mathematics is applicable, although some may take a couple of hundred years before it can actually be applied to a real problem. For example, although the theory of functions of a complex variable was an abstraction when invented by Gauss, it is now an integral part of applied mathematics.

There was an era when an intellectual giant was able to span several fields and contribute to all of them. For example, Descartes not only had a lasting influence on mathematics and medicine but also contributed to optics, chemistry, physics, anatomy, embryology, astronomy, meteorology, philosophy, and religion. Since several disciplines were contained under one skull, any interdisciplinary work was purely intracranial: between right and left brain. This type of interdisciplinary research is not found today. In fact it is rare to find someone who excels in all branches of a single field. I quote Wiener [6] on interdisciplinary research:

> Since Leibniz there has perhaps been no man who has had a full command of all the intellectual activity of his day. Since that time, science has been increasingly the task of specialists, in fields which show a tendency to grow progressively narrower . . . Today there are few scholars who can call themselves mathematicians or physicists or biologists without restriction. A man may be a topologist or an acoustician or a coleopterist. He will be filled with the jargon of his field, and will know all its literature, and all its ramifications, but, more frequently than not, he will regard the next subject as something belonging to his colleague three doors down the corridor.

Unfortunately, real-life problems cross several boundaries. Many important problems lie in the twilight zone untouched and unsolved by researchers in any single discipline. That is where cooperative interdisciplinary research is needed.

Let us look at the problems involved in interdisciplinary biomathematical research.

1. Recognition of need. Interdisciplinary efforts can only be fruitful if there is a feeling of mutual need. If one walks into some well-established biomedical laboratory, one can feel the air of complacency. Why not? Doesn't the half-million-dollar grant reflect the good research being done? Who needs mathematicians? On the other hand, there exist some applied mathematicians who work on nonexistent problems and publish in journals unread and unknown by biologists or physiologists. Don't the publications reflect the good research being done? Who needs biologists? Such noncooperative attitudes certainly do not solve the many problems lying within the twilight zone.

2. Communication. (a) *Language:* There is a minimum foreign vocabulary which must be mastered by interdisciplinary workers before any communication can commence. I estimate a minimum of one year's independent study or course study before one is able to manage "pidgin language."

(b)*Point of view*: There must also be an ability to understand the methods and the thinking processes of the researcher's counterpart. I quote Wiener [**6**]:

> The mathematician need not have the skill to conduct a physiological experiment, but he must have the skill to understand one, to criticize one, and to suggest one. The physiologist need not be able to prove a certain mathematical theorem, but he must be able to grasp its physiological significance and to tell the mathematician for what he should look.

All of these take time and effort.

3. Education. Current educational practices do not encourage interdisciplinary studies. In general, high school science students who like mathematics go into physics, mathematics, or engineering. Those who dislike mathematics go into biology, physiology, or medicine. Even within some mathematics departments there is a tendency to regard pure mathematics as having a higher status than applied mathematics. Good students are goaded into pure mathematics programs without being presented with an alternative. Graduate students spend years passing the prelims and completing course requirements; finally, six months before graduation, they find out that they would like to "do some biomathematics." We have to turn these young aspirants away. In the current system there is simply no time allotted for the 20 or more credits of biology needed before viable research can be attempted.

And if the student does start out on an interdisciplinary track, forfeiting some pure mathematics subjects for a study of another discipline, he then faces a problem in employment. I quote C. C. Lin's opening address at the Conference of Education in Applied Mathematics [**4**]:

> It often happens (although there are exceptions) that when a person is considered for an appointment or promotion in a mathematics department, he must first be a qualified pure mathematician. Since the knowledge and research work of an applied mathematician are usually not that of a pure mathematician, he suffers (in an unfair manner) in comparison with a pure mathematician with marginal interest in applications. A similar situation holds in a department of science, but with opposite qualifications being emphasized.

The last sentence means: If you are in a biological science department, you are judged only on how well you can do the experiments; the more time you spend in mathematics the worse off you become.

4. Significance. There are numerous fields of biomedicine in which mathematics can be of some help. The following is a partial list.

Medicine	*Biology*
anatomy	biochemistry
anesthesiology	biology
dentistry	biophysics
medical engineering	botany
medical technology	cytology
medicine	ecology
nuclear medicine and radiology	embryology
nutrition	entomology
pathology	genetics
pharmacology	microbiology
physical medicine and rehabilitation	zoology
physiology	
psychology	
public health	
sociology	
surgery	
veterinary medicine	

Each field involves many branch areas. For instance, physiology includes reproductive physiology, renal physiology, endocrinology, gastroenterology, cardio-vascular system, respiratory system, nervous-sensory system, musculo-skeletal system, etc.—enough extra studying until the hair turns grey. The number of problems which are amenable to mathematics certainly is not lacking. But some problems have more significance than others. For example, use of the Fibonacci sequence to study the arrangement of sunflower seeds is certainly less significant than the study of enzyme kinetics in order to increase the yield of sunflower seeds. It is imperative to choose an area of significance so that your extra efforts are worthwhile.

5. Difficulties inherent in biomedical problems. (a) *The basic laws are unknown.* Biomedicine involves a living process not too well understood. For example, mechanisms of muscle contraction still cannot be fully explained by current physical laws.

(b) *The phenomena are difficult to isolate.* Unlike the situation in physics or chemistry, where the effect of each variable can be controlled, biological systems involve numerous pathways, feedback loops, and safety mechanisms which make modeling difficult. In many cases oversimplifications of the model yield erroneous results.

(c) *The theory is difficult to test.* Variations occur from species to species, from specimen to specimen, and within a single specimen from time to time. The results require numerous statistics.

(d) *The errors of biological measurements are large.* The high degree of error in biological measurements (as compared, say, to physics) makes certain "solutions" meaningless. For instance, if measurements of biological phenomena can only be made with 20 percent error, it is ridiculous to apply refined mathematical methods to try to correct the theoretical results to 5 percent error.

Realizing the need for interdisciplinary biomedical mathematics and seeing the many difficulties in store, is it still worthwhile to apply mathematics to biomedicine? This is a decision for the individual.

APPENDIX

The following are some famous mathematicians who also became famous in biomedicine. These mathematicians are recognized in the history of biology, physiology, and medicine [1], [2], [3], [5] and actually contributed to the development of biomedicine.

Democritus 470-380 BC

Democritus was a geometer who studied and dissected many animals. He regarded the brain as the organ of thought and attempted to classify animals by the color of blood.

Johannes Kepler 1571-1630

As a mathematician and astronomer to the court of Austria, Kepler established the fundamentals of visual principles using geometry. He studied pupillary contraction, accommodation, and nearsightedness.

René Descartes 1596-1650

Descartes founded analytic geometry and contributed to algebra and the theory of equations. His book *De homine* (1664) was considered the first text in physiology. He envisioned a nervous network emanating from the brain and regarded organs as machines which operate on laws of mathematics.

Giovanni Alfonso Borelli 1608-1679

Borelli, a professor of mathematics at Messina, founded the iatromathematics school. He applied physical principles to organs, especially in connection with muscular action. He theorized that respiration involves a transfer of some vital elements from air to blood and that the kidney serves as a filter.

Daniel Bernoulli 1700-1782

Aside from his contributions to hydrodynamics, probability, theory of equations, and partial differential equations, Bernoulli, a physician who chaired anatomy at Basel, introduced mathematics to epidemiology and also calculated cardiac output by the product of force and the length of impulse.

George-Louis Leclerc Buffon 1707-1788

Buffon invented geometrical probability (Buffon's needle problem). He also compiled 44 volumes of *Natural History*. His ideas on natural relations laid the foundations of genetics and evolution.

Denis Diderot 1713-1784

A mathematician and natural philosopher, Diderot contributed to differential equations, geometry, and the theory of vibrations. He studied psychology, physiology, and advanced theories in natural selection. He advocated a science which is based on facts rather than pure logic.

Pierre Simon de Laplace 1749-1827

Laplace worked in partial differential equations, potential theory, astronomy, perturbation theory, and probability. He studied heat transfer and interpreted respiration as a form of slow biological oxidation.

Hermann Ludwig Ferdinand von Helmholtz 1821-1894

Helmholtz worked in hydrodynamics, postulated the conservation of energy, and formulated the basic equations of electromagnetic field theory. He was also a physician who invented the ophthalmoscope and contributed to the theories of vision, color, hearing, speech, muscle contraction, and the transmission of nervous impulse.

Vito Volterra 1860-1940

Volterra made major contributions to functional analysis and integral equations. He was also famous for his studies on population models in mathematical ecology.

Godfrey Harold Hardy 1877-1947

Hardy worked principally in number theory and analysis, but he is also well known for the Hardy-Weinber law in population genetics.

There are, of course, other scientists who worked in both mathematics and biomedicine. For instance, DaVinc studied anatomy besides plane and solid geometry; Dürer, also an artist-anatomist, laid the basis for descriptive geometry; Newton invented calculus and founded the theory of light and color in optics; Wiener, an analyst, worked on control theory and brain networks. However, their contributions to biomedicine have not been as widely recognized.

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PROBLEMS AND SOLUTIONS

EDITED BY A. P. HILLMAN

The editors receive more problems than they can use, but they are always on the lookout for interesting and novel problems. The task of selection will be much easier if proposers will keep the following guidelines in mind:

Send all problems (both elementary and advanced) to A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, New Mexico 87131, in duplicate if possible. The editors urge proposers to include any solutions or information that will help the editors, including reasons why the problem is interesting. Problems in well-known textbooks and results that appear in generally accessible sources are not acceptable.

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution. If you submit a problem without a solution, you should tell the editors whether or not you know (or somebody else knows) how to solve the problem. If you are uncertain, the problem might be more suitable for the Research Problems section; see the instructions given there.

Proposers are asked to aim for the same audience as for the rest of the MONTHLY: *a rule of thumb is to think of people who have had at least a year of graduate work in Mathematics, very likely some years ago. Proposals that can be both understood and solved by high school students, or by those in the first two years of college, are usually more appropriate for the Mathematics Magazine or the Two-Year College Mathematics Journal. No precise line can be drawn between elementary and advanced problems, but in general an elementary problem will involve notions no more difficult than those encountered in the first year of graduate work and should be solvable by methods common at the same level. An advanced problem may deal with more advanced concepts or require more sophisticated techniques for its solution.*

A problem that takes half a page or more to state, or that involves several ad hoc definitions, is usually unsuitable. The statement of a problem should avoid specialized jargon and symbols whenever the English language is more readily understood: for example, "f is a continuous function" is preferable to "$f \in C$."

Before submitting a problem, be sure that it says exactly what you intend it to say. Solvers are not telepathic and will usually solve what was proposed, not what you had in mind.

PROBLEMS DEDICATED TO EMORY P. STARKE

Solutions of the problems in this issue dedicated to Professor Emory P. Starke should be mailed to Prof. A. P.

Hillman, Dept. of Mathematics and Statistics, University of New Mexico, Albuquerque, N.M. 87131 (USA), before October 31, 1979. To facilitate consideration, solutions should be typed (with double spacing).

S 14*. *Proposed by C. L. Mallows, Bell Laboratories, Murray Hill, N. J.*

Let $n(m,f,r)$ be the number of arrangements $(a_1,a_2,\ldots,a_m)$ of $(1,2,\ldots,m)$ that have f *fixed points* $(a_i=i)$ and r *rises* $(a_i<a_{i+1})$. Prove (a) $n(m,0,r)=n(m,1,r)$ for $1\leqslant r\leqslant m-1$, and (b) $n(m,f,r)=$

$$\sum_{j=2}^{m-r}(-1)^{m-r-j}(j-1)^j j^{m-f-j}\binom{f+j-1}{j-1}\binom{m+1}{m-r-j}+(-1)^{m-f}\frac{\delta_{r+1-f}}{m-f+1}\binom{m}{f}$$

for $0\leqslant f\leqslant r+1\leqslant m, 0\leqslant r$, where $\delta_k=1$, $=0$ if $k=0$, $\neq 0$, respectively.

S 15. *Proposed by Joel L. Brenner, Palo Alto, California.*

Let $f(t)>0$ for $a\leqslant t<b$ and u be a fixed real number. Show that the functional $[\int f^{s+u}/\int f^s]^{1/u}$ increases with s.

For $u=1$, this theorem is due to E. F. Beckenbach (this MONTHLY, 1950, p. 1).

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA (USA) 94303. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (in duplicate with double spacing) and should be mailed before October 31, 1979. Please enclose a self-addressed card or label (for acknowledgement).

E 2744 [1978, 823] (**correction**). *Proposed by H. L. Montgomery, University of Michigan*

Let $a_n\geqslant 0$ and $a_{m+n}\leqslant a_m+a_n$ for $m,n=1,2,\ldots$. Show that $\sum_{k=1}^{n}k^{-2}a_k\geqslant\frac{1}{4}n^{-1}a_n\log n$.

E 2779. *Proposed by H. Schwerdtfeger, McGill University.*

(1) Let $A=(a^{(1)}\ a^{(2)}\ldots a^{(n)})$ be a non-singular matrix, over a field F, whose columns $a^{(j)}$ represent points in the n-dimensional affine space S_n. Let π be the hyperplane passing through the points $a^{(1)},\ldots,a^{(n)}$. Let $b\in S_n$, $b\neq 0$, and B be the matrix $(b\ b\ldots b)$. Show that the determinant $|A-B|=0$ if and only if $b\in\pi$.

(2) Generalize the statement (1) to a more general matrix of rank one, namely $B=(\gamma_1 b\ \gamma_2 b\ldots\gamma_n b)$, $\gamma_1\gamma_2\cdots\gamma_n\neq 0$, $\gamma_j\in F$.

(3) If A is singular and Σ is the subspace of S_n generated by the columns of A, show that there is no b in Σ such that $|A-B|\neq 0$, with $B=(b\ b\ldots b)$.

E 2780. *Proposed by Jim Totten, University of Saskatchewan.*

Let $d(n)$ be the number of (positive integral) divisors of the natural number n and define $S(n)$ as $\Sigma d(k)$, with the sum taken over all divisors k of n. Determine the values of n for which $n=S(n)$.

E 2781. *Proposed by James Propp, student, Harvard College.*

Let S be a set of n integers and $m=n(n+1)/2$. When $n\geqslant 3$, can $S+S$ constitute a complete residue set modulo m? (Here $S+S=\{a+b|a,b\in S\}$.)

E 2782. *Proposed by Robert E. Shafer, Berkeley, California.*

D. S. Mitrinović, in his reference book *Analytic Inequalities*, cites an inequality (p. 190, display 3.1.8) of Ostrowski:

$$\sum_{k=n}^{\infty}\frac{1}{k^2}<\frac{1}{n-\frac{1}{2}} \qquad \text{for } n \text{ a positive integer.}$$

Establish the generalization

$$2\arctan\frac{1}{2x-1}<\sum_{n=0}^{\infty}\frac{1}{(n+x)^2}<\frac{1}{x-\frac{1}{2}} \qquad \text{for } x>\tfrac{1}{2}.$$

E 2783. *Proposed by William Knight and Bruce Lund, University of New Brunswick.*

Find all functions $\phi(z)$ such that ϕ is a one-to-one continuous map of the unit circle $\{z:|z|=1\}$ onto itself and $[\phi(z)]^2=\phi(z^2)$ for all z on the circle.

E 2784. *Proposed by F. S. Cater, Portland State University.*

For each positive integer n and each positive number x, let $F_n(x)=0$ if $x<(n+1)^{-1}$, and let

$$F_n(x)=k^{-1}\left[(n+1)^{-1}+2(n+2)^{-1}+\cdots+k(n+k)^{-1}\right]$$

if $x\geqslant(n+1)^{-1}$, where k is the largest integer satisfying

$$(n+1)^{-1}+(n+2)^{-1}+(n+3)^{-1}+\cdots+(n+k)^{-1}\leqslant x.$$

Let $F(x)=\lim_{n\to\infty}F_n(x)$ for $x>0$. Determine the function $F(x)$. Is the convergence of $F_n(x)$ uniform in x? Find $\sup F(x)$ and $\sup[F(x)/x]$.

SOLUTIONS OF ELEMENTARY PROBLEMS

E 2649 [1977, 294] **(correction)**. The fact that this problem was also solved by O. P. Lossers (Netherlands) was inadvertently omitted when the solution [1978, 596] was printed.

Legendre Polynomials Again

E 2658 [1977, 387]. *Proposed by W. Weston Meyer, General Motors Research Laboratories, Warren, Michigan.*

(1) For $0<\alpha<\pi/2$ and integral $n\geqslant 0$ show that

$$\int_0^{\alpha}\left(\frac{\sin\theta}{\sin\alpha}\right)^{2n}d\theta=\sum_{k=0}^{n}c_{nk}\int_0^{\alpha}\left(\frac{\tan\theta}{\tan\alpha}\right)^{2k}d\theta,$$

where the constants c_{nk} are independent of α.

(2) Find all polynomials P such that the ratio

$$\int_0^{\alpha}P\left(\frac{\sin\theta}{\sin\alpha}\right)d\theta\div\int_0^{\alpha}P\left(\frac{\tan\theta}{\tan\alpha}\right)d\theta$$

is independent of $\alpha\in(0,\pi/2)$.

*Solution by Otto G. Ruehr and Clark Givens, Michigan Technological University (indepe*ı *dently).* We have

$$\int_0^{\alpha}\frac{d\theta}{1-t\sin^2\theta}=\int_0^{\alpha}\frac{\cos\theta\, d\theta}{\sqrt{1-t(\cos^2\theta-\cos^2\alpha)}}$$

since each of these integrals is equal to

$$\frac{1}{\sqrt{1-t}}\sin^{-1}\left(\sin\alpha\cdot\sqrt{\frac{1-t}{1-t\sin^2\alpha}}\right).$$

Expand each integrand in its Maclaurin series, integrate term by term, and equate coefficients of corresponding powers of t to get

$$\int_0^\alpha \sin^{2n}\theta\, d\theta=\binom{1/2}{n}\int_0^\alpha\left(1-\frac{\cos^2\alpha}{\cos^2\theta}\right)^n d\theta$$

$$=2^{-2n}\binom{2n}{n}\sin^{2n}\alpha\int_0^\alpha\left(1-\frac{\tan^2\theta}{\tan^2\alpha}\right)^n d\theta.$$

Hence (1) is valid with

$$c_{nk}=(-1)^k 2^{-2n}\binom{n}{k}\binom{2n}{n}.$$

Since

$$f_k(\alpha)=\int_0^\alpha\left(\frac{\tan\theta}{\tan\alpha}\right)^k d\theta$$

is of the form

$$f_k(\alpha)=\begin{cases}(-1)^r\alpha\cot^k\alpha+P_1(\cot\alpha), & k=2r\\ -(-1)^r\cot^k\alpha\log\cos\alpha+P_2(\cot\alpha), & k=2r+1\end{cases}$$

where P_1, P_2 are polynomials, it is clear that the functions $f_0(\alpha), f_1(\alpha), \ldots$ are linearly independent.

On the other hand

$$g_k(\alpha)=\int_0^\alpha\left(\frac{\sin\theta}{\sin\alpha}\right)^k d\theta$$

contains no logarithmic terms. Hence our polynomial P must be even, say,

$$P(t)=\sum_{i=0}^n a_i t^{2i}.$$

Denote the constant ratio of the integrals by I_n. Then for all $\alpha\in(0,\pi/2)$ we must have

$$I_n\cdot\Sigma a_i f_{2i}(\alpha)=\Sigma a_1 g_{2i}(\alpha).$$

Using part (1) we obtain from here

$$I_n a_i=\sum_{j=i}^n a_j c_{ji}=\sum_{j=i}^n(-1)^i 2^{-2j}\binom{j}{i}\binom{2j}{j}a_j.$$

Thus a_n may be arbitrary, $I_n=c_{nn}=(-1)^n 2^{-2n}\binom{2n}{n}$ and for $i<n$,

$$a_i=(-1)^{n+i}\binom{2n}{n+i}\binom{2n+2i}{2n}a_n.$$

Hence $P(t)$ is a scalar multiple of the $2n$th Legendre polynomial.

Also solved by Fred Dodd, L. E. Mattics, and the proposer.

(0, 1)-Matrices

E 2678 [1977, 738]. *Proposed by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario, Canada.*

Find the maximum number of ones in an $n\times n$ $(0,1)$-matrix whose square is again a $(0,1)$-matrix.

Solution by A. A. Jagers, Technische Hogeschool Twente, Enschede, Netherlands. Denote the number of ones in the (0, 1)-matrix A by $w(A)$. Let $\mathcal{C}_n$ be the set of those $n\times n$ (0, 1)-matrices A such that A^2 is again a (0, 1)-matrix and let α_n be the maximum of $w(A)$ for $A\in\mathcal{C}_n$. Then

(i) $\alpha_n=(n^2+4n-1)/4$ if n is odd,
(ii) $\alpha_n=(n^2+4n-4)/4$ if n is even, $n\neq 4$,
(iii) $\alpha_4=8$.

Proof. Clearly $\alpha_2=2$. Let I_k be the $k\times k$ identity matrix and $J_{m,k}$ be the $m\times k$ matrix with every entry 1. The following matrices $A\in\mathcal{C}_n$ (with $n=2m-1$, $n=2m$, and $n=4$, respectively) have $w(A)=\alpha_n$.

$$\left[\begin{array}{c|c} 0 & I_{m-1} \\ \hline J_{m,m} & 0 \end{array}\right],\quad \left[\begin{array}{c|c} 0 & \begin{array}{c} I_{m-1} \\ \hline 00\cdots0 \end{array} \\ \hline J_{m,m+1} & 0 \end{array}\right],\quad \begin{bmatrix} 1010 \\ 1010 \\ 0101 \\ 0101 \end{bmatrix}.$$

To prove that $w(A)\leqslant\alpha_n$ for all A, let $A=[a_{ij}]\in\mathcal{C}_n$. Let k be the maximal column sum of $A;\Sigma_i a_{ir}=k$, say. Fix r. Divide $\{1,2,\ldots,n\}$ into two subsets S and T given by $S=\{i|a_{ir}=1\},T=\{i|a_{ir}=0\}$. Then $\Sigma_{j\in S}a_{ij}\leqslant 1$ for all i, and $\Sigma_i a_{ij}\leqslant k$ for all $j\in T$. That is, $w(A)\leqslant n+(n-k)k$. This upper bound attains its maximum at $k=n/2$ if n is even, and at $k=(n-1)/2$ if n is odd. Thus $\alpha_n\leqslant(n^2+4n)/4$, if n is even, and $\alpha_n\leqslant(n^2+4n-1)/4$, if n is odd.

This completes the proof of (i) and (iii). For n even, $n\neq 4$, we see that in any case, $(n^2/4)+n-1\leqslant\alpha_n\leqslant(n^2/4)+n$. Suppose it were possible that $w(A)=(n^2/4)+n=m^2+2m$ where $n=2m,m>2$. Then $k=m$ and $\Sigma_i a_{ij}=m$ for all $j\in T$. Since the transpose of A belongs to $\mathcal{C}_n$ too, we may also assume that the row sums of A are not greater than m either. Hence $\Sigma_j a_{ij}\leqslant m$ for all $i\in S$ and thus $\Sigma_j\Sigma_{i\in T}a_{ij}\geqslant w(A)-m^2=n$. However $\Sigma_{i\in T}a_{ir}=0$ by definition of T, so a number j exists such that $a_{pj}=a_{qj}=1$ for some $p,q\in T,p\neq q$. It follows that the pth and qth columns each have m ones and in complementary positions, i.e., that $a_{ip}+a_{iq}=1$ for all i. This in turn shows that each row of A has at most 2 ones. Since $w(A)=m^2+2m>2n$ for $m>2$, this is a contradiction.

Also solved by Robert Breusch, Census Bureau Mathematical Group, Paul Cull, Thomas Foregger, R. W. Hamel, G. A. Heuer & K. W. Heuer, Keith Hodge, G. W. Peck, Martin Schaefer (Germany), C. J. Smyth (Australia), Dean Sturtevant, David Witte, and Shelby Worley. Partially solved by Itshak Borosh & Doug Hensley, Thomas Elsner, JoAnne Growney, and I. A. Sakmar (Canada).

Convexity of a Polyhedron

E 2694 [1978, 48]. *Proposed by I. J. Schoenberg, University of Wisconsin.*

Let Π be a prism inscribed in the sphere S of unit radius and center O. The base of Π is a regular n-gon of radius r. For each face F of Π drop a directed perpendicular from O and let A_F be the point where it intersects S. Let Π^* be the polyhedron obtained by adding to Π, for each face F, the pyramid of base F and apex A_F.

For which values of r is Π^* convex?

Solution by David Witte, undergraduate, University of Wisconsin, Madison. It is clear that Π^* is convex iff for all adjacent faces F and G, the line segment $\overline{A_FA_G}$ lies in Π^* (the line segment joining the apexes of any two neighboring annexed pyramids lies in Π^*).

By symmetry, we need only consider the line segments $\overline{A_BA_F}$ and $\overline{A_FA_G}$ where B is a base of Π and F,G are two adjacent lateral faces of Π.

Let P be the midpoint of the common side of B and F. Since $\measuredangle(OA_BA_F)=\pi/4$, $\overline{A_BA_F}$ lies in Π^* iff $\theta=\measuredangle(OA_BP)\geqslant\pi/4$. We have

$$\tan\theta=r\cos\frac{\pi}{n}/(1-\sqrt{1-r^2}\,),$$

and our condition is equivalent to

$$r \leqslant 2\Big/\left(\cos\frac{\pi}{n}+\sec\frac{\pi}{n}\right). \tag{1}$$

If Q is the midpoint of $\overline{A_F A_G}$ then $\overline{OQ}=\cos\frac{\pi}{n}$. Hence $\overline{A_F A_G}$ lies in Π^* iff

$$r \geqslant \cos\frac{\pi}{n}. \tag{2}$$

Hence Π^* is convex iff (1) and (2) hold.

Also solved by J. Dou (Spain), Barry Monson, and the proposer.

Expected Number of Draws

E 2696 [1978, 116]. *Proposed by William P. Wardlaw, U.S. Naval Academy.*

(a) If numbers are drawn randomly (using uniform distribution with replacements) from the set $\{1,2,\ldots,n\}$ until their sum first exceeds n, what is the expected number of draws?

(b) The same problem for numbers selected from $\{0,1,\ldots,n-1\}$ until their sum exceeds $n-1$.

Solution by Lee Erlebach, Michigan Technological University, and Jerrold W. Grossman, Oakland University, Michigan (independently). For each integer $k \geqslant 0$ let E_k be the expected number of draws until the sum first exceeds k.

(a) We claim that $E_k=(1+1/n)^k$ for $0 \leqslant k \leqslant n$. This is clear for $k=0$. If $1 \leqslant k \leqslant n$, then by considering the outcome of the first draw we obtain a recursion formula

$$E_k=\frac{n-k}{n}+\frac{1}{n}\sum_{i=0}^{k-1}(1+E_i).$$

The result now follows by induction on k.

(b) We shall ignore the trivial case $n=1$. In this case we claim that

$$E_k=\left(\frac{n}{n-1}\right)^{k+1}, \qquad 0 \leqslant k \leqslant n-1.$$

If $0 \leqslant k \leqslant n-1$, then by considering the outcome of the first draw we obtain the relation

$$E_k=\frac{n-k-1}{n}+\frac{1}{n}\sum_{i=0}^{k}(1+E_i).$$

Since E_k is bounded above by

$$(k+1)E_0=(k+1)\sum_{m=1}^{\infty} mn^{1-m}\frac{n-1}{n}=(k+1)\frac{n}{n-1}<\infty,$$

we conclude that

$$\frac{n-1}{n}E_k=1+\frac{1}{n}\sum_{i=0}^{k-1}E_i.$$

For $k=0$ we obtain $E_0=n/(n-1)$. The result now follows by induction on k.

Also solved by forty-one other readers and the proposer.

REMARK. For the continuous version of this problem see this MONTHLY [1961, p. 18].

The sup of an inf

E2707 [1978, 276]. *Proposed by Leonard Shapiro, North Dakota State University.*

Find $\sup\sigma(f)$ where

$$\sigma(f)=\inf_{x>0}\left\{\frac{f(x)}{x}\int_0^x(1-f(t))\,dt\right\}$$

and f ranges over continuous functions on $[0,\infty)$. For which f (if any) is this supremum achieved?

Solution by Tom Jager, Calvin College. The supremum is $1/4$ and $\sigma(f)=1/4$ if and only if $f(x)\equiv 1/2$. Since $f(x)$ is continuous on $[0,\infty)$, by the mean value theorem $(1/x)\int_0^x(1-f(t))dt=1-f(c_x)$ for some $c_x\in[0,x]$. Thus, as $x\to 0+$, $(f(x)/x))\int_0^x(1-f(t))dt$ approaches $f(0)(1-f(0))\leqslant 1/4$. This shows that $\sigma(f)\leqslant 1/4$ and $\sigma(f)=1/4$ implies that $f(0)=1/2$. Elementary integration shows that $\sigma(f)=1/4$ for $f(x)\equiv 1/2$, so that $\sup\sigma(f)=1/4$. Assume now that $\sigma(f)=1/4$. Let

$$F(x)=\begin{cases}\dfrac{1}{x}\displaystyle\int_0^x f(t)\,dt, & x>0\\ 1/2, & x=0\end{cases}$$

Since $f(0)=1/2$, $F(x)$ is continuous for $x\geqslant 0$. Let $[0,c]$ be any interval on which $f(x)$ is non-negative. Since $f(x)(1-F(x))\geqslant\sigma(f)=1/4$ and $f(x)(1-f(x))\leqslant 1/4$ for every $x\geqslant 0$, $f(x)\geqslant F(x)$ on $[0,c]$. Hence for $x\in(0,c)$

$$\begin{aligned}F'(x)&=(1/x^2)\left(xf(x)-\int_0^x f(t)\,dt\right)\\&=(1/x)(f(x)-F(x))\geqslant 0.\end{aligned}$$

Thus, F is increasing on $[0,c]$ and $f(x)\geqslant F(x)\geqslant 1/2$ on $[0,c]$. Since f is continuous, this forces $f(x)\geqslant 0$ on $[0,\infty)$, so that $f(x)\geqslant F(x)$ and $F(x)$ is increasing on all of $[0,\infty)$. Since $f(x)(1-F(x))\geqslant 1/4$ for all x's, $F(x)\leqslant 1$ for all x. Hence $L=\lim_{x\to\infty}F(x)$ exists. There must exist a sequence $x_n\to\infty$ such that $f(x_n)\to L$, for otherwise there would be $\epsilon>0$ and K such that for $x\geqslant K$

$$\begin{aligned}F(x)&=\frac{1}{x}\int_0^x f(t)\,dt\geqslant\frac{1}{x}\left\{\int_0^K f(t)\,dt+\int_K^x(L+\epsilon)\,dt\right\}\\&=\frac{1}{x}\left\{\int_0^K f(t)\,dt-K(L+\epsilon)+x(L+\epsilon)\right\}\end{aligned}$$

so that $\lim F(x)\geqslant L+\epsilon$. Next, for such a sequence x_n

$$\frac{1}{4}=\sigma(f)\leqslant\lim f(x_n)(1-F(x_n))=L(1-L)\leqslant\frac{1}{4}$$

which forces $L=1/2$. Since $F(0)=1/2$, $F(x)\equiv 1/2$ and $f(x)=(d/dx)\int_0^x f(t)\,dt=(d/dx)xF(x)\equiv 1/2$.

Also solved by Aage Bondesen (Denmark), Peter Doffer, M. Dixon, Addison Fischer, Joel Levy, Peter Lindstrom, O. P. Lossers (Netherlands), Robert Scherrer, and the proposers. Many solvers found $\sup\sigma(f)$ but failed to show that $f(x)\equiv 1/2$ is the only function for which the supremum is achieved.

A Subclass of the Absolute Primes

E 2718 [1978, 384]. *Proposed by Gordon D. Pritchett, Hamilton College.*

Find all prime numbers p which have the following two properties:
(i) all numbers obtained from p by permuting its digits are also prime;
(ii) the sum and the product of the digits of p are also prime.

Remark by the editor. The required primes are $2,3,5,7,113,131,311$. These exhaust the set of such primes, as 43 solvers showed. O. P. Lossers (Netherlands) refers to his solution of Aufgabe 773, Elemente der Math., 32 (1977). As to condition (i), Allan Wm. Johnson, Jr., calls attention to his solution of problem 953, Math. Mag., 50 (1977) 100-103, and to the article by T. N. Bhargava and P. H. Doyle, "On the Existence of Absolute Primes," Math. Mag., 47 (1974) 233. Numbers of the form $R_p=(10^p-1)/9$ (rep-units) are eliminated by condition (ii). R_p is prime for $p=2,19,23,317$. (See S. Yates's article, The mystique of repunits, Math. Mag., 51 (1978) 22-28.)

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to Professor Roger C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109. To facilitate their consideration, solutions of Advanced Problems in this issue should be typed (in duplicate with double spacing) and should be mailed before October 31, 1979.

6270. *Proposed by Kenneth S. Williams, Carleton University.*

Let p be a prime congruent to 1 modulo 8. Let ϵ_{2p} denote the fundamental unit of the real quadratic field $Q(\sqrt{2p})$ and let $h(-2p)$ denote the class number of the imaginary quadratic field $Q(\sqrt{-2p})$. Prove that if the norm of ϵ_{2p} is -1, then

$$h(-2p)\equiv 0 \pmod 8), \quad \text{if } p\equiv 1 \pmod{16},$$

and

$$h(-2p)\equiv 4 \pmod 8), \quad \text{if } p\equiv 9 \pmod{16}.$$

6271. *Proposed by Michael Barr, McGill University.*

For positive integers n define

$$a_n=\frac{n-1}{n}+\frac{(n-1)(n-2)}{n^2}+\cdots+\frac{(n-1)!}{n^{n-1}},$$

$$b_n=\frac{n}{n+1}+\frac{n^2}{(n+1)(n+2)}+\cdots+\frac{n^{n-1}}{(n+1)\cdots(2n-1)}.$$

(A) Prove that, for all $n>1$, $0<b_n-a_n<1$.

(B*) Prove or disprove that $\lim_{n\to\infty}(b_n-a_n)=2/3$ and that

$$b_n-a_n-2/3=O(1/n).$$

6272. *Proposed by P. Olin, York University, and Kenneth W. Smith, University of Toronto.*

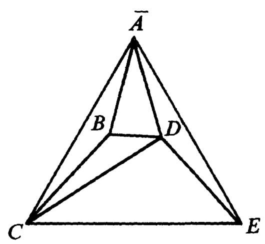
It is known (Waszkiewicz and Weglorz, Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys., 17 (1969) 195–199; see top of page 197) that there is a complete, $\aleph_1$-categorical theory T of first-order logic such that the direct product $T\times T$ is not $\aleph_1$-categorical. Are there complete first-order theories T_1, T_2 with T_1 the theory of a finite model, T_2 $\aleph_1$-categorical, and $T_1\times T_2$ not $\aleph_1$-categorical? If so, find such a pair T_1, T_2 with the cardinality of the model of T_1 as small as possible.

SOLUTIONS OF ADVANCED PROBLEMS

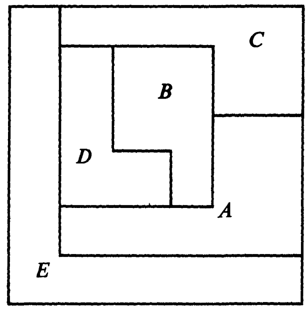
Rectangular Graphs

6182* [1977, 828]. *Proposed by A. K. Austin, University of Sheffield, England.*

Prove or disprove that any finite planar graph can be represented by a map in which all the regions are L-shaped with sides horizontal and vertical. For example



can be represented by



Solution by Stephen E. Wilson, Michigan State University. The statement is false even if the requirement that the regions be L-shaped is dropped. A vertex in a map, all of whose edges are horizontal or vertical, can have valence at most four, and so some embedding of the corresponding graph into the plane must have regions with at most four sides apiece. But the skeleton of the regular dodecahedron is a planar graph whose only embedding in the plane has 5-gonal regions. Thus, it cannot be represented by any map with horizontal and vertical sides.

The Number of Idempotents

6183* [1977, 829]. *Proposed by Albert A. Mullin, Ft. Hood, Texas.*

Let R be a ring with a finite number n of multiplicative identities.
(a) If R is commutative, show that n is a power of 2.
(b) If R has a unit, show that n is even but need not be a power of 2.
(c) Is there an R for which n is an odd prime?

Solution by Tom Jager, Calvin College. Let I be the set of idempotents of the ring R.
(a) Suppose R is commutative. Define the operation $*$ on R by

$$x*y = x+y-2xy.$$

It can easily be verified that I is closed under $*$ and that $(I,*)$ is an abelian group whose exponent is $\leqslant 2$. Since I is finite, $|I| = 2^k$ for some k.
(b) Suppose $1 \in R$ and $1 \neq 0$. If $x \in I$, then

$$(1-x)^2 = 1-2x+x^2 = 1-x,$$

so that $1-x \in I$. Define the mapping A from I to I by $A(x) = 1-x$. Since A^2 is the identity mapping, the sets of the form $\{x, A(x)\}$ partition I. Suppose $A(x) = x$. Then $1-x = x$ so that $1 = 2x$. But $x \in I$, so $2x = 1 = 1 \cdot 1 = 2x \cdot 2x = 4x^2 = 4x$. Hence, $2x = 0$. Since $1 \neq 0$, this is impossible. Thus, each set $\{x, A(x)\}$ contains two members. Hence, I contains an even number of elements.
(c) Let R be any ring. Let $R^* = RxR$ and define addition and multiplication on R^* by: $(a,b)+(c,d) = (a+c, b+d)$; $(a,b)(c,d) = (a(c+d), b(c+d))$. It can easily be verified that R^* is a ring under these operations. Also, (a,b) is an idempotent in R^* iff $a = a(a+b)$ and $b = b(a+b)$. If R is a field, these conditions hold iff $a = b = 0$ or $a = 1-b$. In particular, if R is a field containing 2^k elements, there are $1+2^k$ idempotents in R^*. Hence, there are rings containing an odd prime number of idempotents.

Also solved by Paul R. Chernoff, Douglas L. Costa, Boris Datskovsky, Robert Gilmer, Pat Halpin, F. David Hammer, A. A. Jagers (Netherlands), Thomas McConnell, Barbara Osofsky, John Petro, Douglas F. Rall, Anthony N. Richoux, Samson M. Rozenzweig, Gregory P. Wene, Albert Wilansky, and the proposer. Several showed how to construct a ring with an arbitrary finite positive number of idempotents.

Sums of Reciprocals

6194 [1978, 122]. *Proposed by Erwin Just and Norman Schaumberger, Bronx Community College of CUNY, New York.*

Let N be an arbitrary integer >6 and let $\{a_i\}$, $i=1,2,\ldots,m$, denote the set of positive composite integers less than N which are not powers of primes. Prove that

$$\sum_{i=1}^{m} \frac{1}{a_i} \text{ is not an integer.}$$

Solution by B. Ferrero, Farmington, Connecticut. Let S be the sum in question. If $6<N<19$, we see directly that 3 divides the denominator of S. If $N>18$, then by Bertrand's postulate there is a prime $p>5$ such that $N/4<p<N/2$. Then the only multiples of p less than N which are composite are $2p$ and possibly $3p$. Since $1/2p+1/3p=5/6p$, the denominator of S is divisible by p.

Also solved by Robert Breusch, Heiko Harborth (West Germany), L. Kuipers (Switzerland), O. P. Lossers (Netherlands), N. Miku (Netherlands), William H. Myers III, Ivan Niven, Barry J. Powell, Problem Solving Group of Bern (Switzerland), Blair Spearman, University of South Alabama Problem Group, and the proposers. Several persons noted that it is not necessary to exclude (proper) powers of primes.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR., AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf, Carleton, and Macalester Colleges
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

FILMS

Turning a Sphere Inside Out. Produced by the Topology Films Project under the direction of Nelson Max. Commentary by Nelson Max, Steve Smale, Charles Pugh, and Judith Bregman. 23 minutes color. Sale $350, rental $25.00, from International Film Bureau, Inc.

An *immersion* f of a k-sphere S^k into Euclidean n-space R^n is a differentiable map $f: S^k \to R^n$ with Jacobian matrix of maximal rank. For example, the figure eight in the plane is the range of an immersion of S^1 in R^2. Two immersions f_0, f_1 are *regularly homotopic* if there exists a homotopy $f_t: S^k \to R^n$ such that at each stage t the map f_t is an immersion. (Note that any two immersions $f_0, f_1: S^k \to R^n$ are homotopic; the regularity condition is critical.) An immersion is a way of differentiably mapping S^k into R^n in such a way that the range contains no tears, creases, cusps, etc., although it may contain self-intersections.

In 1958 Stephen Smale (Trans. Amer. Math. Soc., 90 (1958) 281–290) classified regular homotopy classes of immersions of S^2 in $R^n (n>2)$. In particular he showed that any two immersions $S^2 \to R^3$ are regularly homotopic. There are two obvious immersions—putting in S^2 in standard position and putting it in "inside out." So Smale's theorem implies that there is a path of immersions between these—in other words, that the sphere may be turned inside out.

This theorem and its generalizations were the subject of a remarkable paper entitled "Turning a Surface Inside Out" by Anthony Phillips (Scientific American (May 1966) 112–118).

The film has three sections. In the first there is a discussion in the Berkeley coffee room of the mathematical problem, and Pugh describes how he made his chicken-wire models. In the second Bregman plays straightman to Pugh for a tour of the chicken-wire models. The bulk of the film is devoted to computer animation of the regular homotopy.

For whom is such a film intended? We tried it out on several audiences. Prospective elementary school teachers were overwhelmed; they liked the pretty pictures but didn't have the faintest idea of what was being proved. In fact one class regarded the film as "stupid, boring, the same dull thing over and over." Calculus students fared somewhat better. One class, which had been thoroughly prepared in advance, enjoyed the film very much. Faculty were impressed but had mixed reactions.

The film failed to say what was going on and why one should be interested in it. There should have been a few animations in slow motion. The film had too many repeats of the same animation (differing only in use of color or change in perspective). On the other hand, seeing the film repeatedly did not automatically enable one to understand the mathematics. (The most helpful section of the film in that respect was the Pugh-Bregman walk-through with the models.)

Is the film suitable for computer-science students? The graphic techniques in color are quite advanced, but there are other films which are as good at teaching the potential of computer graphics. Further, there is virtually no explanation in the film of how the film was made. (Apparently many measurements were made of each model. The computer was fed this information and instructed to move from one model to the next in a smooth fashion.)

Is the film, then, meant only for the graduate students and faculty in differential topology? Surely not. Whom is it *really* meant for? I don't know.

The film gives no references to the literature in the field (at the expository or research level), nor does it refer to Phillips's article, which seems unfair. Films should interact with teaching; it is hard to imagine this film being used in conjunction with the usual undergraduate mathematics courses. A supplementary booklet might solve the reference problem, but the interaction problem seems unsolvable.

CLAUDE SCHOCHET, Wayne State University

Editor's note. The reviewer's desire for a supplementary booklet has now been satisfied by the International Film Bureau, which has published a 16-page film guide written by Nelson Max. This extensive guide, containing 68 carefully drawn figures, provides relatively detailed mathematical explanations that would have been too heavy a burden to be carried by the film's narration. The guide also contains supplementary material on generic and singular positions, followed by an explanation of Morin's regular homotopy and a classification of the 14 singularities possessed by regular homotopies. The guide concludes with a short historical summary and references. —*S.S.*

MISCELLANEA

26. The good old days? Baccalaureate essays to grade, departmental examinations, lectures to prepare—they have already ruined my Easter vacation. I can't describe the effort I have to put out to understand the scale drawings in Descriptive Geometry, which I hate, and stuff like annuity formulas in Arithmetic, etc. How happy are those who are allowed to think only about Analysis!

Hermite to Stieltjes, May 8, 1890.

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S, P, L. *The Perceptive Eye: Art and Math.* Lillian F. Baker, Doris J. Schattschneider. Allentown Art Museum (Fifth and Court Sts., Allentown, PA 18105), 1979, 69 pp, $9 (P). "With a perceptive eye, mathematics can be used to analyze various works of art; with a deliberate hand, mathematics can be used to create art." This special monograph explains and catalogues an exhibit (at the Allentown Art Museum, April 1-June 7, 1979) which uses objects in the Museum's collection to illustrate two-dimensional geometric patterns: point, line, rotational, kaleidoscopic, translation and glide-reflection symmetry; periodic designs; ratio and proportion; perspective. Also available: supplementary notes (loose bound) on projects and bibliographies in art-math activities for school children (pre-school to college), and examples of computer-generated periodic ornamental design. A rich, imaginative scheme that could well serve as a model for similar cooperative ventures throughout the country. LAS

GENERAL, S, L*. *Mind over Math.* Stanley Kogelman, Joseph Warren. Dial Pr, 1978, xii + 239 pp, $8.95. [ISBN: 0-8037-5658-5] A program for overcoming math anxiety, based on the authors' pioneering experience in this work. "If we were given ten meetings to teach math to people who feared and disliked it, we would schedule no math for the first five. Instead, we would use the time to focus on feelings about math, as we have in this book. We would also develop a realistic view of what math is and how it is done." LCL

GENERAL, T(13), S, L. *A Comprehensive Textbook of Classical Mathematics, A Contemporary Interpretation.* H.B. Griffiths, P.J. Hilton. Springer-Verlag, 1978, xxix + 637 pp, $19.80 (P). [ISBN: 0-387-90342-9; 3-540-90342-9] Identical with the original (hardcover edition, TR, January 1971) except for elimination of misprints and a few additional calculus exercises. LCL

GENERAL, S(13), L. *Mathematics as Diversion.* James E. Foster. Fulton County Pr, 1978, xiii + 220 pp, $10. [ISBN: 0-9601854-0-2] A succession of problems, solutions, queries, and anecdotes, mainly from the mathematical recreations literature, the *Monthly* and the *Mathematics Magazine*. LCL

GENERAL, S(13). *Solve It! A Perplexing Profusion of Puzzles.* James E. Fixx. Doubleday, 1978, 94 pp, $5.95 (P). [ISBN: 0-385-13039-2] About a hundred brainteasers, many of them familiar, written for young people, by the author of the best seller, *The Complete Book of Running*. LCL

GENERAL, S(13-16). *Mathematics: A Psychological Perspective.* John H. Mason. Open U Pr, 1978, 58 pp, (P). [ISBN: 0-335-05836-1] A creative and attractive booklet consisting of exhortation, problems, and gimmicks for Open University correspondence students who suffer from math anxiety. Could be very useful in American clinics for patients with similar ailments. LAS

PRECALCULUS, T(13; 1). *Precalculus Mathematics, New Impression.* Michael Payne. Saunders, 1978, viii + 429 pp, $13.95. [ISBN: 0-7216-7126-8] Reprint of *First Edition* (TR, March 1978). LCL

PRECALCULUS, T(13; 1). *Algebra for College Students.* Richard S. Paul, Ernest F. Haeussler, Jr. Reston, 1978, xii + 443 pp, $14.95. [ISBN: 0-87909-029-4] Inviting format makes this text easy to read and understand. Minimum of theory: emphasis is on manipulative aspects of algebra and functions. Warnings of commonly made errors are inserted at appropriate places throughout the text. Index is incomplete. MW

EDUCATION, S(16-17), P. *Didaktik der Mathematik.* Hans-Georg Steiner. Wissenschaftliche Buchgesellschaft, 1978, xlviii + 478 pp. [ISBN: 3-534-06005-9] A rich collection of articles on the teaching of mathematics, all previously published, but many of them not readily accessible in the U.S. Topics range from the training of teachers through Piaget's theories, to proposals as to how very specific subjects should be taught. JD-B

EDUCATION, P, L*. *Applications in School Mathematics, 1979 Yearbook.* Sidney Sharron, Robert E. Reys. NCTM, 1979, viii + 243 pp, $12. [ISBN: 0-87353-139-6] Twenty articles illustrating the use of "real-world" models in teaching elementary and secondary mathematics: examples of effective classroom applications, discussion of motivation and curricular balance, and a bibliography of nearly 200 items cross-classified by mathematical subject and area of application. A timely volume, sure to be of immense value to teachers seeking appropriate responses to the current pressure for increased "relevance" of school mathematics. LAS

HISTORY, P. *History of the Department of Mathematics of The University of Kansas 1866-1970.* G. Baley Price. Kansas U Endowment Assoc, 1976, xii + 788 pp. A departmental history pays tribute to the contributions of those who preceded us and provides an identity for those who follow. In this volume, G. Baley Price, 1970 recipient of the MAA Distinguished Service Award, gives an absorbing account of the forces and circumstances that controlled events at the University of Kansas, as well as a wealth of original documents and reports; a model for others to emulate. LCL

HISTORY, S, P, L*. *Georg Cantor: His Mathematics and Philosophy of the Infinite.* Joseph Warren Dauben. Harvard U Pr, 1979, ix + 404 pp, $27.50. [ISBN: 0-674-34871-0] The first full scholarly

biography of the man whom Bertrand Russell described as "one of the greatest intellects of the nineteenth century" and his ideas, which caused one of the greatest revolutions in the history of mathematics. Dauben carefully documents a holistic view in which mathematical creation emerges from philosophical and personal drives, and concludes that despite popular views to the contrary ("none has treated his case with less reliability than E.T. Bell"), Cantor's breakdown was endogenous. Dauben also documents deep ties between Cantor's mathematics and his (Lutheran) religious faith: "Why should he have been the mathematician most likely to defend transfinite set theory, despite tremendous criticism...? ...Because he believed that set theory [was] divinely inspired from God... '...I have followed its roots, so to speak, to the first infallible cause of all created things.'" LAS

HISTORY, S, P, L*. *A Critical Study of the Yang Hui Suan Fa: A Thirteenth-century Chinese Mathematical Treatise.* Lam Lay Yong. Singapore U Pr, 1977, xvii + 360 pp, $55. A complete translation of Yang Hui's *Methods of Computation,* together with a comparison of the development of similar topics (arithmetic, mensuration formulas, elementary algebra and number theory) in other parts of the world. An interesting reference, accessible also to mathematics education students. Note price. LCL

HISTORY, L**. *The Rhind Mathematical Papyrus.* Arnold Buffum Chace. NCTM, 1979, xi + 140 pp, $15. [ISBN: 0-87353-133-7] Reprint in one volume of a two-volume M.A.A. publication, originally published in 1927 and 1929, with a new introduction, reorganized and updated bibliography, and lists of additions and corrections. The volume contains a survey of the content and significance of the papyrus, description of the process of translation, free translation and commentary, facsimile plates of selected problems, and a few photographs of the papyrus. LAS

FOUNDATIONS, P. *Bounds on Transfer Principles for Algebraically Closed and Complete Discretely Valued Fields.* Scott Shorey Brown. Memoirs No. 204. AMS, 1978, iv + 92 pp, $7.60 (P). [ISBN: 0-8218-2204-7] A version of the author's doctoral dissertation under Simon B. Kochen at Princeton. It is a study of the replacement of quantifiers in statements about algebraically closed and complete discretely valued fields, by qualifiers which range over finite sets of algebraic numbers, thus yielding decision procedures. LCL

FOUNDATIONS, S(16-18), P, L. *The Unprovability of Consistency: An Essay in Modal Logic.* George Boolos. Cambridge U Pr, 1979, viii + 184 pp, $19.95. [ISBN: 0-521-21879-9] A very welcome exposition of recent work of de Jongh and Sambin, Boolos, Solovay and others, linking modal logic and proof theory. A complete axiomatization, an elegant proof method by analytic tableaux, and a Kripke semantics are given for a new system of modal propositional logic G in which necessity [$\Box$] is interpreted as the Gödelian proof predicate Bew(x): "x is (the Gödel number of) a theorem of Peano arithmetic." (For example, $\sim\Box(0=1) \to \sim\Box\sim\Box(0=1)$ is the statement of Gödel's Second Incompleteness Theorem). Analysis of the system G gives significant insight into the proof theory of Peano arithmetic. Essential reading for anyone desiring a deeper understanding of Gödel's theorems. GHM

COMBINATORICS, S(16), P, L. *Relations Between Combinatorics and Other Parts of Mathematics.* Ed: D.K. Ray-Chaudhuri. Proc. of Symp. in Pure Math., V. 34. AMS, 1979, xiii + 378 pp, $23.20. [ISBN: 0-8218-1434-6] Proceedings of the AMS Symposium which took place in Columbus, Ohio in March, 1978. Includes interconnections between combinatorics and geometry, group theory, number theory, special functions, lattice packings, logic, topological embeddings, games, experimental designs, sociological and biological applications. CEC

NUMBER THEORY, T(15: 1), S, P, L*. *Solved and Unsolved Problems in Number Theory.* Daniel Shanks. Chelsea, 1978, xiii + 258 pp, $11.95. [ISBN: 0-8284-0297-3] The first three chapters of this book are a reprint, with typographical corrections, of the author's outstanding 1962 edition. A fourth chapter has been added which discusses advances that are pertinent to the first part of the book. Includes some questionable "conjectures" in the new section on pseudoprimes. CEC

FINITE MATHEMATICS, T(13: 1, 2). *The Power of Mathematics, Applications to Management and the Social Sciences.* Kenneth L. Whipkey, Mary Nell Whipkey, George W. Conway, Jr. Wiley, 1978, xiii + 462 pp, $14.95. [ISBN: 0-471-93785-1] A careful presentation with a nice blend of theory and application. Matrices, linear programming, probability and statistics, exponentials and logarithms, the derivative (including partial derivatives) and the integral (including differential equations). LCL

FINITE MATHEMATICS, T(13: 1). *Finite Mathematics and Its Applications.* Robert E. Rector, Earl J. Zwick. HM, 1979, ix + 470 pp, $14.95. [ISBN: 0-395-27206-8] Serves as an introduction to mathematics: sets, matrices, linear programming, counting, probability, Markov chains, theory of games. Highly structured with minimal academic demand. Prerequisite: one year of high school algebra. Plain format compared to the competition. LCL

FINITE MATHEMATICS, T(13-14: 1), L. *Mathematics, Models and Applications.* Lawrence C. Eggan, Charles L. Vanden Eynden. Heath, 1979, xiii + 525 pp, $12.95. [ISBN: 0-669-01051-0] Finite mathematics at an elementary level, including linear programming (but not the simplex method), probability and statistics, compound interest, Markov chains. Somewhat less standard are Chapter Two--The Computer, and Chapter Six--Graph Theory. Encourages use of calculators. JS

FINITE MATHEMATICS, T(13-15: 1, 2), S. *Finite Mathematics with Applications, Third Edition.* A.W. Goodman, J.S. Ratti. Macmillan, 1979, xiv + 584 pp, $15.95. [ISBN: 0-02-344760-5] This new edition has a chapter on statistics, more on linear programming, and more review problems. (*First Edition,* TR, May 1971; ER, January 1973; *Second Edition,* TR, May 1975). FLW

FINITE MATHEMATICS, T(13: 1). *Finite Mathematics with Applications.* Laurence D. Hoffmann, Michael Orkin. McGraw, 1979, viii + 295 pp, $14.95. [ISBN: 0-07-029310-4] Brief treatment of linear programming, network models, probability, statistics, theory of games, and mathematics of finance. LLK

FINITE MATHEMATICS, T. *Finite Mathematics with Applications for Business and Social Sciences, Third Edition.* Abe Mizrahi, Michael Sullivan. Wiley, 1979, xiv + 638 pp, $16.50. [ISBN: 0-471-03336-7] Essentially the same topics as in previous editions (*First Edition,*TR, October 1973) with some revisions in organization and presentation. Questions from recent CPA, CMA, and Actuarial Examinations have been reproduced at the ends of appropriate chapters. LCL

FINITE MATHEMATICS, T(13: 1). *The Nature and Application of Mathematics.* George J. Kertz. Goodyear, 1979, xiv + 402 pp, $15.95. [ISBN: 0-87620-614-3] Topics introduced in a manner designed to develop the appreciation of liberal arts students for mathematics: graph theory, probability, statistics, logic, and some consumer finance. LLK

FINITE MATHEMATICS, T(13: 1). *Topics in Contemporary Mathematics, Second Edition.* Jack R. Britton, Ignacio Bello. Har-Row, 1979, xvi + 608 pp, $16.95. [ISBN: 0-06-040953-3] One of those texts for liberal arts students that supplies cartoons in the margins. Contains an introduction to logic, linear programming, probability, statistics, the metric system, calculators and computers. (*First Edition,* TR, August 1975.) LLK

FINITE MATHEMATICS, T(13: 1). *Mathematics for Management and Social Sciences.* A.W. Goodman, J.S. Ratti. HR&W, 1979, xi + 496 pp, $15.95. [ISBN: 0-03-022161-7] A text covering topics for a course for business and social science majors, including topics in finite mathematics, precalculus, and calculus. LLK

CALCULUS, T*(14: 2). *Multivariable Mathematics: Linear Algebra, Calculus, Differential Equations, Second Edition.* Richard E. Williamson, Hale F. Trotter. P-H, 1979, x + 575 pp, $21. [ISBN: 0-13-604850-1] In this somewhat trimmer *Second Edition* sections and chapters have been rearranged. Some topics, e.g., Newton's method and existence-uniqueness theorems for differential equations, have been dropped. Answers to almost all computational problems are at book's end. Skimpy index. A textbook with appeal. (*First Edition,* TR, October 1974; ER, March 1976.) JK

CALCULUS, T(13-14: 1-3). *Calculus.* Stanley I. Grossman. Acad Pr, 1977, xviii + 1141 pp, $22.95. [ISBN: 0-12-304350-6] Emphasis on problem solving with an unusually large and complete collection of examples and exercises, including interesting applications from a wide variety of fields, several with "real" data requiring a hand calculator. LCL

CALCULUS, T(13-14: 1-3). *Calculus with Analytic Geometry.* Roland E. Larson, Robert P. Hostetler. Heath, 1979, xv + 974 pp, $17.95. [ISBN: 0-669-01301-3] Appealing page design--open and uncluttered with attractive graphics. Highly readable text; definitions and theorems are stated clearly and simply. Coverage for third term is not as extensive as some other texts; does not include a chapter on differential equations. LCL

CALCULUS, T(13: 1, 2). *Calculus and Mathematical Models.* Nathaniel A. Friedman. Prindle, 1979, ix + 541 pp, $17.95. [ISBN: 0-87150-265-8] Intuitive calculus, honestly presented, requiring two years of high school mathematics. Over 500 drawings, photographs and newspaper clippings. Over 3000 graded exercises. Over 90 applications to economics, biology, ecology and the social sciences. Summaries at end of each section or sub-section. Flexible with respect to course length and emphasis. Attractive. JK

CALCULUS, T*(13-14: 1-3). *Calculus With Analytic Geometry.* Daniel J. Fleming, James J. Kaput. Har-Row, 1979, xiii + 857 pp, $12.95. [ISBN: 0-06-382672-0] All of the standard topics are included with the possible exception of an introduction to differential equations. A clearly written text with lots of excellent exercises. Several interesting and unusual applications are included. This text could be easily adapted for use in a P.S.I. course since objectives and progress tests appear with each section and an *Instructor's Resource Guide* and Test Bank are available. CEC

CALCULUS, T*(13-14: 1-3). *Calculus With Analytic Geometry, Second Edition.* Earl W. Swokowski. Prindle, 1979, xvi + 1085 pp, $24.95. [ISBN: 0-87150-268-2] This edition contains 1500 new exercises. The number of solved examples has also increased. This is a well-organized text which is clearly written and has an appropriate degree of rigor. Several sections have been rewritten for this edition for increased clarity. (*First Edition,* TR, December 1975.) CEC

CALCULUS, T(13-14: 4). *Calculus and Analytic Geometry, Fifth Edition.* George B. Thomas, Jr., Ross L. Finney. A-W, 1979, xiv + 961 pp, $23.95. [ISBN: 0-201-07540-7] Still the grandfather of them all--adapted from the *Alternate Edition* (TR, June/July 1973) but using the vector analysis from the *Fourth Edition* (TR, December 1968; ER, June/July 1970). There has been a general rewriting and addition of some applications to disciplines other than natural science. LLK

CALCULUS, T(14: 2). *Calculus of Several Variables, Second Edition.* Serge Lang. A-W, 1979, xii + 479 pp, $16.95. [ISBN: 0-201-04299-1] Excellent text for the second year of calculus. Interweaves the needed linear algebra with multivariable topics. (There is not enough linear algebra to substitute for a course in linear algebra.) Changes from the first edition (TR, October 1974) include some physical applications and more worked-out examples. LLK

CALCULUS, T(13: 1). *Calculus: An Historical Approach.* William McGowen Priestley. Springer-Verlag, 1979, xvii + 441 pp, $14.80. [ISBN: 0-387-90349-6; 3-540-90349-6] A "short" calculus designed for humanities students, emphasizing the cultural context of calculus: "Mathematics is romance in reason." Standard elementary topics are mixed with diverse quotations, historical exercises and general "liberal arts" exposition. Appendices on popular writings about mathematics and on style in writing up calculus problems; the latter alone may be worth the price of the book! LAS

CALCULUS, T(13: 1). *Applied Calculus, A Goals Approach.* Shirley O. Hockett, Martin Sternstein. D. Van N-Rein, 1979, ix + 645 pp, $14.95. [ISBN: 0-442-23428-7]; *Instructor's Manual,* 80 pp, (P). [ISBN: 0-442-23582-8] A text for a short course in calculus emphasizing applications and using an

intuitive approach. Intended for use in a regular classroom but with an instructor's manual containing extra tests useful for a PSI approach. LLK

CALCULUS, T(13: 1), L. *The Power of Calculus, Third Edition.* Kenneth L. Whipkey, Mary Nell Whipkey. Wiley, 1979, xvii + 370 pp, $15.95. [ISBN: 0-471-03140-2] Earlier editions (*First Edition,* TR, August/September 1972; *Second Edition,* TR, December 1975) have been among the more successful of the short calculus texts and the *Third Edition* should not change the pattern. Changes are relatively minor; a brief section on multiple integrals has been added. JS

REAL ANALYSIS, T(17-18), P. *Measure and Integral.* Konrad Jacobs. Prob. and Math. Stat. Acad Pr, 1978, xv + 575 pp, $45. [ISBN: 0-12-378550-2] First three chapters constitute a graduate text with basic ideas done in detail; applications, special cases and extensions in the form of remarks and exercises. The final thirteen chapters, more summary in nature, constitute a reference for the researcher. LCL

REAL ANALYSIS, T(15-17: 1), S, L. *An Introduction to Lebesgue Integration and Fourier Series.* Howard Wilcox, David L. Myers. Krieger, 1978, viii + 159 pp, $13.50 (P). [ISBN: 0-88275-614-1] A gentle, well-motivated treatment designed as a transition between advanced calculus and graduate-level real analysis. The integral is defined by partitioning the range, is developed on R^1 exclusively, and is applied to the classical theory of L^2. Many exercises. LAS

DIFFERENTIAL EQUATIONS, T(15-16: 1, 2), L. *Ordinary Differential Equations: A Computational Approach.* Charles E. Roberts, Jr. P-H, 1979, xiv + 400 pp, $16.95. [ISBN: 0-13-639757-3] A unique introductory text that fully integrates numerical and theoretical methods: each topic involves procedures for finding explicit or implicit solutions, theory that guarantees existence and uniqueness of solutions, and algorithms (with Fortran programs) for numerical approximations. Covers first order equations, linear equations, and systems of first order equations. LAS

DIFFERENTIAL EQUATIONS, P*, *Elliptic Systems in the Plane.* W.L. Wendland. Fearon-Pitman, 1979, xi + 404 pp, $57.50. [ISBN: 0-273-01013-1] "...unified treatment of both solvability theory and computational methods for elliptic systems of first order partial differential equations in two variables ...normal forms, representation theorems, boundary value problems...three basic computational methods ...with their asymptotic error analysis: integral equation methods, finite difference methods and finite element approximations." Extensive reference lists. JK

OPTIMIZATION, T*(14-15: 1), S*, L. *An Introduction to Linear Programming and Game Theory.* Paul R. Thie. Wiley, 1979, xiii + 335 pp, $17.95. [ISBN: 0-471-04248-X] The treatment is mathematically sound, without appearing rigorous. Very nice treatments of duality, the dual simplex, and sensitivity. The game theory is somewhat elementary. An attractive choice for a one-semester "baby operations research" course. TAV

OPTIMIZATION, T(17: 1), P. *Numerical Methods in Extremal Problems.* B.N. Pshenichny, Yu. M. Danilin. Trans: V. Zhitomirsky. MIR (US Distr: Imported Pub, 320 W. Ohio St., Chicago, IL 60610), 1978, 276 pp, $8.25. Foundations in convexity theory. Unconstrained problems: gradient, dual directions, conjugate directions and Newton's method. Constrained problems: feasible directions, linearization, penalty function and projection methods. Consideration of accuracy and roles of convergence. RWN

ANALYSIS, T?(13-14: 1, 2), S. *Mathematics for Physical Geographers.* Graham N. Sumner. Halsted Pr, 1978, xv + 236 pp, $15.95. [ISBN: 0-470-26557-4] A superficial look at a lot of topics intended for students with no prior mathematical knowledge (!?): algebra and trigonometry, vectors, line fitting, matrices (through eigenvalues), differential calculus (including transcendental functions), series, probability (including gamma distribution), integral calculus, Fourier series, elementary differential equations (constant coefficients), and much more. This, however, represents a "working minimum for most present-day undergraduate geography courses." Only a tiny scattering of exercises. LCL

ANALYSIS, T(15-17: 1-3), L. *Advanced Engineering Mathematics, Fourth Edition.* Erwin Kreyszig. Wiley, 1979, xvii + 1031 pp, $24.95. [ISBN: 0-471-02140-7] Notable changes from previous editions (*Second Edition,* TR, December 1967; *Third Edition,* TR, October 1972; ER, June/July 1974) include greater emphasis on modelling, more applications in the problems, new chapter on phase plane methods, a streamlined treatment of complex variables, new section on splines, modernization of chapter on matrices, updated references. Book remains flexible with enough material for two or three courses. A bargain at the price. JK

ANALYSIS, P. *Multivariate Approximation.* Ed: D.C. Handscomb. Acad Pr, 1978, xiii + 353 pp, $25.75. [ISBN: 0-12-323350-X] Contains most of the invited and contributed lectures given at an international symposium held at the University of Durham in July, 1977. Concerned with surface fitting, or the representation of functions of several variables. RSK

GEOMETRY, S(14), P, L. *A Simple Non-Euclidean Geometry and Its Physical Basis.* I.M. Yaglom. Trans: Abe Shenitzer. Springer-Verlag, 1979, xviii + 307 pp, $19.80 (P). [ISBN: 0-387-90332-1; 3-540-90332-1] An easily readable, annotated presentation of Galilean or isotropic geometry (geometry of two-dimensional manifolds of events subjected to Galilean transformations of kinematics) and Minkowskian geometry; presumes only high school mathematics. JNC

TOPOLOGY, T(17-18: 1, 2), S, P. *Dimension Theory.* Ryszard Engelking. Math. Lib., V. 19. North-Holland, 1978, x + 314 pp, $44.50. [ISBN: 0-444-85176-3] This book presents dimension theory in the context of its historical development. The first half is devoted to separable metric spaces, followed by chapters on the large inductive and covering dimensions. The last chapter is devoted to the theory on metrizable spaces. Sections conclude with historical and bibliographical notes and several exercises. TLS

PROBABILITY, T(16-17: 1). *Conditional Independence in Applied Probability.* Paul E. Pfeiffer. EDC/UMAP, 1978, vii + 132 pp, $4 (P). A rigorous exposition of basic probability in the context of conditional independence (P(AB/C) = P(A/C)P(B/C) for some event C which is a common influence of both A and B) leading to applications in Markov processes. Presumes a solid measure-theoretic knowledge of probability. A preliminary edition, the first monograph in EDC's Undergraduate Mathematics and its Applications Project (UMAP). LAS

PROBABILITY, P. *Branching Processes.* Ed: Anatole Joffe, Peter Ney. Dekker, 1978, x + 322 pp, $34.50. [ISBN: 0-8247-6800-0] Volume 5 of the Advances in Probability Series; eleven papers, covering many of the lines of work of the past decade, presented at a conference on branching processes (Quebec, August 1976). LCL

STATISTICS, T(13: 1), S. *Basic Statistics for Medical and Social Science Students.* A.E. Maxwell. Chapman and Hall, 1978, 126 pp, $4.95 (P). [ISBN: 0-470-99374-X] Brief, nontechnical primer: an inviting overview, with enough examples and discussion to make the subject look natural and sensible. Descriptive statistics, contingency tables, correlation and regression, tests of significance, analysis of variance. First published in 1972 under the title *Basic Statistics in Behavioral Research.* No exercises. LCL

STATISTICS, T(14-15: 1). *An Introduction to Probability and Statistics Using BASIC.* Richard A. Groeneveld. Statistics, V. 26. Dekker, 1979, xiv + 446 pp, $17.50. [ISBN: 0-8247-6543-5] A true computer-oriented approach, assuming a background of single variable calculus. 48 Basic programs are used both to do computations and to generate examples to illustrate theoretical ideas. Content includes topics recommended by CUPM for course Mathematics 2P, except that there is a greater emphasis on statistical methodology. RSK

STATISTICS, T(14: 1). *A Basic Course in Statistics.* G.M. Clarke, D. Cooke. Halsted Pr, 1978, xvi + 368 pp, $16.95. [ISBN: 0-470-26527-2] Designed for British students. First half deals only with discrete random variables, but introduces estimation and hypothesis testing. Second half, which requires some calculus, treats continuous random variables, the Poisson distribution, and most of the usual statistical topics. RSK

STATISTICS, T(14-17: 1, 2), S*, P, L*. *Statistics for Experimenters: An Introduction to Design, Data Analysis, and Model Building.* George E.P. Box, William G. Hunter, J. Stuart Hunter. Wiley, 1978, xviii + 653 pp, $23.95. [ISBN: 0-471-09315-7] "An introduction to the philosophy of experimentation and the part that statistics plays." Comparing two treatments, comparing more than two treatments, measuring the effects of variables, and building and using models. Introduces statistical ideas as they are needed. Uses no calculus. FLW

STATISTICS, T(14-16: 1, 2), L. *Statistical Analysis: A Computer Oriented Approach, Second Edition.* A.A. Afifi, S.P. Azen. Acad Pr, 1979, xx + 442 pp, $19.50. [ISBN: 0-12-044460-7] Updated version of the authors' 1972 *First Edition* (TR, April 1973), designed to follow a course in elementary statistics, presenting features of the latest versions of the statistical packages BMDP, SPSS, and SAS. New sections include screening of data sets, robust estimation, handling of multivariate missing observations, new measures of association in contingency tables, and multivariate analysis of variance. RSK

STATISTICS, T(13: 1), S. *Inferential Statistics for Geographers.* G.B. Norcliffe. Halsted Pr, 1977, 272 pp, $19.50. [ISBN: 0-470-99206-9] A first course; emphasis on intuition with minimal mathematical derivation. Geographical perspective with more than the usual emphasis on the Poisson distribution and topics in spatial analysis. Discussion includes difficulties and pitfalls in using certain techniques. Nice examples, but no exercises. Large bibliography. LCL

COMPUTER PROGRAMMING, P, L. *LINPACK Users' Guide.* J.J. Dongarra, *et al.* SIAM, 1979, viii + 341 pp, $14 (P). Thorough discussion of a recently completed machine-independent package of Fortran programs for the solution of linear systems and related problems. Developed at Argonne National Laboratory, the package has been tested (for speed and accuracy) on two dozen university systems; reports on these tests are included in this *Guide.* The team effort that led to this package was intended in part to provide "a yardstick against which future mathematical software projects can be measured." LAS

COMPUTER PROGRAMMING, T(13), S. *Instant Freeze-Dried Computer Programming in BASIC.* Jerald R. Brown. Dymax, 1977, 159 pp, $9.95 (P). [ISBN: 0-918398-21-5] In workbook form with many illustrations, cartoons and graphics. For ALTAIR style Basic primarily, although it can be used for DEC's Basic Plus. The format is non-threatening and could easily be used to help the beginner over "computer anxiety." TAV

COMPUTER PROGRAMMING, T*(13-18: 1), S. *PASCAL: An Introduction to Methodical Programming.* William Findlay, David A. Watt. Computer Sci Pr, 1978, xii + 306 pp, $10.95 (P). [ISBN: 0-914894-19-6] A text devoted to imparting structured programming principles, good style, and a modular approach to program development in the context of teaching the language Pascal. Divided into 5 parts: simple program constructs; elementary data types; control structures; functions and procedures; structured data types (records, strings, files, sets, pointers). Conveniently organized; good, complete explanations; many examples and exercises, some with answers; extensive appendices; index. RJA

COMPUTER PROGRAMMING, T(13-18: 1), S. *A Guide to PL/M Programming for Microcomputer Applications.* Daniel D. McCracken. A-W, 1978, x + 262 pp, $9.95 (P). [ISBN: 0-201-04575-3] PL/M is a high-level language designed for systems and applications work with microcomputers. Each chapter is built around specific microcomputer applications programs. Emphasizes use of procedures, block structuring, program understandability. Also includes PL/M statements and data structures, PL/M interrupt processing, and system programming facilities. Chapter exercises. Answers to selected exercises. Index. RJA

COMPUTER PROGRAMMING. *32 BASIC Programs for the PET Computer.* Tom Rugg, Phil Feldman. Dilithium Pr, 1979, xviii + 267 pp, $15.95 (P). [ISBN: 0-918398-25-8] Listings of 32 Basic Programs for an 8K Commodore Pet 2001 along with explanations and suggestions for changing the programs. Includes applications, teaching programs, games, graphics, mathematical, engineering and statistical programs. CEC

COMPUTER PROGRAMMING, T(13-14: 1), S, L. *A Practical Introduction to Pascal.* I.R. Wilson, A. M. Addyman. Springer-Verlag, 1979, xii + 148 pp, $7.90 (P). [ISBN: 0-387-91136-7] A straightforward (no frills) introduction to programming in Pascal, complete with appendices giving Pascal syntax diagrams, bibliography and addresses of user groups. Intended to supplement the "definitive reference work" *Pascal User Manual and Report* by Jensen and Wirth (TR, December 1976). LAS

COMPUTER PROGRAMMING, S(13-18). *Schaum's Outline of Theory and Problems of Programming with FORTRAN.* Seymour Lipschutz, Arthur Poe. McGraw, 1978, 314 pp, $4.95 (P). [ISBN: 0-07-037984-X] Contains all the features of standard Fortran. Each chapter contains text presentation with examples, solved problems, and supplementary problems with answers. Special chapter on structured Fortran. Appendix on library functions; another appendix on the internal representation of data. Index. RJA

COMPUTER SCIENCE, T(15-18: 1, 2), S, P, L. *Combinatorial Algorithms for Computers and Calculators, Second Edition.* Albert Nijenhuis, Herbert S. Wilf. Comp. Sci. and Appl. Math. Acad Pr, 1978, xv + 302 pp, $19.50. [ISBN: 0-12-519260-6] Text is composed of two parts: combinatorial families and combinatorial structures. The former contains algorithms of basically two types: exhaustive search, termed "next algorithms," and random sampling, termed "random algorithms." The latter part contains a collection of algorithms employing a variety of ideas. Each chapter concerns a specific problem, its mathematical basis, a formal algorithm, a Fortran subroutine with a specifications list, and sample program output. Exercises. Bibliographic notes. References. Index. (*First Edition,* TR, February 1976.) RJA

COMPUTER SCIENCE, T*(13-18: 1, 2), S, L. *Computer Organization and Assembly Language Programming.* James L. Peterson. Comp. Sci. and Appl. Math. Acad Pr, 1978, xii + 448 pp, $14.95. [ISBN: 0-12-552250-9] Text commences with general discussion of computer organization and then proceeds to a description of the MIX computer, its machine language and the MIXAL assembly language. Assembly language programming techniques are presented for the MIX machine. Very extensive treatment is given to systems programs--loaders, assemblers, language translators, operating systems. Finally, a chapter with brief descriptions of several real computers--PDP-8, HP2100, PDP-11, IBM 360 and 370, Burroughs B5500. Chapter exercises. References. Appendices on MIX instructions. Index. RJA

COMPUTER SCIENCE, P. *Mechanizing Hypothesis Formation: Mathematical Foundations for a General Theory.* P. Hájek, T. Havránek. Springer-Verlag, 1978, xv + 396 pp, $24 (P). [ISBN: 0-387-08738-9; 3-540-08738-9] A formal "logic of discovery" addressing the ultimate question: can computers formulate and justify scientific hypotheses based on empirical data? This predominantly technical treatise applies methods of mathematical logic and computational statistics to develop a logic of suggestion (how to select reasonable hypotheses) and of induction (how to justify hypotheses). Treatment includes consideration of computational complexity. Will be of interest to students of artificial intelligence and computer science. GHM

COMPUTER SCIENCE, P. *Numerical Software--Needs and Availability.* Ed: D. Jacobs. Acad Pr, 1978, xviii + 408 pp, $24. [ISBN: 0-12-378660-6] Proceedings of a conference held at the University of Sussex in September 1977: libraries of numerical software, linear and non-linear algebra, data fitting, differential and integral equations, optimization. LAS

APPLICATIONS (ARTIFICIAL INTELLIGENCE), S(14-18), L. *An Introduction to Artificial Intelligence: Can Computers Think?* Richard Bellman. Boyd & Fraser, 1978, x + 147 pp. [ISBN: 0-87835-066-7] Computer thinking is modelled around the method of tracing a path through a network. Text includes discussion of underlying mathematical questions, useful properties of digital computers, decision making, puzzle solving, simulation, learning, consciousness, paradoxes in logic, local logics, mathematical models of the mind, and communication. A good, concise introductory overview of the field. Chapter bibliographies. Author and subject indices. RJA

APPLICATIONS (BIOLOGY), T(15-16: 1), S. *Introduction to Population Modeling.* James C. Frauenthal. EDC/UMAP, 1979, xvi + 186 pp, $4 (P). Lecture notes from a course at Stony Brook in the formulation and solution of mathematical models. "It makes no pretense of being a text in ecology. The idea of a population is employed mainly as a pedagogic tool, providing unity and intuitive appeal..." Models for the dynamics of a single species are followed by models for interaction among several species. Each of the 11 short chapters concludes with a few exercises and references to primary sources. This typescript paperbound preliminary edition is part of EDC's Undergraduate Mathematics and its Applications Project (UMAP). LAS

APPLICATIONS (BUSINESS), T(13: 1). *Management Science for Management Decisions.* Ulysses S. Knotts, Jr., Ernest W. Swift. Allyn, 1978, xvi + 397 pp, $15.95. [ISBN: 0-205-06039-0] Low-level book for students with weak mathematics skills. Descriptive presentation using very simply problems. Limited discussion of applications and philosophy. WC

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; William Carlson, St. Olaf; Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Lorraine L. Keller, St. Olaf; Richard S. Kleber, St. Olaf; Joseph Konhauser, Macalester; Loren C. Larson, St. Olaf; George H. Mills, St. Olaf; R.W. Nau, Carleton; John Schue, Macalester; Lynn Arthur Steen, St. Olaf; Timothy L. Strotman, St. Olaf; T.A. Vessey, St. Olaf; Martha Wallace, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036

SUGGESTION BOX

Members of the MAA are encouraged to send in suggestions, questions, etc., about the operations of the Association. Communications will be referred to the appropriate officer of the Association for answering; from time to time, those of general interest may also be answered in one or both of the official journals. Communications should be addressed to: Suggestion Box, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036

PERSONAL ITEMS

University of Hawaii: Dr. Yuk J. Leung, University of Michigan, has been appointed Assistant Professor. Associate Professor Ralph S. Freese was on leave at Vanderbilt University during the 1978-79 academic year.

Dartmouth College: Thomas Tucker, Colgate University, is a Visiting Associate Professor. Arlene Ash, Boston University, is a John Wesley Young Research Instructor. Albert J. Nijenhuis, University of Pennsylvania, is a Visiting Professor.

University of Notre Dame: Associate Professor Alan Howard has been promoted to Professor. Dr. Ronald Sverdlove, formerly a post-doctoral fellow at the Claremont Mathematics Clinic, has been appointed Assistant Professor.

University of Wisconsin-Oshkosh: Associate Professor John F. Lucas has been promoted to Professor. Assistant Professor Jeanne A. Wright has been promoted to Associate Professor.

Oregon State University: Professor Richard M. Schori, formerly of Louisiana State University, has been appointed Chairman of the Mathematics Department. Professor Emeritus Nachman Aronszajn, Kansas State University, has joined the Mathematics Department as Adjunct Professor.

California State University, Long Beach: Associate Professors M. Shafqat Ali and Ruth H. Afflack have been promoted to Professors.

Northern Michigan University: Drs. Agnes Andreassian and Kevin Gallagher have joined the staff as Professors. Associate Professor William S. Mutch has been promoted to Professor and Department Head. Dr. Jane O. Swafford is Director of an NSF Institute in Geometry and Measurement for elementary teachers in Upper Michigan.

Messiah College, Grantham, Pennsylvania: Associate Professor Dwight Paine has been promoted to Professor. Assistant Professor Gene Chase has been promoted to Associate Professor.

Temple Junior College, Temple, Texas: Professor and Chairman Ethel J. Haag, Mathematics Education, has retired with the title of Professor Emeritus. Instructor Bill W. Vannatta has been appointed Chairman.

Illinois State University, Normal: Edna F. Bazik, Concordia Teachers College, and John D. Bradburn, Elgin Community College, are Visiting Assistant Professors. Dr. Charles S. Weaver, formerly with the Computer-based Education Research Laboratory, University of Illinois, has been appointed Assistant Professor. Associate Professor Kenneth Berk has been promoted to Professor.

St. Peter's College, Jersey City, New Jersey: Assistant Professor Maryam S. Hastings has left to become an Assistant Professor at William Paterson College. Assistant Professor Gerard P. Protomastro has been promoted to Associate Professor.

Youngstown State University, Ohio: Dr. Stephen E. Rodabaugh, Kansas Wesleyan University, has been appointed Assistant Professor. Associate Professor Dean R. Brown has been appointed Chairman, Department of Mathematical and Computer Sciences.

Jamestown College, North Dakota: Dr. Douglas Swan, Detroit Institute of Technology, has been appointed Associate Professor. Douglas Dokken, North Dakota State University, has been appointed Instructor.

Oakland University, Rochester, Michigan: Dr. Hai-Ping Ko, Ohio State University, has been appointed Assistant Professor. Dr. Stuart Wang, Texas Tech University, and Dr. Wanda Mourant, Wayne State University, are Visiting Assistant Professors. Edward Moylan has been appointed Supervisor in the Computer Applications Department, Ford Motor Company. Professor George Feeman has been appointed Chairman.

University of Tennessee, Chattanooga: Dr. Stephen W. Kuhn, University of Georgia, has been appointed Assistant Professor. Assistant Professors Richard C. Detmer and Edward R. Rozema have been promoted to Associate Professors.

Lycoming College, Williamsport, Pennsylvania: Richard Troxel, University of Indiana, has been appointed Instructor. Assistant Professor Rita Cantor has accepted other employment. Assistant Professor John R. Hubbard has been promoted to Associate Professor and has been appointed Chairman.

Regis College, Denver: Dr. Diana Wagner, Flint, Michigan, has been appointed Associate Professor. Assistant Professor Charles Brase has been promoted to Associate Professor. Professor Fred T. Daly has retired with the title of Professor Emeritus.

Eastern Michigan University: Assistant Professors Enoch Tse and John Remmers have been promoted to Associate Professors. Associate Professor Nelly Ullman has been promoted to Professor.

Monmouth College, West Long Branch, New Jersey: Assistant Professor John Carson has been promoted to Associate Professor. Associate Professor G. Boyd Swartz has been appointed Chairman.

Professor Charles Hagopian, California State University, Sacramento, was invited to Warsaw, Poland, for one month through the National Academy of Sciences. He gave his paper, *Uniquely Arcwise Connected Plain Continua Have the Fixed Point Property,* at the International Conference on Geometric Topology and lectured at the Institute of Mathematics of the Polish Academy of Sciences.

Dr. Juhn-Hsiong Uong was appointed Assistant Professor at Eastern Oregon State College.

Assistant Professor Carolyn Blaine, Paul D. Camp Community College, Franklin, Virginia, has taken other employment.

Associate Professor F. J. Papp, University of Lethbridge, Alberta, has been appointed Chairman.

Dr. William Edelson, formerly Senior Systems Analyst at the Lederle Laboratory, has been appointed Assistant Professor at Pace University, Pleasantville, New York.

Associate Professor Johnny A. Johnson, University of Houston, has been promoted to Professor.

Dr. Alan Durfee, University of Washington, is a Visiting Assistant Professor at Columbia U.

Assistant Professor Johnny W. Lott, University of Montana, has been promoted to Associate Professor.

Assistant Professor W.D. Kaigh, University of Texas at El Paso, has been promoted to Associate Professor.

Dr. Carolyn Eisele, Past President of the Peirce Society and Visiting Scholar at the Texas Tech University Institute for Studies in Pragmaticism, is working at Tech on a history of science as developed by Charles Sanders Peirce. She is the editor of the five-volume edition of *New Elements of Mathematics* by Peirce.

Professor Boris Korenbaum, SUNY, Albany, gave an invited address at the International Congress of Mathematicians in Helsinki.

Dr. Robert H. DeVore, Voorhees College, has been appointed to a one year replacement as Assistant Professor at Newberry College, South Carolina.

Dr. James M. Whitehead, Kent State University, has been appointed Assistant Professor at Denison University, Granville, Ohio.

Ms. Judith Grice, SUNY, Albany, has been appointed Instructor at Schenectady County Community College, New York.

Associate Professor Elwyn H. Davis, Pittsburg State University, Kansas, has been promoted to Professor.

Dr. Paul R. Patten, University of Oklahoma, has been appointed Instructor at Brewton-Parker College, Mount Vernon, Georgia.

Dr. Richard C. Vrem, University of Oregon, has been appointed Assistant Professor at Bucknell U.

Dr. Dennis P. Geoffroy, University of South Carolina—Sumter, has been appointed Assistant Professor at Armstrong State College, Savannah, Georgia.

Associate Professor Robert E. Dressler, Kansas State University, has been promoted to Professor.

Dr. Chris Smith has been appointed Assistant Professor at Saginaw Valley State College, University Center, Michigan.

Associate Professor Allan B. Cruse, University of San Francisco, has been promoted to Professor.

Dr. Chi-Shang Soong, LeHigh University, has been appointed Assistant Professor at Villanova U.

Assistant Professor Milton P. Eisner, J. Sargeant Reynolds Community College, Richmond, Virginia, has been named Mathematics Department Head.

John Grant, formerly Associate Professor of Computer and Information Sciences at the University of Florida, has been appointed Associate Professor and Computer Science Coordinator at Towson State University, Towson, Maryland.

Mr. Ahmed Anter, Nisantas, Istanbul, Turkey, died on October 31, 1978. He was a member of the Association for four years.

Professor Arthur E. Hallerberg, Valparaiso, Indiana, died on November 23, 1978, at the age of sixty. He was a member of the Association for thirty five years.

Mr. Roland Brossard, Montreal, Quebec, Canada, died in December, 1978. He was a member of the Association for eighteen years.

Dr. Allen F. Strehler, Pittsburgh, Pennsylvania, died in August, 1978. He was a member of the Association for nineteen years.

Dr. Ernest W. Anderson, Iowa State University Aerospace Engineering Department, died in 1978 (date not given). He was a member of the Association for thirty five years.

Professor James L. Shawn, Arlington, Texas died on January 31, 1979. He was a member of the Association for nineteen years.

Professor Carl Cohen, Harvard University, died on January 31, 1979, at the age of seventy seven. He was a member of the Association for twenty nine years.

Colonel Gerald Medsger, El Paso, Texas, died in 1979 (date not given). He was a member of the Association for ten years.

Mr. Joseph W. Kemme, Merrimack College, North Andover, Massachusetts, died in 1979 (date not given). He was a member of the Association for six years.

Dr. Nancy J. Winkelman, Bay Village, Ohio, died on March 12, 1979. She was a member of the Association for fourteen years.

Dr. William R. Zigler, Las Vegas, Nevada, died in 1979 (date not given) at the age of thirty seven. He was a member of the Association for one year.

A CALL FOR MODULE WRITERS

The Modules and Monographs in Undergraduate Mathematics and its Applications Project (UMAP) has been operating under a grant from the National Science Foundation to Education Development Center, Inc., of Newton, Massachusetts, since 1976.

One goal of UMAP is to develop, through a community of users and developers, 300 instructional modules and 40 monographs for use in mathematics classes and classes where mathematics is applied. A UMAP module is a self-contained piece of curriculum material; i.e., each module stands on its own provided that certain explicitly stated prerequisites have been met by the student. No specific length is prescribed.

UMAP is also looking for reviewers and field-testers. Reviewers get a chance to lend a hand in the development of materials that can be used by undergraduate students, make a professional application of mathematics easily accessible to undergraduates, and present an applicable mathematical theory that undergraduates usually do not see. Field-testers may use UMAP materials in their classes free of charge in return for field-test data which is incorporated into the final revision of the unit tested.

The UMAP Catalog and further information about the Project are available from EDC/UMAP, 55 Chapel Street, Newton, Massachusetts 02160.

MEETING OF ASSOCIATION FOR WOMEN IN MATHEMATICS

The AWM will meet at the August 1979 Joint Mathematics Meeting in Duluth, Minnesota. All activities are on Thursday, August 23. At 4 p.m. there will be a panel, *Math education: a feminist perspective.* Speakers will include Diane Resek on elementary education, Lenore Blum on secondary and college education, and a speaker to be announced on adult education. At 5 p.m. there will be a business meeting. The main item on the agenda is the new by-laws, and everyone is urged to attend.

MARGARET T. MUNROE, *Adminstrative Assistant*

PITT ESTABLISHES INSTITUTE

In July, 1978 the University of Pittsburgh established an Institute for Computational Mathematics and Applications (ICMA) in the Department of Mathematics and Statistics. The Institute will coordinate departmental research activities in the area of scientific computing, including current projects in multidimensional two phase fluid flow, contour dynamics, numerical solution of nonlinear algebraic and differential systems, data analysis of flight control and applications of computer graphics. ICMA has a limited number of Graduate Research Assistantships and visiting faculty positions. The University appointed Charles Hall as the Executive Director of ICMA.

PI MU EPSILON MEETING

The annual meeting of Pi Mu Epsilon will be held August 21-23 at the University of Minnesota, Duluth, Minnesota. Details are as follows:

Tuesday, August 21	7:00 p.m.	Reception
Wednesday, August 22	12:00 Noon	Council luncheon
	3:00-5:30 p.m.	Contributed papers
	7:00 p.m.	Banquet
	8:30 p.m.	Frame Lecture
Thursday, August 23	8:00 a.m.	Pi Mu Epsilon Breakfast
	9:00-10:30 a.m.	Contributed papers
	3:00-5:30 a.m.	Contributed papers

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

FEBRUARY MEETING OF THE NORTHERN CALIFORNIA SECTION

The Annual Meeting of the Northern California Section was held on February 24, 1979, jointly with the Northern California Section of SIAM, at Sonoma State University, Rohnert Park.

At the business meeting, with William Chinn, Section Chairman, presiding, Stephen T. Tschantz and Daniel Knierim were honored for their performance on the 1977 Putnam Competition. Professor Bacon of Stanford delivered a eulogy for Professor Karel de Leeuw who was killed in August, 1978.

The membership elected the following section officer: Vice Chairman—Sister Madeleine Rose Ashton, College of Holy Names. Succeeding William Chinn as Chairman is Carroll Wilde and succeeding Jane Day as Program Chairman is William Chinn.

The following invited papers were presented:

The Unreasonable Effectiveness of Mathematics, Richard Hamming, Naval Postgraduate School.

Properties of the Symmetric Group, Derrick Lehmer, U. C. Berkeley.
Isoperimetric Inequalities, Pure and Applied, Robert Osserman, Stanford University.
From Tiling to Algebra and Combinatorics, Sherman Steen, U. C. Davis

A luncheon was held at the college dining hall and featured a talk by George Polya on *Some Mathematicians I Have Known.*

MARCH MEETING OF THE FLORIDA SECTION

The Twelfth Annual Spring Meeting of the Florida Section of the MAA was held on March 2-3, 1979 on the Dale Mabry Campus of Hillsborough Community College. There were 151 registrants attending the meeting.

Six invited addresses were presented as follows:

Peer Group Instruction, Professor Calvin Lathan, Monroe Community College
Infinite Games and Borel Sets, Professor David Blackwell, University of California at Berkeley
The Applied Catastrophe Theory Controversy, Professor Peter Hilton, Case Western Reserve University
The Generalized Matrix Inverse, Professor David Sherry, University of West Florida
PRIME 80: A Conference on Prospects in Mathematics Education for the Next Decade, Professor Don Hill, Florida A & M University
Developmental Mathematics, Professor Robert Alwin, St. Petersburg Junior College

In conjunction with the meeting there was a State Articulation Conference. The following talks were presented to the Conference:

Qualifications for Two Year College Teachers of Mathematics, Patricia Dyer, Dean of Academic Affairs, Broward Community College
Mathematics Anxiety, Debby Levinson, University of South Florida
Competency Based Certification for Teachers - An Update, Professor Lawrence Couvillon, Florida State University
Solving 2nd, 3rd, and 4th Degree Equations by Synthetic Division, Professor Bill Carpenter, Central Florida Community College.

A Saturday morning session was sponsored by Pi Mu Epsilon, the Mathematics Honorary Fraternity. The following talks were presented:

Limits Arising from Infinitely Repeated Integration/Differentiation of Real Valued Polynomial/Reciprocal Polynomial Functions, Rodney S. Smith, Florida Delta Chapter, University of South Florida
Mathematics and Transformational Grammers, Calvin Williamson, Florida Epsilon Chapter, University of South Florida
Two Problems in Number Theory, Janet Reid, Florida Eta Chapter, University of North Florida.

The following papers were presented to the section:

Mind Sets—Positive or Negative?, Edwin G. Landauer, Naval Nuclear Power School
Natural Numbers Snack Bar, Alan Wayne, Pasco-Hernando Community College
The Coefficient Conjecture for Functions All of Whose Derivatives are Univalent, Michael Lachance, University of South Florida
Countable Connected Almost Regular Spaces, Stephanie M. Boyles, University of Florida
Semiprime Semigroup Rings, Eleanor Geis Turman, University of Florida
Permutation Tests Based on Robust Estimates of Locations, Sallie Keller, University of South Florida
A Continued Fraction for π^2, J. Sutherland Frame, Michigan State University
Cyclic Actions on Euclidean 3-Space, David G. Winslow, University of Florida
Functional Composition Preserving Polynomials: Another Illustration of the Difference between R and C, Don Blevins, University of Florida
A Characterization of Distinguished Subfields, Tim Morrison, Florida State University
The Math Room in the Hillsborough County Museum of Science and Industry, Dave Snider, University of South Florida
Functional Literacy, Progress or Demise?, Eugene D. Nichols, Florida State University
Addition Sets and Their Properties, John S. Sumner, University of Miami
Continua in $\beta R^1 - R^1$, Alicia B. Winslow, University of Florida
Torus Knots and Periodic Transformations of Euclidean Space, Gerhard Ritter, University of Florida
Special Functions Encountered in a Study of Efficiency of Some Loss Functions, Christine H. Deans, University of South Florida
Computer Generated Color Graphis, E. P. Miles, Jr., Florida State University
Linear Ordering of Convex Ideals, James McKnight, University of Miami
Sums of Cubes, Hermann Simon, University of Miami.

The Association for Women in Mathematics held a meeting.

The Florida Section has divided the State into seven areas and in the fall of 1978 Mini-Sectional Meetings were held in six of the seven areas. These local meetings were organized so that everyone in the State could attend a meeting without extensive traveling. Teachers from Junior High Schools, High Schools and Colleges were invited to attend. The programs dealt primarily with teaching and articulation.

The luncheon-business meeting was held Saturday, March 3, 1979. Chairman George Lofquist presided at the meeting. Committee reports were presented. Professor Edwin Duda of University of Miami was elected chairman-elect; Professors Patricia Dyer of Broward Community College and Bruce Edwards of University of Florida were elected as Vice-Chairmen. Frank L. Cleaver was re-elected Secretary of the Section.

FRANK L. CLEAVER, *Secretary*

MARCH MEETING OF THE SOUTHERN CALIFORNIA SECTION

The 61st Regular Meeting of the Southern California Section of the MAA was held at the University of Southern California on March 10, 1979. There were 135 persons in attendance. The following election results were announced:

Chairman: John McGhee, California State University, Northridge; 1st Vice-Chairman: Dena Patterson, Santa Monica City College; 2nd Vice-Chairman: Judy Grabiner, California State University, Dominguez Hills; Program Chairman: Sandy Grabiner, Pomona College; Secretary-Treasurer: Myron Hood, California State Polytechnic University, San Luis Obispo. James Murphy of California State College, San Bernardino becomes Past-Chairman and Alice M. King of California State University, Pomona continues as the Section Governor.

The program for this meeting was arranged by the committee headed by Susan Montgomery of the University of Southern California. Local arrangements were made by Jacob Matijevic.

The featured speaker was Bradley Efron of Stanford University who talked on *The Jacknife, the Bootstrap, and other Useful Instruments of Statistical Theory.*

A panel consisting of Herbert Gindler, San Diego State University (moderator), Jan Ford, San Diego State, David Schochat, Santa Monica City College and Chuck Lanski, USC discussed *Mathematical Competency: Issues Without Answers.*

Professor Olga Taussky-Todd of the California Institute of Technology was the luncheon speaker and she presented a very interesting address on *My Career in Number Theory.*

Invited addresses during the afternoon sessions included:

Strong Axioms of Infinity and Their Applications, Everett Bull, Pomona College
Uniform Approximation by Analytic Functions, Stephen Scheinberg, University of California, Irvine
Mathematical Problems in Applied Pharmacokinetics, Alan Schumitsky, University of Southern California
The Role of Time Lags in Some Mathematical Models, Stavros Busenberg, Harvey Mudd College

The program concluded with a reception hosted by the Assocation for Women in Mathematics and the Mathematics Science Interchange.

EDMUND I. DEATON, *Secretary-Treasurer*

APRIL MEETING OF THE SOUTHWESTERN SECTION

The 39th annual meeting of the Southwestern Section MAA was held at the University of Texas at El Paso on April 6 and 7, 1979. Sixty-eight members and seven students registered their attendance.

There were three invited addresses. John Brillhart, University of Arizona spoke on *An Irreducibility Theorem of A. Cohn.* Jerome Goldstein, Tulane University gave a talk entitled *Semigroups of Operators and What They are Good For.* John Selfridge, Executive Editor of *Mathematical Reviews* gave the banquet talk entitled *Mathematical Reviews: Where are We Going?* and an invited address entitled, *Number Theory Computing with Large Numbers.*

An additional feature of the banquet program was a slide presentation of the newly acquired MAA Headquarters by our governor, Ralph Ball of New Mexico Institute of Mining and Technology.

The contributed papers were presented in four sessions chaired by Delmar Boyer, Frederick Strauss, Michael Gray and Eugene Schuster all of the host institution.

The following papers were contributed:

Why Equivalence Classes?, Fred Richman, New Mexico State University
A Note on a Theorem of Lindstrom (How to Locate Bad Lightbulbs), William Leahey, University of Texas at El Paso
About the ...537 in the "Last" (?) Fermat Prime, F_4=65,537., Justin MacCarthy, Deming, New Mexico
Uniquely Hamiltonian Graphs, Roger Entringer, University of New Mexico
How to Play the Game of Life Without a Checkerboard, Douglas Jackson, Eastern New Mexico University
Applications of Non-Linear Mathematics to the Army Sound Ranging Problem, Bernard Engebos, White Sands Missile Range
The Disconnected Connection, Charles Mumma II, New Mexico Institute of Mining and Technology
Affirmative Action is Best, Ray Mines, New Mexico State University
The Current State of Cryptography Research, Neal Wagner, University of Texas as El Paso
The Case for Bilingual Mathematics, J.R. Provencio, University of Texas at El Paso
Unified Field Theory Via the Principle of Duality, A. Swimmer, Arizona State University
Plotting and Goodness of Fit Tests Using Order Statistics—A Review, Gerald Rogers, New Mexico State University
A Point Set Characterization for a Metric Space to be Isomorphic to R, Ronald Brashear, New Mexico Mining Institute
A Variant of Lebesgue's Differentiation Theorem, Richard Bagby, New Mexico State University
Some Conjectures on Approximation by Rationals, Everett Walter, Northern Arizona University
A Generalization of a Result of Sylvester's, Melvyn Knight, University of Texas at El Paso

Carl Hall, Chairman of the Southwestern Section acted as meeting coordinator.

A. SWIMMER, *Secretary*

APRIL MEETING OF THE SOUTHEASTERN SECTION

The fifty-eighth annual meeting of the Southeastern Section was held on April 6-7, 1979 at The University of Tennessee at Chattanooga. A total of 312 persons attended the meeting, including 60 students and 204 members of the Association. The local arrangements were handled by Professor DeWayne Nymann.

Three invited addresses were given: Peter J. Hilton (First Vice-President of the Association) of Western Reserve University on *The applied catastrophe theory controversy*; Leonard Carlitz of Duke University on *Dedikind Sums*; and Trevor Evans (Section Lecturer) of Emory University on *Combinatorics without Counting.*

There were eight sessions for contributed papers. The presiders were John Kenelly (Chairman of the Section), DeWayne S. Nymann and John Neff for the general sessions and E. Hayashi, D. Clanton, J. Baxley, H. Taylor, E. Kirkman, V. Perry, C. Brown, and C. Huff for the special sessions.

Officers elected for 1979-80 are: Chairman, John D. Neff, Georgia Tech; Chairman-elect, Lida K. Barrett, University of Tennessee, Knoxville; Vice-Chairman, Cathrine C. Aust, Clayton Junior College; Section Lecturer, Billy F. Bryant, Vanderbilt University, all for a one year term.

At the business meeting, it was announced that the winner of the Section prize for the best performance on the Putman examination was Hartwig P. Arenstorf of Vanderbilt University.

The section voted to hold its 1981 meeting at the University of Alabama in Birmingham.

The following papers were presented:

A note on Ramsey multiplicity, Michael Jacobson, Emory University
Square-separable primes: A research problem, Jacob T. B. Beard, Jr., Emory University
Linear congruences with a GCD constraint, Richard L. Robinson, Wofford College.
A mathematical analysis of world championship chess match systems, Paul W. Lawrence, Jr., Dyersburg State Community College.
Electronic mail, a mathematical smörgåsbord, Lee O. Hagglund, Wofford College
Dice tossing and the Pascal triangle, John D. Neff, Georgia Institute of Technology
Description of a self-paced mastery format for introductory calculus, H. Tom Mathews, The University of Tennessee, Knoxville
A comparison of the effectiveness of three instructional formats in introductory calculus for students from different ability levels, Herman E. Jackson, Walters State Community College
Teaching elementary proofs in algebra, J. Pelham Thomas, Western Carolina University
Models developed in an NSF LOCI Project, James R. Smith, Appalachian State University
Better bicycling through the calculus of variations, Mark Harris, Appalachian State University
A few new "Box Problems" for the calculus student, G. Marvin Eargle, Appalachian State University
Graduate and undergraduate mathematics education in Colombia and Venezuela, Bill Watson, The College of Charleston
On the summation of slowly convergent alternating series, F. Virginia Rohde, Mississippi State U.
Convergence of Walsh-Fourier Series, Lyndell Kerley, East Tennessee State University
An example of a cellular set which does not have the fixed-point property, Paul R. Patten, Brewton Parker College
On a girdle of equal temperature antipodal points around the earth, H. S. Hahn, West Georgia College
The convergence of two interpolation series for ln x, C. Ray Wylie, Furman University
Calculation of the Hurt-Waid generalized inverse of a matrix, Eleanor L. White, University of Tennessee, Knoxville
Pattern from the Past-Greek mathematics 600BC - 300AD, Kenneth Farrell, The Citadel
An improved derivation of inverse hyperbolic sine and cosine formulas, Chris Light, Tennessee Tech U
Closed form sums of certain infinite series, Gary Thompson, Wake Forest University
P-adic number systems for error-free computation, Ruth Ann Lewis, University of Tennessee, Knoxville
On the associativity of certain alternative rings, Tae-il Suh, East Tennessee State University
On Prony's method for fitting a linear combination of exponentials to a set of data points, Joseph M. Garber and James C. Pleasant, East Tennessee State University
Some elementary extensions of the Weierstrass approximation theorem, John V. Baxley, Wake Forest U
Conduit based materials in mathematics, David Smith, Duke University
Concerns in the evaluation of Audio-Visual material, Julia P. Kennedy and Janice T. Astin, Georgia State University
The conclusion of one on one mathematics—A five year project, Don M. Jordan, University of South Carolina

IVEY C. GENTRY, *Secretary-Treasurer*

KENTUCKY SECTION ANNUAL MEETING

The sixty-second annual meeting of the Kentucky section of the Mathematical Association of America was held at Morehead State University on April 6-7, 1979.

Invited speakers were David Roselle, V.P.I., secretary of the M.A.A., and B.F. Bryant of Vanderbilt University. On Friday night, Professor Roselle spoke on *Combinatorial Problems-Solved and Unsolved.* On Saturday afternoon, he spoke on *The Langford-Skolen Problem.* Professor Bryant's address was entitled *Sir Isaac Newton: The Man and His Papers.*

Student papers given on Friday night were:

Eudoxus' Method of Exhaustion, Roxanne Bow, Eastern Kentucky University;
The Games Mathematicians Play, James T. Snodgrass, Western Kentucky University; and
A Theorem Concerning Primitive Roots of p^n, Cecil Andrew Ellard, Eastern Kentucky University.

Contributed papers given on Saturday were:

Can You Find a Surface with Prescribed Tangent Planes?, Professor Michael Freeman, University of Kentucky

The Neighborly Tree Property in Banach Spaces, Professor Philip McCartney, Northern Kentucky U

The High School Mathematics Program, Brother Emeric, C.F.X., St. Xavier High School

Revolving Faculty Exchange—An Approach to Faculty Development, Professor Carroll Wells, Western Kentucky University

Some Properties of Inverse Clifford Rings, Professor Philip Breshear, Eastern Kentucky University

Covers and Envelopes, Professor Edgar E. Enochs, University of Kentucky

Grading Answer-Until-Correct Tests, Professor Thomas M. Lamm, Eastern Kentucky University

A Surveyor's Formula for the Area of a Polygon, and an Application to Calculus, Professor Bart Braden, Northern Kentucky University,

Of the 58 persons who registered for the meeting, 50 were members of the section.

Professor Gordon Nolen, Morehead State University, was appointed Vice Chairman for one year to complete the term of Professor Christine Parker, Murray State University, who has retired. Professor Richard Davitt, University of Louisville, was elected Chairman-Elect. Professor Kyle Wallace, Western Kentucky University, assumed the chairmanship, succeeding Professor Bennie Lane, Eastern Kentucky University.

The 1980 meeting of the section will be held at Western Kentucky University, Bowling Green. The tentative dates are April 11-12, 1980.

JOE K. SMITH, *Secretary*

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I.e.

CALENDAR OF FUTURE MEETINGS

Fifty-ninth Summer Meeting, University of Minnesota, Duluth, August 21–23, 1979.

Sixty-third Annual Meeting, San Antonio, Texas, January 5–7, 1980.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, last weekend in April or first weekend in May. Deadline for papers 6 weeks before meeting.

FLORIDA, early March. Deadline for paper titles 2 weeks before meeting.

ILLINOIS, first Friday/Saturday in May.

INDIANA

INTERMOUNTAIN

IOWA, third weekend in April. Deadline for papers February 1.

KANSAS, March or April. Deadline for papers January 1.

KENTUCKY, early April. Deadline for papers 6 weeks before meeting.

LOUISIANA-MISSISSIPPI, Louisiana Tech University, Ruston, February 15–16, 1980.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK, spring. Deadline for papers 2 weeks before meeting.

MICHIGAN, first Friday and Saturday in May. Deadline for papers 6 weeks before meeting.

MISSOURI, late March/early April. Deadline for papers January 31.

NEBRASKA, April.

NEW JERSEY, early November and early May.

NORTH CENTRAL, end of October and April. Deadline for papers October 1 and April 1.

NORTHEASTERN, University of Maine, Orono, June 22–23, 1979.

NORTHERN CALIFORNIA, first or second Saturday in February.

OHIO, College of Wooster, Wooster, October 19–20, 1979.

OKLAHOMA-ARKANSAS, (approx.) Friday and Saturday of first weekend in April. Deadline for papers 3 weeks before meeting.

PACIFIC NORTHWEST, University of British Columbia, Vancouver, June 15–16, 1979.

PHILADELPHIA, Drexel University, Philadelphia, November 17, 1979.

ROCKY MOUNTAIN, last weekend in April or first in May. Deadline for papers 8 weeks before meeting.

SEAWAY, first Saturday in November and Saturday in late April. Deadline for papers 6 weeks before meeting.

SOUTHEASTERN

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, usually in April. Deadline for papers 2 weeks before meeting.

TEXAS, Friday and Saturday in early April. Deadline for papers March 1.

WISCONSIN, Friday and Saturday between mid-April and first week in May. Deadline for papers 6 weeks before meeting.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, San Francisco, January 3–8, 1980.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES

AMERICAN MATHEMATICAL SOCIETY, University of Minnesota, Duluth, August 22–25, 1979.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Louisiana State University, Baton Rouge, June 25–28, 1979.

ASSOCIATION FOR COMPUTING MACHINERY, Plaza Hotel, Detroit, Michigan, October 29–31, 1979.

ASSOCIATION FOR SYMBOLIC LOGIC, New York City, December 28–29, 1979.

ASSOCIATION FOR WOMEN IN MATHEMATICS

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES, University of Saskatchewan, Saskatoon, June 4–5, 1979.

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS, Washington, D.C., August 13–16, 1979.

MU ALPHA THETA, Athens State College, Athens, Alabama, August 5–8, 1979.

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Seattle, Washington, April 16–19, 1980.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Marc Plaza Hotel, Milwaukee, Wisconsin, October 15–17, 1979.

PI MU EPSILON, University of Minnesota, Duluth, August 21–23, 1979.

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Radisson Muehlbach, Kansas City, Missouri, November 1–3, 1979.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Royal York Hotel, Toronto, Canada, June 11–13, 1979.

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1979

THE AMERICAN MATHEMATICAL MONTHLY

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SEARCH AND ITS OPTIMIZATION

BERNARD O. KOOPMAN

This paper treats the problem of how best to search for an object when the amount of searching effort is limited and only probabilities of the object's possible positions are given. The subject is not only considered mathematically but is made to serve as an example of what a new application of mathematics to the outside world may require—the steps the author had to take in solving a naval problem in World War II. Mathematics, science, and technology were young together; but with age and professionalism they have tended to fall apart. Then in the grim calculus of the Battle of the Atlantic they regained some of their youth and, with it, the freshness of vision and the sense of balance between reality and abstraction. To the many who shared this experience, a fragment of which is presented here, it seemed as if time had been turned back to the days when new sciences were born. Operations research itself may be said to have grown from seeds sown at this epoch—from problems of the sort presented here.

During World War II one of the author's tasks for the U.S. Navy was the design of screening formations to protect our Atlantic convoys. The number of available defensive forces being tragically small, the problem was one of their most economical use. Careful examination of U-boat tactics, combined with common-sense reasoning, showed that they had preferential directions of approach to a convoy—more often from the two forward flanks than from due ahead. Rearward approaches were prevented by their low relative submerged speed. This rough predictability of directions of closing to torpedo range suggested the possibility of an optimal placement of screening units, an opportunity that had up to then been neglected.

Further examination showed not only how to formulate and solve the mathematical problem involved but also that it belonged to a broad class of similar ones, both military and civilian—occurring, in fact, in everyday life: the *problem of optimal search*. The situation is characterized by three data: (i) the probabilities of the object of search's (the "target's") being in its various possible positions; (ii) the local *detection probability* that a particular amount of local searching effort (time spent, concentration of searching units, and the like) should detect the target; and (iii) the total amount of searching effort available. The problem is to find the *optimal* distribution of this total effort, most obviously the one that maximizes the probability of detection.

This is evidently a problem of the calculus of variations with *unilateral* contraints. The total effort is fixed (an "isoperimetric constraint"); but, furthermore, the density of searching effort is non-negative (the "unilateral" condition)—setting a variation equal to zero and then solving the resulting equations can lead to such absurdities as negative probabilities. On the other hand, a method first applied by J. Willard Gibbs [6] to physical problems in 1875 has led us to the correct answer.

There are two parts to our problem: its mathematical formulation by seeking out what is crucial and simple amid the complexities of the apparent facts; and the mathematical solution of the problem posed in this process. The former part proved more difficult than the latter; accordingly, we develop it in detail in Part I below. In Part II the intuitive ideas in Gibbs's

This paper is from the invited address by the same title presented at the joint annual meeting of the AMS and the MAA, St. Louis, Missouri, January 29, 1977.

The author received his Ph.D. from Harvard in 1926; he was a National Research Fellow in 1926–27 and was then a member of the Department of Mathematics at Columbia until his retirement in 1968 as Adrain Professor Emeritus. From 1968–75 he was a part-time member of A. D. Little, Inc. During World War II he was a member of the Operations Research Group, U.S. Navy. He was a founding member of the Operations Research Society of America and has served as its president. His interests have included analysis, probability, dynamical systems, operations research, information theory applied to statistical mechanics, and the mathematical theory of search, of which he was a founder.—*Editors*

method are presented and their mathematical implementation given, followed by geometrical interpretations and practical applications. The paper closes with some general conclusions and with citations of the literature for extensions of the subject. References to the first publications are **[8]**, **[10]**; the former has long been out of print and virtually unavailable but is appearing in a new edition **[11]**.

Part I: Mathematical Formulation of the Problem

Three general ideas bearing on the practical situation must be put into precise mathematical form: the *a priori probability distribution* of the object of search (the "target"); the *density of searching effort*; and the *detection probability* for a given density of effort applied in the locality of the target. The possibility of a scientific formulation of these concepts reposes on a set of assumptions concerning the physical and practical state of affairs involved in the search. It is emphasized that these assumptions are not self-evident; they characterize only a particular class of searches.

1. The probability distribution of targets. A searcher would be in a bad way indeed if the area A containing the object searched for were very great and the searcher had no idea whatsoever where it might be. If A is a large lawn in which you have lost a pocketbook and have but a limited time to find it, your chance of success would depend on how well you could reason out where you might have dropped it and your use of time looking for it in the more likely places. In searching for a submarine in a vast region A of ocean by the use of detection equipment of very moderate range, the task would not be worth the effort if there were not enough clues to show that certain relatively restricted regions had a much higher probability of containing the target than others. During World War II the clues were many and various; interception of radio emissions by different receivers giving a *cross fix*; intercepted messages in a code that could be broken; general reasoning concerning the probable tactics of the enemy; a preliminary rough sighting by long-range equipment.

In searching for a life-raft released from a downed aircraft, the coordinates of position radioed from the latter at the time of the accident have served as the center of a normal distribution with variances corresponding to the known accuracies of navigational estimates. The very expensive step of drilling for oil or mining for minerals is always guided by prolonged preliminary geological studies to establish the quantitative chances of the drilling's being fruitful. Probabilities are not mentioned as such by Sherlock Holmes or Dr. Watson, but they are an essential element in police operations and in medical diagnosis, both of which involve a process of search.

The mathematical concept needed in the formulation in all these examples is, of course, the a priori probabilities (i.e., before starting the search) of the target's being in the various possible positions in the region A, usually its "probability per unit area" or *probability density* $p(x,y)$. When integrated over various subregions B of A, it gives the probability of the sought object's being in B. The fact that such a function can be found—even approximately—by reasoning based on the situation is the precondition for the scientific theory of search.

The design of an anti-submarine screen (or of any type of barrier) requires an estimate of the *probability flux* and is complicated by the facts of relative motion. It can often be reduced to the case of search for a fixed target in a plane moving with respect to the ocean surface with the constant mean velocity of the target—if that can be determined.

2. The density of local searching effort. We are assuming a situation common to many types of search, in which the detection range is so small in comparison with the dimensions of A that the searcher, like the person looking for the lost pocketbook, must explore various parts of A in detail: viz., a search made up of *localized* parts. In formulating the principles that should govern our search, we must have an expression for the amount of searching done in various places: the

density of searching effort. In a reasonably large subregion B of A, the total searching done can be measured in various simple ways: the time T spent in B; or if the searchers are moving at uniform speed v, the total length of track(s) $L=vT$ contained therein. If searchlights or radar sets are used and can be turned off to save energy, or if the number of men in the watch can be varied, these factors also must be taken into account in the measurement of the searching effort in B. Whatever the measure adopted, whether T, L, or other, we shall denote the quantity of searching effort by S.

When A can be subdivided into a limited number of regions B_i (with $p_i = \text{prob}\ (B_i)$) throughout each of which the target probability density is constant and thus equal to p_i/B_i (these quantities p_i being supposed known), there is no conceptual difficulty. But when $p(x,y)$ varies continuously throughout A, we are obliged to define the density of searching effort at each individual point (x,y). The physical situation rules out an imitation of the elementary definition of surface density of matter as the limit of the mass of a piece of matter containing (x,y) divided by its area, as the piece shrinks up to (x,y). We have to base the definition of "density of search at (x,y)" upon a far more complicated physical picture, one describable only by using the word "very"—in a mathematically undefined, yet physically meaningful, sense.

In describing the operation of search in the present connection we must recognize that it involves regions having dimensions of *three different orders of size*: "very small," "very large," and "intermediate."

By a "very small region" is meant one of dimensions not exceeding the presumable ranges of detection. For the lost pocketbook, this could be a few yards; for the submarine, a few thousand yards; for the life-raft, several miles. (Satellite and other long-range detection methods are not being considered.) Since a searcher moving about in such a small region is capable of detecting targets within it and, about as easily, outside it—as he nears the boundary—we cannot describe his searching effectiveness as *confined* to such a region.

At the other extreme are "very large regions," such as the whole expanse of A (of the lawn or the ocean) in which the target lies. Major pieces of A are also to be considered as "very large." They are characterized by the fact that a search at one point cannot detect targets at most points of such regions. Another characteristic of very large regions is that $p(x,y)$ usually has very different values at different widely spaced points (except of course in the special case of a uniform distribution: $p(x,y) = \text{constant}$).

Between these extremes lie distances and regions of "intermediate" size, characterized more precisely by two properties: First, a search conducted within such a region will have but a negligible chance of detecting a target lying outside it; second, throughout a region of intermediate size $p(x,y)$ is always very nearly constant.

Let ΔA be a region of intermediate size containing a fixed point (x,y), and suppose that on a particular occasion (past or planned) the total effort ΔS is expended in it. Obviously we are interested in the ratio $\Delta S/\Delta A$. This, however, cannot serve as the required density at (x,y), since it can change abruptly if ΔA—always containing the fixed (x,y)—is subjected to minor rigid displacements. These changes are produced by dropping or adding pieces of the searcher's track. The way out of this difficulty is to replace ΔS by its mean or expected value $\overline{\Delta S}$, which is not subject to such sudden variations and which, moreover, is all that can be known precisely in planning any actual search. We are then justified by general physical considerations in saying that $\overline{\Delta S}/\Delta A$ is sensibly unchanged by such types of changes of ΔA. Therefore it essentially is determined when (x,y) is given: this means that $\overline{\Delta S}/\Delta A = \phi(x,y)$, a function of the point (x,y). One thing we *cannot* do: let the region ΔA shrink up so that it approaches the point (x,y); for as soon as it reaches the range of small sizes, the numerator $\overline{\Delta S}$ loses its physical meaning. Nevertheless we can obtain a sufficiently approximate value of the expected search in any intermediate or large region B of A by integrating $\phi(x,y)$ over B: We may describe the function as an "interpolated approximation."

The use of the mathematically undefined but physically meaningful concepts of the three

orders of size is just as necessary here as it is in many other branches of applied mathematics, longer established and better known. We cite the example of the modern kinetic theory of matter, in which three vastly different *time scales* are considered in molecular motion. Cf., e.g., Bogolubov's treatment as discussed in Cohen [3].

3. The elementary detection probability. The third element needed to complete the formulation of the problem of optimal search is the *elementary detection function* $D(x,y,z)$, where $z=\phi(x,y)$. This is defined as the conditional probability that a search in a region of intermediate size containing (x,y) and with the local density of effort z shall detect a target *given* to be at (x,y):

$$D(x,y,z)=\text{prob}\{\text{targ. detected}|\text{targ. at } (x,y), \text{ search eff. } z\}. \tag{3.1}$$

Evidently the function notation in (3.1) implies that the probability of detection at (x,y) is *determined* by the value z of the density of effort at the one point (x,y)—or what is the same thing, in view of the considerations of the preceding section, by its value at any other point of the region ΔA of intermediate size containing (x,y). But many examples can be given in which this is not true and where, on the contrary, this probability depends on the whole spread of values of $\phi(x,y)$, so that it is a *functional* of this function. First, there are cases in which the act of searching outside of ΔA can alert the target at (x,y) and enable it to react before ΔA is reached. In World War II, surfaced U-boats submerged when their search-receivers recorded distant airborne radar, this being possible because they could detect the searcher long before it could detect them. An opposite sort of target reaction occurs when people on a life-raft discharge rockets to signal a distant searcher. Second, the process of search may physically alter the searcher's state (by fatigue, depletion of equipment, etc.); or that of the target (covering or uncovering the lost pocketbook with grass or sand); or, finally, of the environment ("muddying the waters"). Evidently these possibilities show that the probability in (3.1) may depend on the whole course of the events and cannot in such cases be regarded as determined when only (x,y) and z are given.

We shall say that a search is by *passive observations* when none of the possibilities above occurs: The target does not react, and the search causes no material change in the target, the searcher, or their common environment. In the contrary case, we shall say the the search involves *active observations*. This distinction, and its mathematical consequences, were first set forth by J. M. Dobbie [5]. We may say that we have here an analogue, in the theory of search, of Heisenberg's principle in the quantum mechanics of elementary particles: The physical act of observation can change the system observed [7]. Henceforth we consider only the cases of search by passive observations.

In the early history of search, further assumptions were made: that the *detectability* of the target at each point (x,y) is a fixed quantity; also that the *effectiveness* of the searcher, his equipment, and the factors in the environment are given at each point; and, finally, that these quantities are known before the search starts. Experience has shown that these assumptions represent very useful but pretty rough approximations. Thus, if listening to target sounds or other radiation is important to detection, these may vary. If detection is by echo-ranging (by radar or sound pulses), the target's *aspect* is known to play a decisive role. This effect was studied extensively during World War II, and polar diagrams showing the effect of angle on echo strength were plotted under laboratory conditions. Other details of the position of the target can be important, such as degree of submergence of a submarine; or when the target is a wreck, its condition and position on the bottom.

The practical question, on the other hand, has always been how to take these possibilities into quantitative account. For the most part, they represent fleeting and ever changing factors; and even when constant, it has not proved possible to measure them or to give a valid basis for their probability distributions in any actual case. We shall accordingly treat the problem on the basis

of the simplifying assumption of a detectability determined by position (x,y) and not changing with time.

Part II: The Mathematical Consequences

Mathematically, the attack on the problem is in three stages: the derivation of the form of $D(x,y,z)$; the formulation of the optimization problem; and the mathematical solution. Interpretations and applications then follow.

4. The exponential detection law.

THEOREM I. *In a localized search as defined in Section 2, using entirely passive observation, and with a given and fixed target detectability and search effectiveness,*

$$D(x,y,z)=1-\exp[-w(x,y)z], \qquad w(x,y)\geqslant 0. \tag{4.1}$$

This is the "exponential law of random search," the latter term expressing the disorderly and somewhat unpredictable nature of the details of most real searching processes. For its proof we denote complementary probabilities with dashes, writing $\bar{D}=1-D$, and suppose that two searches are made, either simultaneously or in quick succession, one with the value $z_1=\phi_1(x,y)$, the other with $z_2=\phi_2(x,y)$. Our definition of search density allows us to add these quantities, expressing by z_1+z_2 the combined searching at (x,y). Consider the probability that both searches fail to detect the target, *given* to be at (x,y). By general compound probability,

$$\bar{D}(x,y,z_1+z_2)=\bar{D}(x,y,z_1)\bar{D}(x,y,z_2|z_1),$$

where $\bar{D}(x,y,z_2|z_1)$ is doubly conditional probability of non-detection of the target given to be at (x,y), searched by the amount z_2, and given that the z_1 search has failed to detect it.

The crucial fact is that under our assumptions these two failures *are independent events*, so that

$$\bar{D}(x,y,z_2|z_1)=\bar{D}(x,y,z_2),$$

and hence our previous equation becomes

$$\bar{D}(x,y,z_1+z_2)=\bar{D}(x,y,z_1)\bar{D}(x,y,z_2) \qquad (z_i\geqslant 0). \tag{4.2}$$

Indeed, consider what the datum that "the z_1 search failed" could contribute to the knowledge regarding the outcome of the z_2 search: It could not cast doubt on the target's being at the point (x,y), since that is postulated in all these conditional probabilities; it could not alter our knowledge concerning the target's detectability, the searcher's performance, etc., since these factors are supposed *given*, by hypothesis. Finally, the assumed passive nature of the observations guarantees that the act of the z_1 search does not itself cause any material change. The independence, and therewith (4.2), follow.

In giving the steps in the solution of (4.2) we leave out of the notation the letters x,y, writing it in the form $\bar{D}(z_1+z_2)=\bar{D}(z_1)\bar{D}(z_2)$. We can see at once that if $\bar{D}(z_1)=0$ for some $z_1>0$, then $\bar{D}(z)=0$ for all $z>0$: In the case $z>z_1$, we have but to apply (4.2) with $z_2=z-z_1$; hence the set of values of z for which $\bar{D}(z)=0$ is a connected semi-infinite segment. We have but to show that its lower limit $z_0=0$. If $z_0>0$, we could find a positive integer n making $z_1/n<z_0$, and an n-fold application of our equation would give $D(z_1/n)^n=0$, so that $D(z_1/n)=0$, contrary to the assumption that z_0 is the lower limit of such z values.

In all other cases we can take logarithms: On setting $L(z)=\log\bar{D}(z)$, (4.2) gives $L(z_1+z_2)=L(z_1)+L(z_2)$. This is one of the simplest and oldest functional equations. If we assume that $L(z)$ is continuous at $z=0$ on the right, it will follow that $L(z)=-wz$. The traditional method of proof is to establish the latter expression successively, first for integral z (defining w as $L(1)$);

next for fractions; then extending the result to all rational values; finally, proving the equation for irrationals, by continuity.

Restoring the coordinates so that $w=w(x,y)$, the equation (4.1) is established. If $w(x,y)<0$, (4.1) would give a negative probability for all $z>0$. If at some points $w(x,y)=0$, $D(x,y,z)=0$, however great the searching effort: The target is invisible there. The case excluded at the outset, in which $\bar{D}(x,y,z)=0$, corresponds with points (x,y) with $w(x,y)=\infty$: absolute certainty of detection with arbitrarily little search. As $w(x,y)$ increases indefinitely there is an indefinitely increasing drop from $\bar{D}(x,y,0)=1$ (no search: no detection) to $\bar{D}(x,y,\infty)=0$. In general, the larger the $w(x,y)$, the greater the probability of detection. Hence this function can be regarded as a *local measure of detectability at* (x,y).

Under the useful but drastically simplified assumption of the *definite range law*, according to which every target within a fixed range R of the searcher is detected, and no other, the quantity $W=2R$ is called the *sweep width*, because of its obvious meaning when the searcher moves in a straight line. If he does this through a region of area ΔA in which his length of track is $\Delta L=\Delta S$, his probability of detecting a point target randomly placed in ΔA is the ratio of areas, very close to $W\Delta L/\Delta A$ or $W\phi(x,y)$, where $\phi(x,y)=\overline{\Delta S}/\Delta A$ is essentially $\Delta L/\Delta A$. If instead of a straight path the searcher moves in an irregular one, it is easy to show directly that (4.1) is a good approximation, with $w(x,y)=W$. This approach to the formulation was the original one given by the author [**8**]. Consequently, $w(x,y)$ is usually called the sweep width, even though neither the definite range law nor the picture of a random path need be assumed.

The case of the protective screen for a convoy or other surface formation moving with a mean vector velocity $\vec{v}$ is treated by introducing "convoy space," a plane moving along with the velocity $\vec{v}$, in which the convoy can be regarded as at rest. A target moving with velocity $\vec{u}$ with respect to the ocean will have the velocity $\vec{w}=\vec{u}-\vec{v}$ in convoy space. It can approach only when $\vec{w}$ has a certain set direction (when $u<v$), meaning, in the case of slow targets, only from forward directions. Because of the irregular relative track, (4.1) remains applicable.

5. The optimal distribution of searching effort. An obvious formulation of the mathematical problem of finding the optimal distribution of a fixed total amount of searching effort Φ over a region A of the plane is the following: *Find that search density function* $\phi(x,y)$ *which renders maximum the probability of detection*

$$\begin{aligned} P[\phi] &= \int\int p(x,y) D(x,y,\phi(x,y))\,dx\,dy \\ &= \int\int p(x,y)\{1-\exp[-w(x,y)(x,y)]\}\,dx\,dy \qquad (5.1) \end{aligned}$$

subject to the two constraints

$$\int\int \phi(x,y)\,dx\,dy=\Phi, \qquad \phi(x,y)\geqslant 0. \qquad (5.2)$$

The integrations are over the whole region A.

The first constraint expresses the fixed nature of the total effort available, which in view of the discussion in Section 2 can be written as an integral. The second states the obvious fact that searching effort as defined cannot be negative. Obvious as the latter fact may seem, it plays a decisive role in the mathematical derivation of the answer, since it renders our problem a *unilateral* one of the calculus of variations, not to be solved by setting variations equal to zero without more ado.

The solution of the problem, i.e., establishing a *necessary* condition for a maximum, from which an explicit formula for $\phi(x,y)$ can be obtained, proceeds by the method of J. W. Gibbs [**6**]; it takes the place of the classical derivation of Euler's equations as necessary conditions, applied when unilateral (inequality) conditions are absent. It may be described as a process of

"taking from the haves and giving to the have-nots"; and, on the supposition that we started with the optimal $\phi(x,y)$, stating mathematically that this alteration cannot increase $P[\phi]$. From this inequality, others are derived that lead to the required formula for $\phi(x,y)$.

In carrying out this process we may assume that the given functions $p(x,y)$ and $w(x,y)$ are continuous. This is because they are approximations to values in the physical world which (on the present scale) "is continuous," i.e., is most closely approximated by continuous—and, in fact, highly continuously differentiable, even analytic—functions. We shall *assume* that the optimal $\phi(x,y)$ is continuous *in general*; possible exceptions only on loci of lower dimension on A. *This represents a constraint in addition to* (5.2).

Let $\phi(x,y)$ be the optimal density and let (x',y') be any general point (i.e., of continuity) where $\phi(x',y')>0$; further, let (x'',y'') be any other general point. Our method is to pick a small bit of the searching from (x',y')—which this point "doesn't need" (i.e., to keep its density positive)—and transfer it to (x'',y''); this alteration will lower the value of $P[\phi]$ or leave it unchanged.

Write $F(x,y,z)=p(x,y)\{1-\exp[-zw(x,y)]\}$ and $F_3=\partial F/\partial z$ for brevity; then (5.1) becomes $P[\phi]=\int\int F(x,y,\phi(x,y))dx\,dy$. To alter ϕ as described above, we observe that by its continuity at (x',y') we can find a neighborhood N' of this point throughout which it is positive. Then let $\psi'=\psi(x,y)$ be a function that is continuous everywhere, zero outside N', and positive but less than $\phi(x,y)$ throughout N'. This is what is being transferred from the first point to the second: Let N'' be a neighborhood of (x'',y'') congruent to N', i.e., under the vector translation of components $(x''-x',y''-y')$; and let $\psi''=\psi''(x,y)$ be the correspondingly translated function obtained from ψ': $\psi(x,y)=\psi'(x-x''+x',y-y''+y')$. Then our modified ϕ is $\phi-\psi'+\psi''$; it satisfies (5.2) since the integrals produced in the modification cancel, and by the choice of ψ' it is non-negative. Evidently the same will be true of the 1-parameter family of functions

$$\phi_t=\phi-t\psi'+t\psi'', \qquad 0\leqslant t\leqslant 1.$$

Since the modification cannot increase $P[\phi]$, we have

$$\int\int F(x,y,\phi)dx\,dy \geqslant \int\int F(x,y,\phi-t\psi'+t\psi'')dx\,dy. \tag{5.3}$$

Because of the way in which $F(x,y,z)$ depends on z, the second member in (5.3) can be expanded in powers of t (e.g., by Taylor's theorem with the remainder of the second order). The term independent of t is equal to the first member of (5.3) and cancels it. On dividing the result by t and then letting $t\to 0$ through positive values, the higher order terms approach zero. This process maintains the inequality relations of (5.3), and yields a result that can be written as

$$\int\int F_3(x,y,\phi)\psi'dx\,dy \geqslant \int\int F_3(x,y,\phi)\psi''dx\,dy. \tag{5.4}$$

By the construction of ψ' and N', the integral on the left reduces to an integral over N' only, inside which the factor $\psi'>0$. Applying the first law of the mean for integrals, the corresponding integral in (5.4) can be written in the form $F_3(\bar{x},\bar{y},\phi(\bar{x},\bar{y}))$ times $\int\int_{N'}\psi'dx\,dy$, where $(\bar{x},\bar{y})$ is some point in N'. A corresponding expression is given to the integral on the right in (5.4), the point in N'' being denoted by $(\bar{\bar{x}},\bar{\bar{y}})$, the integral factor being $\int\int_{N''}\psi''dx\,dy$. Now obviously these two integral factors are positive and (by their relations of congruence) are equal. Hence they may be canceled from the inequality, given the same $(\leqslant)$ between the values of the function F_3 at the points $(\bar{x},\bar{y})$ and $(\bar{\bar{x}},\bar{\bar{y}})$. Let N' (and hence N'') shrink to the points (x',y') and (x'',y''). By the continuity of F_3 at these points, the former inequality will lead to

$$F_3(x',y',\phi(x',y'))\geqslant F_3(x'',y'',\phi(x'',y'')). \tag{5.5}$$

Now suppose that $\phi(x'',y'')>0$: The same proof applies with the roles of (x',y') and (x'',y'')

interchanged, giving (5.5) but with ($\geqslant$) replaced by ($\leqslant$)—and hence by ($=$). From this follows that for $\phi(x,y)$ to be optimal it is necessary that at all points where it is positive the values of $F(x,y,\phi(x,y))$ be the same: we shall denote this common value by λ. Furthermore, at all other points $F_3(x,y,\phi(x,y))=F_3(x,y,0)\leqslant\lambda$.

Returning to the original form for F in (5.1), we have the resulting necessary property for an optimal ϕ, established as above for points where it is continuous, and extended by obvious arguments by limits to possible points of discontinuity (which lie by hypothesis on manifolds of lower dimension):

THEOREM II. *For an optimal $\phi=\phi(x,y)$, there is a constant $\lambda>0$ such that*

$$\begin{aligned} &\text{(i) } \textit{if } \phi(x,y)>0, \qquad p(x,y)w(x,y)\exp[-w(x,y)\phi(x,y)]=\lambda,\\ &\text{(ii) } \textit{if } \phi(x,y)=0, \qquad p(x,y)w(x,y)\leqslant\lambda. \end{aligned} \tag{5.6}$$

That $\lambda>0$ is an immediate consequence of these expressions. That, further, it is a function of the given Φ, the total effort available, is easily seen. Moreover, the locus in A of positive ϕ also is determined when Φ is given and may vary when the latter does. We denote it and its complement by

$$A(\Phi)=\{(x,y)\colon \phi(x,y)>0\}; \qquad \bar{A}(\Phi)=A-A(\Phi). \tag{5.7}$$

In these terms, the optimal $\phi(x,y)$ is easily expressed; we need only solve (5.6) on $A(\Phi)$:

$$\phi(x,y)=\frac{1}{w(x,y)}\log\frac{p(x,y)w(x,y)}{\lambda} \quad \text{for } (x,y)\in A(\Phi). \tag{5.8}$$

Thus *there will usually be some places where it is unprofitable to search at all*; *but in the others, the density of search is a linear function of the logarithm of the probability*. From (5.2) and (5.8) we obtain

$$\Phi=\int\int_{A(\Phi)}\{\log p[x,y)w(x,y)]-\log\lambda\}\,dx\,dy/w(w,y), \tag{5.9}$$

a relation that evidently determines λ in terms of Φ and vice versa.

6. Geometrical interpretation and properties of the solution. These rather complicated relations can be given a simple intuitive interpretation in terms of a mechanical model (simplifying to a purely geometrical construction in important special cases).

Consider the surface having, in the coordinates (x,y,z), the equation $z=\log[p(x,y)w(x,y)]$. Suppose that the *solid* under this surface (which may extend upwards indefinitely in some places and downwards indefinitely in others) is made of a material substance whose density is $1/w(x,y)$; it will be constant along each vertical line, but will in general vary from line to line. Next, pass the horizontal plane $z=\text{constant}=\log\lambda$ through the solid: it will cut a piece of matter, above the plane but under the surface, determined by the inequalities

$$\log\lambda\leqslant z\leqslant\log[p(x,y)w(x,y)]. \tag{6.1}$$

Evidently its *mass* is the integral in (5.9). It should be intuitively evident that this mass decreases continuously as λ increases, and fairly clear that it approaches zero as λ increases without limit, and lastly, that it becomes infinite as $\lambda\to 0$ (so that $\log\lambda\to-\infty$). Evidently it passes through every intermediate value (e.g., $\log\lambda$) just once. These facts are established analytically by a routine study of inequalities **[10]**. Thus for just one height of the plane—viz., for just one value of λ—*the mass of the solid equals* Φ.

This construction gives the two elements needed to determine the search: the value of λ and

the region $A(\Phi)$ to be searched, evidently being the projection upon A of the region cut from the horizontal plane by the logarithmic surface. Thus (5.8) gives the required $\phi(x,y)$, everything in it having become known. This formula also shows that this density is $1/w(x,y)$ times the length of the segment cut off from the vertical line through (x,y) between the horizontal plane and the logarithmic surface.

In many important cases the spatial variation in the coefficient of detectability can be ignored, so that we can set $w(x,y) = W$, constant. This can then be combined with the search density factor in the exponent of (4.1), (5.1), etc., so that instead of $W(x,y)\phi(w,y) = W\phi(x,y)$ we write simply $\phi(x,y)$. This amounts to *a change of units of searching effort*: Suppose that it had originally been an expected length of track $\overline{\Delta L}$ in ΔA; then since the original $w\phi$ in the exponent must be dimensionless, the new $\phi = W\Delta\bar{L}/\Delta A$ must be so likewise; hence W is of the dimension of length: It is called the *sweep width*, and $W\Delta\bar{L}$, the *swept area*—both terms corresponding to the highly simplified notion mentioned late in Section 4—detecting targets if and only if they are in this sub-area of ΔA. When $\phi(x,y)$ occurs alone in the exponents, the preceding construction reduces to a simple geometrical one, in which volumes are matched.

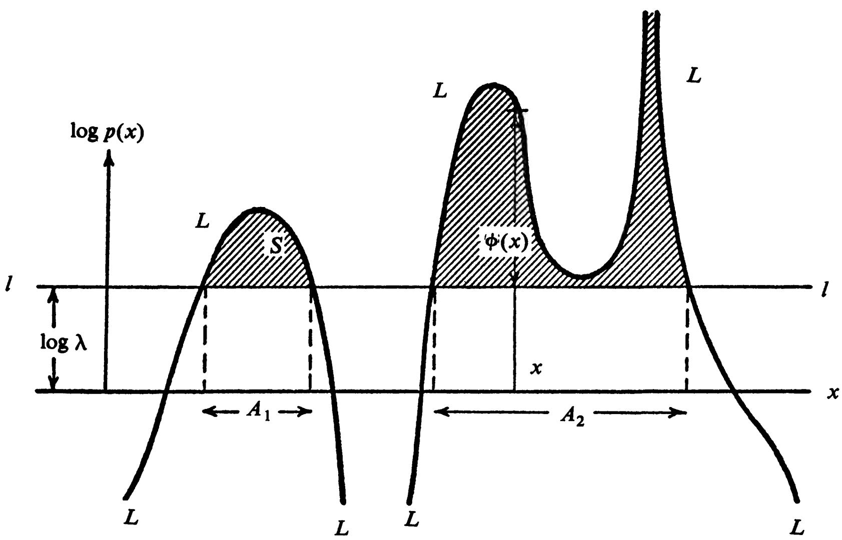

FIG. 1

Figure 1 shows the construction in the case of one-dimensional search, the x-axis replacing the xy-plane; single integrals replacing double; areas replacing volumes. The subregion $A(\Phi) = A_1 + A_2$ is determined by finding the λ making the shaded area equal to the given Φ. Then the search density $\phi(x)$ is the length cut off between the curve and the horizontal lines. No searching is done outside $A_1 + A_2$.

Figure 2 shows the corresponding plot of the a priori probability density and the construction, by truncation and re-normalization, of the a posteriori probability $\bar{p}(x)$—given failure to detect after using up Φ, an event of probability $1 - P$, where P is the shaded area in this figure.

Even in the general case of a non-constant $W(x,y)$, the above simplification can be brought about; but since it involves a non-linear change of coordinates, from (x,y) to (x',y'), it is somewhat removed from the original physical picture. Assuming, as we may in approximations to physical reality, the required regularity of our two given functions, all we have to do is to take for x' and y' a pair of regular functions of (x,y) having a Jacobian $J = \partial(x',y')/\partial(x,y) = 1/w(x,y)$. This determines the change in the two densities, which must be divided by J, i.e., multiplied by $w(x,y)$, in order to guarantee that the formal relations $p'dx'dy' = p\,dx\,dy$ and

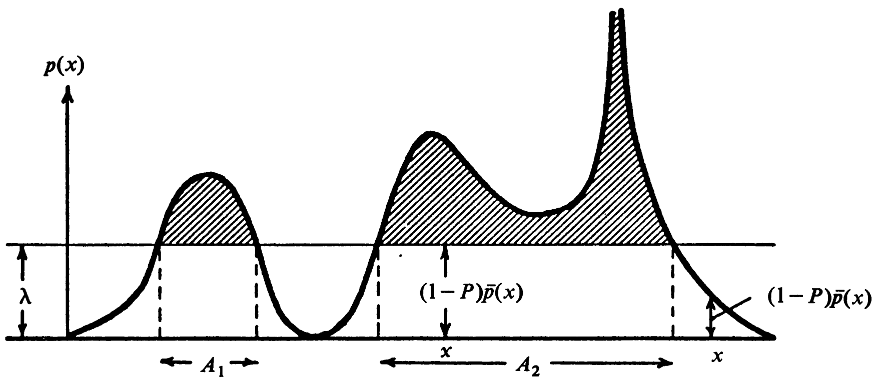

FIG. 2

$\phi' dx' dy' = \phi\, dx\, dy$ shall hold. This leads to the same expressions and simple geometrical interpretation of the preceding paragraph. It shows also how places of lowered target detectability are reflected in the optimized search in the same way as are those of lowered target probability.

Suppose that the searching effort Φ has been used up, having been distributed optimally, *but has failed to find the target*. How should any further searches be conducted? Assuming that nothing has been changed physically, the answer is given by the same distribution of the new effort $\bar{\Phi}$ as before, but starting with the "updated" density $\bar{p}(x,y)$ given by an elementary application of Bayes's formula. We describe the updating in terms of the geometrical interpretation applying directly to the case of constant $w(x,y)$ (taken as unity). We proceed in three steps, illustrated by Figure 2 in the one-dimensional case.

Step one: Using the λ and $A(\Phi)$ obtained as before for the first part of the search, construct the horizontal plane $z = \lambda$ (shown one-dimensionally in Fig. 2).

Step two: Truncate the solid under the surface $z = p(x,y)$ by the horizontal plane just constructed. This gives a flat-and-curved surface which is the graph of the function $\min\{\lambda, p(x,y)\}$; in the case of Figure 2, it lies under the shaded part removed.

Step three: "Normalize" the above function by dividing it by its integral over the whole region A. Since the volume under the original surface is unity and as an easy calculation based on the optimal search formulas shows that the probability P of detection is the volume cut off, the normalizing factor is $(1-P)^{-1}$; hence $\bar{P}(x,y) = \min\{\lambda, p(x,y)\}/(1-P)$. In Figure 2, P is the shaded area.

Now suppose that those responsible for the operation were to declare (indignantly): "If we had known beforehand that $\Phi + \bar{\Phi}$ were to be available, we might have done better than we were able to do when it was given out piecemeal: first Φ, then $\bar{\Phi}$." The answer is in the *negative*: The result would have been no better—would in fact have coincided with the piecemeal result. This is simply proved by calculating the two probabilities and observing their equality. This property is unique to the exponential detection function—among those types of search having continuous detection functions less than unity. It would be true of the "definite range law" in which $\lambda = \infty$ within range R of the searcher and $\lambda = 0$ elsewhere.

This last property has led J. M. Dobbie [5] and his successors (cf. [13]) to treat search as a continuous sequential operation, the key random variable being the amount of effort Φ used up by the time the target is detected, Φ having any non-negative value. The optimal search procedure is the one which *minimizes* the expected value $\bar{\Phi}$. It turns out that this leads at each stage to the law of distribution of effort given above.

Such considerations evoke many long-known methods in the calculus of variations, convexity, and the geometrical interpretation of Lagrange multipliers. They were recalled in the present connection in 1956 by the author [9] and later were made the basis of a systematic mathematical treatment of search by L. D. Stone [13].

In essence, we consider the set of points in the uv-plane given by $u = \Phi$ and $v = P[\Phi]$ in (5.2) and (5.1). For every "search strategy" $\phi(x,y) \geqslant 0$, good or bad. This set is convex, bounded by the u-axis and by the optimal curve (5.8), plotted against Φ, and lies in the first quadrant, extending to the right indefinitely, and asymptotic from below to $v = 1$ (except if there is an appreciable probability of the target's being in a region of undetectability: $w = 0$). We change the problem if, instead of maximizing v for fixed u, we are able to determine a "trade-off" between the former (a desirable quantity) and the latter (undesirable, as representing expenditure). Mathematically this means that we can determine constants a and b and maximize $-au + bv$, or equivalently, $v - \lambda u$. Evidently we must choose the point of tangency (u_0, v_0) of the line $v - \lambda u = \text{const.}$ to the upper boundary. Clearly λ is the Lagrange multiplier.

7. Application to the normal distribution. One of the earliest and most useful applications of the preceding results was to cases of stationary targets under conditions in which strong objective evidence existed for approximating to $p(x,y)$ by a Gaussian density $p(x,y) = K\exp[-Q(x,y)]$. Here $Q(x,y)$ is a positive definite quadratic form plus lower-order terms. In most of these cases the further assumption of a constant detectability $w(x,y) = W$ could be justified physically, entailing the simplifications described in the previous section. The graph of $z = \log p(x,y)$ is a quadric surface, the paraboloid $z = \log K - Q(x,y)$, concave downward; and its intersection with the horizontal plane $z = \log\lambda$ is an ellipse. The integration called for in (5.8) is an elementary exercise; and its explicit result, which contains λ, can be matched to Φ, thus obtaining the value of λ, and therewith the solution (5.8): No searching is done outside the ellipse; a quadratic density of search, within it.

The most usual case, to which the general one can always be transformed by a simple change of variables of integration, is the circular normal distribution of variance σ^2:

$$p(x,y) = \frac{1}{2\pi\sigma^2} e^{-r^2/2\sigma^2}, \qquad r^2 = x^2 + y^2. \tag{7.1}$$

The elliptic intersection $A(\Phi)$ reduces to the circle $r = a$. The relations between the radius a, Φ, and λ are found by integrating $\log p(x,y)$ over this circle and applying the simplified formulas developed before. We obtain

$$a^4 = 4\sigma^2\Phi/\pi, \qquad \lambda = e^{-a^2/2\sigma^2}(2\pi\sigma^2)^{-1}. \tag{7.2}$$

The first shows that the area to be searched is $2\sigma\sqrt{\pi\Phi}$, which is dimensionally correct, Φ in the present units being of the dimension of area. Finally, the law of search within the circle is given by

$$\phi(x,y) = (a^2 - r^2)/2\sigma^2, \tag{7.3}$$

while the probability of detection is

$$P[\phi] = 1 - \left(1 + \frac{1}{\sigma}\sqrt{\frac{\Phi}{\pi}}\right)\exp\left(\frac{-1}{\sigma}\sqrt{\frac{\Phi}{\pi}}\right). \tag{7.4}$$

To give a numerical application, let $\sigma = 100$ miles (half a chance of the target within 118 miles of the origin), and let the target be a floating object, e.g., a life-raft. This is to be searched for by five aircraft, flying at 130 miles an hour, each available for five hours. Finally, suppose that the "sweep width" W (double of average detection radius when flying past the target) is five miles. Thus that $\Phi = WL = 16{,}250$ square miles of total coverage is possible, but inevitably distributed with such irregularity that the exponential law (4.1) (with wz replaced by ϕ) is the appropriate approximation.

The preceding formulas give a radius $a = 121$ miles to be searched, with a mean track length per square mile $\phi W = (121^2 - r^2)10^{-5}$ at the distance r (0.15 at the center). The probability of

detection is given by (7.4) to be 0.163. If the knowledge of the target's position had been twice as precise in the sense that $\sigma = 50$ miles, the circle searched would have its radius divided by $\sqrt{2}$ (half the area), and the probability of detection would rise to 0.423—a striking improvement. If, on the other hand, with the same $\sigma = 100$ but twice the effort available, the probability would be 0.272, showing the advantage of increased accuracy over increased effort—a fact clear enough to common sense.

Naturally it is impossible to have a search performed by a moving person or vehicle that conforms precisely to any given continuous density, because of the physical limitations of motion in a realizable path. The practical value of calculation of the sort above is, on the one hand, in planning the actual layout of the paths to approach the ideal as closely as practicable; and, on the other hand, to test various plans to see how far they fall short of the optimum. Often they can be within 75 percent of it, which may be very good—in the rough world of practical operations.

8. Generalizations and conclusions. Since the first development of the theory of search in 1946 and its numerous military and civilian applications soon afterward, it has been made the subject of generalizations too numerous to list in any detail here. The greater number of these have been "purely mathematical," in the sense that various forms of the detection function $D(x,y,z)$ more general than the exponential one of Theorem I have been assumed on an ad hoc basis, usually with little analysis of the physical actualities of the search. The first generalization was made in 1961 by J. deGuenin [4] who required of $D(x,y,z)$ only that it be what he terms a "regular detection function," in the sense that its graph as a function of z be strictly concave downward, pass through the origin with a tangent of positive slope, and increase monotonically to a horizontal asymptote not higher than unity above the z-axis, as z increases indefinitely. Many detection functions established in 1946 by the present author [8, Chap. 2] have this property. Largely by analogy with these, most of the subsequent mathematical developments have been based on regular (deGuenin) functions. See, for example, the book by L. D. Stone [13], with its full references.

Unfortunately, as J. M. Dobbie [5] first pointed out, in many very important cases, such as in search by active methods, the detection function is not regular (deGuenin). He also noted (1. (c)) that it is often not even a *function* of (x,y,z), but a *functional* of $z=\phi(x,y)$. A simple case of an S-shaped detection function, tangent to the horizontal z-axis at the origin, occurs when the target is presented on an electronic scope in such a way that its recognizability increases in the course of the time that the detection device progressively builds up its image (the effect of *integration time* (see [11, §11] and [12, §7]). Thus the physical assumptions have a critical effect on the form of any mathematical theory proposed as a representation of the phenomena of search: hence our meticulous physical analysis in Part I.

There are, however, physical reasons, intimated in Section 3, for going beyond the exponential form for $D(x,y,z)$ of Theorem I. The simplest extension modifies just one item in the hypothesis of the latter: Instead of assuming that the detectability is determined by the position (x,y) of the target, it may be more realistic to assume that it depends on other elements, identifiable by a finite set of parameters (u). Then the proof of Theorem I is not altered; but its conclusion requires the presence of these parameters in (4.1); e.g., $w=w(x,y,u)$. Of course, if the values of (u) were known at each position (x,y), the problem of optimal search would coincide with the one treated before. But if these values are unknown and no probability distribution for them can be established, *the problem is indeterminate*. Finally, if there is a valid basis for such a distribution—more generally, for a joint distribution $P(x,y,u)$—the problem becomes determinate but can take two quite different forms. The first case occurs when it is possible to distribute the searching effort not only to different geographical localities but to different parts of the parameter space (u): $z=\phi(x,y,u)$. Then its optimal distribution is found exactly as before, extended trivially to the higher dimensional space of (x,y,u). Only a very few situations of this sort arise in practice.

The second possibility is that our search can only be distributed *geographically* and is unable to explore parameter space: $\phi = \phi(x,y)$. Then the formulation of the optimal search problem replaces (5.1) by a similar expression but with a multiple integral extended over the higher dimensional space of (x,y,u), whereas (5.2) is unchanged, the integral being over (x,y). The same method of Gibbs used before gives a valid result but in the form of a complicated integral equation involving the unknown $\phi(x,y)$. It is possible to rewrite the equations in the form (5.1) and (5.2) after replacing the exponential $D(x,y,z)$ by a *partial detection function* $b(x,y,z)$ in terms of a Laplace-Stieltjes transform:

$$b(x,y,z) = 1 - \int_0^\infty e^{-wz} dG(w). \tag{7.1}$$

Here $G(w)$ in general involves (x,y); it goes non-decreasingly from 0 to 1 as w goes from 0 to $+\infty$. It is determined by rather obvious formulas in terms of $P(x,y,u)$. (Its simple proof is given in [**12**, §6].) By Bernstein's theorem applied to such transforms (see Widder [**14**, p. 160]), such functions of z are not only regular (deGuenin) but satisfy infinitely many conditions besides: They are analytic and have non-negative derivatives of odd order and non-positive ones of even order.

The practical problem of finding $G(w)$ (or, equivalently, $P(x,y,z)$) has so far only been solved by guesswork, without subsequent verification. As early as 1946, statistical variations of sweep-width were examined to account for inconsistencies in wartime operational data, and both a normal and a Pearson type III distribution were suggested [**8**, Chap. 2, §9]. The latter, under its present name of the "gamma distribution," is made the subject of detailed study, resulting in elegant explicit formulas, by Stone, Richardson, and Belkin (see [**13**] for an account in (2.3) and for references). It is to be hoped that these results will be tested by controlled and reproducible observations.

To give perspective to the applications of the theories of optimal search, we recall the closing statement of Section 7: that even the most perfect solution gives an optimal $\phi(x,y)$ that can only be crudely approached by any set of searchers' paths. With so many untouched problems in the field, would not a better allocation of research effort be to study these to the point of useful first approximation, rather than to continue to refine existing theories?

In conclusion, the *operational* requirements of P. W. Bridgman [**1**], [**2**] as a *necessary condition* for linking any abstract mathematical structure to concrete reality have been conscientiously followed in this presentation. While his "operationalism" has had wide repercussions in the philosophy of science, the parts essential to our problem are that, in introducing any abstract concept or relation, the objective (physical) preconditions assumed, and the type of observation and measurement upon which the definitions of the abstract elements in the structure rest, must be set forth in all explicitness. In particular, the concept of "event" in probability must be given a meaning recognizable physically (as well as in terms of the theory of measure). The incompatibility of certain trials is an issue which has challenged the bases of probability since the work of Heisenberg [7]—and continues a challenge to all presentations of this theory, in its scientific context.

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MATHEMATICS AS AN OBJECTIVE SCIENCE

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1. Introduction. Morris Kline has written that "mathematics is a body of knowledge. But it contains no truths." [**13**, p. 9] Views of this general kind, which deny that mathematics has objective scientific content, are widely held by mathematicians and are disseminated in classrooms and in popular books such as Kline's. I believe that such views are false and that their dissemination does no good for our own or others' respect for our subject. Below I shall examine four views which, though they do not exhaust the current range of opinion in the philosophy of mathematics, are nevertheless sufficiently representative to raise what seem to me to be the main issues about the objectivity of mathematics. I shall argue that each of these views arises from an oversimplification of what happens when we do mathematics.

2. Surfacism. In order to bring out some of the features which the views I want to oppose have in common, let me begin with an imaginary analogous view in the philosophy of physics. Many of the qualities we associate with material objects—such as definite shape, hardness, color—can be thought of as qualities of their surfaces. Consider a philosopher who is misled by this simple observation and believes that *all* qualities of material objects are qualities of their surfaces. He holds, let us say, that material objects are not solid, as we usually suppose, but instead are infinitely thin surfaces. It is meaningless, on his view, to speak of the inside of a material object. Since no one would refer to his own position as "superficialism," we may imagine that our philosopher calls his view "surfacism." Asked to explain the fact that when we cut into an object we do not just find a void, our surfacist says that the edge of the knife pulls on the surface to which it is applied, thereby stretching that surface so as to create two new surfaces. Asked to give an account of a quality which is difficult to treat consistently as a quality of surfaces, such as weight, he asserts that the quality is illusory. What is actually going on, he claims, is that certain qualities of the surfaces of our bodies, or of our interactions with other surfaces, are being projected into the external world. For example, suppose we consider the case of weight more carefully. The weight of an object is really just the difficulty I have in lifting it. That difficulty must, strictly speaking, be located in those points at which the object and my body interact. Hence the weight must reside in the common surface of the object and my body. It is a gratuitous oversimplification to think of the weight as a quality of the material object in and of itself.

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We need not suppose that our surfacist philosopher is always on the defensive. He may maintain, for instance, that the conventional view first violates the principle of parsimony by creating an entirely unnecessary entity—the inside of the object—and then goes on to give that entity absurd qualities. The inside, for one example, is supposed to be material but invisible. Why should insides be so different from outsides? Whence this asymmetry? If there really is space inside material objects, would it not be more reasonable to suppose that, like the space outside material objects, the space inside is filled with air? Occasionally, we may suppose, our surfacist complains about the unscientific and superstitious character of his opponents' views. Belief in the solid inside of a material object, he asserts, is a remnant of belief in the immortal soul, which was the "solid" inside of a human being. As a matter of fact, he argues, the usual account is simply incomprehensible. Who can visualize a material object except by visualizing its surface? Who, when visualizing a material object, can visualize anything in addition to its surface?

It seems to me that the views about the nature of mathematics that I wish to discuss are forms, more or less disguised, of surfacism. Hence it will be useful for me to consider how one might refute surfacism in the pure form just described.

The purpose of having a view about the nature of material objects is to order our experiences of those objects in a way which is useful in our dealings with them. Such a view is a social artifact which serves a variety of social functions. Material objects are themselves public in character, and most of my interactions with my material environment are, directly or indirectly, also interactions with my social environment. It follows that the most important function which such a view must serve is to facilitate both those of our interactions with material objects which have public significance and those of our interactions with each other which are mediated by material objects or which concern material objects. Hence a view about the nature of material objects which is intended to be more than a debating position should satisfy the following *Principle of Objectivity*: *Anything which is practically real should be taken as objectively real.*

Let me make this clearer. When I say that an attribute like weight is practically real, I mean that the attribute plays a role, and that there exists a consensus that the attribute does play a role and should play a role, in our interactions with the objects that have the attribute. It will follow that there is at least a rough consensus on the degree or kind of presence of the attribute in a particular object. For, to repeat what I said above, our interactions with objects are generally also interactions with each other. On the other hand, when I say that an attribute is taken as objectively real, I mean that it is taken to reside in the observed object rather than in the subjective experience of the observer or in the subjective relationship between the observer and the observed object. A theory about the nature of material objects, then, is only serious if it accepts as its data all those attributes which have a commonly accepted role in our ordinary social dealings with the objects. It must take those data and unite them into a coherent account, explaining some in terms of others no doubt, but not explaining any of them away. In particular, a theory will undermine our ordinary activities, rather than support them, if it treats attributes which are important in those activities as mere subjective illusion. Of course one can find examples in which entities that formerly appeared to play a role in our practical activities were later shown not to exist. Nevertheless, in an argument about the objective reality of something whose practical reality is evident, the whole burden of proof should fall on the proponent of the negative position. After all, the simplest explanation for the apparent practical importance of an entity is that the entity actually exists and actually plays a role in our practice. In the absence of a strong argument to the contrary, then, the presumption must be that anything practically real is objectively real.

To avoid possible misunderstandings, let me consider a case in which the principle of objectivity is satisfied. An argument one sometimes hears against taking physics literally is that in the world of the physicist there is no such thing as yellow. If that were true, it would be a powerful argument. Fortunately, it is not true.

First of all, it is important to distinguish our experience of yellow from the color itself. What is relevant to our public dealings with a material object is not how it appears to this observer or that observer under these conditions or those conditions. What is relevant is the actual color of the object—roughly speaking, how it appears to a normal observer under standard conditions. Thus it cannot be the task of physics, as opposed to psychology, to give an account of our experience of yellow.

It remains, however, that physics does not take color as an ingredient in its description of the world. Nevertheless, the usual account in terms of wave lengths of light does give objective content to talk about yellow. Colors are not denigrated or explained away. They are not made to reside in our eyes or in our minds. On the contrary, our ability to deal with color is enriched. Not only does the theory account for the observed properties of colors, but it makes possible their manipulation in new ways. The physicists have even found new colors (for example, in the infrared) which we cannot see.

Thus in this case the principle of objectivity is amply satisfied. The merely private aspects of our experience of color are dismissed as subjective. The practically real color itself, on the other hand, is supplied with objective content.

The principle of objectivity, then, may be used to refute surfacism as follows. The weight of a homogeneous material object is proportional to its volume and not to its surface area. It is reasonable to conclude that the weight of the object is distributed through it. Hence the surfacist must hold that weight is merely an illusion—not objectively real. But since weight is important in our dealings with material objects, and since it can be measured in a way which is interpersonally valid, the surfacist who declares weight to be illusory thereby trivializes his theory.

In order to apply these ideas to the philosophy of mathematics, we must observe that mathematics is a public activity. It occurs in a social context and has social consequences. Posing a problem, formulating a definition, proving a theorem are none of them private acts. They are all part of that larger social process we call science. A functioning mathematician is aware of the work of other mathematicians, publishes his own work, and expects other mathematicians to take his work into account. Thus a philosophy of mathematics is closely analogous to a view about the nature of material objects. Its main function should be to facilitate the ongoing social process of doing mathematics. It follows that a serious philosophy of mathematics must satisfy the principle of objectivity. That is, it must not deny objective reality to any aspects of mathematical activity which have practical reality.

3. Formalism. No one who observes the behavior of mathematicians can fail to notice that they manipulate symbols in accordance with rules. Thus our first attempt at a philosophy of mathematics might be to hold that mathematics is the rule-governed, or *formal,* manipulation of symbols and nothing else. (The phrase "and nothing else" is the mark of the surfacist.) This view is often called *formalism.* Positions more or less like this may be found in Haskell Curry [5], Abraham Robinson [17], and Paul Cohen [4]. (The views of David Hilbert, though often called "formalism," are quite different from the position we are discussing here, since Hilbert takes at least the finite, combinatorial part of mathematics to be meaningful and true. See, for example, Hilbert [12] or Kreisel [15].) An example of a different sort is provided by some computer scientists interested in artificial intelligence. They naturally want to think that human intelligence is not in principle different from what their computing machines are doing. Thus the human brain is assimilated to a computer, theories are assimilated to programs, and thought is assimilated to the operation of a Turing machine. After all, says the formalist, what else could mathematics be? Can you imagine a mathematician working in any way other than by manipulating symbols?

To make this somewhat more concrete, let us imagine asking a formalist what he takes to be the content of the fundamental theorem of arithmetic. If he is really a strict formalist, he must reply that, standing alone, it has no content at all. The theorem is, after all, just a string of symbols. What makes us feel that it has content is only that it plays a definite role in certain

activities we engage in. It is like a frequently encountered position in chess. If we give a more precise description of our symbolic activities, say by giving a particular formal system which codifies some part of mathematics, then we can also give a precise account of the role of the fundamental theorem of arithmetic. We might specify one or more formal proofs of the theorem in our system, and we might give some examples of uses of the theorem in formal proofs of other theorems. For the formalist, however, the theorem has no meaning apart from its role in our symbolic activities. For the strict formalist, the theorem does not make any assertion about natural numbers, since for him no such objects exist.

Now I agree that mathematics almost always involves the formal manipulation of symbols. I agree that a mathematician can usually be viewed as working inside some formal system. This seems to me an important insight. There is a branch of mathematical logic whose subject is just this aspect of mathematical activity. I mean the theory of recursive functions. That theory has contributed more than any other part of mathematical logic to our understanding of the inherent limitations of mathematics. Let me state this quite strongly. I do not believe that mathematicians will ever compute a nonrecursive function, solve a recursively unsolvable problem, or work in a theory which is not recursively axiomatizable. But all of that is not to concede that human minds are algorithmic devices in the sense of recursive function theory. Rather, it is analogous to the harmless concession we might make to surfacism that no one will ever see a material object without a surface.

It is easy to understand how a philosopher who never actually did any mathematics might hold a formalistic view of its foundations. After all, what is there for him to see but the outer play of symbols? On the other hand, I must admit that I find it difficult to understand when, as happens occasionally, a creative mathematician is a formalist. Introspection shows that when I am actually doing mathematics, when I am wrestling with a problem that I do not know how to solve, then I am hardly dealing with symbols at all, but rather with ideas and constructions. Some of the hardest work a mathematician does occurs when he has an idea but is, for the moment, unable to express that idea in a formal way. Often such ideas first manifest themselves as visual or kinesthetic images. As the mathematician becomes clearer about them, as they become more formal, he may discover that they manifest considerable internal structure which is, so to speak, not yet symbolically encoded. This point is hard to discuss in a way which avoids purely psychological categories not directly relevant to the epistemological point I am trying to make. Still, mathematicians customarily talk about ideas, constructions, and proofs in a way which makes it clear that they have in mind something other than the symbols they use. Thus mathematicians may discuss whether two distinct papers embody the same idea, whether two distinct strings of symbols express the same construction, or whether two distinct lectures expound the same proof. Every mathematician knows that the same construction can be used in quite different parts of mathematics and that, if you find a new proof of an old theorem, you had better check that it is not just an old proof in a new form.

As has been customary since Brouwer, let me use the word "construction" to refer generically to all of these entities which lie behind the symbols the mathematician writes and which give those symbols life and content. I think there can be no doubt that constructions are practically real in the sense I introduced above. Mathematicians discuss them constantly, agree on their general properties, and agree that they are what is important in mathematical creation. It follows that an adequate philosophy of mathematics cannot just treat constructions as subjective illusion. Most formalist philosophers, however, either do not mention them at all or else dismiss them under some such name as "heuristics" without giving any account that would explain the properties that mathematicians agree constructions have. Indeed, the formalist cannot give a theory of constructions, since he denies they exist. For example, even if there could be a program which could recursively recognize whether or not two strings of symbols embody the same idea, the formalist could not admit that that is what the program does. What could it even mean to say that a computing machine had an idea for a proof but was having trouble formalizing it?

In order to state this argument more carefully, let me introduce the word "intuitive." In the sense that is relevant here, "intuitive" is used to contrast with the word "formal." Thus an argument may be called intuitive if it is natural and easy to follow. This is roughly the sense in which the word "intuitive" seems to be used in intuitionism. Thus an intuitive proof, in that context, is one which is unformalized, independent of symbols, and perhaps not even entirely communicable. At any rate, there certainly are constructions which are *intuitive*, in the sense that they are not formal and not symbolic, but which do have internal structure, do enable us to see new facts, and can be formalized so as to give correct proofs.

Now my argument may be summarized as follows. Intuitive constructions are practically real. They are vital to the practice of mathematics. It is of the essence of formalism that it denies their objective reality. Therefore, by the principle of objectivity, formalism cannot be an adequate philosophy of mathematics.

4. Intuitionism. If formalism must be rejected because it neglects the intuitive content of mathematics, then it is natural to make a second attempt at a philosophy of mathematics as follows. Let us hold that mathematics consists of intuitive constructions, of the formal manipulation of symbols which is their external expression, and of nothing else. This seems to me to be the essence of the view usually called *intuitionism*. It was worked out by L. E. J. Brouwer and Arend Heyting. A good introduction is Heyting **[11]**. A more recent introduction is Dummett **[6]**. Perhaps the clearest general statement by Brouwer himself is his **[3]**. A related, but definitely distinct, point of view is that of Errett Bishop **[2]**. I should say that very few of my remarks about intuitionism apply directly to Bishop's philosophy of mathematics, since Bishop has little of Brouwer's subjectivistic tendency.

It is characteristic of intuitionism that it denies the existence of any mathematical reality external to the mathematician or even of any mathematical truth beyond what the mathematician has actually proved or could actually prove. Mathematical objects exist for me only as the results of my constructions, and mathematical facts are true for me only insofar as they are the conclusions of arguments I can make. Thus the sequence of natural numbers, being infinite and hence not surveyable, is only potentially real. Statements which have so far been neither proved nor refuted, like Fermat's conjecture, have no definite truth-value. The logical law of the excluded middle, which asserts that every statement is either true or false, is rejected as inapplicable to statements about infinite sets, and indirect proofs of such statements are rejected as invalid.

To take an example, let us again consider the fundamental theorem of arithmetic. The intuitionist, unlike the formalist, does not take this to be a mere string of symbols. The theorem has a meaning. Nevertheless, he also does not take the theorem to be a truth about an externally existing domain of natural numbers. Rather he thinks of it as expressing a certain ability that we have—namely, our ability to factor an arbitrary natural number into primes and to see, given two such decompositions, that they consist of the same primes with the same multiplicities. Like the formalist, the intuitionist takes the meaning of the theorem to reside in our practice, not in any external reality to which the statement might refer.

Let us examine Brouwer's rejection of the law of the excluded middle somewhat more closely. Brouwer does not have available any concept of truth which could be used to justify, or even to explain, a truth-functional interpretation of the logical connectives. Moreover, for Brouwer it only makes sense to assert a mathematical statement as the conclusion of an intuitive proof. But a proof that either A is true or B is true ought to contain an indication as to which of the two alternatives is being proved. Otherwise we could assert the existence of a number n such that if $n=0$ then A, and if $n=1$ then B; but we would not know the value of any such number. Surely, however, we know the value of a number we have actually constructed. Thus we would be asserting the existence of a number without having constructed it. Hence a proof that the Fermat conjecture is either true or false would have to contain either a proof or a refutation of

the conjecture. Since I can supply neither, it follows from Brouwer's point of view that I am not in a position to assert that the conjecture is either true or false. Thus the law of the excluded middle is "refuted" not by finding a third possibility but by making an additional demand. An assertion is only to be considered justified if an intuitive construction can be supplied which justifies it.

As an intellectual movement, mathematical intuitionism is similar to other positions, like existentialism, which emphasize our isolation from each other and which conclude from that isolation that we are epistemically reduced to our own individual resources. That is to say, it is characteristic of all of these views that they hold that our inner experience, as such, is the only source of knowledge available to us and that they deny that our inner experience essentially entails an external reality to which it refers. In consequence, these views tend to collapse into irrationalism and solipsism. When Brouwer emphasizes the absolute freedom of the creative subject in mathematics, he is taking a stance related to that of the existentialist emphasizing the absolute freedom of that same creative subject in aesthetics, in ethics, or in politics.

Looked at in our context, however, intuitionism is a fairly typical form of surfacism. Its characteristic rhetorical gesture is to ask what a mathematician could possibly have access to other than his own constructions. Put differently, try to think something other than one of your thoughts, or try to visualize something other than one of your images.

As in the case of formalism, it seems to me important not to overlook the contributions that intuitionism has made to our understanding of the practice of mathematics. The writings of the intuitionists are a rich source of ideas about the internal process of mathematical creation. Here again there is a branch of mathematical logic devoted to trying to extract and develop the precise content of these insights. The various realizability notions, functional interpretations, Kripke structures, and the like, seem to me to give promise of a mathematical theory, perhaps yet to come, of the experience of doing mathematics.

I myself have been attracted by intuitionism. But I have gradually come to see that, in the long term, strong intuitionistic convictions undermine one's actually doing mathematics. By embracing intuitionism the mathematician is giving up the most powerful motivation for his work—the search for publicly validated truth. Mathematics, after all, is a part of science. The main purpose of doing mathematics is to discover new truths. If that conception is given up, as it is in intuitionism, then mathematics is reduced to an esoteric art form—to a kind of play. There is a sense in which intuitionism is inadequate in its own terms, for it overlooks what is introspectively obvious: that I am interested in my constructions not for their own sake but for the new truths they enable me to find. The constructions derive their significance from their epistemic role. Who would be interested in a proof that established nothing? Just as the constructions lie behind the symbols and give them their interest and meaning, so there is something behind the constructions—mathematical truth.

In this respect mathematical creation is not at all free. A mathematical argument often gives a feeling of inevitability. The concept of rigor, which plays such a great role in the mathematician's talking and thinking about his work, is a restriction on his freedom which he accepts in order that his theorems may be true and in order that his arguments may genuinely establish their truth.

Mathematical truth, unlike a mathematical construction, is not something I can hope to find by introspection. It does not exist in my mind. A mathematical theory, like any other scientific theory, is a social product. It is created and developed by the dialectical interplay of many minds, not just one mind. When we study the history of mathematics, we do not find a mere accumulation of new definitions, new techniques, and new theorems. Instead, we find a repeated refinement and sharpening of old concepts and old formulations, a gradually rising standard of rigor, and an impressive secular increase in generality and depth. Each generation of mathematicians rethinks the mathematics of the previous generation, discarding what was faddish or superficial or false and recasting what is still fertile into new and sharper forms. What guides

this entire process is a common conception of truth and a common faith that, just as we clarified and corrected the work of our teachers, so our students will clarify and correct our work.

In order to formulate a more careful argument, I need to say a few words about the concept of *rigor*. It is widely believed that this notion changes. Arguments that seemed rigorous to Euler seemed inadequate to Cauchy. Arguments that seemed rigorous to Cauchy seem to us to contain obvious gaps. But it is not really the case that the concept of rigor has changed—only the standard of rigor. That is to say, a rigorous argument is always an argument which suffices to establish the truth of its conclusion. As our insight grows, we see that more is required to establish truth, and therefore arguments that once seemed rigorous are now seen to have gaps. But the concept of rigor itself has not changed since at least the time of Euclid.

More is true than that the concept of rigor presupposes the concept of truth. Actually, when we evaluate a mathematical argument, we do not check to see whether it accords with some set of rules taken, let us say, from a logic text. Rather, we try to determine whether the argument works—that is, whether it convinces us, and ought to convince us, of the truth of its conclusion. Thus the concept of mathematical truth is directly involved in the practice of mathematical rigor. It functions as an indispensible ingredient in the very criterion of rigor.

Now I may formulate my argument against intuitionism as follows. Mathematical truth is practically real. Indeed, without the practical reality of mathematical truth, there would be no such thing as mathematical rigor. But it is of the essence of intuitionism that it denies the objective reality of mathematical truth. Therefore, by the principle of objectivity, intuitionism cannot be an adequate philosophy of mathematics.

5. Logicism. If we reject intuitionism because it neglects mathematical truth, then we may be led to make a third attempt at a philosophy of mathematics as follows. Let us hold that mathematics consists of certain truths, of the arguments that establish these truths, of the constructions underlying those arguments, of the formal manipulation of symbols that expresses those arguments and truths, and of nothing else. It seems to me that this is the central thrust of what has traditionally been called *logicism*. Views of this sort have been advocated most prominently by Gottlob Frege and by Bertrand Russell. Classical statements of logicism may be found, for example, in Frege [7] or Russell [**18**]. A somewhat more recent statement is in Hempel [**9**].

A logicist, unlike a formalist or an intuitionist, would take the fundamental theorem of arithmetic as a truth whose content is quite independent of our activity. For the logicist, however, there are no natural numbers which exist as independent entities and which happen to have the property expressed by the theorem. Instead, the theorem is to be understood on the basis of a long sequence of definitions. When all the expressions used in the theorem are expanded out in accordance with these definitions, then, according to the logicist, the theorem will turn out to be merely a very complex logical truth. The fundamental theorem of arithmetic, for a logicist, is on a par with an assertion like, "if all A's are both B's and C's, then all A's are C's."

What the logicist denies is that there is any subject matter for mathematical truths to be about. Mathematical terms, for the logicist, do not refer—or at least do not refer uniquely. It follows that mathematical truths are not true by virtue of successfully describing any actual state of affairs. They are empty of factual content. Hence mathematical truths must be true solely by virtue of their own internal structure and of their relations to one another. That is the way in which logical truths are true: Hence the logicist thesis that mathematics is merely logic. In practice, of course, logicists have tended to use the term "logic" rather loosely, sometimes including all of set theory under that name. But the basic idea is always to deny that mathematical assertions have factual content—that is, to deny that their truth rests on anything outside of the structure of the mathematical statements themselves. That is presumably also what Kline means to deny by the words quoted at the beginning of this essay. (For an explicit statement of Kline's views, see [**13**, pp. 424–431]. For more details, see Kline [**14**, pp. 1028–1039].)

Logicism motivated much of the early work in mathematical logic. I think that logicism has made greater contributions than any other philosophy of mathematics to our understanding, not so much of the practice of mathematics, but of its foundations. The desire to reduce all of mathematics to "logic"—that is, to merely conceptual reasoning—has provided a strong impetus to simplify and unify the basic mathematical notions and to find and make explicit the fundamental principles upon which mathematics is based. Moreover, logicism is still making such contributions today. Much of what is now called proof theory can be seen as an effort to view larger and larger parts of mathematics as consisting of logical truths by extending the concept of logic in various directions. To mention only one example, the past twenty-five years have seen the development of a theory of infinitely long formulas and proofs so as to give a "logical" analysis of arithmetic and of increasingly extensive fragments of mathematical analysis.

Unlike formalism or intuitionism, logicism does provide an adequate account of a significant part of actual mathematical practice. Much of mathematics really is just logic. We reason from clearly formulated premises, trying to find an argument that will settle some previously formulated question. I doubt, however, that any work a mathematician would consider deep can be accounted for in terms the logicist would accept. Every mathematician knows that his best work is based not on mere reasoning but on the characteristic kind of insight he calls "intuition." In this sense, the word "intuition" refers to a faculty by which the mathematician is able to perceive properties of a structure which, at the time, he is not in a position to deduce. This perception can be trained, and it is often quite reliable. Sometimes, when trying to work deductively, one feels like a man trying to find his way around an unfamiliar room in the dark. The mind is full of details that fail to cohere into a pattern. But then, either gradually or suddenly, one's eyes adjust to the dark, one sees dimly how the room is arranged, one knows about chairs one has not yet bumped into, and one is able to get about comfortably. It is an everyday occurrence that a mathematician "knows intuitively" that thus and so must be the case but does not have the vaguest idea how to go about proving it. Often, of course, he is wrong. But far more often than not he is right. Certainly, if I respect a particular mathematician and if he has had extensive experience with a particular structure, I will be willing to rely on his intuitions about the structure even in the absence of a proof—not absolutely, but to a very large extent.

Let me say at once that I am not urging the existence of an occult faculty whereby we have direct knowledge of platonic objects. Rather, I think that the mathematician's intuition is a special case of the general human ability to recognize patterns or, more specifically, to synthesize complex structures from scattered cues. Thus I think the mathematician's intuition about a particular structure is simply the result of long experience with that structure. It is not different in kind from a carpenter's "feel" for his wood. The fact is that mathematicians are able to arrive at more or less reliable conclusions about mathematical objects without having to deduce those conclusions. Indeed, mathematical creativity is much more a matter of intuition than it is of logic. (For essentially the same view, see Wilder **[19]** or Resnik **[16]**.) It follows that a logicist account of mathematics cannot be adequate.

But what is missing? The logicist holds that mathematics is a body of truths that are not about anything. They are true just by virtue of their internal logical structure, not by virtue of any external objects to which they refer. But if that were true, then the phenomenon of mathematical intuition would be incomprehensible. For if the logicist is right, then there are no structures for the mathematician to become familiar with or to have insight into.

An interesting special case of this difficulty is the problem, from a logicist point of view, of the status of axioms. A principle which is neither a logical truth nor deduced from antecedently accepted principles is not being accepted merely by virtue of reasoning. Logicists, therefore, often deny that such principles are being accepted at all. Thus they tend to think of geometry, for example, as a hypothetical discipline. If physical space satisfies the axioms, then it satisfies the theorems. (For this opinion see the references to Kline above, or see Hempel **[10]**.) But, as a matter of fact, we have a clear intuition of Euclidean space, and the theorems of Euclidean

geometry are outright true about that structure. It is generally held that the earliest geometrical knowledge was arrived at empirically. If so, then that knowledge does not have a hypothetical character. The non-Euclidean geometries only show the logical consistency of denying the parallel postulate. They do not show that the parallel postulate is false. The general theory of relativity shows that certain esoteric observations are well described by treating space-time as a four-dimensional manifold of non-constant curvature. It may follow from this, though I am not sure that it does, that the space of our intuition does not correspond perfectly to physical space. It certainly does not follow that we do not have a clear spatial intuition. Moreover, Euclidean geometry remains an excellent description of the space we actually live in and actually experience. It is not as though the use of figures in geometrical demonstrations were derivative from purely logical proofs based on the axioms. On the contrary, some of the axioms, such as the axioms of order, are so evident to the intuition that the need for them was not noticed until the nineteenth century. It seems implausible that all the geometers before Moritz Pasch were guilty of the same systematic logical errors. It seems much more likely that they were engaged in some activity other than deducing the logical consequences of a set of axioms. I think they were studying space.

Let me summarize the argument. Mathematical intuition is practically real. It is only comprehensible as a non-deductive insight into structures external to the mathematics itself. Hence such external mathematical structures are practically real. But it is essential to logicism that it denies the objective reality of any such structure. Therefore, by the principle of objectivity, logicism cannot be an adequate philosophy of mathematics.

6. Platonism. Logicism, in other words, must be rejected as an incomplete philosophy of mathematics because it omits the objects that mathematics is about. Thus we may make a fourth attempt at a philosophy of mathematics as follows: Mathematics consists of truths about abstract structures existing independently of us, of the logical arguments that establish those truths, of the constructions underlying those arguments, of the formal manipulation of symbols that expresses those arguments and truths, and of nothing else. This is the philosophy of mathematics that I think ought properly to be called *platonism*. Its most distinguished contemporary proponent was Kurt Gödel. (For example in his **[8]**.)

A platonist would interpret the fundamental theorem of arithmetic literally. For the platonist there are such things as natural numbers existing independently of us, and it is as a matter of fact true that they are all uniquely decomposable into prime factors.

The most characteristic expression of platonism within mathematical logic is model theory. This discipline is the study of the semantic content of mathematical theories. Of course formalism, intuitionism, and logicism all deny that mathematical theories have semantic content. The central problem of model theory is the question of what properties of structures can be expressed in particular languages. This question only arises if structures are assumed to exist and to have properties independently of their description.

Let me try to summarize quickly the picture of mathematical activity that platonism offers. The mathematician, on this view, is confronted by a wide variety of abstract structures which themselves precede his mathematical activity. He does not create these structures; he finds them. In the course of his training, and then as he develops his powers, he forms and refines an intuition about these structures. Typically, of course, he will have much more insight into some of them than into others. His intuition is formed by the truths about the mathematical world that have been discovered by his predecessors and by his colleagues, and then his intuition, in turn, enables him to find new structures and to make new conjectures about the old structures. In order to verify these conjectures, to answer the questions that occur to him, he performs constructions, makes arguments, defines new concepts. These constructions, in turn, get expressed in mathematical English, are bolstered by computations, are made rigorous and formal. Thereby they are made publicly accessible and verifiable and become part of the larger social dialectic through which mathematics develops.

This seems to me a fairly satisfactory account of what the pure mathematician is doing. Indeed, I think that most contemporary mathematicians, even if they have not bothered to articulate it for themselves, would accept some variant of this view. So satisfactory is platonism that very few recent mathematicians or philosophers of mathematics have felt any need to go beyond it. Just in the past few years, however, there have been signs of discontent. To indicate their source, let me pause for some brief historical remarks.

In the eighteenth century, mathematics was considered a science distinguished from the other sciences only in being more certain and more fundamental. Its special province was the laws governing space and quantity. In the course of the nineteenth century, this conception of the nature of mathematics was strongly undermined. First the non-Euclidean geometries were used to deny the existence of a unique spatial structure for our intuitions to be about. Then analytic geometry was used to undercut the view that there was an intuition of space at all apart from our intuition of the numerical continuum. The end product of this development is the contemporary mathematician who tells his undergraduate students that by three-dimensional Euclidean space he *means* the set of all ordered triples of real numbers. Obviously, that is not what Euclid meant. Toward the end of the nineteenth century, even the intuitive conception of quantity or magnitude was replaced, at least officially, by the purely conceptual structures introduced by Weierstrass, Dedekind, and Cantor. Again, a contemporary mathematician is likely to tell his students that by a real number he *means* a Dedekind cut. Obviously, that is not what Euler meant.

One effect of these changes was to produce what might be called a foundational vacuum—a situation in which mathematicians were without any systematic account of the nature of the structures they were dealing with. Axiomatic set theory rushed in to fill this void. The set-theoretic view of foundations, however, is platonism in its most narrowly reductionistic form. All the objects of the set-theorist's world are abstract. Even if individuals are allowed, and they are usually excluded, these individuals are taken to have neither internal structure nor intensional relationships. They are mere abstract points. Thus the reduction of all of mathematics to set theory entails a narrowing of the subject matter of mathematics so as to exclude all of concrete reality.

For about two generations axiomatic set theory was a great success. I think there can be little doubt that set theory provides an elegant and convenient framework within which to do pure mathematics. It is wonderfully simple in conception, almost never gets in the way of mathematical practice, gives smoothly reassuring answers to questions like "But what are numbers, really?" and provides a wealth of interesting structures of which no one before Cantor could have dreamed.

In the past decade, however, set theory has been undermined roughly in the same way that geometry was undermined about a hundred years earlier. The independence results, the proliferation of large cardinal axioms, and the construction of increasingly bizarre models for set theory have made mathematicians realize how weak their set-theoretic intuition actually is. In the absence of new insight, the views of set-theorists begin to diverge. Some still follow Cantor in thinking the continuum hypothesis plausible, but others follow Gödel in believing more and more strongly that it must be false. It is becoming truistic that we need a new concept, one more fundamental than that of a set. Unfortunately, no one can imagine where to look for such a concept.

None of this is incompatible with a sufficiently liberal platonism. Increasingly one hears the suggestion that there is not just one set-theoretic universe, but many. You work in a world in which the continuum hypothesis holds, and I will work in one in which Martin's axiom holds but the continuum hypothesis fails. He will work in a universe containing a measurable cardinal, and she will work in one in which, since all sets are constructible, a measurable cardinal is impossible. These are all just different structures, all equally entitled to be considered interesting and worthy of study. Where is the problem?

The problem, of course, is the same as it was in 1890. How do these different structures

interact? What are they? What are the laws that govern the mathematical universe as a whole, if none of these set-theoretic "universes" can any longer be regarded as including all of the structures mathematicians concern themselves with? None of these questions have generally accepted answers. I think it is out of despair at this situation that some mathematicians retreat to formalism, intuitionism, or logicism—positions from which such questions cannot arise.

Let me put the problem differently. It seems to me that mathematics can only flourish if there is a common conception of what we are about, if there is an agreement that the different structures we study are aspects of one reality. Without a foundational consensus, it seems to me, mathematics will tend to break apart into schools.

Actually, not only is set theory tending to split into pieces, but mathematical platonism itself is the result of a split in the larger structure of science. The traditional view of the nature of science, for example in the time of Newton, was that there is only one reality and therefore only one science. On this view the several special sciences—mathematics, physics, chemistry, biology—share a common reality but ask different questions about it and use different methods to study it. Of course, each special science will reveal its own particular aspect of the world; it remains a fundamental assumption of science as traditionally conceived that these various aspects are complementary, mutually illuminating aspects of one world. As a matter of fact, most branches of mathematics cast light fairly directly on some part of nature. Geometry concerns space. Probability theory teaches us about random processes. Group theory illuminates symmetry. Logic describes rational inference. Many parts of analysis were created to study particular physical processes and are still indispensable for the study of those processes. The list could be extended almost indefinitely. From the point of view of the platonist, however, only pure mathematics is really mathematics. For, according to platonism, the objects which mathematics studies are necessarily abstract. How can the theory of finite groups tell us about the structure of crystals if the only groups we consider are built up out of sets of sets of sets?

When the foundations of mathematics became completely abstract and ceased to have anything to do with the world of the senses, the connection between mathematics and the other sciences became obscure. Recently, as economic circumstances have forced mathematicians to look around for new means of support, this divorce of mathematics from the other sciences has ceased to be a matter for pride and become a matter of concern. Set theory, however, provides no clue as to how a reconciliation with the rest of science is to be effected.

Thus I think that mathematical platonism is again a form of surfacism. It is a practical reality that our best theorems give information about the concrete world. It is a practical reality that there is no clear boundary between pure and applied mathematics. There is only one science. It follows from the principle of objectivity that an adequate philosophy of mathematics would identify the objective content of these facts. Such a philosophy of mathematics would be only one chapter in a larger philosophy of science. That philosophy would make it clear in what sense there is only one objective world and how it is that the objects studied by the mathematician, many of which are not realized in physical reality, can nevertheless be seen as part of that world. Unfortunately, that philosophy has yet to be formulated.

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WHAT IS MATHEMATICS?

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1. Introduction. Mathematics is sometimes defined in a strictly mathematical way by saying, "Mathematics consists of everything which can be formulated and proved by means of the language and axioms of ZF." Here, ZF stands for Zermelo and Fraenkel; any other axiom system of set theory would of course have done just as well.

How good is this definition? In the sense that all of classical mathematics, except perhaps for some material on the fringes of category theory, can be developed in terms of ZF, the definition is pretty good. But how useful is it? We would like, from any definition of mathematics, to obtain at least a hint of how one might answer the question, "Why is mathematics free of contradictions?" The above definition turns this question into: "Why is ZF free of contradictions?" Since no one knows how to prove the consistency of ZF, the above definition of mathematics is not very useful. But worse, this definition suffers from a serious defect. Namely, it flatly denies that intuitionism is part of mathematics. The author does not accept the intuitionistic thesis that intuitionism is all of mathematics and that anything that is not intuitionism is meaningless (see Section 6). On the other hand, he feels equally strongly that intuitionism is part of mathematics and that any definition that denies this is wrong.

From the above discussion emerge the following three necessary criteria a useful definition of mathematics must satisfy. (1) It must recognize all of classical mathematics as mathematics. (2) It must give some hint as to why mathematics is free of contradictions. (3) It must recognize intuitionism as a branch of mathematics. Is such a definition possible?

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There are several reasons why a strictly mathematical definition, like the one in the opening sentence of this paper, will never fulfill these three criteria. One reason is that it is known from Gödel's work [4] that a strictly mathematical proof of the consistency of mathematics is not possible. If mathematics alone cannot provide a proof of consistency, where then should we turn for help? The author is convinced that we must seek help from philosophy. And if philosophy is necessary to prove the consistency of mathematics, philosophy is equally necessary to give a definition of mathematics that satisfies the above three criteria. Let us say more precisely what the role of philosophy should be in such a definition.

The basic trouble with the "ZF-definition" of mathematics is that it describes only the end result of mathematical activity, namely, the set of mathematical theorems. In fact, that definition says that a theorem is a mathematical theorem if and only if it can be formulated and proved by means of the language and axioms of ZF. The definition says nothing about the mathematical activity that produces these theorems. Modern mathematical logic points strongly to the fact that, as long as we study only the set of mathematical theorems and shy away from studying mathematical activity, the problems that beset the foundations of mathematics will not be solved. However, the study of mathematical activity cannot be entirely mathematical but will have to rely heavily on philosophy. It is precisely when we make the step of adding the study of mathematical activity to the study of mathematical theorems that philosophy comes in. It is important to make this step before one attempts to define mathematics. The reason is that a definition of mathematics, of which mathematical activity is an integral part, rather than a definition in terms of only the set of mathematical theorems, at least has a chance to satisfy the above three criteria and thereby be useful. It is the purpose of this paper to attempt such a definition.

2. The world of realities. When we practice mathematics, we always have a world of realities in front of us. This world depends on the branch of mathematics being practiced, be it Euclidean geometry, number theory, set theory, or what have you.

EXAMPLE 2.1. *Three-dimensional Euclidean geometry*. The world of realities that underlies this branch of mathematics consists of the space that surrounds us, its points, lines, planes, spheres, etc.

EXAMPLE 2.2. *Number theory*. In this paper, "number theory" means the study of the natural numbers $0,1,2,3,\ldots$. When we do number theory, the world of realities consists of the natural numbers, the successor function, the operations of addition and multiplication, the relation of "greater than," etc.

EXAMPLE 2.3. *Set Theory*. Here the world of realities consists of the "naive" sets and the "naive" membership relation $\in$.

The first step in the practice of any branch of mathematics always consists in analyzing the underlying world of realities as thoroughly as possible. This world is usually perceived by us directly and, although it can and should be *analyzed philosophically*, it can usually not be *defined mathematically*. Let us reflect for a moment on the difference between mathematical definition and philosophical analysis.

Mathematical definitions are always relative in nature in the sense that they define in terms of previously accepted notions. And when a mathematical definition uses the absolute minimum of such previously accepted notions, these notions themselves cannot be defined mathematically any longer. They can only be given philosophical justification and it is precisely by ignoring these philosophical justifications that mathematical definitions achieve their vaunted rigor. In contrast, a philosophical analysis of an idea does not have such a relative character. In such an analysis, everything that is in the least composite is analyzed into its constituents until constituents are reached that are clearly not composite any longer. Since "to analyze" is not the same as "to define," we follow Bertrand Russell by referring to "philosophical analysis" rather than "philosphical definition" [7, pp. 111, 112]. In particular, since any useful definition of mathematics will have to be philosophical in character (Introduction), we will from now on

refrain from saying "definition of mathematics" and say "analysis of mathematical activity" instead.

EXAMPLE 2.4. *Three-dimensional Euclidean geometry*. When Euclid said, "A point is that which has no part," "A line is breadthless length," etc., he was giving a philosophical analysis of the world of realities underlying Euclidean geometry. He could not define points and lines mathematically in terms of previously accepted notions, since there were no such notions to be found. The fact that philosophical analyses lack the reassuring rigor of mathematical definitions should not cause the mathematician to look down on them. These analyses must be the starting point of the practice of any branch of mathematics. Who can have confidence in the axiom that a unique line passes through two distinct points unless a satisfactory description of points and lines has been given first?

EXAMPLE 2.5. *Number theory*. This branch of mathematics must begin with an analysis of the idea of natural number. Even if the plan is to define these numbers later axiomatically, say by means of the axioms of set theory, it remains an absolute necessity to say first what one is trying to define. Examples of philosophical analyses of the idea of natural number are to be found for instance in elementary school teaching and in intuitionistic mathematics.

EXAMPLE 2.6. *Set theory*. Who can believe in the axioms of set theory unless it has first been made clear what sets are and what the membership relation $\in$ signifies? When Cantor **[1]** said things like, "By a set we are to understand any collection into a whole of definite and separate objects of our intuition or our thought," he was making a philosohpical analysis of the world of realities underlying the theory of sets.

The word "realities" in the phrase "world of realities" does not signify that this world necessarily consists of objects in the physical world outside of us as in the case of three-dimensional Euclidean geometry. These objects may be mental constructs, as in the case of number theory. It is also well known that one and the same branch of mathematics can often be based on different worlds of realities. An example is the theory of complex numbers for which many such worlds have been constructed. One of them is the set of ordered pairs of real numbers, added and multiplied in the usual way. Another one is the familiar representation of complex numbers by vectors in the Euclidean plane, and there are of course several others.

The evolution of non-Euclidean geometry seems to contradict our statement that the practice of a branch of mathematics always begins with an analysis of the underlying world of realities. After all, Gauss, Bolyai, and Lobachevski had developed hyperbolic geometry well before Poincaré, Beltrami and Klein constructed their various worlds of realities for this geometry. This seeming contradiction will be resolved in Section 5.

Finally, logicians may wonder why the author is using the phrase "world of realities" instead of the standard term "model" of mathematical logic. The reason is that, although a world of realities is often a model in the logician's sense, this in not always the case. For example, the world of realities of intuitionism (Section 6) is not such a model.

3. Truth. Suppose we have decided to study a certain branch of mathematics and are trying to analyze philosophically its underlying world of realities W. How far should we push this analysis? What should we be able to gain from it?

The analysis should give us enough insight into W so that we can investigate W mathematically and not only philosophically. Section 5 will try to say what it means to investigate W mathematically. Right now we point out that, in order to make a mathematical investigation at all possible, the philosophical analysis of W must first give us two things: (1) the domain of the "truth predicate" of W and (2) a sufficient amount of "intuitively evident sentences" concerning W. Let us look closer at both.

(1) *The truth predicate*. Some sentences make meaningful statements about W, and these sentences are either true or false but never both. Let us call a sentence which makes a meaningful statement about W a **declarative** W**-sentence** (dW-sentence). The **truth predicate** of

W is the function that assigns the number 0 to each true dW-sentence, and 1 to each false one. Hence the domain of this function is the set of dW-sentences and the philosophical analysis of W must tell us what this domain is. More precisely, this analysis must be profound enough to enable us to say of every given sentence whether it is a dW-sentence or not. This does of course not mean that, having recognized a dW-sentence, we can necessarily decide whether it is true or false. It means only that we can recognize dW-sentences, i.e., those sentences which make statements about W which are either true or false.

(2) *Intuitively evident sentences*. In order to be able to investigate W mathematically, the philosophical analysis of W must also enable us to recognize *some* dW-sentences as true sentences. We shall call a dW-sentence whose truth is evident from the philosophical analysis of W alone, an **intuitively evident dW-sentence** (eW-sentence). In order to investigate W by classical mathematics, the set of eW-sentences cannot be empty (Section 4). However, in the case of intuitionism, the set of eW-sentences is empty (Section 6).

EXAMPLE 3.1. *Three-dimensional Euclidean geometry*. The philosophical analysis of 3-space must enable us to recognize such sentences as "The three bisectors of a triangle pass through a point" as dW-sentences. The analysis must also convince us of the truth of at least the familiar Euclidean axioms.

EXAMPLE 3.2. *Number theory*. The philosophical analysis of the world of numbers must tell us that such sentences as "0 is the smallest natural number" are dW-sentences. The analysis must also convince us of the truth of at least the Peano axioms.

EXAMPLE 3.3. *Set theory*. The philosophical analysis of the universe of sets must enable us to recognize such sentences as "There is a set without elements" as dW-sentences. The analysis should also convince us of the truth of at least the axioms of ZF.

4. Proof. Suppose the world of realities W underlying some branch of mathematics has been worked through philosophically to the point where its set of dW-sentences and a goodly amount of eW-sentences are in our possession. We are then ready to start mathematics proper.

What does it mean to investigate W mathematically? It means to choose an appropriate method of proof and then *to prove* that certain dW-sentences that happen to interest us are true. A proof method is a method that enables one to deduce from a given set of true dW-sentences another true dW-sentence. There are many proof methods—for instance, the many classical ones, i.e., those that are the subject matter of mathematical logic except intuitionistic logic. When applying a classical proof method, one always starts with some eW-sentences, applies the proof method to them, and in this way tries to deduce the dW-sentence that is of interest to us. This is why the application of a classical proof method requires a nonempty set of eW-sentences. See, for instance, [3] for the classical proof methods. The intuitionistic proof method is different and will be discussed in Section 6.

Suppose then that an appropriate proof method P, classical or otherwise, has been chosen in order that we may investigate the given world W mathematically. We then have, besides the truth predicate of W, also its proof predicate. This means that a dW-sentence is not only true or false but not both, but is also provable or not provable but not both. There are many interconnections between these two predicates, the most important being that every provable sentence is true. As is well known, there are in general true sentences that are not provable.

It is of the utmost importance for the philosophy of mathematics to realize the difference in character between the truth predicate and the proof predicate. The truth predicate is not man made. It is intrinsically built into every world of realities, and the best man can hope for is to observe its general properties, such as the law of the excluded middle. This is the law which says that, if σ is a dW-sentence and we use customary logical notation, then $\sigma \vee \neg \sigma$ is always true.

The proof predicate, on the other hand, is entirely man made. Its properties depend on the chosen proof method. For instance, the classical proof methods are all artfully designed in such a way that their corresponding proof predicates approximate the truth predicate as closely as

possible. Hence, as part of this design, we craftily arrange things so that the law of the excluded middle holds for every classical proof method. This means that we see to it that, whenever σ is a dW-sentence, then $\sigma \vee \neg \sigma$ is always provable. If, however, the purpose of the classical proof methods had not been to approximate the truth predicate a closely as possible, man could easily have constructed proof methods that do not satisfy the law of the excluded middle. This is in fact what happens in intuitionism (Section 6).

There is hence never anything mysterious about it when someone presents us with a proof method that does not satisfy the law of the excluded middle or any other familiar law of logic. The laws obeyed by a proof method are man made and can be changed at will. In contrast, the laws obeyed by the truth predicate are immutable and cannot be changed by man. Man can at best observe them with the help of philosophy.

The necessity of separating truth and proof is beautifully argued in Tarski's inspiring article Truth and Proof [**8**].

5. Mathematical activity. We can now analyze mathematical activity. It is complex and consists of two constituents, each of which can be further analyzed. The first constituent consists of the philosophical analysis of a world of realities W. This analysis must be thorough enough to enable us to recognize dW-sentences and to give us a sufficient amount of eW-sentences. The second constituent consists of choosing an appropriate proof method and using it to prove the truth of interesting dW-sentences. Mathematical rigor is present only in the activity of the second constituent. This is why mathematicians often ignore the first constituent when discussing their science. The various worlds give rise to the various branches of mathematics. If the world W is such that one of the classical proof methods (Section 4) is appropriate, the resulting branch of mathematics belongs to classical mathematics.

In this light, let us now examine non-Euclidean geometry, since, as mentioned in Section 2, the evolution of this geometry seems to contradict our claim that philosophical analysis of a world of realities is the first constituent of mathematical activity. We shall concentrate on plane hyperbolic geometry, but analogous remarks can be made about the other non-Euclidean geometries as well.

Plane hyperbolic geometry arose from the study of plane Euclidean geometry whose world of realities and proof method were well established. In the latter geometry, the fifth parallel axiom of Euclid is an eW-sentence and the classical question was: "Can the fifth parallel axiom be proved from the other axioms?" In order to show that the answer to this question is negative, Gauss, Bolyai and Lobachevski all had the idea of replacing the fifth parallel axiom by the new axiom, "Through a given point not on a given line, more than one line can be drawn not meeting the given line." The resulting hyperbolic geometry was then investigated axiomatically. Where was the world of realities for hyperbolic geometry from which, supposedly, all mathematical activity springs?

As long as these axiomatic investigations were not backed up by an appropriate world of realities, they could not be considered as constituting a new branch of mathematics, but had to be considered as strictly axiomatic investigations arising from the world underlying plane Euclidean geometry. Since an eW-sentence of that world (Euclid's fifth parallel axiom) had been replaced by a sentence false in that world, the possibility that a contradiction would turn up and show the new geometry to be no geometry at all hung over these investigations like a sword of Damocles. The sword was removed only when Poincaré, Beltrami, and Klein found appropriate worlds for the new set of axioms, and only then could one claim that a new branch of mathematics had been born.

In general, strictly axiomatic investigations arising from replacing some eW-sentences of a given world of realities W by sentences false in that world must be considered as belonging to the same branch B of mathematics to which W gave rise. These axiomatic investigations can only be considered as constituting a new branch of mathematics, different from B, after an

appropriate world of realities has been found for the new axioms. Mathematical activity always springs from a world of realities.

6. Intuitionism. This section is meant for readers who know intuitionism. Several of our conclusions will be controversial, but whenever this happens we will say so explicitly.

We begin by locating in intuitionism the two constituents of mathematical activity (Section 5). For intuitionism, the world of realities W_i consists of those objects that are the result of a "mental construction" (mc). Examples of such objects are sequences defined by a law, choice sequences, spreads, and species; see, for example, [2] or [5] for the meaning of "mental construction."

The declarative W_i-sentences (dW_i-sentences) can be clearly recognized. For instance, if S is a species and c some object of W_i whose mc is independent of the mc of S, then the sentence "c is a member of S" is a dW_i-sentence. And here comes the first controversial conclusion: For us, every declarative sentence concerning any world of realities and in particular every dW_i-sentence is either true or false but not both. Hence for us, the truth predicate of W_i stands before us just as bright and shiny as the truth predicate of any other world of realities. The intuitionist, on the contrary, finds this truth predicate meaningless. We shall refer to this truth predicate as "the classical truth predicate."

Turning now to the proof method P_i the intuitionists select, they use the same mc's to construct proofs as to construct the other objects of W_i. Every completed intuitionistic proof is itself an object of W_i, but proofs can be clearly distinguished from the other objects of W_i. For example, the species of the real numbers is an object of W_i which is not a proof; but the mc which proves the fan theorem constructs a proof. A proof always refers to a given dW_i-sentence σ and either proves σ or proves its negation $\neg\sigma$; see [2] or [5] for the intuitionistic meaning of proving $\neg\sigma$. In short, the proof method P_i consists of those mc's which construct the special objects of W_i one recognizes as proofs.

We said in Section 4 that a proof predicate is associated to every proof method. This means in the present case that every dW_i-sentence is either provable or not provable but not both. This is again a very nonintuitionistic attitude. It is based on a platonistic realm of proofs, something which is denied by the intuitionists.

The intuitionists define their own notions of Truth and Falsity which, in order to distinguish them from the classical truth predicate of W_i (above), we spell with capital letters. For them, a dW_i-sentence σ is True if and only if σ has been proved, and σ is False if and only if $\neg\sigma$ has been proved. In particular, there are no eW_i-sentences since there is no Truth without proof.

We have now described the world of realities W_i, the set of dW_i-sentences, the (empty) set of eW_i-sentences and the proof method P_i of intuitionism. Hence in the case of intuitionism, the two constituents of mathematical activity stand clearly before us and we recognize intuitionism as a branch of mathematics. It is the branch that studies the objects constructed by mc's. Intuitionists cannot agree with this conclusion either, since, for them, intuitionism is all of mathematics.

The author feels that it is an error in philosophy to define Truth and Falsity the intuitionistic way (above). The classical truth predicate is built into every world of realities W as soon as its dW-sentences can be recognized. Anything that can be defined only after a proof method for W has been selected does not deserve the name of "truth." All mysteries of intuitionistic logic disappear if this error is not committed.

For instance, there are dW_i-sentences σ galore such that it cannot be guaranteed that the dW_i-sentence $\sigma \vee \neg\sigma$ can be proved by P_i ([2] or [5]; $\vee$ and $\neg$ denote of course the intuitionistic "or" and "not," respectively). There is nothing mysterious in this; it is simply caused by the fact that the selected proof method P_i is not made in order to approximate the classical truth predicate of W_i as closely as possible. This would indeed be impossible since this predicate does not even make sense to intuitionists. But if we now commit the philosophical error in question,

the lucid statement "In general, it cannot be guaranteed that $\sigma \vee \neg \sigma$ can be proved by P_i" becomes the mysterious statement "In general, it cannot be guaranteed that $\sigma \vee \neg \sigma$ is True, and hence the law of the excluded middle does not hold for Truth." If this philosophical error is avoided, the extremely interesting intuitionistic logic is seen to deal only with man-made proof and thereby loses all its mystery.

The above discussion of intuitionism can be adapted to any of the other constructivist schools. Each of them is then recognized as a branch of mathematics, and none of them is all of mathematics.

7. Consistency. Does our analysis of mathematical activity (Section 5) give at least a hint as to why mathematics is free of contradictions? Suppose that W is a world of realities, underlying some branch of mathematics, and that a proof method P for W has been chosen. If σ is a dW-sentence, we denote "σ is true" by $W \models \sigma$ and "σ is provable" by $P \vdash \sigma$. We recall that, if $P \vdash \sigma$ then $W \models \sigma$, but not necessarily conversely. To say that mathematics is free of contradictions—i.e., that mathematics is consistent—means that $P \vdash \sigma$ and $P \vdash \neg \sigma$ will never happen simultaneously. Again, $\neg \sigma$ denotes the negation of σ.

Why then can't we prove the consistency of mathematics simply by observing that, if $P \vdash \sigma$ and $P \vdash \neg \sigma$ did happen simultaneously, then $W \models \sigma$ and $W \models \neg \sigma$ would happen simultaneously and the latter is absurd?

The trouble with the above proof is not so much that it is wrong as that it is unconvincing. The unconvincing step is the use of the implication "if $P \vdash \sigma$, then $W \models \sigma$". Since this implication goes from proof to truth, we shall refer to it as the "pt-implication".

Mathematicians who have complete confidence in their world W and are consequently willing to accept the pt-implication without question, do not fear contradictions. This includes the "working mathematicians," the intuitionists and other constructivists. Formalists, on the other hand, distrust every world of realities except those that can be described by finitary means. However, it is known from [4] that trust in only finitary worlds cannot give us a proof of consistency for all worlds. Even the finitary proof that if intuitionistic arithmetic is consistent so is classical arithmetic ([6, section 81]), does not prove to formalists that classical arithmetic is free of contradictions. The reason is that the world W_i of the intuitionists cannot be described by finitary means.

In the author's opinion, all this shows that the proof of the consistency of mathematics will have to consist in justifying the pt-implication. The main problem will be to answer basically philosophical questions. What really is a world of realities? What special properties must such a world have in order that an appropriate proof method can be chosen for it? What really is an appropriate proof method acceptable to mathematicians? In short, the two constituents of mathematical activity (Section 5) must be further analyzed.

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THE CANTOR-LEBESGUE THEOREM

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1. Introduction. The theorem named in the title of this paper has many properties which make its study worthwhile for advanced undergraduate or beginning graduate students. First, it has great intrinsic value; it is an important fact about trigonometric series. Second, it is intimately involved with the basic problem of termwise integration of a sequence of integrable functions; it provides a concrete situation which illustrates the need for, and the limitations of, the standard theorems on termwise integration. Third, as is the case with many fully-developed theorems, its history shows how a theorem with a long and laborious proof was gradually generalized and given a short elegant proof once the necessary concepts were identified and clearly formulated. Finally, important extensions of this theorem remain to be proved, and there is reason to believe that interested students could contribute to the resolution of these unanswered questions.

2. The setting for the theorem. Although the Cantor-Lebesgue Theorem is part of a problem which is the inverse of the main topic of Fourier analysis, the difficulty it overcomes is also encountered in the direct problem; and some of the results established in the direct problem are needed in solving the inverse problem. We begin, therefore, with a brief synopsis of the basic problem of Fourier analysis, the representation of a function by a Fourier series.

Suppose $f(x)$ is a function of period 2π which is integrable over the interval $[-\pi,\pi]$. (At this point it makes no difference whether integrals are taken in the sense of Riemann or Lebesgue.) The Fourier coefficients of $f(x)$ are the numbers

$$c_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx,$$

defined for all integers, n, positive, negative, and zero. The primary problem is to determine in what sense, if any, the function $f(x)$ is represented by the formal series $\sum_{n=-\infty}^{\infty} c_n e^{inx}$, called the Fourier series of f. Taking this question in its broadest sense, one might ask only if the transformation $f \to \{c_n\}_{n=-\infty}^{\infty}$ is one-to-one, without asking for the explicit procedure by which f is to be recovered from the sequence $\{c_n\}_{n=-\infty}^{\infty}$. A basic tool, and the first assertion made in most textbooks, is the *Riemann-Lebesgue lemma*:

$$\lim_{n\to\infty} c_n = 0 = \lim_{n\to-\infty} c_n.$$

The proof of the Riemann-Lebesgue lemma is quite simple. First, if f is the indicator (characteristic) function of an interval, one can compute c_n explicitly and verify the relation. Using the linear property of integrals, one proves the relation for step functions. Finally, it is a well-known fact of both Riemann and Lebesgue integration that if f is integrable and $\varepsilon > 0$, there is a step function g such that $\int_{-\pi}^{\pi}|f(x)-g(x)|\,dx < \varepsilon$. The lemma now follows from the inequality

$$\left|\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx\right| \leqslant \left|\int_{-\pi}^{\pi} g(x)e^{-inx}\,dx\right| + \int_{\pi}^{\pi}|f(x)-g(x)||e^{-inx}|\,dx$$

and the fact that $|e^{-inx}|=1$.

The basic convergence theorems of Fourier analysis, which assert that $f(x)$ is in some sense the sum of its Fourier series, depend for their proofs on applying this lemma to some modification of $f(x)$ in order to show that the difference between $f(x)$ and a "partial sum" tends to 0. A particular consequence of any of these theorems is that only one function can have a

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given Fourier series. This result leads naturally to the converse question: Can a given function be represented by more than one trigonometric series? The answer depends on the interpretation of the word "represent," and it turns out that the assertion of the Riemann-Lebesgue lemma plays an important role in the answer.

To formulate this problem precisely, suppose that

$$\sum_{n=-\infty}^{\infty} c_n e^{inx} = \lim_{N\to\infty} \sum_{n=-N}^{+N} c_n e^{inx} = 0$$

for all x. This equation clearly holds if $c_n = 0$ for all n, and we would like to show that this sufficient condition is also necessary. Notice that if we could multiply the series by e^{-imx} and integrate termwise, we would have

$$0 = \sum_{n=-\infty}^{\infty} c_n \int_{-\pi}^{\pi} e^{i(n-m)x}\,dx = 2\pi c_m,$$

since $\int_{-\pi}^{\pi} e^{ikx}\,dx = 0$ if k is any integer except 0.

This termwise integration is justified in Riemann integration if the sequence $S_N(x) = \sum_{n=-N}^{N} c_n e^{inx}$ converges uniformly. It is justified in Lebesgue integration if, for example, $|S_N(x)| \leqslant g(x)$ for all x, N and some integrable function $g(x)$. In general, however, even the most refined theorem on termwise integration will fail for some sequences of trigonometric functions. Therefore, some substitute must be found for this process. Such an approach was found by Riemann in 1854 and published in 1867.

Riemann's idea was to study the series

$$\sum_{n=1}^{\infty} \frac{c_n e^{inx} + c_{-n} e^{-inx}}{-n^2}$$

obtained by integrating the original series twice termwise. If this series converges to a function $g(x)$, we ought to have $g''(x) = -c_0$ since $c_0 + \sum_{n=1}^{\infty}(c_n e^{inx} + c_{-n} e^{-inx}) = 0$. If indeed $g''(x) = -c_0$, then $g(x)$ is a quadratic polynomial in x. However, because of its definition, $g(x)$ is periodic. Since the only periodic polynomial is a constant, we have

$$c = \sum_{n=1}^{\infty} \frac{c_n e^{inx} + c_n e^{-inx}}{n^2}.$$

This series has obviously better convergence properties than the original series, and one ought to be able to justify the termwise integration described above to prove that $c_m = 0$ for all m. Two difficulties should be observed in this program. First, although Riemann showed that $g(x)$ is continuous (more on this subject in a moment), it cannot be known at the outset that $g''(x)$ exists. The profundity of Riemann's work is seen in his skillful use of a generalized second derivative

$$Dg(x) = \lim_{t\to 0} \frac{g(x+t) + g(x-t) - 2g(x)}{t^2},$$

which does exist. Second, to prove that $g(x)$ is continuous and to justify termwise integration at the end of the proof, one needs to use the Weierstrass M-test [3, p. 184]; and, in order to do that, one must establish that the coefficients c_n are bounded. This task is accomplished by the Cantor-Lebesgue Theorem.

3. The Theorem and its Proofs. The modern Cantor-Lebesgue Theorem asserts much more than is needed for the modest program outlined above. It reads as follows:

Let $A_n(x) = c_n e^{inx} + c_{-n} e^{-inx}$. If $A_n(x)$ tends to zero as n tends to infinity for all x in a set E of positive Lebesgue measure, then c_n and c_{-n} tend to zero as n tends to infinity.

Cantor gave a proof of this theorem for the special case in which E is an interval. His proof

required several delicate lemmas and occupied eight pages (cf. [4]). Today there are many proofs of this theorem, none of them more than half a page long. The one most commonly given involves Egorov's theorem, which asserts that E contains a subset F, also of positive measure, on which $A_n(x)$ tends uniformly to zero. Let $\chi_F(x)$ be 1 or 0 according as x belongs to F or its complement, so that $\chi_F(x)|A_n(x)|^2$ tends uniformly to 0 on $[-\pi,\pi]$. Further, let $p_n=(|c_n|^2+|c_{-n}|^2)^{1/2}$. The theorem asserts that $p_n\to 0$, and it is easy to see that one may assume p_n is bounded. For, suppose there is a counterexample $\{A_n(x)\}_{n=1}^{\infty}$ to the theorem. Thus for this sequence we have $p_{n_k}\geqslant\alpha>0$ for an infinite number of indices n_k. If $B_n(x)=0$ for $n\neq n_1, n_2,\ldots, B_{n_k}(x)=p_{n_k}^{-1}A_{n_k}(x)$, then $B_n(x)$ tends to 0 on E, yet if $B_n(x)=d_ne^{inx}+d_{-n}e^{-inx}$, then $|d_n|^2+|d_{-n}|^2\leqslant 1$. Thus, if there is a counterexample to this theorem, there is a counterexample with p_n bounded.

Now since $|A_n(x)|^2=|c_n|^2+|c_{-n}|^2+c_n\bar{c}_{-n}e^{2inx}+\bar{c}_nc_{-n}e^{-2inx}$, we deduce two things: First, Parseval's identity

$$\int_{-\pi}^{\pi}|A_n(x)|^2dx=2\pi(|c_n|^2+|c_{-n}|^2), \tag{1}$$

second

$$\int_{\pi}^{\pi}\chi_F(x)|A_n(x)|^2dx=p_n^2\int_{-\pi}^{\pi}\chi_F(x)\,dx+R_n, \tag{2}$$

where

$$R_n=\left(\int_{-\pi}^{\pi}\chi_F(x)e^{2inx}dx\right)(c_n\bar{c}_{-n})+\left(\int_{-\pi}^{\pi}\chi_F(x)e^{-2inx}dx\right)(\bar{c}_nc_{-n}).$$

As mentioned above $c_n\bar{c}_{-n}$ may be assumed bounded. It thus follows from the Riemann-Lebesgue lemma that R_n tends to 0.

Now the Cantor-Lebesgue theorem follows from equation (2) and the fact that R_n tends to 0. Notice that the proof implies another interesting fact, namely, that there is a constant K depending only on the set F such that for all n

$$\int_{\pi}^{\pi}|A_n(x)|^2dx\leqslant K\int_{-\pi}^{\pi}\chi_F(x)|A_n(x)|^2dx. \tag{3}$$

This inequality may be interpreted as asserting that the values of $A_n(x)$ are evenly distributed, so that the values on the complement of F can be inferred (on the average) from the values on F. To illustrate this point further, we give a second proof of the theorem, which we believe to be new. This proof is based on the identity

$$\begin{aligned}|A_n(x)|^2+|A_n(y)|^2=&\,2\sin^2n(x-y)(|c_n|^2+|c_{-n}|^2)\\&+2\,\mathrm{Re}\big(A_n(x)\,\overline{A_n(y)}\,\cos n(x-y)\big),\end{aligned} \tag{4}$$

whose proof is a tedious computation.* We again invoke Egorov's theorem to obtain a set F of positive measure on which $A_n(x)$ tends uniformly to 0. Now we make a special choice of x and y in F. To do so we recall a theorem of Steinhaus [12] which asserts that the set of differences

*For readers who grow tired proving trigonometric identities, here is as brief a derivation of equation (4) as one can reasonably expect.

$$\begin{aligned}A_n(x)\,\overline{A_n(y)}&=(c_ne^{inx}+c_{-n}e^{-inx})(\bar{c}_ne^{-iny}+\bar{c}_{-n}e^{iny})\\&=c_n\bar{c}_ne^{in(x-y)}+c_n\bar{c}_{-n}e^{in(x+y)}+c_{-n}\bar{c}_{-n}e^{-in(x-y)}+\bar{c}_nc_{-n}e^{-in(x+y)}\\&=|c_n|^2e^{in(x-y)}+|c_{-n}|^2e^{-in(x-y)}+2\,\mathrm{Re}(c_n\bar{c}_{-n}e^{in(x+y)}).\end{aligned}$$

Similarly, $\overline{A_n(x)}A_n(y)=|c_n|^2e^{-in(x-y)}+|c_{-n}|^2e^{in(x-y)}+2\,\mathrm{Re}(c_n\bar{c}_{-n}e^{in(x+y)})$. Adding, we get

$$2\,\mathrm{Re}\big(A_n(x)\,\overline{A_n(y)}\big)=(|c_n|^2+|c_n|^2)\cdot 2\cos n(x-y)+4\,\mathrm{Re}(c_n\bar{c}_{-n}e^{in(x+y)});$$

$x-y$ ranges over an interval $(-d,d)$ for some $d>0$ as x and y range over F. In particular, if $n>\pi/2d$, we can choose x_n, y_n in F such that $x_n-y_n=\pi/2n$.

The identity (4) becomes

$$|A_n(x_n)|^2+|A_n(y_n)|^2=2(|c_n|^2+|c_{-n}|^2). \tag{5}$$

Equation (5), together with the fact that A_n tends uniformly to 0 on F, implies the Cantor-Lebesgue theorem.

We now make three remarks on the preceding proof.

REMARK 1. Equation (5) asserts that the continuous average $1/2\pi\int_{-\pi}^{\pi}|A_n(x)|^2dx=|c_n|^2+|c_{-n}|^2$ is equal to the discrete average $(|A_n(x_n)|^2+|A_n(y_n)|^2)/2$ at two points of F. This fact reinforces the point that the Cantor-Lebesgue theorem depends on the smooth distribution of values of $A_n(x)$.

REMARK 2. The perceptive reader may have noticed that the basic problem in this theorem involves inferring c_n and c_{-n} from values $A_n(x)$, and that this can be done merely by solving a pair of simultaneous linear equations. As an exercise in understanding, it would be well to construct a third proof along these lines. The identity (4) gives merely a shortened version of such a proof.

REMARK 3. Although Steinhaus's theorem had originally a long proof, like the Cantor-Lebesgue theorem, it, too, can be given a short proof. The basis of such a proof is the fact that translation is a continuous operation. Specifically, if $f_t(x)=f(x+t)$, then the mapping $t\to f_t$ is continuous from the real numbers to the space of square-integrable functions whose metric is $d(f,g)=(\int|f-g|^2)^{1/2}$. (This fact, in turn, can be proved by imitating the proof of the Riemann-Lebesgue lemma given above.) It follows that $\phi(x)=\int\chi_F(t)\chi_F(t-x)dt$ is a continuous real valued function. Now $\phi(0)$, being the measure of F, is positive. Hence $\phi(x)>0$ for x in an interval $(-d,d)$. For any such x, there must be at least one t such that $\chi_F(t)\chi_F(t-x)$ is not zero; hence $x=t-(t-x)$, which is in the difference set of F. This proof generalizes to any locally compact abelian group with its Haar measure.

4. Generalization and Counterexamples. In attempting to extend Riemann's uniqueness theorem to more than one variable, one is naturally led to seek a two-variable Cantor-Lebesgue Theorem. Here a new difficulty arises which considerably complicates the picture. A trigonometric series in two variables is, of course, a formal sum $\Sigma_{n=-\infty}^{\infty}\Sigma_{m=-\infty}^{\infty}c_{mn}e^{i(mx+ny)}$. If Riemann's work is to be extended, one hopes to prove that if the series converges to 0 at every point, then all the coefficients are zero. The difficulty alluded to is the following: How is the double series to be arranged as a single limit, or should we leave it as a double limit?

A natural definition of the partial sum of such a series is $S_N(x,y)=\Sigma_{n=-N}^{N}\Sigma_{m=-N}^{N}c_{mn}e^{i(mx+ny)}$. Unfortunately, with this natural definition, Riemann's program cannot even be started, because the Cantor-Lebesgue theorem fails spectacularly [1]. In fact, if we set

$$c_{mn}=\begin{cases}0 & \text{if } |m|>n\\ (-1/4)^n n^{\alpha}\binom{2n}{n-m} & \text{if } |m|\leqslant n,\end{cases}$$

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if we multiply by $\cos n(x-y)$ we get

$$2\,\mathrm{Re}\big(A_n(x)\,\overline{A_n(y)}\,\cos n(x-y)\big)=(|c_n|^2+|c_{-n}|^2)\cdot 2\cos^2 n(x-y)+4\,\mathrm{Re}(c_n\bar{c}_{-n}e^{in(x+y)}\cos n(x-y)).$$

Since $\cos n(x-y)=\frac{1}{2}[e^{in(x-y)}+e^{in(y-x)}]$, we have $e^{in(x+y)}\cos n(x-y)=\frac{1}{2}(e^{2inx}+e^{2iny})$. Thus,

$$2\,\mathrm{Re}\big(A_n(x)\,\overline{A_n(y)}\,\cos n(x-y)\big)=(|c_n|^2+|c_{-n}|^2)\cdot 2\cos^2 n(x-y)+2\,\mathrm{Re}(c_n\bar{c}_{-n}(e^{2inx}+e^{2iny})).$$

Now

$$|A_n(x)|^2+|A_n(y)|^2=(|c_n|^2+|c_{-n}|^2)\cdot 2+2\,\mathrm{Re}(c_n\bar{c}_{-n}(e^{2inx}+e^{2iny})).$$

Subtracting this last equation from the one preceding gives the desired result.

where α is any positive number, the partial sum S_N is given by

$$S_N(x,y) = c_{00} + \sum_{n=1}^{N} n^\alpha(\sin^2(x/2))^n e^{iny}.$$

Since $\sum_{n=1}^\infty n^\alpha r^n < \infty$ if $r<1$, the sequence $S_N(x,y)$ converges if $\sin^2(x/2)<1$, that is, at any point (x,y) where x is not an odd multiple of π. In particular $S_N(x,y)$ converges almost everywhere. Nevertheless, Stirling's formula, $\lim_{k\to\infty}(\sqrt{2\pi k}\,(k/e)^k)/k! = 1$, applied to the binomial coefficient $\binom{2n}{n} = (2n)!/(n!)^2$ shows that $\lim_{n\to\infty} c_{0n} n^{(1/2)-\alpha} = 1/\sqrt{\pi}$. Thus c_{0n} increases like $n^{\alpha-(1/2)}$. Since α is arbitrary here, the coefficients of a square-convergent double trigonometric series may grow as fast as any power of n. (This, as it happens, is about the worst that can be said. Paul Cohen showed in a 1958 dissertation at the University of Chicago **[5]** that c_{0n} cannot grow like an exponential function.) Thus, if we hope to get a Cantor-Lebesgue theorem in two variables, we must look for another way to arrange the terms of a double trigonometric series. Fortunately, another natural arrangement of terms was studied by Bochner **[2]**, who showed that direct analogues of many one-variable theorems can be obtained. In Bochner's method, the double series is arranged "circularly" as the single series $\sum_{r=0}^\infty A_r(x,y)$, where $A_r(x,y) = \sum_{m^2+n^2=r} c_{mn} e^{i(mx+ny)}$. At this point, it will help to avoid confusion if we write $X=(x,y)$, $M=(m,n)$, and $MX = mx+ny$, $M^2 = m^2+n^2$. Thus $A_r(x,y) = A_r(X) = \sum_{M^2=r} c_M e^{iMX}$. Also $\int_{T^2} f(X)\,dX = 1/4\pi^2 \int_{-\pi}^{\pi}\int_{-\pi}^{\pi} f(x,y)\,dx\,dy$; and T^2 is the set of pairs for which $-\pi \leqslant x \leqslant \pi$, $-\pi \leqslant y \leqslant \pi$.

The author **[7]** discovered the following partial generalization of the Cantor-Lebesgue theorem:

If $A_r(X)$ tends to zero almost everywhere, then $\int_{T^2} |A_r(X)|^2 dX = \sum_{M^2=r} |c_M|^2$ tends to zero.

In this form the theorem is clearly a matter of termwise integration of a sequence of functions. Nevertheless, the basic principle—that the values of $A_r(X)$ must be evenly distributed—is the same as in the proofs given above. In fact, if $f_r(X)$ is a sequence of integrable non-negative functions tending to 0 almost everywhere on T^2, the following criterion is easily seen to be necessary and sufficient in order that the integrals $\int_{T^2} f_r(X)\,dX$ tend to 0: For every $\varepsilon>0$ there exists $\delta>0$ such that, for all r, $\int_{T^2}\chi_E(X) f_r(X)\,dX < \varepsilon$ whenever E has measure less than δ. The integrals $\int_{T^2} f_r(X)\,dX$ are said to be *uniformly absolutely continuous* if this criterion is met, the word *uniformly* referring to the fact that δ can be chosen without knowing r.

To prove this criterion, one uses Egorov's theorem to find a set E whose measure is less than δ and such that $\chi_{T^2-E}(X) f_r(X)$ tends uniformly to 0. It is clear then that

$$\int_{T^2} f_r(X)\,dX = \int_{T^2} \chi_E(X) f_r(X)\,dX + \int_{T^2} \chi_{T^2-E}(X) f_r(X)\,dX$$

is less than ε for all large r, since the first term on the right is always less than ε and the second term tends to 0 as r tends to infinity. This proves the sufficiency of the criterion. Its necessity is trivial, granted that each integral $\int_{T^2} f_r(X)\,dX$ is absolutely continuous.

This criterion reflects once again the importance of a smooth distribution of values in a sequence $f_r(X)$ if termwise integrability is to be obtained.

One sufficient condition for uniform absolute continuity of a sequence such as $f_r(X)$ is the following, due to de La Vallée Poussin.

If $\int_{T^2} f_r(X)^p\,dX$ is bounded for some $p>1$, then the integrals $\int_{T^2} f_r(X)\,dX$ are uniformly absolutely continuous.

This theorem is an immediate consequence of Hölder's inequality:

$$\int_{T^2} \chi_E f_r(X)\,dX \leqslant \left(\int_{T^2} f_r(X)^p\,dX\right)^{1/p} \left(\int_{T^2} \chi_E(X)^{p/p-1}\,dX\right)^{(p-1)/p},$$

since the first factor on the right is bounded by assumption and the second is precisely the measure of E raised to power $(p-1)/p$.

After all these reflections, the proof of our two-variable theorem is simple. First we may assume that $\int_{T^2}|A_r(X)|^2 dX$ remains bounded since the argument given in the proof of the one-variable theorem applies equally well in this case. Now we need only take $f_r(X)=|A_r(X)|^2$ and attempt to apply de La Vallée Poussin's theorem. We shall use $p=2$ in this theorem.

We note that $f_r(X)$ is a finite sum of exponentials. In fact

$$f_r(X)=\sum_{M^2=r}|c_M|^2+\sum_{\substack{M\neq N\\ M^2=N^2=r}} c_M\bar{c}_N e^{i(M-N)X}=\sum_{M^2=r}|c_M|^2+\sum_{Q\neq 0} d_Q e^{iQX}$$

where

$$d_Q=\sum_{\substack{M^2=N^2=r\\ M-N=Q}} c_M\bar{c}_N.$$

To study the coefficients d_Q we draw the following picture.

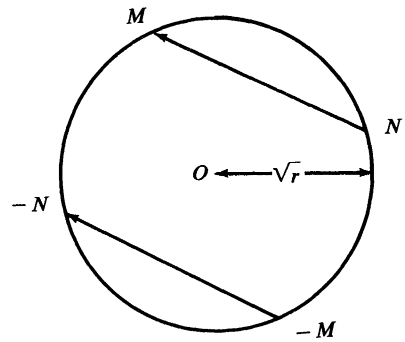

FIG. 1

From Figure 1 it is clear that Q is represented by the directed line segment from N to M. Since a circle has at most two chords of given length and direction, it is clear that the sum making up d_Q contains at most two terms, so that

$$|d_Q|^2\leqslant 2\sum_{\substack{M-N=Q\\ M^2=N^2=r}}|c_M|^2|c_N|^2.$$

Having established this crucial fact, we can use a technique which is often used in one-variable trigonometric series [10, p. 112]. By Parseval's equation

$$\begin{aligned}\int_{T^2}|f_r(X)|^2&=\Big(\sum_{M^2=r}|c_M|^2\Big)^2+\sum_{Q\neq 0}|d_Q|^2\\ &\leqslant\Big(\sum_{M^2=r}|c_M|^2\Big)^2+2\sum_{Q\neq 0}\sum_{\substack{M-N=Q\\ M^2=N^2=r}}|c_M|^2|c_N|^2\\ &\leqslant 3\Big(\sum_{M^2=r}|c_M|^2\Big)^2=3\Big(\int_{T^2}|A_r(X)|^2dX\Big)^2.\end{aligned}$$

As remarked above, this last expression may be assumed bounded. This verifies de La Vallée Poussin's criterion and hence proves our Cantor-Lebesgue Theorem.

In the preceding proof the inequality

$$\int_{T^2}|A_r(X)|^4\,dX \leqslant 3\left(\int_{T^2}|A_r|(X)|^2\,dX\right)^2 \tag{6}$$

plays the role which inequality (3) plays in the one-variable case. In the case of square sums, where we could have $A_r(x,y)=\Sigma_{\max(|m|,|n|)=r}c_{mn}e^{i(mx+ny)}$, this inequality definitely does not hold. In fact, the number of terms in the corresponding sum for d_Q does not remain bounded, and the proof breaks down at that point.

Actually, by a more careful use of inequality (6) one can prove the Cantor-Lebesgue theorem assuming only that $A_r(X)$ tends to zero on a set E whose measure is more than two-thirds of the measure of T^2 (see [**11**]). However Zygmund [**15**] showed that one need only assume convergence on any set of positive measure, thereby giving a full generalization of the Cantor-Lebesgue theorem. Zygmund went on to show that the circular grouping of terms is not essential, that it is also permissible to group all the terms c_M for which M lies on an ellipse, and that, furthermore, the eccentricity of the ellipse may increase to unity as the ellipse expands to infinity. Subsequently, Panferov [**9**] proved that if one groups all the terms on curves obtained by dilating a fixed curve γ, then the Cantor-Lebesgue theorem holds if and only if γ contains no line segments of rational slope. This result makes our knowledge of the two-variable case fairly complete. The most general framework for this type of question is the following. Suppose the set of points $M=(m,n)$ with integer coordinates is decomposed into pairwise disjoint subsets U_r, $r=0,1,2,\dots$, and $A_r(X)=\Sigma_{M\in U_r}c_Me^{iMX}$. Such a decomposition may be said to have the Cantor-Lebesgue property if for every sequence $A_r(X)$ which tends to zero on a set of positive measure $\int_{T^2}|A_r(X)|^2dX=\Sigma_{M\in U_r}|c_M|^2$ tends to zero. It is a simple exercise to show that this property holds if and only if for each set F of positive measure there is a constant K such that for all r

$$\int_{T^2}|A_r(X)|^2\,dX \leqslant K\int_{T^2}\chi_F(X)|A_r(X)|^2\,dX.$$

It is possible to show in this way that the decomposition $\{U_r\}_{r=0}^{\infty}$ has the Cantor-Lebesgue property if the number of points in U_r is a bounded function of r; see [**14**, vol. 1, p. 370] for details. This case however, has little interest, except in one variable.

If we ask whether the conclusion $\Sigma_{M\in U_r}|c_M|^p\to 0$ can be obtained with $p<2$, the answer, again an easy exercise in analysis, is, "Only if the number of terms in U_r remains bounded as r increases." In Panferov's theorem, as in our proof above, one finds $\int_{T^2}|A_r(X)|^4dX\to 0$. Whether the exponent 4 can be increased is unknown to the present author.

We end our discussion of the two-variable case with a brief mention of a different approach to double trigonometric series. The double series $\Sigma\Sigma c_{mn}e^{i(mx+ny)}$ is said to converge unrestrictedly (rectangularly) to $f(x,y)$ if for each $\varepsilon>0$ there exist m_0n_0 such that $|f(x,y)-\Sigma_{n=-\nu}^{\nu}\Sigma_{m=-\mu}^{\mu}c_{mn}e^{i(mx+ny)}|<\varepsilon$ for all $\mu>m_0$, $\nu>n_0$. This assumption is much stronger than the square convergence considered earlier. It is not merely a matter of arranging the terms in a linear sum. For this process Ash and Welland [**1**] established a Cantor-Lebesgue Theorem.

5. Recent Work and Conjectures. The two-variable work presented above was all done during the years 1970–1973. It would seem natural to press on and see what can be said in the case of three or more variables. Quite recently Connes [**6**] has shown that the theorem can be extended to any number of variables if the terms are arranged spherically. Connes has assumed that a sequence $A_n(X)=\Sigma_{M^2=n}c_Me^{iMX}$ tends to zero on a non-empty open set U. From this assumption he has shown that $\Sigma_{M^2=n}|c_M|^2$ tends to zero. Here $M=(m_1,\dots,m_u)$ and $MX=m_1x_1+\cdots+m_ux_u$, etc. Actually, as Connes points out, one may except a set E of measure zero in U, provided $U-E$ is of second category. Meaney [**8**] has shown that Connes's result extends to all compact Lie groups.

It would still be of interest to replace the open set U by an arbitrary set of positive measure. Recalling our new proof of the one-variable result, we suggest considering the identity

$$|A_r(X)|^2+|A_r(Y)|^2=2\sum_{M^2=r}|c_M|^2 +2\sum_{\substack{M\neq N\\ M^2=N^2=r}} c_M\bar{c}_N e^{i(M-N)((X+Y)/2)}\cos((M-N)(X-Y)/2).$$

By Steinhaus' theorem, $X-Y$ fills up an open set as X and Y range over a set of positive measure. It may be that by a judicious choice of X and Y the second sum on the right can be made to go to zero. Or perhaps Connes's result can be generalized to apply to the second term on the right. A variant of this method would be to consider the spherical sum in $2u$ variables

$$B_{4r}(X,Y)=A_r(X+Y)\overline{A_r(X-Y)}=\sum_{N^2=r}\sum_{M^2=r} c_M\bar{c}_N e^{i(M-N,M+N)(X,Y)}.$$

This sum is spherical because of the parallelogram law: $(M-N,M+N)^2=2(M^2+N^2)$.

Interested readers are invited to use these suggestions as a starting point or to supply their own ideas to solve this important and interesting problem.

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27. The Caterpillar in Chapter V of Alice in Wonderland is a typical teacher of mathematics, and Alice is a typical pupil, in this case doubly surprised and bewildered because she is at the same time the figure to be understood.

Scott Buchanan, *Poetry and Mathematics*,
New York, 1929

THE BOUNDED CONSISTENCY THEOREM

WILLIAM H. RUCKLE

Introduction. The Bounded Consistency Theorem (BCT, Theorem 1.2 below) is a principal result of summability theory. Indeed, it holds a claim to a second position in the theory just after the Silverman-Toeplitz Theorem (Theorem 1.1 below). My purpose here is to make the BCT and a proof of it accessible to the largest possible readership. If you have taken a course in functional analysis, you should find the narrative easy to follow, and a nice application of what you have learned. In fact, I hope that this article will be accessible to many who have the prerequisites for reading the second part of [11], i.e., a good course in advanced calculus and some linear algebra. Such a reader can preview some results of functional analysis and see how they apply to summability theory. Of course the harder you must work to read this article, the more you can expect to benefit from it. The expert on summability theory need only browse through the proof of Lemma 3.2 in order to grasp the main thrust of our development. This proof, based on the Schur Property of l^1, is original, and apparently the easiest functional analytic proof. The basis for our discussion will be the text of Wilansky [17].

Remarkable for its history as well as its content, the Bounded Consistency Theorem was first announced without proof by S. Mazur and W. Orlicz [7] in 1933. Their investigation on the application of functional analysis to summability theory was a portion of the intense activity by the "school of Lwów," a group of functional analysts flourishing around S. Banach prior to the outbreak of World War II. A special case of the BCT had already appeared on page 95 of Banach's famous monograph [1]. Mazur and Orlicz were not able to publish a complete report of their investigation until 1954 [8]. Meanwhile, the BCT had been independently rediscovered by the Russian mathematician A. L. Brudno [3] and published with a different (nonfunctional analytic) proof requiring seven pages of calculations.

In his 1955 book [4] R. G. Cooke described the work of Brudno on pages 130–131 and asked whether a shorter proof of the BCT could be constructed. This challenge was met by P. Erdős and G. Piranian writing jointly [5] and by G. M. Petersen [10]. Both of these proofs are streamlined versions of Brudno's. Petersen attacked the problem directly and completed the calculations in one page. Erdős and Piranian developed a general technique based on the methods of Brudno and called it "the principle of aping sequences." They used this technique to study in depth the bounded convergence field of a matrix, and derived the BCT as a quick application. The BCT is stated on page 67 of Zeller's monograph on summability theory [18]. Other proofs of the BCT or versions thereof can be found in [2], [9], [13] and [19].

We shall now explain our notation and some elements of summability theory leading up to the statement of the BCT. The jth coordinate of a sequence s of real or complex numbers is written $s(j)$; thus $s=(s(j))$. We prefer the functional to the subscript notation because we often need to discuss sequences of sequences. The sum and scalar multiple of sequences are defined coordinatewise. For any sequence s, $s[\leqslant n]$ denotes the nth section of s; i.e., $s[\leqslant n]$ denotes that sequence t for which $t(i)=s(i)$ for $i\leqslant n$ and $t(i)=0$ for $i>n$; $s[>n]=s-s[\leqslant n]$. The sequence with 1 in the nth place and 0's elsewhere is denoted by e_n. The space of all sequences with only finitely many nonzero coordinates is denoted by ϕ. The spaces of sequences which are bounded, convergent, or convergent to zero are denoted, respectively, by m, c and c_0.

The i,j element of an infinite matrix is denoted by $A(i,j)$. For any matrix A, we denote by c_A

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the space of all sequences s such that

$$\Sigma_j A(n,j)s(j)$$

converges for each n and $\lim_A s$ defined by $\lim_A s = \lim_n \Sigma_j A(n,j)s(j)$ exists. A study of the nature of c_A and A is part of the subject matter of summability theory. A matrix A is called *regular* if c_A contains c and $\lim_A$ coincides with the ordinary limit on c, i.e., $\lim_A s = \lim s$ for $s \in c$. A main theorem of summability theory is the Silverman-Toeplitz Theorem **[14]**, **[16]**, which characterizes regular matrices.

THEOREM 1.1. *A matrix A is regular if and only if the following conditions hold*:

(a) $\sup_n \Sigma_j |A(n,j)| < \infty$
(b) $\lim_n A(n,j) = 0$ *for* $j = 1,2,\ldots$
(c) $\lim_n \Sigma_j A(n,j) = 1$.

An easy functional analytic proof of this theorem is given on page 91 of **[1]**; see also Application 2, page 118 of **[17]**.

The BCT can easily be stated in the terms explained above.

THEOREM 1.2. *If A and B are regular matrices such that $c_A \cap m \subset c_B$ then $\lim_A s = \lim_B s$ for all s in $c_A \cap m$.*

In other words, the BCT asserts that under the most natural possible conditions $\lim_A$ and $\lim_B$ are consistent on bounded sequences.

2. Spaces. In this section we review the ideas from functional analysis used to prove the BCT. An F-space is defined to be a linear space X with a complete metric topology determined by an increasing sequence $p_1 \leqslant p_2 \leqslant \cdots$ of seminorms. (This would be called a locally convex Fréchet space in **[17]** or a B_0-space by the successors of Banach **[8]**.) Three important principles of functional analysis which can be stated in terms of F-spaces are the Banach-Steinhaus Closure Theorem, the Closed Graph Theorem, and the Hahn-Banach Theorem. We need a version of each of these principles in our proof of the BCT.

The form of the Banach-Steinhaus Theorem we use is quoted in **[17]** on page 200 as Corollary 4. A linear map from an F-space into the scalar field is called a *functional.*

THEOREM 2.1. *Let $\{f_n\}$ be a pointwise convergent sequence of continuous linear functionals on an F-space X. Then f defined by $f(x) = \lim_n f_n(x)$ (for x in X) is continuous.*

The Closed Graph Theorem appears on page 200 of **[17]** as Theorem 2. A linear map T from an F-space X into an F-space Y is called *closed* if whenever $x_n \to x$ in X and $Tx_n \to y$ in Y it follows that $Tx = y$. The condition "T is closed" is easier to verify than the condition "T is continuous." This is why the following theorem is valuable.

THEOREM 2.2. *Every closed linear map T from an F-space X to an F-space Y is continuous.*

The Hahn-Banach Theorem appears in many versions which often bear little resemblance to one another. In order to describe the form we need, we first recall that for X an F-space, X^* denotes the space of all continuous linear functionals on X. If $\mathfrak{S}$ and $\mathfrak{T}$ are two topologies on a set S, $\mathfrak{S}$ is said to be *weaker* then $\mathfrak{T}$ if each $\mathfrak{S}$-open set is also $\mathfrak{T}$-open. The *weak topology* on X is the weakest topology on X such that each f in X^* is still continuous on X. A basic weak neighborhood of a point x in X has the form

$$\{y: |f_i(x-y)| < \varepsilon_i, i-1,2,\ldots,n\}$$

where $\{f_1,f_2,\ldots,f_n\}$ is any finite subset of X^* and $\varepsilon_i > 0$ for $i = 1,2,\ldots,n$. For infinite dimensional F-spaces the weak topology is always strictly weaker than the metric topology. However, we

have the following form of the Hahn-Banach theorem which asserts that the closed convex sets are the same in both topologies.

THEOREM 2.3. *A convex subset of an F-space X is closed if and only if it is weakly closed.* (See Corollary 5 on p. 221 of **[17]**.)

The Closed Graph Theorem is applied in summability theory through the introduction of *FK*-spaces. An *FK*-space is an *F*-space of sequences such that the functional $x \to x(i)$ is continuous for each i. If X and Y are *FK*-spaces and $X \subset Y$, it follows immediately from the Closed Graph Theorem that the inclusion is continuous. Examples of *FK*-spaces are c_0, c, m and l^1 (the space of all sequences x such that $\Sigma_n |x(n)| < \infty$) all of which are normed spaces. It is known (p. 91 of **[17]**) that the dual space $(c_0)^*$ of c_0 is represented by l^1. This means that for each f in $(c_0)^*$ the sequence $(f(e_n))$ is in l^1,

$$f(x) = \Sigma_n x(n) f(e_n) \qquad x \in c_0, \tag{2-1}$$

and that every series (2-1) with $(f(e_n))$ in l^1 determines a member of $(c_0)^*$. A consequence of this representation for $(c_0)^*$ is the following fact.

THEOREM 2.4. *If X is an FK-space containing c_0 then for each f in X^*, the sequence $(f(e_n))$ is in l^1.*

Just as l^1 represents the dual space of c_0, m represents the dual space of l^1. The *FK*-space l^1 has a peculiar property discovered by I. Schur in 1920 **[12]**, which is the core of our proof of the BCT.

THEOREM 2.5. (The Schur Property). *Let (x_n) be a sequence in l^1 for which* $\lim_n \Sigma_i x_n(i) u_n(i)$ *exists for each u in m. Then there is x in l^1 such that* $\lim_n \Sigma_i |x(i) - x_n(i)| = 0$.

Theorem 2.5 can be restated: "Weak and strong convergences of sequences coincide in l^1." A proof of the theorem, essentially that given by Schur, can be found on page 281 of **[6]**.

Let X be a subspace of m. One says that l^1 is *X-sequentially complete* if whenever (t_n) is a sequence in l^1 such that $\lim_n \Sigma_j t_n(j) x(j)$ exists for all x in X there is t in l^1 such that $\lim_n \Sigma_j t_n(j) x(j) = \Sigma_j t(j) x(j)$ for all x in X. It follows from the Schur property that l^1 is m-sequentially complete and from the Banach-Steinhaus Theorem that l^1 is c_0-sequentially complete.

For any matrix A the space c_A is an *FK*-space. The following representation of $(c_A)^*$, of considerable importance in summability theory, was first given by Zeller and is stated on page 230 of **[17]**.

THEOREM 2.6. *Let A be a matrix and f be in $(c_A)^*$. Then f can be represented in the form*

$$f(x) = a \lim_A x + \Sigma_n t(n) \Sigma_j A(n,j) x(j) + \Sigma_n u(n) x(n), \qquad x \in c_A \tag{2-2}$$

where t is in l^1 and u is in $(c_A)^\beta$, the space of all sequences v such that $\Sigma_j v(j) x(j)$ converges for all $x \in c_A$.

3. Proof of the Theorem. Given a regular matrix A, let W_A denote the set of all s in c_A for which

$$f(s) = \Sigma_n s(n) f(e_n) \quad \text{for all} \quad f \in c_A^*. \tag{3-1}$$

It is not hard to see that W_A is a linear subspace of c_A.

LEMMA 3.1. *If A is a regular matrix, then $W_A \cap m$ has codimension one in $c_A \cap m$.*

Proof. We show first that for s in $c_A \cap m$, s is in W_A if and only if

$$\lim_A s = 0. \tag{3-2}$$

Condition (3-2) is necessary for s to be in W_A since $\lim_A e_n = 0$ for each n and $\lim_A$ is continuous on c_A. To verify that (3-2) is sufficient, let f in c_A^* be given, and assume it has the representation (2-2). Thus $f(e_k) = \Sigma_i A(i,k)t(i) + u(k)$ for each $k = 1, 2, \ldots$. If s satisfies (3-2), we have

$$\begin{aligned}\Sigma_k s(k) f(e_k) &= \Sigma_k s(k) \Sigma_i A(i,k) t(i) \\ &\quad + \Sigma_k s(k) u(k) \\ &= \Sigma_i t(i) \Sigma_k A(i,k) s(k) \\ &\quad + \Sigma_k s(k) u(k) \\ &= f(s).\end{aligned}$$

The change of order of summation of the double series is justified because the series converges absolutely; i.e.,

$$\Sigma_i \Sigma_k |s(k) A(i,k) t(i)| \leqslant \sup_k |s(k)| (\sup_i \Sigma_k |A(i,k)|) \Sigma_i |t(i)| < \infty$$

by the Silverman-Toeplitz Theorem and the fact that t is in $(c_A)^B \subset l^1$.

The above argument shows that $W_A \cap m$ is the null space of the linear functional $\lim_A$ restricted to $c_A \cap m$. This linear functional is non-zero since $\lim_A e = 1$. Therefore, $W_A \cap m$ has codimension one in $c_A \cap m$.

The following lemma is a special case of Theorem 3 of **[2]**, but our proof applies in the more general case with only formal modifications. The proof in **[2]** is involved and based on several theorems of functional analysis regarded as "deep"—among others, the Completeness and Precompactness Theorems of Grothendieck and the Orlicz-Pettis Theorem. Our proof relies only on the Schur property of l^1 which antedates these theorems by at least twenty years.

LEMMA 3.2. *If A is a regular matrix, then l^1 is $W_A \cap m$ sequentially complete.*

Proof. Let (s_n) be a sequence in l^1 such that $\lim_n \Sigma_j s_n(j) t(j)$ exists for each t in $W_A \cap m$. Since $c_0 \subset W_A \cap m$, a continuous linear functional g is defined on c_0 by $g(v) = \lim_n \Sigma_j s_n(j) v(j)$ (Theorem 2.1). This implies that $(g(e_j)) = (\lim_n s_n(j))$ is in l^1. Thus it suffices to prove that $\lim_n \Sigma_j s_n(j) t(j) = \Sigma_j g(e_j) t(j)$ for each t in $W_A \cap m$. We may, therefore, assume that $\lim_n s_n(j) = 0$ for all j and from this prove that

$$\lim_n \Sigma_j s_n(j) t(j) = 0 \qquad t \in W_A \cap m. \tag{3-3}$$

Each t in $W_A \cap m$ is contained in the closure of

$$\operatorname{conv}\{ t[\leqslant k]: \qquad k = 1, 2, \ldots \}$$

by which we denote the convex hull of the sections of t. Let $\{p_n\}$ be an increasing sequence of semi-norms which determines the topology of c_A. If (v_n) converges to t in c_A, so does $(t[\leqslant k] + v_n[>k])$ for each fixed $k = 1, 2, \ldots$. Thus, given t in $W_A \cap m$, there is a sequence (u_n) in ϕ such that

(a) u_n is in $\operatorname{conv}\{t[\leqslant k]: k = 1, 2, \ldots\}$,

(b) $p_n(u_n - t) < \dfrac{1}{n}$,

(c) $u_n(i) = t(i), \qquad i \leqslant n.$

We construct an increasing sequence $\{m_k\}$ of indices as follows. Let $m_0 = 0$, and suppose $m_1, m_2, \ldots, m_k$ have been determined. Choose $m_{k+1} > m_k$ such that

(d) $u_{m_k}(i) = 0$ for $i \geqslant m_{k+1}$,

(e) $p_{k+1}(u_r - u_s) < 2^{-k}$ for $r \geqslant s \geqslant m_{k+1}$.

Let $x_k = u_{m_k} - u_{m_{k+1}}$ for $k = 1, 2, \ldots$. Then $\Sigma_k p_n(x_k) < \infty$ for each n so that $\Sigma_k a_k x_k$ converges absolutely in c_A for all (a_k) in m. Moreover $\Sigma_k a_k x_k$ is in m since if $m_k \leqslant i < m_{k+1}$

$$\begin{aligned}\Sigma_k |a_k x_k(i)| &\leqslant \sup_k |a_k| \Sigma_k |x_k(i)| \\ &\leqslant \sup_k |a_k| (|u_{m_h}(i)| + |u_{m_{h+1}}(i) - u_{m_h}(i)|) \\ &\leqslant (\sup_k |a_k|)(3 \sup_i |t(i)|).\end{aligned} \tag{3-4}$$

That $|u_{m_h}(i)| \leqslant |t(i)|$ for all h follows from the fact that u_{m_h} is in conv$\{t[\leqslant k]: k=1,2,\ldots\}$. We have now verified that $\Sigma_k a_k x_k$ is in $c_A \cap m$ for all (a_k) in m. This sum is also in W_A because for each f in c_A^*

$$\begin{aligned} f(\Sigma_k a_k x_k) &= \Sigma_k a_k f(x_k) \\ &= \Sigma_k \Sigma_j a_k x_k(j) f(e_j) \\ &= \Sigma_j \Sigma_k a_k x_k(j) f(e_j) \\ &= \Sigma_j z(j) f(e_j) \end{aligned}$$

where $z = \Sigma_k a_k x_k$. The permutation of summation above is legitimate because the double series converges absolutely:

$$\begin{aligned} \Sigma_k \Sigma_j |a_k x_k(j) f(e_j)| &\leqslant \sup|a_k| \Sigma_j (\Sigma_k |x_k(j)|) |f(e_j)| \\ &\leqslant 3 \sup|a_k| \Sigma_j |t(j)| \, |f(e_j)| < \infty \end{aligned}$$

by (3-4) and the fact that $(f(e_j))$ is in l^1 (Lemma 2.4).

Since $\Sigma_k a_k x_k$ is in $W_A \cap m$ for all (a_k) in m, it follows that

$$\lim_n \Sigma_j \Sigma_k a_k x_k(j) s_n(j) = \lim_n \Sigma_{k=1}^{\infty} a_k \Sigma_{j=1}^{\infty} x_k(j) s_n(j)$$

exists for each (a_k) in m. Here (s_n) is the sequence in l^1 mentioned in the first sentence of the proof. Because of the Schur Property (2.5), $\lim_n (\Sigma_j x_k(j) s_n(j))_{k=1}^{\infty}$ exists in l^1. But since $\lim_n s_n(j) = 0$ for each j and x_k is in ϕ $\lim_n \Sigma_j x_k(j) s_n(j) = 0$ for each k. Thus

$$\lim_n \Sigma_{k=1}^{\infty} |\Sigma_{j=1}^{\infty} x_k(j) s_n(j)| = 0.$$

But since

$$\begin{aligned} |\Sigma_j s_n(j) t(j)| &= |\Sigma_j s_n(j) \Sigma_k x_k(j)| \\ &= |\Sigma_j \Sigma_k s_n(j) x_k(j)| \\ &= |\Sigma_k \Sigma_j s_n(j) x_k(j)| \\ &\leqslant \Sigma_k |\Sigma_j s_n(j) x_k(j)| \end{aligned}$$

for each n, it follows that $\lim_n \Sigma_j s_n(j) t(j) = 0$.

To complete the proof of the Bounded Consistency Theorem, let A and B be regular matrices with $c_A \cap m \subset c_B$. If t is in $c_A \cap m$, then by Lemma 3.1 $t = u + (\lim_A t)e$ where u is in $W_A \cap m$. Since $c_A \cap m \subset c_B$, it follows that $\lim_n \Sigma_j B(n,j) u(j)$ exists for each u in $c_A \cap m$, thus for each u in $W_A \cap m$. By the Silverman-Toeplitz Theorem (1.1) $(B(n,j): j = 1,2,\ldots)$ is in l^1 for each n. Thus by Lemma 3.2, there is s in l^1 such that $\lim_n \Sigma_j B(n,j) u(j) = \Sigma_j s(j) u(j)$ for each u in $W_A \cap m$. But since $\lim_n, B(n,j) = 0$ for all j, $s = 0$. This implies that $\lim_B u = 0$ for all u in $W_A \cap m$ so that

$$\begin{aligned} \lim_B t &= \lim_B u + (\lim_A t)(\lim_B e) \\ &= \lim_A t \end{aligned}$$

for t in $c_A \cap m$.

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CORRECTION TO

"The 'Why-Don't-You-Just...?' Barrier in Discrete Algorithms"

(This MONTHLY, 86 (1979) 30-36)

HERBERT S. WILF

The average number of comparisons in the binary search tree method is $O(k\log k)$, and not $Ck^{3/2}$ as stated. The reason is that the various trees are not equally likely to occur. A proof is in [2, vol. 3, p. 427]. This does not affect the ranking of this method in the hierarchy of the methods discussed.

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MATHEMATICAL NOTES

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

Material for this department should be sent to Professor Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis, MO 63121.

Advice to prospective authors: The editors have recently been receiving about **ten times** as many Mathematical Notes as can be used. It will simplify our work if authors will submit only especially interesting manuscripts. Mathematical Notes should be short papers of one to four printed pages which give new insights, new and improved proofs of old theorems, brief bits of mathematical folklore that have not found a home in the literature, or (occasionally!) new results that are not too technical. The topics should be of wide current interest. Papers that have already been rejected by a research journal are only very rarely suitable as Mathematical Notes.

R.P.B.

A PROOF OF THE HAIRY BALL THEOREM

MURRAY EISENBERG AND ROBERT GUY

Introduction. On the 2-sphere

$$S^2=\{\mathbf{p}\in\mathbf{R}^3:\mathbf{p}=(x,y,z),x^2+y^2+z^2=1\}$$

a **continuous tangent vector field** is a continuous vector-valued map V: $S^2\to\mathbf{R}^3$ such that at each $\mathbf{p}\in S^2$ the dot product $\mathbf{p}\cdot V(\mathbf{p})=0$. The vector field **vanishes** at $\mathbf{p}$ when $V(\mathbf{p})=\mathbf{0}$ (the zero vector); V is **nonvanishing** when it vanishes at no $\mathbf{p}\in S^2$.

The theorem of the title is:

THEOREM. *Any continuous tangent vector field on S^2 must vanish somewhere.*

This theorem is due to Poincaré [**9**, Chap. 13], who deduced it from his two-dimensional precursor of the Poincaré-Hopf theorem, which equates the sum of the indices of the vector field at its singular points with the Euler characteristic (cf. [**6**, p. 35]). The well-known proof due to Brouwer [**2**], valid for all even-dimensional spheres, uses the degree of a map between spheres, defined by means of homology groups or directly by means of simplicial subdivision (e.g., see [**5**, p. 70] and [**4**, p. 343]). Milnor [**6**, p. 31] also uses degree, but as defined by methods of differential topology. The editor has informed us that Milnor has another proof, analytic in nature, which appeared in this MONTHLY [**7**]. Boothby [**1**] gives a proof involving differential forms that avoids any overt use of algebraic topology.

Our proof will be elementary in that it uses only the fundamental group of the circle. Other elementary proofs are known. Munkres [**8**, pp. 367–368] also uses that fundamental group, but in a different way. Chinn and Steenrod [**3**, pp. 123–126] give a proof quite similar to ours, but employing the index of a plane vector field with respect to a closed curve.

Further discussions of the theorem in the context of related results appear in [**10**] and [**11**].

Prerequisites. We shall use the degree, $\deg f$, of a map f: $S^1\to S^1$ from the unit circle

$$S^1=\{(x,y)\in\mathbf{R}^2:x^2+y^2=1\}$$

to itself. This degree is an integer defined by means of the homomorphism that f induces between the fundamental groups of S^1 at a point and at its image (see [**8**, p. 373]). However, we do not need the explicit definition, but only three properties of degree together with two simple facts about homotopy:

(1) If continuous maps $f,g:S^1\to S^1$ are homotopic, then $\deg f=\deg g$.
(2) Any constant map $c:S^1\to S^1$ has degree 0.
(3) The map

$$f_2:S^1\to S^1:(x,y)\mapsto(x^2-y^2,2xy)$$

(which doubles the polar angle of each point on S^1) has degree 2.

(4) A continuous map $f:S^1\to S^1$ is homotopic to a constant map if it has a continuous extension $F:D^2\to S^1$ to the closed 2-disk $D^2=\{(x,y)\in\mathbf{R}^2:x^2+y^2\leqslant 1\}$.

(5) Any two constant maps $S^1\to S^1$ are homotopic.

Proof of Theorem. Suppose V is a continuous tangent vector field on S^2 that is nonvanishing. Express V in components as

$$V(\mathbf{p})=(A(\mathbf{p}),B(\mathbf{p}),C(\mathbf{p})).$$

Identify $\mathbf{R}^2$ with the plane $\mathbf{R}^2\times\{0\}$ in $\mathbf{R}^3$ that intersects S^2 in its equator; thereby identify D^2 with $D^2\times\{0\}$, and the equator $S^1\times\{0\}$ with the circle S^1.

Construct a continuous nonvanishing map

$$W:D^2\to\mathbf{R}^2$$

as follows. The stereographic projection through the north pole $(0,0,1)$ of S^2 maps the complement of $(0,0,1)$ onto $\mathbf{R}^2$ and maps the lower hemisphere

$$S^2_- = \{(x,y,z) \in S^2 : z \leqslant 0\}$$

homeomorphically onto D^2; let $h: D^2 \to S^2_-$ be the inverse homeomorphism. Explicitly,

$$h(x,y) = \left(\frac{2x}{x^2+y^2+1}, \frac{2y}{x^2+y^2+1}, \frac{x^2+y^2-1}{x^2+y^2+1} \right).$$

For $\mathbf{r} = (x,y) \in D^2$, define

$$W(\mathbf{r}) = (A(h(\mathbf{r})) + C(h(\mathbf{r}))x, \quad B(h(\mathbf{r})) + C(h(\mathbf{r}))y).$$

(In geometric terms, $W(\mathbf{r})$ is the vector in $\mathbf{R}^2$ from $\mathbf{r}$ to $\mathbf{p}$, where $\mathbf{p}$ is the point at which the plane $\mathbf{R}^2$ meets the line passing through $h(\mathbf{r}) + V(h(\mathbf{r}))$ and parallel to the line joining $\mathbf{r}$ to the north pole.) Clearly W is continuous. It is nonvanishing because, by direct computation, $h(\mathbf{r}) \cdot V(h(\mathbf{r})) = 0$ and $W(\mathbf{r}) = \mathbf{0}$ imply that $A(h(\mathbf{r})) = B(h(\mathbf{r})) = C(h(\mathbf{r})) = 0$, contrary to the assumption that V is nonvanishing.

Repeat the preceding construction, but starting with stereographic projection from the south pole, to get another continuous nonvanishing map

$$W^*: D^2 \to \mathbf{R}^2$$

with

$$W^*(\mathbf{r}) = (A(h^*(\mathbf{r})) - C(h^*(\mathbf{r}))x, \quad B(h^*(\mathbf{r})) - C(h^*(\mathbf{r}))y)$$

for $\mathbf{r} = (x,y) \in D^2$. Here $h^*: D^2 \to S^2_+$ is the homeomorphism onto the upper hemisphere obtained by following $h: D^2 \to S^2_-$ with the reflection $(x,y,z) \mapsto (x,y,-z)$ about the equatorial plane.

Now consider the positively oriented unit tangent vector field $T: S^1 \to \mathbf{R}^2$ on the circle, given by

$$T(x,y) = (-y,x).$$

For $\mathbf{r} = (x,y) \in S^1$, reflect each vector $\mathbf{v} \in \mathbf{R}^2$ about $T(\mathbf{r})$ to get a vector $R_\mathbf{r}(\mathbf{v})$. Then

$$W^*(\mathbf{r}) = R_\mathbf{r}(W(\mathbf{r})) \qquad (\mathbf{r} \in S^1). \tag{6}$$

In fact, for $\mathbf{r} \in S^1$ the equality $h^*(\mathbf{r}) = h(\mathbf{r})$ together with the definitions of W and W^* yield $W(\mathbf{r}) \cdot T(\mathbf{r}) = W^*(\mathbf{r}) \cdot T(\mathbf{r})$. Then $W^*(\mathbf{r}) = R_\mathbf{r}(W(\mathbf{r}))$ or $W(\mathbf{r}) = W^*(\mathbf{r})$. But in the latter case $C(h(\mathbf{r})) = 0 = C(h^*(\mathbf{r}))$ and so $W(\mathbf{r}) = W^*(\mathbf{r}) = T(\mathbf{r})$.

Since neither W nor W^* vanishes, the formulas

$$F(\mathbf{r}) = W(\mathbf{r})/|W(\mathbf{r})|, \qquad F^*(\mathbf{r}) = W^*(\mathbf{r})/|W^*(\mathbf{r})|$$

define continuous maps $F, F^*: D^2 \to S^1$. Let $f, f^*: S^1 \to S^1$ be their restrictions to the circle. From (1), (2), and (4),

$$\deg f = 0 = \deg f^* \tag{7}$$

and there is a homotopy $(H_t : 0 \leqslant t \leqslant 1)$ from f to a constant map $c: S^1 \to S^1$. In view of (5) we may assume the constant value $\mathbf{v}$ of the map c is $\mathbf{v} = (-1,0)$. By (6) we have $f^*(\mathbf{r}) = R_\mathbf{r}(f(\mathbf{r}))$ for all $\mathbf{r} \in S^1$. Hence the formula

$$H_t^*(\mathbf{r}) = R_\mathbf{r}(H_t(\mathbf{r}))$$

defines a homotopy $(H_t^* : 0 \leqslant t \leqslant 1)$ from f^* to the map

$$c^*: S^1 \to S^1 : \mathbf{r} \mapsto R_\mathbf{r}(c(\mathbf{r})) = R_\mathbf{r}(\mathbf{v}).$$

From (1) and (7), $\deg c^* = 0$.

To conclude, we show, to the contrary, that $\deg c^* = 2$. We show, in fact, that $c^* = f_2$, the map

defined in (3). Let $\mathbf{r} \in S^1$. Recall that $c^*(\mathbf{r}) = R_{\mathbf{r}}(\mathbf{v})$. By direct computation,

$$\mathbf{v} \cdot T(\mathbf{r}) = f_2(\mathbf{r}) \cdot T(\mathbf{r}).$$

Hence $f_2(\mathbf{r}) = \mathbf{v}$ or $f_2(\mathbf{r}) = R_{\mathbf{r}}(\mathbf{v})$. But the former case occurs only when $\mathbf{r} = (0, \pm 1)$, and then $R_{\mathbf{r}}(\mathbf{v}) = \mathbf{v} = f_2(\mathbf{r})$.

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MINIMAL INFINITE TOPOLOGICAL SPACES

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In this note we will describe a collection of five infinite topological spaces having the property that every infinite space contains one of the members of the collection as a subspace. We denote the set of natural numbers by ω. Consider the following five topologies with underlying set ω:

(i) *discrete*: all subsets of ω are open;
(ii) *indiscrete*: the only open sets are ω and $\varnothing$;
(iii) *cofinite*: the open sets are ω, $\varnothing$, and all subsets of ω whose complements are finite;
(iv) *initial segment*: the open sets are ω, $\varnothing$, and all sets of the form $[0,n] = \{k \in \omega : k \leqslant n\}$ where $n \in \omega$;
(v) *final segment*: the open sets are ω, $\varnothing$, and all sets of the form $[n,\omega] = \{k \in \omega : n \leqslant k\}$ where $n \in \omega$.

We will establish the following result.

THEOREM. *Every infinite topological space contains one of the preceding five spaces as a subspace.*

Note that no two of the five spaces are homeomorphic, and each of the five spaces is homeomorphic to all of its infinite subspaces. It follows that these five spaces form the smallest collection of infinite spaces satisfying the conclusion of the theorem.

Before proceeding with the proof, let us compare our result with the analogous situation in some other mathematical structures.

(a) Let G be a graph with infinitely many vertices. Ramsey ([3], cf. [2, page 15]) showed that

G must contain as a subgraph either the complete graph on countably many vertices or the totally disconnected graph on countably many vertices.

(b) Let P be a partially ordered set with infinitely many elements. From Ramsey's theorem it follows that P must contain either an infinite chain or an infinite antichain. Since any infinite chain will contain ω if it is well ordered and ω^* (the dual of ω) if it is not, P must contain one of the partially ordered sets of Figure 1 as a subset.

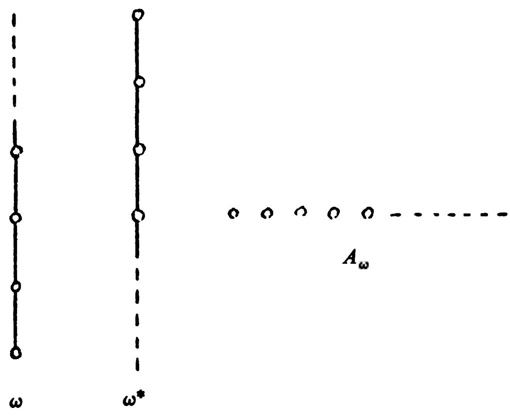
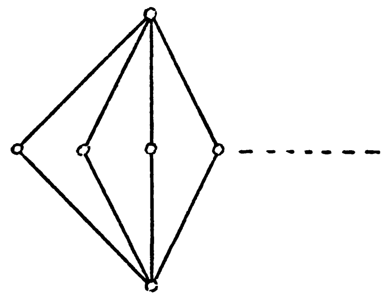


FIG. 1.



M_ω

FIG. 2.

(c) Let L be a lattice with infinitely many elements. T. P. Whaley [4, Corollary 2.4] showed that L must contain a sublattice isomorphic to ω, ω^*, or the lattice M_ω of Figure 2. We feel it worthwhile to include a proof of this result. If L contains an infinite chain, then L contains ω or ω^*, as desired; therefore we assume that all chains in L are finite. (This is a very strong assumption. For example, it implies that L is complete.) Recall that for $a,b \in L$, a is said to be a *lower cover* of b if $a<b$ and no element of L lies strictly between a and b. Every non-empty subset of L contains a maximal and a minimal element; thus every element of L other than the least element 0 contains a lower cover, and we may choose $a \in L$ such that the interval $[0,a]$ is infinite but $[0,c]$ is finite for all lower covers c of a. The set C of all lower covers of a must then be infinite. Let $c \in C$. Since $\{x \wedge c : x \in C\} \subseteq [0,c]$ is finite, there are an element b of L and an infinite subset C_1 of C such that $c \wedge x = b$ for all x in C_1. Choose an element d of L which is maximal with respect to the property that there exists $c \in C$ and an infinite subset C_1 of C such that $c \wedge x = d$ for all $x \in C_1$. The preceding argument shows that elements having this property exist, hence such a choice is possible. Choose $c_0 \in C$ and an infinite subset C_1 of C such that $c_0 \wedge x = d$ for all $x \in C_1$. Now choose $c_1 \in C_1$. Since $[0,c_1]$ is finite, there is an infinite subset C_2 of C_1 such that $c_1 \wedge x = c_1 \wedge y$ for all x and y in C_2. Furthermore, by the choice of d, we must have $c_1 \wedge x = d$ for all x in C_2. Choosing $c_2 \in C_2$ and continuing this procedure, we obtain a sublattice $\{a,d,c_0,c_1,c_2,\dots\}$ isomorphic to M_ω.

(d) In striking contrast is the situation for groups. Here, the problem of finding all "minimal" infinite groups seems to be very difficult. In fact, this problem may not even be meaningful! For example, there might exist non-isomorphic infinite groups G and H such that G contains a subgroup isomorphic to H, H contains a subgroup isomorphic to G, and every infinite subgroup of G or H is isomorphic to G or H; in this case no infinite subgroup of G or H would seem to qualify as a member of the list of "minimal" infinite groups. If such a list exists, it must be infinite (for each prime p, consider the group of rational numbers between 0 and 1 with denominators a power of p, under addition modulo 1). Furthermore, it must contain at least one nonabelian group, since infinite Burnside groups have the property that every abelian subgroup is finite (cf. [1, page 34]).

Proof of the theorem. Let X be an infinite topological space. For a subset S of X we denote the closure of S in X by $\mathrm{cl}\, S$. Consider the equivalence relation $\sim$ on X defined by $a \sim b$ iff $\mathrm{cl}\{a\} = \mathrm{cl}\{b\}$. The subspace topology on each equivalence class is the indiscrete topology. Therefore if some equivalence class modulo $\sim$ is infinite, X contains a subspace homeomorphic to (ii) above. So assume each equivalence class is finite. In particular, there are infinitely many distinct equivalence classes. Choosing one element from each class gives rise to an infinite subspace of X which satisfies the T_0 separation axiom. We may therefore assume, without loss of generality, that X itself is T_0.

Now consider the relation $\leqslant$ on X defined by $a \leqslant b$ iff $a \in \mathrm{cl}\{b\}$. Notice that since X is T_0, $\leqslant$ is antisymmetric, and it follows that $\leqslant$ is a partial order on X. If $x < y$, then every open subset of X which contains x also contains y; furthermore, $y \notin \mathrm{cl}\{x\}$, so there is an open subset of X containing y but not x. The partially ordered set $(X, \leqslant)$ is infinite, and hence contains a subset isomorphic to one of the three partially ordered sets of Figure 1. Let $A = \{a_1, a_2, \ldots\}$ with $a_1 < a_2 < \cdots$ be a subset of X isomorphic to ω. Let G be a nonempty open subset of A, and set $m = \min\{k \in \omega \mid a_k \in G\}$. Then $G\{a_k \mid k \geqslant m\}$; moreover, for each n there is an open subset of X containing a_n but not a_{n-1}, whence $\{a_k \mid k \geqslant n\}$ is open in A for each n. Thus, A with the subspace topology is homeomorphic to ω with the final segment topology (v). Similarly, let $B = \{b_1, b_2, \ldots\}, b_1 > b_2 > \cdots$, be a subset of X isomorphic to ω^*, and let $G \neq B$ be a nonempty open subset of B. It follows that $m' = \max\{k \in \omega \mid b_k \in G\}$ exists and that $G = \{b_k \mid k \leqslant m'\}$; moreover, for each n there is an open subset of X containing b_n but not b_{n+1}, whence $\{b_k \mid k \leqslant n\}$ is open in B for each n. Thus, B with the subspace topology is homeomorphic to ω with the initial segment topology (iv). Therefore we assume $(X, \leqslant)$ contains an infinite antichain, and it follows that the subspace topology on this antichain will be T_1. So, without loss of generality, we assume X is T_1.

Thus points are closed, and so every cofinite subset of X is open in X. If the topology on X is the cofinite topology, then any countable subspace of X is homeomorphic to (iii) above. Thus we may assume that no infinite subspace of X carries the cofinite topology. In particular, there is a non-empty open set U_0 in X whose complement is infinite. Choose $x_0 \in U_0$. Since the subspace topology on $X - U_0$ is not the cofinite topology, there is an open set U_1' in X such that $U_1' - U_0$ is non-empty and such that $X - (U_0 \cup U_1')$ is infinite. Choose $x_1 \in U_1' - U_0$ and let $U_1 = U_1' - \{x_0\}$. Then U_0, U_1 are open in X, $x_0 \in U_0 - U_1$, $x_1 \in U_1 - U_0$, and $X - (U_0 \cup U_1)$ is infinite. Suppose we have chosen open sets $U_0, U_1, \ldots, U_n$ in X and points $x_0, x_1, \ldots, x_n$ of X such that

(a) for all $i, j \leqslant n$, $x_i \in U_j$ iff $i = j$, and

(b) $X - \cup_{i=1}^{n} U_i$ is infinite.

Then $X - \cup_{i=1}^{n} U_i$ does not carry the cofinite topology, so there is an open set U_{n+1}' in X such that $(X - \cup_{i=1}^{n} U_i) \cap U_{n+1}' \neq \varnothing$ and $(X - \cup_{i=1}^{n} U_i) - U_{n+1}'$ is infinite. Choose a point x_{n+1} in $U_{n+1}' - \cup_{i=1}^{n} U_i$ and let $U_{n+1} = U_{n+1}' - \{x_0, x_1, \ldots, x_n\}$. Then $U_0, U_1, \ldots, U_{n+1}$ and $x_0, x_1, \ldots, x_{n+1}$ satisfy (a) and (b) above for $n+1$. By induction, we obtain a sequence of open sets $\{U_0, U_1, U_2, \ldots\}$ in X and points $\{x_0, x_1, x_2, \ldots\}$ in X satisfying (a) and (b) for all n. Let $Y = \{x_0, x_1, x_2, \ldots\}$. By (a), points of Y are open in Y. Thus Y is discrete and homeomorphic to (i) above. ■

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A PROPERTY OF QUADRATIC POLYNOMIALS

SHIGERU HARUKI

1. Let $f: R \to R$. By the mean value theorem of differential calculus it is well known (e.g., [2]) that if the quadratic polynomial $f(x)=ax^2+bx+c$, $a \neq 0$, is a solution of the functional differential equation

$$\frac{f(x+h)-f(x)}{h}=f'(x+\theta h) \qquad (0<\theta<1) \tag{1}$$

assumed for all $x \in R$ and $h \in R-\{0\}$, then $\theta=\frac{1}{2}$. Conversely, it is also well known (e.g., [4, p. 122]) that if a function f satisfies (1) with $\theta=\frac{1}{2}$ and if f''' exists, then the only solution is a quadratic polynomial. Note that simple transformations of x and h in (1) with $\theta=\frac{1}{2}$ imply the equation

$$\frac{f(x)-f(y)}{x-y}=f'\left(\frac{x+y}{2}\right). \tag{2}$$

To solve (2) the existence of the assumption f''' is rather artificial.

We will show that the general solution of (2) is given by a quadratic polynomial even if no regularity assumptions (differentiability, continuity, measurability, etc.) are imposed on f and f'.

2. We first consider the following more general equation.

THEOREM 1. *Functions $f,g,\phi: R \to R$ satisfy the equation*

$$\frac{f(x)-g(y)}{x-y}=\phi\left(\frac{x+y}{2}\right) \tag{3}$$

for all $x,y \in R$, $x \neq y$ if and only if there exist constants a, b, and c such that

$$f(x)=ax^2+bx+c, \qquad g(x)=ax^2+bx+c, \qquad \phi(x)=2ax+b \quad (=f'(x)=g'(x)) \tag{4}$$

for all $x \in R$.

Proof. Replace x by $x+y$ and y by $x-y$ in (3). Then clearly

$$\frac{f(x+y)-g(x-y)}{2y}=\phi(x) \tag{5}$$

for all $x,y \in R$, $y \neq 0$. Further, replace y by $-y$ in (5) to obtain $(f(x-y)-g(x+y))/(-2y)=\phi(x)$ so that for all u, $v \in R$,

$$\begin{aligned}
\phi(u+v)+\phi(u-v) &= \frac{1}{2y}(f(u+v+y)-g(u+v-y)+f(u-v+y)-g(u-v-y)) \\
&= \frac{1}{2y}(f(u+v+y)-g(u-v-y))+\frac{1}{2y}(f(u-(v-y))-g(u+(v-y))) \\
&= \frac{1}{2y}(2(v+y)\phi(u)-2(v-y)\phi(u)) \\
&= 2\phi(u)
\end{aligned}$$

and so $\phi(u+v)+\phi(u-v)=2\phi(u)$. If $u+v$ and $u-v$ are replaced by s and t, then

$$\phi\left(\frac{s+t}{2}\right)=\frac{\phi(s)+\phi(t)}{2} \tag{6}$$

for all s, $t \in R$, which is known as Jensen's equation [1].

Define a function $A: R \to R$ by $\phi(s)-\phi(0)=A(s)$ for all $s \in R$. Then we have $A(s)+A(t)=\phi(s)+\phi(t)-2\phi(0)=2\phi(\frac{1}{2}(s+t))-2\phi(0)=2A(\frac{1}{2}(s+t))$ for all s, $t \in R$. Set $t=0$. We find $A(0)=0$ and

$A(s)=2A(s/2)$ which by replacing s by $s+t$ implies $A(s+t)=2A((s+t)/2)$. Therefore $A(s+t)=A(s)+A(t)$ for all $s, t\in R$ ([3, p. 112]).

Thus there exist an additive function A and a constant $B=\phi(0)$ such that (5) becomes

$$\frac{f(x+y)-g(x-y)}{2y}=B+A(x). \tag{7}$$

Set $y=x$ and $y=-x$ in (7). Then we have two equations

$$f(2x)=g(0)+2Bx+2xA(x), \qquad g(2x)=f(0)+2Bx+2xA(x).$$

Further, replace $2x$ by x. Then, by the additivity of A,

$$f(x)=g(0)+Bx+\tfrac{1}{2}xA(x), \qquad g(x)=f(0)+Bx+\tfrac{1}{2}xA(x).$$

If the above two functions are substituted in (5), then one obtains $(1/(2y))(g(0)-f(0)+2By+yA(x)+xA(y))=\phi(x)$, which with $x=1$ implies $A(y)=dy+f(0)-g(0)$ where $d=2\phi(1)-A(1)-2B$. But A is additive, hence $f(0)-g(0)$ must vanish. Therefore we obtain $A(x)=dx$ and $f(x)=g(x)=f(0)+Bx+\frac{1}{2}dx^2$ as required.

Conversely, (4) always satisfies equation (3). This proves Theorem 1.

So Theorem 1 immediately implies that the general solution of (2) is given by a quadratic polynomial.

Note that in general it is not true that if $A:R\to R$ satisfies $A(x+y)=A(x)+A(y)$ for all $x, y\in R$, then $A(x)=dx$ for some constant d, when no regularity assumptions are imposed on A (cf. [1], [5]).

3. As an application of equation (3) we solve the equation

$$\frac{f(x)-f(y)}{x-y}=\frac{\phi(x)+\phi(y)}{2} \tag{8}$$

for all $x, y\in R$, $x\neq y$. This equation sometimes appears in analysis.

THEOREM 2. *In R equations* (8) *and*

$$\frac{f(x)-f(y)}{x-y}=\phi\left(\frac{x+y}{2}\right) \tag{9}$$

for all $x, y\in R$, $x\neq y$ are equivalent to each other.

Proof. It follows from Theorem 1 with $f\equiv g$ that (9) implies (8), since ϕ in (9) satisfies Jensen's equation (6).

Conversely, we show that if (8) is satisfied, then ϕ also satisfies Jensen's equation (6). To show this we replace x by $x+y$ and y by $x-y$ in (8). Then for $y\neq 0$,

$$f(x+y)-f(x-y)=y(\phi(x+y)+\phi(x-y)). \tag{10}$$

Further, replace x by $x+y$, x by $x-y$, and y by $-2y$, respectively, in (10) to obtain the following three equations

$$f(x+2y)-f(x)=y(\phi(x+2y)+\phi(x)),$$
$$f(x)-f(x-2y)=y(\phi(x)+\phi(x-2y)),$$
$$f(x-2y)-f(x+2y)=-2y(\phi(x+2y)+\phi(x-2y)).$$

If the above three equations are added, then $0=y(2\phi(x)-\phi(x-2y)-\phi(x+2y))$ for all $x, y\in R$, $y\neq 0$, which implies $0=2\phi(x)-\phi(x-2y)-\phi(x+2y)$. With $2y$ replaced by y we have $\phi(x+y)+\phi(x-y)=2\phi(x)$ for $y\neq 0$; since the equation holds trivially for $y=0$, as before we have Jensen's equation (6) for all $x, y\in R$, and hence (8) clearly implies (9), yielding Theorem 2.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

SOME QUESTIONS ABOUT ARITHMETIC PROGRESSIONS

SAMUEL S. WAGSTAFF, JR.

For integers $k \geqslant 3$ and $m \geqslant 1$, let $A_k(m)$ denote the cardinality of the largest subset of $\{0,1,\ldots,m-1\}$ which contains no arithmetic progression of k (distinct) terms. The reader may easily verify that $A_4(10)=8$ because $\{0,1,2,4,5,7,8,9\}$ contains no 4-term arithmetic progression, while every nine-element subset of $\{0,1,\ldots,9\}$ does contain one. The same example shows that $A_5(10)=8$. The elements of the forbidden arithmetic progressions are not required to be consecutive elements of the sequence. Let $a_k(m)=A_k(m)/m$.

Many of the known values of $A_k(m)$ for $k \leqslant 12$ are shown in Table 1. The isolated values in the A_5 column were produced by the greedy algorithm discussed below and the obvious triangle inequality $A_k(m+n) \leqslant A_k(m)+A_k(n)$. The former provides the lower bound $64 \leqslant A_5(94)$, while the latter and $A_k(m) \leqslant A_k(m+1)$ give

$$64 \leqslant A_5(94) \leqslant A_5(95) \leqslant A_5(96) \leqslant A_5(97) \leqslant A_5(74)+A_5(23)=64.$$

The other three isolated values may be obtained similarly. Can you find any others?

Do the fractions $a_k(m)$ tend to zero as $m \to \infty$? Erdős and Turán [6] conjectured that they do for each k. Behrend [2] proved that $\tau_k = \lim_{m\to\infty} a_k(m)$ always exists and $\tau_k = \inf_m a_k(m)$. Roth made a major breakthrough when he showed [12] that $\tau_3=0$ and moreover [13] that $A_3(m)=O(m/\log\log m)$. Next [16] the young Hungarian mathematician Szemerédi proved that $\tau_4=0$ and finally [17] that $\tau_k=0$ for every k. For this achievement, he collected the \$1,000 prize which Erdős offered for settling the problem. Recently [7] Furstenburg has proved Szemerédi's theorem using ergodic theory.

Although we now know that $A_k(m)=o(m)$, the precise growth rate of $A_k(m)$ remains a packing problem of some interest. For example, if we could prove that $A_k(m)=o(m/\log m)$, then it would follow that there are arbitrarily long arithmetic progressions of primes. Weintraub [20] found an arithmetic progression of 17 primes. The first term is 3430751869 and the common difference is 87297210. Chowla [4] proved that there are infinitely many triplets of primes in arithmetic progression.

TABLE 1

m	A_3	A_4	A_5	A_6	A_7	A_8	A_9	A_{10}	A_{11}	A_{12}	m
10	5	8	8	9	9	9	9	9	10	10	10
20	9	12	16	17	18	18	18	18	19	19	20
23	9	14	16	19	20	20	21	21	21	21	23
30	12	18	21	23	26	26	27	27	28	28	30
40	15	22	28	30	35	35	35	36	37	37	40
50	16	26	33	37	39	44	44	44	46	46	50
60	19	30	40	42	47	48	51	53	55	55	60
62	19	30	42	44	48	50	52	54	57	57	62
63	20	--	43	44	49	51	53	55	58	58	63
66	20	--	45	47	52	53	55	57	60	61	66
70	20	--	48	49	54	56	59	60	64	64	70
71	21	--	48	50	55	57	60	61	65	65	71
72	21	--	48	50	56	58	60	62	66	66	72
73	21	--	48	51	57	58	61	63	67	67	73
74	22	--	48	52	58	59	61	63	68	68	74
75	22	--	--	53	59	60	62	64	69	69	75
76	22	--	--	54	60	61	63	65	70	70	76
77	22	--	--	54	60	62	--	66	70	71	77
78	22	--	--	54	61	63	--	67	71	71	78
79	22	--	52	55	62	64	--	67	72	72	79
80	--	--	--	55	63	64	--	68	73	73	80
81	--	--	--	--	64	--	--	69	74	74	81
83	--	--	--	--	66	--	--	71	76	76	83
84	--	--	56	--	66	--	--	72	77	77	84
85	--	--	--	--	67	--	--	72	78	78	85
87	--	--	--	--	69	--	--	74	80	80	87
88	--	--	--	--	70	--	--	75	80	81	88
89	--	--	60	--	71	--	--	75	81	81	89
90	--	--	--	--	72	--	--	76	82	82	90
91	--	--	--	--	72	--	--	77	83	83	91
92	--	--	--	--	72	--	--	--	84	84	92
94	--	--	64	--	72	--	--	--	86	86	94
95	--	--	64	--	--	--	--	--	87	87	95
96	--	--	64	--	--	--	--	--	88	88	96
97	--	--	64	--	--	--	--	--	89	89	97
109	--	--	--	--	--	--	--	--	100	100	109
114	--	--	--	--	--	--	--	--	100	104	114
115	--	--	--	--	--	--	--	--	--	105	115
122	--	--	--	--	--	--	--	--	--	112	122
127	--	--	--	--	--	--	--	--	--	112	127
Least $a_k(m)$	$\frac{22}{79}$	$\frac{30}{62}$	$\frac{48}{74}$	$\frac{55}{80}$	$\frac{72}{94}$	$\frac{58}{73}$	$\frac{61}{74}$	$\frac{75}{89}$	$\frac{100}{114}$	$\frac{112}{127}$	

Analysis of Szemerédi's proof [17] shows that $A_k(m)=O(m/\log_{f(k)}m)$ as $m\to\infty$, where $\log_r m$ is the r-fold iterated logarithm and $f(k)$ is a swiftly increasing function of k. Several authors [6], [14], [3], [10] have found lower bounds on $A_k(m)$, the greatest being that of Rankin [11]: $A_k(m)>m\exp(-c\log^{\kappa}m)$, where $\kappa=1/\lceil\log k/\log 2\rceil$ and $\lceil x\rceil$ is the least integer $\geqslant x$. All the theoretical lower bounds on $A_k(m)$ are proved by considering numbers which, when written in a certain radix, have restrictions on their digits which prevent long arithmetic progressions. There is still a considerable gap between the best known upper and lower bounds for $A_k(m)$.

Erdős [5] has offered \$3,000 for a proof or disproof of the statement: If $\sum_{i=1}^{\infty}1/a_i$ diverges,

where $\{a_i\}$ is an increasing sequence of positive integers, then the sequence contains arbitrarily long arithmetic progressions. Since the sum of the reciprocals of the primes diverges, the statement would imply that there are arbitrarily long arithmetic progressions of primes. Let $M_k(i)$ denote the least m such that $A_k(m) \geqslant i$. If $\{a_i\}$ contained no arithmetic progression of k terms, then $A_k(a_i) \geqslant i$ and hence $a_i \geqslant M_k(i)$. Therefore Erdős's statement would hold if $\sum_{i=1}^{\infty} 1/M_k(i) < \infty$ for each k. As a first step, one might try to prove (or disprove) $\sum_{i=1}^{\infty} 1/M_3(i) < \infty$. Gerver **[8]** has broken a lance on this problem of Erdős.

The greedy algorithm provides a lower bound for $A_k(m)$: Begin a sequence with 0 and consider the numbers $1,2,\ldots,m-1$ seriatim. Insert each one into the sequence if and only if it is not the last term of a k-term progression whose other members are already in the sequence. If k is prime, this sequence consists of exactly those numbers which lack the digit $k-1$ when expressed in radix k. In any case, $A_k(m)$ is at least the cardinality of our sequence. For large m, the construction of Rankin **[11]** produces a much denser sequence. But, for $m<k^2$, the greedy algorithm gives a bound which often equals $A_k(m)$. For prime k, the two are the same for $m \leqslant k(k-1)+1$. Probably it would not be too hard to determine $A_k(m)$ for all k and $m \leqslant k^2$. We have $A_k(m)=m$ for $1 \leqslant m < k$ and $A_k(k)=k-1$. How dense is the greedy algorithm sequence for composite k?

Szekeres (see **[6]**) once conjectured that

$$A_k(k^h-(k^h-1)/(k-1))=(k-1)^h$$

for prime k and positive integers h. (The right side is the greedy algorithm lower bound for the left side.) Equality always holds for $h=1$ and 2 and also for the other values for which the left side is known (see **[19]**). Although the conjecture has been disproved for each k (see **[14]** and **[11]**), no one has ever given an explicit counterexample. For each k, what is the least h for which equality fails?

Several people have used a computer (as in **[18]**) to calculate $A_k(m)$ for small k and m. In 1971, Tom Brown found $A_3(m)$ for $54 \leqslant m \leqslant 79$. A year later, Karl Rubin calculated $A_6(m)$, $A_7(m)$, and $A_8(m)$ for m from 53, 54, and 58, respectively, to the end of those columns in the Table. The other entries in the Table with $m \geqslant 50$ or $k \geqslant 4$ were done by the author. The first part of the A_3 column was done by hand (**[6]**, **[9]**, **[10]**).

Using the fastest computers available today, it takes only a few seconds to compute $A_k(m)$ for all $m \leqslant 50$. However, there is a dramatic increase in running time as one approaches the last entry in each column. Using the standard backtracking algorithm **[18]**, we attempted to calculate the first missing entry in each column, but each of these jobs remained unfinished after more than an hour of CPU time. Our next question is whether there is a faster algorithm for calculating $A_k(m)$ or is the function really difficult to compute in some sense? For example, is there a way to determine $A_k(m)$ whose running time is a polynomial in m or is the function NP-hard? (There is a large class of functions, called NP-complete, widely believed to be not computable in polynomial time, and all equally hard to compute in the sense that if one of them could be computed in polynomial time, then all could be. A function is called NP-hard if it is at least as hard to compute as the NP-complete ones. See Chapter 10 of **[1]**.)

All known ways of computing $A_k(m)$ produce **witnesses**, that is, sequences of length $A_k(m)$ from $0,1,\ldots,m-1$ which contain no progression of k terms. For example,

$$0,1,2,4,5,7,9,12,14,15,16,24,26,27,28,31,$$
$$32,33,35,43,44,45,47,50,52,54,55,57,58,59$$

is a witness for $A_4(60)=30$. How many witnesses are there? The full symmetric group on m letters acts in an obvious way on the set of all sequences of length $A_k(m)$ from $0,1,\ldots,m-1$. What is the largest subgroup which stabilizes the set of all witnesses? It always has at least two elements (because if $\{a_i\}$ is a witness, then so is $\{m-1-a_i\}$) and frequently contains many more.

See [5] and [15] and the references there for more problems about arithmetic progressions. The author will send a copy of the full Table to anyone interested.

The author is grateful to Carl Pomerance for supplying references [8] and [15]. He thanks Richard Guy for suggestions which improved the readability of the article.

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CLASSROOM NOTES

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

Material for this department should be sent to Professor Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis, MO 63121.

Advice to prospective authors: The editors have recently been receiving about **ten times** as many Classroom Notes as can be used. It will simplify our work if authors will submit only especially interesting manuscripts.

R.P.B.

LESS THAN OR EQUAL TO AN EXERCISE

HARLEY FLANDERS

In high school one stiffens muscles with hundreds of exercises of the substitution form, e.g., starting with $x^2-y^2=(x+y)(x-y)$, factor $4x^2-9y^2, x^2y^2-16z^2$, etc.; or starting with the binomial formula $(x+y)^3=x^3+3x^2y+3xy^2+y^3$, expand $(x+3y)^3$, $(y-2x)^3$, etc. The results are hardly noteworthy!

In college, one can also substitute in basic relations; the results may appear exotic, but again are hardly noteworthy. For example,

$$e^{\sin x} \geqslant 1+\sin x, \qquad e^{2x}+(\ln x)^2 \geqslant e^x \ln(x^2)$$

are merely substitutions in

$$e^x \geqslant 1+x \quad \text{and} \quad x^2+y^2 \geqslant 2xy;$$

fine in an exercise set, but hardly more.

Now how shall we evaluate a slightly more sophisticated substitution? Suppose we have a relation established for a broad class of functions. Is it interesting to publish particular cases, or are they merely exercises?

For example, a C'' function $f(t)$ on an interval $a<t<b$ satisfying $f''(t)>0$ is strictly convex. Hence (Jensen's inequality) if $p_1,\ldots,p_n>0$, $\Sigma p_i=1$, and $a<x_i<b$, then

$$f(\Sigma p_i x_i) \leqslant \Sigma p_i f(x_i),$$

with equality if and only if $x_1=\cdots=x_n$.

This is an important result, about as central for inequalities as the binomial theorem is for high school algebra. And some of its specializations are really important, e.g., take $f(x)=-\ln x$. Then the A-G inequality falls out. Three exericises:

(1) $$f(t)=\ln\left(\frac{1}{t}+1\right), \qquad 0<t.$$

Then

$$1+\frac{1}{\Sigma p_i x_i} \leqslant \Pi\left(\frac{1+x_i}{x_i}\right)^{p_i}.$$

If $p_i=1/n$ and it happens that $\Sigma x_i=1$, then

$$(1+n)^n \leqslant \frac{\Pi(1+x_i)}{\Pi x_i}.$$

(2) $$f(t)=\ln\left(\frac{1}{t}-1\right), \qquad 0<t<1$$

Then

$$\frac{1}{\Sigma p_i x_i}-1 \leqslant \Pi\left(\frac{1-x_i}{x_i}\right)^{p_i}.$$

If $p_i=1/n$ and we suppose $\Sigma x_i=1$, then

$$(n-1)^n \leqslant \frac{\Pi(1-x_i)}{\Pi x_i}.$$

(3) $$f(t)=\ln\left(\frac{1+t}{1-t}\right), \qquad 0<t<1.$$

Then

$$\frac{1+\Sigma p_i x_i}{1-\Sigma p_i x_i} \leqslant \Pi\left(\frac{1+x_i}{1-x_i}\right)^{p_i}.$$

Once again, if $p_i = 1/n$ and $\Sigma x_i = 1$, then the result specializes to

$$\left(\frac{n+1}{n-1}\right)^n \leqslant \frac{\Pi(1+x_i)}{\Pi(1-x_i)}.$$

Moral: When you find a "new" inequality, check to see if it isn't just a routine specialization of Jensen's inequality or something equally common.

The third example was given in [1] with a proof analogous to one of the more inelegant proofs of the A-G inequality. For a more complicated proof, see [2].

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THE CURIOUS SUBSTITUTION $z = \tan\theta/2$ AND THE PYTHAGOREAN THEOREM

A. E. STRATTON

In an article with the above title, appearing in this section [1], Alan H. Schoenfeld remarks that the usual approach in calculus texts to the substitution $z = \tan\theta/2$, in order to transform any rational function of $\sin\theta$ and $\cos\theta$ to a rational function in z, "deprives students of a means for deriving the substitution and a frame of reference for it." This is certainly true!

Surely the best frame of reference for this substitution is that from whence I presume it derived. Namely, the problem of finding a rational parametrization of the circle.

Consider the unit circle $x^2 + y^2 = 1$ and the problem of projecting this circle onto the y-axis from the point $(-1,0)$. The straight line $y = m(x+1)$, of slope m, passing through $(-1,0)$, intersects the circle in two points, one of which is $(-1,0)$, the x-coordinate of the other being found by solving the equation

$$0 = x^2 + m^2(x+1)^2 - 1 = (x+1)\{(1+m^2)x + m^2 - 1\}.$$

Thus the second point of intersection is given by

$$x = \frac{1-m^2}{1+m^2}, \qquad y = \frac{2m}{1+m^2} \tag{1}$$

yielding a rational parametrization of the unit circle with the point $(-1,0)$ omitted.

Since the more usual parametrization of the circle is found by putting $x = \cos\theta$, $y = \sin\theta(-\pi < \theta \leqslant \pi)$ equations (1) express $\cos\theta$ and $\sin\theta$ as rational functions of m.

In order to relate m and θ consider Figure 1. Clearly $m = \tan\phi$. But since the angle subtended by a chord at the center of a circle is twice that subtended by the same chord on the circumference of the circle it is clear that $\phi = \theta/2$. Thus equations (1) become

$$\cos\theta = \frac{1-\tan^2\theta/2}{1+\tan^2\theta/2}, \sin\theta = \frac{2\tan\theta/2}{1+\tan^2\theta/2}.$$

Finally we see that we have a bijection between all points of the circle with the point $(-1,0)$ omitted and the points on the y axis, given by

$$(\cos\theta, \sin\theta) \leftrightarrow \tan\theta/2(-\pi < \theta < \pi).$$

We may, of course, find many other substitutions which transform $\cos\theta$ and $\sin\theta$ into rational functions by considering the projections of an ellipse $(x^2/a^2) + (y^2/b^2) = 1$ onto the

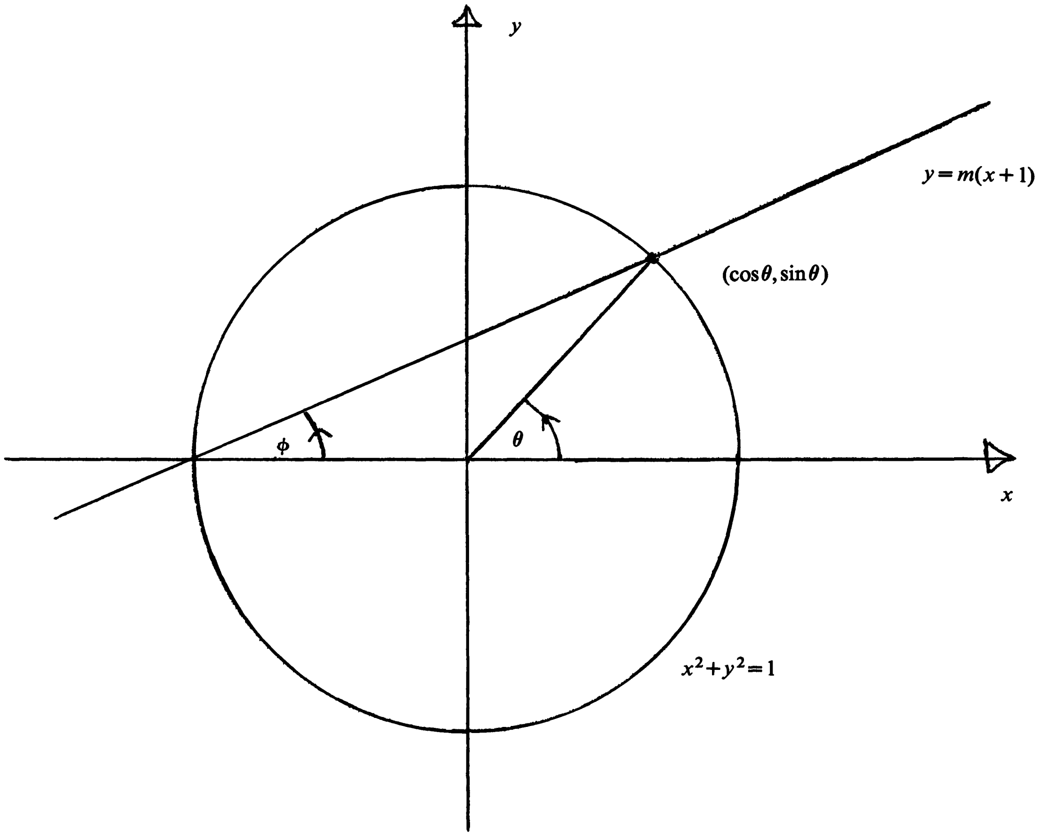

FIG. 1

y-axis and utilizing the parametrization

$$(a\cos\theta, b\sin\theta) \qquad -\pi<\theta\leqslant\pi$$

for this curve.

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THERE IS NO DIFFERENTIABLE METRIC FOR R^n

IRA ROSENHOLTZ

The usual metric on R^n is unfortunately not differentiable. This makes the proof of the "smooth" Urysohn Lemma, for example, somewhat more difficult than the "metric proof." Often the function e^{-1/x^2} is employed. It would be very nice to have a differentiable metric for R^n. A reasonable try might be $d(x,y)=e^{-1/|x-y|^2}$. Unhappily, this doesn't work, and in fact we show that all attempts must fail. In addition to this result's being somewhat surprising in itself, perhaps it can be used to show that certain trianglelike inequalities do not hold.

THEOREM. *There is no differentiable metric for R^n.*

Proof. It suffices to show that there is no differentiable metric for R. (For if $\rho: R^n \times R^n \to R$ were a differentiable metric for R^n, then $d: R\times R \to R$ defined by $d(x,y) = \rho((x,0,0,\ldots),(y,0,0,\ldots))$ would be a differentiable metric for R.)

So suppose $d: R\times R\to R$ were a differentiable metric. Firstly, the first partial derivatives of d must be 0 on the diagonal, roughly because d attains an absolute minimum at each point of the diagonal. More precisely,

$$(D_1 d)(x,x) = \lim_{z\to x} \frac{d(z,x)-d(x,x)}{z-x},$$

and this limit must be 0 by considering $z>x$ and $z<x$. Similarly, $(D_2 d)(x,x)=0$. (Notice that we have used the hypotheses that $d(z,x)\geqslant 0$ and $d(x,x)=0$.)

Next, since $d(x,z)\leqslant d(x,y)+d(y,z)$, we see that for $z>y$,

$$\frac{d(x,z)-d(x,y)}{z-y} \leqslant \frac{d(y,z)-d(y,y)}{z-y}.$$

Thus,

$$\lim_{z\to y+} \frac{d(x,z)-d(x,y)}{z-y} \leqslant \lim_{z\to y+} \frac{d(y,z)-d(y,y)}{z-y}$$

and therefore $(D_2 d)(x,y)\leqslant (D_2 d)(y,y)=0$.

If we consider $z<y$, we reverse all the inequalities and obtain $(D_2 d)(x,y)=0$. A similar argument implies that $(D_1 d)(x,y)=0$, and so the partials are 0 at every point. The mean value theorem implies that d is identically 0, and this is a contradiction. (Here we used the triangle inequality and the fact that if x and y are distinct, then $d(x,y)>0$.)

This completes the proof.

In contrast to this theorem, it is interesting to note that there is a differentiable metric for the usual middle-thirds Cantor set. The reader is invited to find it.

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF WYOMING, LARAMIE, WY 82071.

MATHEMATICAL EDUCATION

EDITED BY W. E. MASTROCOLA

Material for this department should be sent to W. E. Mastrocola, Department of Mathematics, Colgate University, Hamilton, NY 13346.

A COMPUTER-GRADED EXAMINATION TECHNIQUE WITH A HUMAN FACE

W. F. PFEFFER, D. P. MELCON, A. P. FENECH

This paper describes an experiment on methods of testing in a service course which in one form or another is offered by many large universities and colleges. In the UC-Davis catalog the course (Mathematics 16A, B, C) is described as "a short course in analytic geometry and differential and integral calculus. Not recommended for students who may wish to major in the mathematical sciences." The purpose of the course is to give appropriate mathematical background to students whose need for mathematics is peripheral, providing them with basic

technical skills. Classes consist mainly of students who enroll for reasons other than interest in mathematics. Large class sizes (up to 200) encourage the use of computer-graded tests. These tests are extremely unpopular with students, who not only consider the tests unfair and impersonal but attribute to them many other evils. We do not want to argue about the validity of students' objections to computer-graded tests. The dissatisfaction, whether justified or not, seemed to us sufficient reason to try to develop a more satisfactory testing procedure.

The experiment, which involved Mathematics 16A, was carried out in the Fall Quarter of 1976. We used [3] as a textbook, covering Chapters 1 and 2 and Sections 1–3 of Chapter 3. We prepared 234 multiple-choice, multiple-answer (abbreviated MCMA) questions, which, together with their answers, were made available to the students at the beginning of the quarter (see [1]). The following is a typical question from our collection (see [1], problem 160).

Let $s(t)=\frac{t^5}{5}+\frac{2}{3}t^3+t,\ t\geqslant 0$, be the position of a particle at time t. Which of the following answers are correct?

(a) At time $t=1$ the acceleration of the particle is 4.
(b) When the acceleration is 0, the velocity is 0.
(c) The average acceleration of the particle from time $t=1$ to $t=2$ is 21.
(d) When the velocity of the particle is 4, the acceleration of the particle is 8.
(e) $\frac{d^5}{dt^5}[s(t)]=24$.

We explained to the students that all tests would be composed entirely of questions from the list of MCMA questions [1] or from the exercises in that part of [3] which was covered in the lectures. Thus no test contained a problem which students were unable to anticipate. Course grades were based on twenty-minute weekly quizzes, two one-hour midterms, and a two-hour final. The quizzes, which were hand-graded, consisted of not more than two problems taken directly from [3]. Midterms and the final were made up of MCMA questions and were computer-graded. MCMA questions were scored by the method of "correct response," i.e., a student received one point for each correct answer identified and for each incorrect answer left unmarked. To minimize the benefit of guessing, we constructed the MCMA tests so that the total numbers of correct and incorrect statements were approximately the same. Students who were not satisfied with their midterm scores could take up to two makeups for each midterm. The midterm score used in the grade determination was that of the last makeup the student took. There was no makeup for the final. Homework was regularly assigned but not graded. Each examination, including quizzes and makeup midterms, was graded on a scale from 0 to 100. A student's course score was given by

$$S=\frac{1}{4}\left(m_1+m_2+f+\frac{q}{2}+50\right),$$

where

m_1 = the score of the first midterm or its makeup;
m_2 = the score of the second midterm or its makeup;
f = the score of the final;
q = the average score of all quizzes.

Finally, a student's course grade was determined according to the following table:

Score	$\geqslant 97$	$\geqslant 94$	$\geqslant 91$	$\geqslant 88$	$\geqslant 85$	$\geqslant 82$	$\geqslant 78$	$\geqslant 72$	$\geqslant 68$	$\geqslant 65$	$\geqslant 62$	$\geqslant 59$	< 59
Grade	A+	A	A−	B+	B	B−	C+	C	C−	D+	D	D−	F

To see the reason for this distribution of grades, observe that the student can achieve $S=50$

without any work. Since on each MCMA examination approximately one-half of the alternatives are correct, the student can achieve $m_1 = m_2 = f = 50$ simply by marking all or none of the alternatives. So the relevant scores for m_1, m_2, and f are between 50 and 100. On the other hand, the relevant scores for q are between 0 and 100. This observation indicates why the simpler formula $S = \frac{1}{4}(m_1 + m_2 + f + q)$ was not used. All of this, including the formula for S and the grade table, was carefully explained to the students at the beginning of the quarter.

This grading system was quite successful. The students' response was overwhelmingly enthusiastic. Among the 69 students who filled in the class evaluation questionnaire, only one made a negative comment, complaining of a lack of challenge. The atmosphere in the classroom was relaxed and friendly throughout the quarter, something which is not always the case in this type of course. The opportunity to take makeup midterms reduced examination pressure, and students realized that they did not need to compete against one another for a good grade. Examination construction was quite simple; with the set of MCMA questions in hand, it was a matter of minutes to construct a good test. Finally, the distribution of final grades in this class was normal; the mean and standard deviation of the scores S were 80.18 (corresponding to a grade of C+) and 16.07, respectively.

In spite of these positive features, this grading system cannot be endorsed without answering the important question: Will students who successfully complete this course have learned as much as students who have taken this course using a traditional computer-grading system? In other words, is it possible that we merely "taught" our students how to answer 234 MCMA questions from [1] plus another 200 or so problems from the textbook [3]? Before dealing with this question, a clarification of our use of the word "learning" is in order. For our purposes, learning is measured by performance on standard classroom examinations. Although its appropriateness can be questioned, this measure of learning is well defined, adequately reliable, and widely used.

Two assumptions led us to expect that students would learn as well under the experimental system as under the traditional one.

(i) The number of questions (approximately 1,370) is so large that no normal student can mechanically memorize all of the answers.

(ii) Students will learn the techniques of calculus by solving the MCMA questions and the text questions.

With respect to the first assumption, it quickly became clear that while students could not memorize all of the answers, they certainly memorized some. Most students tried to memorize the answers to those problems which they found too difficult to solve. Thus their approach to the tests was based partially on knowledge and understanding and partially on memorization.

In order to assess the validity of the second assumption, we compared the experimental class with a control section of Math 16A which used the traditional computer-graded testing system. The comparison was made by looking at the performance of both classes on a special examination given just prior to the Math 16A final examination and during the next quarter in Math 16B. The classes met at the same time and were tested in the same manner, i.e., with weekly quizzes, two midterms, and a final. The only differences were that the examination questions for the control class were chosen arbitrarily, not from any prescribed set of problems, and there were no makeup midterms. Final grades in the control class were assigned using the standard curve system.

To measure the initial level of both classes, we gave them the same homework, consisting of 28 MCMA questions, on the first day of instruction. The homework was described as a diagnostic test which would also give the students some practice with MCMA questions. We called this homework the "initial control test" (ICT). A few days before finals, we gave both classes an identical test consisting of 10 MCMA questions. We called this test the "final control test" (FCT). The questions on the FCT were prepared by a third person and they did not coincide with any questions in [1] or [3]. And so, to the control class, this was just another

midterm, while to the experimental class the FCT was an entirely new experience. Prior to taking the FCT, both classes were told that the FCT was a required third midterm which would be incorporated into the course grade of only those students who would benefit by it. In order to avoid the Hawthorne effect (see [2]), we explained the real purpose of the FCT only afterwards.

Unfortunately, in the comparison of the classes the influence of the instructor is an uncontrolled variable whose importance is hard to estimate. However, student evaluations over a period of several years have given the instructors comparable ratings.

An extensive statistical analysis was carried out, comparing the FCT scores of the two classes and the Z scores in Mathematics 16B of those students from the two classes who completed Mathematics 16B. The analysis of covariance model was employed, using ICT scores and Z scores in Mathematics 16A as covariates. All of the analyses support the formal assertion that there is no significant difference in learning between the experimental class and the control class. However, parameter estimates in the covariance model consistently suggest the students learn slightly less under our testing method than under the control testing method. In terms of Z scores in 16B, for example, the estimated difference in performance between a student in our class and a comparable student in the control class is one quarter of a standard unit, with an estimated standard error of the same size. In light of these facts, we draw the following conclusion: The data collected in this experiment support our second assumption; students learn about as well under the experimental testing method as they do under the traditional one. Since the experimental testing method also creates a teaching situation which is very comfortable for both students and instructor in Mathematics 16A, our testing method becomes an attractive alternative to the traditional.

We examined another interesting point. In the ICT we asked the students to write down the grades which they expected to earn in the course. We called a student "successful" if the grade actually received was better than or equal to the expected grade with a minus sign (e.g., a student who expected a C and received a C− or better was "successful"). It turned out that the percentages of "successful" students in our class and in the control class were very close; 49% in our class versus 46% in the control class.

Acknowledgements. In designing the experiment the first author benefited from discussions with Ohmer Milton and John Vohs. The FCT was prepared by Jim Diederich. Throughout the experiment we were assisted by Peter Linz, who taught the control class. During the preparation of the experiment, the first and second authors were partially supported by a Summer Planning Award from the University of California at Davis.

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TEACHING 200 STUDENTS IN A PERSONAL WAY

G. T. SALLEE

With enrollments in mathematics service courses continually increasing, but the size of the faculty rarely enlarged, the trend toward large classes is practically irresistible. Classes of 100, 200, or even 500 students are not unheard of, if a lecture hall that size can be found. In such a case, how can you keep the students there from remaining mere passive note-takers, uninvolved

observers of the scene? At the same time, how can you try to answer the questions of a reasonable percentage, so that they do not get too lost in the advancing logic? Here is one approach that I have used successfully several times; it seems to keep students in large classes both interested and learning.

The basic idea is to break up your lecture at strategic points to allow the students to discuss *with each other* what is going on. Then you can circulate in the classroom, or at least around its periphery, observing, listening, and answering an occasional question. You get immediate feedback as to whether you can continue or whether a concept needs further explanation.

Students seem to be very pleased with this process. In the dozen or so times I have tried the technique, I have never had one student request a change—and I get anonymous feedback from everyone twice each quarter. Moreover, it seems to be an efficient way of teaching; my classes always are a day or two ahead of the syllabus, and no one seems to feel pressured. A further psychological bonus is that the class feels in charge of the pace; this has another positive effect.

Let me describe a typical class presentation of the product and quotient rules. First I describe the topics for that lecture and motivate them (in this case, it is easy). Then I state the product rule and do one or two examples. Next I put three examples on the board (easy, medium, and fairly hard) for the students to do in groups. While they are working, I go around the room listening, observing, asking some questions, and answering some questions. When most groups are done, I call the class back together, ask for answers, write them down, and go on to give the proof of the product rule. There is another short break while they discuss the proof. Then we go on to the quotient rule and a similar treatment, except that there probably won't be time for the proof.

Some warnings. First, this technique is not for every class. It works best in courses with small concepts, such as beginning statistics or calculus. Asking students to construct a proof in groups is a sure-fire failure (although explaining proofs to each other after seeing them is valuable). Roughly speaking, group discussions are a help in assimilating algorithms and understanding ideas; they are useless for creative work.

Second, in order to use this approach successfully you must be in complete control of your class. Some students will want to keep on discussing things when it is time for you to lecture again, or they will begin talking early if they miss a point. If this gets started, you will find it hard to stop.

Third, you must use the technique wholeheartedly, especially at the beginning. The first time you explain it, give the students a problem to do in groups of three or four, and walk around the room to *be sure* they are working in a group. If someone is working alone, ask him which group he is in; tell people sitting alone to move. This is *very* important, because students tend to be shy about exposing their ignorance. For the same reason, things get started more smoothly if the first few class sessions involve easy questions which are not threatening and which allow for some socializing (the time is well spent).

With these caveats, however, I recommend the technique. Students enjoy class, they learn, and teaching a large class can be fun for you.

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PROBLEMS AND SOLUTIONS

EDITED BY A. P. HILLMAN

The editors receive more problems than they can use, but they are always on the lookout for interesting and novel problems. The task of selection will be much easier if proposers will keep the following guidelines in mind:

Send all proposed problems, in duplicate if possible, to Prof. A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, N.M. 87131. Please include solutions and any information that will help the editors, including reasons that the problem is interesting. Problems in well-known textbooks and results in generally accessible sources are not acceptable.

Solutions should be sent to the addresses given at the head of each problem set.

An asterisk (*) indicates that neither the proposer nor the editors supplied a solution. If you submit a problem without a solution, you should tell the editors whether you know (or somebody else knows) how to solve the problem. If you are uncertain, the problem might be more suitable for the Research Problems section; see the instructions given there.

Proposers are asked to aim for the same audience as for the rest of the MONTHLY; *a rule of thumb is to think of people who have had at least a year of graduate work in Mathematics, very likely some years ago. Proposals that can be both understood and solved by high school students, or by those in the first two years of college, are usually more appropriate for the Mathematics Magazine or the Two-Year College Mathematics Journal. No precise line can be drawn between elementary and advanced problems, but in general an elementary problem will involve notions no more difficult than those encountered in the first year of graduate work and should be solvable by methods common at the same level. An advanced problem may deal with more advanced concepts or require more sophisticated techniques for its solution.*

A problem that takes half a page or more to state, or that involves several ad hoc definitions, is usually unsuitable. The statement of a problem should avoid specialized jargon and symbols whenever the English language is more readily understood: for example, "f is a continuous function" is preferable to "$f \in C$."

Before submitting a problem, be sure that it says exactly what you intend it to say. Solvers are not telepathic and will usually solve what was proposed, not what you had in mind.

PROBLEMS DEDICATED TO EMORY P. STARKE

Solutions of the problems in this issue dedicated to Professor Emory P. Starke should be mailed to Prof. A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, N.M. 87131 (USA) before January 31, 1980. To facilitate consideration, solutions should be typed (with double spacing).

S 16. *Proposed by I. J. Schoenberg, University of Wisconsin, Madison.*

Characterize the closed sets S of the complex plane such that $d(z + w) \leqslant d(z) + d(w)$ for all complex numbers z and w, where $d(z)$ denotes the euclidean distance from z to S.

S 17. *Proposed by Leonard Gillman, University of Texas, Austin.*

When the upstairs switch is in one position, the downstairs switch turns the stairway light on and off as it should, but when the upstairs switch is in the other position the stairway light remains off irrespective of the position of the downstairs switch. Which is the defective switch?

S 18. *Proposed by V. E. Hoggatt, Jr., San Jose State University, and P. L. Mana, Albuquerque, New Mexico.*

Let $\{a_n\}$ be defined by $a_1=1$, $a_{n+1}=2+a_n$ if n is in $A_n=\{a_1,a_2,\ldots,a_n\}$, and $a_{n+1}=1+a_n$ if n is not in A_n. Also let $a_0=0$. For integers k and n with $0\leqslant k\leqslant n$, let $\left[\begin{matrix} n \\ k \end{matrix}\right]=a_n-a_k-a_{n-k}$. Prove that:

(a) There are an infinite number of integers m such that $\left[\begin{matrix} m \\ k \end{matrix}\right]=1$ for $0<k<m$.

(b) There are an infinite number of integers r such that

$$\left[\begin{matrix} r-s+t \\ t \end{matrix}\right]=\left[\begin{matrix} s \\ t \end{matrix}\right] \text{ for } 0\leqslant t\leqslant s\leqslant r.$$

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be mailed in duplicate to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA 94393 (USA) by January 31, 1980. Please type with double spacing and place the solver's name and mailing address on each sheet. If acknowledgment is desired, include a self-addressed card or label.

E 2747 [1978, 824] (correction). *Proposed by H. L. Krall, Pennsylvania State University, and Emil Grosswald, Temple University.*

The following part of the problem was inadvertently omitted:

(b) Compute the determinant of the matrix $B=(b_{ij})$ with

$$b_{ij}=(-1)^{i+j+1}2^{i+j+1}/[(i+j+1)!] \quad \text{for } i,j\in\{1,2,\ldots,n\}.$$

E 2785. *Proposed by Stephen M. Gagola, Jr., Texas A & M University.*

A *flat* X in a vector space V over a field F is defined to be a coset of a maximal subspace of V. Assume that F is finite with q elements. If V has dimension n and $V-\{0\}$ is the union of m flats, prove that $m\geqslant n(q-1)$.

E 2786. *Proposed by Walter Stromquist, George Washington University.*

The consecutive integers 31 and 32 have these properties: the larger one is twice a square, and the sum of the digits in both numbers is a square.

(a) How many pairs of consecutive integers have the same properties?

(b) Would there exist such a pair if we used base 3 instead of decimal notation?

(c) Does such a pair exist in any odd base other than 3?

E 2787. *Proposed by James V. Whittaker, University of British Columbia.*

Show that if $k\geqslant 3$, then the equation $(\log x)^k=x$ for $x\geqslant 1$ has just two solutions r_k and s_k, where $r_k\to e$ and $s_k\to\infty$ as $k\to\infty$.

E 2788. *Proposed by Kwang-Nan Chow and David Protas, California State University, Northridge.*

Let $\{u_n\}$ be any sequence of real numbers such that $\{u_n\}\to\infty$ and $\{\cos u_n\}$ converges. Does there always exist a real number c such that $\{\cos cu_n\}$ diverges?

E 2789. *Proposed by Doug Hensley, Texas A & M University.*

Suppose $\gcd(n,30)=1$ and $n\geqslant 13$. Let S_n be a set of n points equally spaced around a circle.

Show that there are $(n^2-1)/12$ incongruent triangles with vertices in S_n. Show further that their areas are distinct when n is a prime.

E 2790. *Proposed by Mark D. Meyerson, United States Naval Academy.*

Suppose we have a collection of squares, one each of area $1/n$ for $n=1, 2, 3, \ldots$, and any open set, G, in the plane. Show that we can cover all of G except a set of area 0 by placing some of the squares inside G without overlap. (The edges of the squares are allowed to touch.)

SOLUTIONS OF ELEMENTARY PROBLEMS

Product of a Number and Its Reverse

E 1243 [1956, 724; 1957, 434; 1977, 58]. *Proposed by M. A. Rashid and M. A. Uppal, Lahore, Pakistan.*

Prove that the product of a number consisting of two digits and its reverse is never a square except when the two integers are equal.

Comment by Rolf Winkel, high school student, Hirschberg, Germany. There are examples that show that the product of a number with four or more digits and its reverse can be a square, even when neither of the two factors is a square [see 1977, 58]. There are precisely three examples involving numbers with three digits. They are: $288\cdot 882=504^2$, $528\cdot 825=660^2$, $768\cdot 867=816^2$.

An Unbounded Integral

E 2706 [1978, 198]. *Proposed by David L. Lovelady, Florida State University.*

Let $f(t)=g(t)\int_0^t g(s)^{-\alpha}ds$, where $\alpha>1$ and g is a positive continuous function on $[0,\infty)$. Prove that f is unbounded. Is this true if $\alpha=1$?

Solution I by Ellen Hertz, Belford, N.J. Let $h(s)=g(s)^{-\alpha}$ and $H(t)=\int_0^t h(s)ds$. Then $H'(t)=h(t)$ and $f(t)=h(t)^{-1/\alpha}H(t)$. Suppose $f(t)\leqslant B$ for some constant $B>0$. Then $h(t)H(t)^{-\alpha}\geqslant C$ for all $t>0$ where $0<C=B^{-\alpha}<\infty$. Equivalently,

$$\frac{1}{1-\alpha}\frac{d}{dt}\left[H(t)^{1-\alpha}+C(\alpha-1)t\right]\geqslant 0,$$

so that

$$\frac{d}{dt}\left[H(t)^{1-\alpha}+C(\alpha-1)t\right]\leqslant 0.$$

Hence if $t\geqslant 1$, then

$$H(t)^{1-\alpha}+C(\alpha-1)t\leqslant H(1)^{1-\alpha}+C(\alpha-1);$$

i.e.,

$$H(t)^{1-\alpha}\leqslant H(1)^{1-\alpha}+C(\alpha-1)(1-t)$$

which implies that $H(t)$ is eventually negative, a contradiction. If $\alpha=1$, the statement need not be true; take, e.g., $g(s)=e^{-s}$.

Solution II by Theodore S. Bolis, State University College at Oneonta; O. P. Lossers, Technological University, Eindhoven, Netherlands; and A. Meir, University of Alberta (independently). Let $G(t)=\int_0^t g(s)^{-\alpha}ds$. Then $G'(t)=g(t)^{-\alpha}$ and $f(t)=g(t)G(t)$. Assuming that $f(t)$ remains bounded as $t\to\infty$, then there exists some constant $C>0$ such that $f(t)^{\alpha}=\dfrac{G(t)^{\alpha}}{G'(t)}\leqslant C<$

∞ for all $t>0$. Hence

$$\int_1^t \frac{G'(s)}{G(s)^\alpha}\,ds \geqslant \frac{1}{C}(t-1), \text{ or } \frac{1}{1-\alpha}\left[G(t)^{1-\alpha}-G(1)^{1-\alpha}\right] \geqslant \frac{1}{C}(t-1).$$

Therefore $G(1)^{1-\alpha} > \frac{1}{C}(t-1)(\alpha-1)$ which yields a contradiction as $t\to\infty$. So f is unbounded. If $\alpha=1$, the statement need not be true; take, e.g., $g(s)=e^{-s}$.

Also solved by P. Addor (Switzerland), Theodore A. Bick, Robert Breusch, Andreas Brunnschweiler (Switzerland), D. Cohoon, Columbia University Problems Group, Gustaf Gripenberg (Finland), D. Hensley, A. A. Jagers (Netherlands), L. Kuipers (Switzerland), John S. Lew, Peter W. Lindstrom, L. E. Mattics, Curt McMullen, Dana Nance, Stephen Noltie, Adam Riese, Michael Skalsky, St. Olaf Problems Group, and the proposer.

Editor's note: Mattics shows that one may replace continuity by measurability of $g(t)$ and integrability of $g(t)^{-\alpha}$; and one may conclude that $f(t)$ is in fact unbounded on every subset E of the positive real axis such that $m(E)=\infty$.

Groups Generated by n-Cycles

E 2708 [1978, 276]. *Proposed by Edward T. H. Wang, Wilfrid Laurier University, Ontario.*

Find all n for which the symmetric group S_n has the following property: If $\alpha,\beta\in S_n$ are n-cycles then either $\langle\alpha\rangle = \langle\beta\rangle$ or $\langle\alpha\rangle\cap\langle\beta\rangle = \{1\}$.

Solution by D. M. Bloom, Brooklyn College, Aage Bondesen, Espergaerde, Denmark, Lorraine L. Foster, California State University, Northridge, N. Miku, Catholic University, Nymegen, Netherlands, and Michael Ward, University of Utah (independently). The condition is easily verified if $n\leqslant 4$ or if n is prime. For any other n, we construct the following counterexample. Let $\alpha=(12\cdots n)$. Let $n=kd$, where $k>2$ and $d>1$. Then α^k is the product of the disjoint d-cycles $C_i=(i,i+k,i+2k,\ldots,i+(d-1)k), i=1,2,\ldots,k$. Let t be the involution $(1,2)(1+k,2+k)\cdots(1+(d-1)k,\ 2+(d-1)k)$. Clearly $tC_1t^{-1}=C_2, tC_it^{-1}=C_i (i>2)$. Thus if $\beta=t\alpha t^{-1}$ then β is also an n-cycle and $\beta^k=\alpha^k$, so that $\langle\alpha\rangle\cap\langle\beta\rangle$ is non-trivial. We now show β is not a power of α. For if $\beta=\alpha^j$ for some $1\leqslant j<n$, then $3=\beta(1)=\alpha^j(1)=1+j$, so that $j=2$; and $1=\beta(2)=\alpha^2(2)=4$, an impossibility.

Also solved by Theodore Bolis, D. Broline, F. S. Cater, Jeffrey Mitchell Cohen, Columbia University Problem Group, Neal Felsinger, Robert Gilmer, JoAnne Growney, Yasuhiko Ikeda, Manfred Leitz (Germany), Robert Patenaude, Reinhard Razen, Erik Schreiner, Leland Scott, Mike Slattery, Anita Solow & Donald Goldberg, Stuart Wang, Paul Zwier, and the proposer.

Outer Measures of Choice Sets

E 2710 [1978, 276]. *Proposed by J. A. Andrews.*

Call two real numbers *equivalent* if their difference is rational. Call $S\subseteq\mathbf{R}$ a *choice set* if S is a set of representatives of the equivalence classes in $\mathbf{R}$. Let $\mathcal{F}$ be the family of all choice sets contained in $[0,1]$. Show that the numbers $m^*(S)$ $(S\in\mathcal{F})$ are dense in $[0,1]$. (m^* is the usual outer measure.)

Solution by Stanley Wagon, Smith College, Massachusetts. We show that, in fact, for any a in $(0,1]$ there is a choice set $S\subseteq[0,1]$ such that $m^*(S)=a$. Let a in $(0,1]$ be given and let c denote the cardinality of the continuum. Let $(C_\alpha:\alpha<c)$ be a well-ordering of all sets of the form $\cup_{n=1}^{\infty}I_n$, where I_n is an open subinterval of $[0,a]$ with rational endpoints, and such that $\Sigma_{n=1}^{\infty}l(I_n)<a$, where l denotes length. Such a well-ordering exists because the cardinality of the set of countable subsets of a countable set is c. Note that $m^*([0,a]-C_\alpha)>0$.

We shall define a set $X=\{x_\alpha:\alpha<c\}$ by transfinite induction so that $X\subseteq(0,a)$, no two

elements of X are equivalent, and X is not contained in any C_α. Let x_0 be any real in $(0,a)-C_0$. Assume that x_β for $\beta<\alpha$ has been defined and choose $x_\alpha\in(0,a)-C_\alpha$ such that x_α is not equivalent to any $x_\beta,\beta<\alpha$. Such a choice is possible for the following reasons.

(1) $[0,a]-C_\alpha$ is a closed set of reals which has positive outer measure and so is uncountable. The Cantor-Bendixson Theorem (Hausdorff, Set Theory, §26.3) states that an uncountable closed set of reals has cardinality c.

(2) The cardinality of the set of reals equivalent to some $x_\beta,\beta<\alpha$, is $|\alpha|\cdot\aleph_0<c$, since each real is equivalent to countably many others.

Since $X\subseteq[0,a], m^*(X)\leqslant a$, but since X is not contained in any $C_\alpha, m^*(X)\geqslant a$, hence $m^*(X)=a$. Furthermore, X consists of inequivalent reals and every real is equivalent to one in $[0,a]$; therefore, using the axiom of choice we may extend X to a choice set $S\subseteq[0,a]$. Then $m^*(S)=a$ as required.

Also solved by Ken Brown, Luis Cabrera & Kenneth Schilling, Neal Felsinger, Gustaf Gripenberg (Finland), Joel Levy, Peter Lindstrom, Mark Meyerson, William Myers, N. Miku (Netherlands), Peter Pappas, and the proposer.

Editors note. Cabrera & Schilling, Meyerson, and Myers all proved the stronger result, of Wagon's solution above, that $m^*(S)$ takes all values in $(0,1]$.

Irreducible Characteristic Polynomial

E 2711 [1978, 277]. *Proposed by Frank Uhlig, Aachen, Germany.*

Let A and B be $m\times m$ matrices over a field. If the characteristic polynomial of A is irreducible, show that rank $(AB-BA)\neq 1$.

Solution by Robert M. Guralnick, California Institute of Technology. The key to this is the following:

LEMMA. *Suppose A and B are $m\times m$ matrices over a field. If $C=AB-BA$ has rank 1 and $v\in$ range (C), then $A^iv\in\ker C$, for $i=0,1,2,\ldots$.*

Since CA^i has rank at most 1 and trace $(CA^i)=$ trace $(ABA^i-BA^{i+1})=0, CA^i$ is nilpotent. As v is an eigenvector of $CA^i, CA^iv=0$, as desired.

Thus if rank $C=1$, choose $0\neq v\in$ range (C), and set $W=$ span $\{A^iv|i=0,1,2,\ldots\}$. Then W is a non-zero A-invariant subspace and as $W\subseteq\ker C$, W is a proper subspace. Hence the characteristic polynomial of A is reducible. In fact, more is true.

THEOREM. *Let A, B, and C be as above. If the minimal polynomial of A is irreducible, then rank $C\neq 1$.*

Proof. Since the minimal polynomial of A is irreducible, by a change of a basis, we can assume $A=$ diag $[D,D,\ldots,D]$, and $B=(B_{ij})$, where D and B_{ij} are $n\times n$ matrices and the characteristic polynomial of D is the minimal polynomial of A (e.g., put A in rational canonical form). Then $C=(C_{ij})$, where $C_{ij}=DB_{ij}-B_{ij}D$. If rank $C\leqslant 1$, then rank $C_{ij}\leqslant 1$, and since the characteristic polynomial of D is irreducible, $C_{ij}=0$. Thus $C=0$, as desired.

In fact the converse is also true as we now show.

THEOREM. *Suppose A is an $m\times m$ matrix over a field F. Then there exists an $m\times m$ matrix B over F such that rank $(AB-BA)=1$ if and only if $m(x)$, the minimal polynomial of A, is reducible.*

Proof. Let V be the underlying space. By the previous theorem, we can assume that fg divides m, where f and g are irreducible monic polynomials with coefficients in F. Then there exists an A-invariant subspace U such that fg is the minimal and characteristic polynomial of A restricted to U (first consider ker $f(A)g(A)$, and then put A in rational canonical form on this space). If

$U \neq V$, by induction there exists a B acting on U such that $(AB - BA)|_U$ has rank 1. Then we can extend B to V such that rank $(AB - BA) = 1$. Hence we can assume $U = V$.

Case 1. $f \neq g$. Then $V = V_1 \oplus V_2$, where with respect to this decomposition,

$$A = \operatorname{diag}[A_1, A_2], \qquad f(A_1) = 0 = g(A_2).$$

Let B be the partitioned matrix $[0, C; 0, 0]$. We can consider C to be a map from V_2 to V_1. Now $A_1C - CA_2 = 0$ if and only if C is an $F[x]$-module homomorphism, where V_1 and V_2 are the $F[x]$-modules defined by setting $xv = A_iv$ for $v \in V_i, i = 1, 2$. Since f and g are irreducible and unequal, V_1 and V_2 are simple, nonisomorphic modules. Thus by Schur's Lemma, the only module homomorphism is the zero map. Hence $A_1C - CA_2 = 0$ if and only if $C = 0$. So the linear map that sends C to $A_1C - CA_2$ is injective and hence surjective. Now choose C such that $A_1C - CA_2$ has rank 1.

Then

$$AB - BA = [0, A_1C - CA_2; 0, 0],$$

and so rank $(AB - BA) = 1$.

Case 2. $f = g$. Thus $m = f^2$. Let $W = \ker f(A)$. Pick $v_1 \in V - W$. Then $v_1 + W$ is a cyclic vector for V/W. Set $v_2 = Av_1, \ldots, v_n = A^{n-1}v_1$, where n is the degree of f. Then

$$Av_n = -\sum_{j=1}^{n} a_j v_j + w_1, \quad \text{where } w_1 \in W \quad \text{and} \quad f(x) = x^n + \sum_{j=0}^{n-1} a_{j+1} x^j.$$

Since $f(A)v_1 \neq 0, w_1 \neq 0$. Choose a similar basis starting with w_1 for W. With respect to this basis, $A = [A_1, D; 0, A_1]$, where D has 1 in the upper right corner and 0 elsewhere. Set $B = \operatorname{diag}[0, I]$. Then $AB - BA = [0, D; 0, 0]$ has rank 1, as claimed.

Also solved by E. D. Dixon, Fergus J. Gaines (Ireland), W. Gustafson, Tom Jager, A. A. Jagers (Netherlands), J. C. Lagarias, N. Miku (Netherlands), Donald W. Robinson, and the proposer.

Intersection of Moving Convex Bodies

E 2714 [1978, 384]. *Proposed by M. J. Pelling, Balliol College, Oxford, England.*

Let G_1, G_2 be two bounded convex regions in $\mathbf{R}^2$ and suppose G_1 is translated to $G_1(t)$ by the transformation $x \to x + ta$ where a is a fixed unit vector. Consider the area $A(t)$ of $G_1(t) \cap G_2$ as a function of t. Is it always true that there is a constant c such that $A(t)$ is monotonic increasing for $t \leqslant c$ and monotonic decreasing for $t \geqslant c$?

What happens in $\mathbf{R}^n$?

Note by G. D. Chakerian, University of California at Davis. The solution (for all dimensions) is contained in a theorem of Fáry and Rédei in Math. Annalen 122 (1950) 205–220. They prove that the nth root of the volume of $G_1(t) \cap G_2$ is a concave function over those values of t where the intersection is nonempty.

Also solved by Leonard J. Wallen.

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to Professor Roger C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109. To facilitate their consideration, solutions of Advanced Problems in this issue should be typed (in duplicate, with double spacing) and should be mailed before January 31, 1980.

6273. *Proposed by K. L. Chung, Stanford University.*

Let f be a real-valued function defined on $(-\infty, +\infty)$ and continuous from the right everywhere. Suppose also that the following is true:

$$\lim_{n\to\infty}\left[\max_{-\infty<k<\infty}\left|f\left(\frac{k+1}{n}\right)-f\left(\frac{k}{n}\right)\right|\right]=0$$

where n and k are integers, $n \geqslant 1$. Is f continuous in $(-\infty, +\infty)$?

6274. *Proposed by F. S. Cater, Portland State University.*

Let S denote a topological space in which every compact set is closed, and let x and y be distinct points of S.

(1) Prove that x and y have disjoint neighborhoods if each of x and y has a countable local base.

(2) Show by example that x and y need not have disjoint neighborhoods if each element of S, other than x, has a countable local base.

6275. *Proposed by S. Foldes and E. Howorka, Grenoble, France, and University of Florida.*

Let r be a metric on R^n giving the same topology as the usual Euclidean metric d. Let $I(r), I(d)$ denote their groups of isometries. S. Ulam conjectured recently that, if $I(r)$ contains an isomorphic copy of $I(d)$, then $I(r) \cong I(d)$. This conjecture has not been settled yet. Show that $I(d) \subseteq I(r)$ implies $I(r) = I(d)$.

SOLUTIONS OF ADVANCED PROBLEMS

Rational Function Solutions of $x^n - y^2 = 1$

6082 [1976, 205, and 1978, 503]. *Proposed by Thomas C. Craven, University of Hawaii.*

Let $K(t)$ be the rational function field in one variable over a field K of arbitrary characteristic. Does the equation $x^n - y^2 = 1$ have a nonconstant solution in $K(t)$ when $n > 2$?

Addition to previous note. A reference should have been made to a paper of Melvin B. Nathanson, this MONTHLY, 81 (1974) 371–373, "On Catalan's Equation in $K(t)$," where the same problem is posed at the end of the paper.

Bijection $2^\omega \leftrightarrow N^\omega$

6128 [1977, 62; 1978, 688]. *Proposed by Martin Schechter and Peter Borwein, University of British Columbia.*

Let 2^ω be the set of all sequences with entries 0 or 1 and let N^ω be the set of all sequences with entries from the nonnegative integers. Can one construct a bijection f from 2^ω onto N^ω with the property that for any sequence X in 2^ω one can compute the first n entries of $f(X)$ given only the first m entries of X (where m may depend on X and n)?

Alternate solution by A. M. Dawes, University of Western Ontario, London, Ontario. The given condition on the bijection f from 2^ω to N^ω is easily seen to be equivalent to the requirement that f be continuous (when 2^ω and N^ω have the usual topologies). 2^ω is compact, N^ω is not; and so no such bijection can exist.

Divisors of $\phi(m)$

6160 [1977, 491]. *Proposed by Robert E. Shafer, Berkeley, California.*

(1) If m is the largest odd divisor of n, then with the exception of (2),

$$2^{\nu(n)}m^{\nu(m)/2}|\phi(a^n+b^n)$$

for $a>b\geqslant 1$, $\nu(n)$ is the number of divisors of n, ϕ is the Euler function.

(2) If $a=2$, $b=1$, $n=3^d c$, c odd, $d\geqslant 1$, $2^{\nu(n)-1}m^{\nu(m)/2}3^{d-1}|\phi(a^n+b^n)$.

Comment by Andrzej Makowski, Warszawa, Poland. Almost the same result is contained in the paper, "On the numbers $\phi(a^n\pm b^n)$," by A. Rotkiewicz, *Proc. Amer. Math. Soc.*, 12 (1961) 419–421.

Solutions also submitted by L. E. Mattics and the proposer.

Labeling Lattice Points

6192* [1978, 121]. *Proposed by Harry D. Ruderman, Hunter College Campus School.*

Let R be a rectangular array of lattice points having at least 2 rows and 2 columns. Let each lattice point of R be labeled by one of the numbers: 1,2,3, or 4. Suppose that the boundary points of R contain at least one of each of the four numbers and the boundary is oriented, say counterclockwise, with repetitions permitted, and with possibly more than one cycle (1 is allowed to follow 4). Call two lattice points adjacent if they are vertices of a common small square. Call two lattice points opposite if they are labeled either 1 and 3, or 2 and 4. Prove that for every such R there is a square containing two lattice points that are both opposite and adjacent.

The first proof below is direct. The second proof is of interest in that it derives the solution from a related known result.

I. *Solution by O. P. Lossers, Eindhoven University of Technology, Eindhoven, the Netherlands.* Let $\phi: R\to\{1,2,3,4\}$ be a labeling of the lattice R. For any pair (A,B) of neighboring points in R we define $F(A,B)$ to be the unique integer satisfying $k\equiv\phi(B)-\phi(A)\pmod 4$, $k\in\{-1,0,1,2\}$. With every counterclockwise oriented "small" square $ABCD$ we associate

$$F(A,B)+F(B,C)+F(C,D)+F(D,A). \qquad (*)$$

Summing expression $(*)$ over all small squares one obtains $S=\sum_i F(A_i,A_{i+1})$, where A_i, A_{i+1} runs over the counterclockwise oriented boundary of R. So

$$S=4\times(\text{the number of cycles over the boundary})\geqslant 4. \qquad (**)$$

If R does not contain two opposite, adjacent points, i.e., R does not contain a small square with two adjacent points, then expression $(*)$ is zero for every small square $ABCD$, since then apart from reorderings the only labelings of $ABCD$ are $1,1,1,1$; $1,1,1,2$; $1,1,2,2$ and $1,2,1,2$. So if R does not contain two opposite, adjacent points, then also the sum of expression $(*)$ over all small squares, i.e., S, would be zero, contradicting $(**)$.

II. *Solution by Eric S. Rosenthal, Princeton University.* Let Q be the graph whose vertices are the points of R and whose edges are the sides of the small squares of R together with the diagonal from lower left to upper right of each small square of R. If a point of R is labeled by 1, 2, 3, or 4, then label the corresponding vertex of Q by 1, 2, 3, or 3, respectively. Then a lemma of H. W. Kuhn [H. W. Kuhn, A new proof of the fundamental theorem of algebra, *Mathematical Programming Study 1*, North-Holland, 1974, pp. 148–158] implies that Q has a small triangle with vertices labeled 1, 2, and 3. So the small square of R containing this small triangle of Q has three vertices labeled 1, 2, and 3, or 1, 2, and 4. In either case, two vertices of this small square are adjacent and opposite points of R.

Query. The proposed theorem, established by Lossers's proof, appears to imply a (more simply stated) version of Kuhn's Lemma 2.1. Can one now use Kuhn's argument to derive from this a proof of the Fundamental Theorem of Algebra?

Also solved by Kurt Luoto, who established a wide-ranging generalization.

EDITED BY J. ARTHUR SEEBACH, JR., AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf, Carleton, and Macalester Colleges
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Elementary Differential Equations and Boundary Value Problems, Third Edition. By William E. Boyce and Richard C. DiPrima. John Wiley & Sons, Inc., New York, 1977. xiv+582 pp. \$19.95. *Solutions Manual,* Wiley, 1979. \$6.95.

The first seven chapters of this book can be used to give an excellent introductory course on differential equations. (There is a shortened edition of this book called *Elementary Differential Equations* (TR, November 1977) which contains only these first seven chapters.) The only prerequisite for a course based on the book is calculus, although a knowledge of linear algebra would make this book or any book on differential equations more meaningful. Nonetheless, linear algebra is a prerequisite for the study of the latter half of Chapter 7, because the topic is the solution to systems of differential equations using matrix algebra.

The first seven chapters contain the traditional topics associated with a first course on ordinary differential equations. These chapters include a chapter on the solution to differential equations by infinite series and also a chapter called "The Laplace Transform." The book concludes with two additional topics: Chapters 8 and 9 are on numerical methods for solving differential equations; Chapters 10 and 11 are concerned with partial differential equations and Fourier series.

The last four chapters of the book are interesting but we have not made much use of them at our university. We use part of Chapter 10 to introduce the students to Fourier series. Also, Chapters 10 and 11 provide a good reference for the students who take our course on partial differential equations and Fourier series. In the Preface to the book the authors indicate that they use the first seven sections of Chapter 8, which concludes with a discussion of the Runge-Kutta method, for finding an approximate solution to an initial value problem.

Various editions of the book have been used here for the past six or seven years. This is a remarkable record for longevity. Indeed, this book deserves a departmental book award for long and meritorious service. The book is a pleasure to use.

In our one quarter course (with about 44 class meetings) on ordinary differential equations we primarily attempt to give the student some facility in solving differential equations. The book gives sufficient emphasis to the theory required for the student to understand the solutions, and there are a substantial number of theorems in the book. There are also numerous applications. The important thing is that neither the theory nor the applications hinder the flow of the book.

The first set of applications is appealing. There are two problems about the rate of growth of the human population. There is an interesting problem about mixing two substances—suppose a room of a given volume is free of carbon monoxide and cigarette smoke enters at a rate, etc. In this section there is also a problem concerning the amount of pollutants in Lake Superior at time t. This is followed by a well-written section on elementary mechanics and another stimulating group of problems.

The book has some flaws. The section on elementary mechanics in Chapter 2 provides an excellent application of the first topic of Chapter 3—the solution of "two special types of second order equations." The section on finding a family of curves orthogonal to a given family should have included some stimulating applications. There are interesting applications that could have been mentioned. For example, in meteorology the orthogonal trajectories of the isobars (curves

that connect points with the same barometric pressure) give the direction of the wind which flows from areas with high pressure to those of low pressure. I consider the latter part of Chapter 4 on series solutions at regular singular points the weakest part of the book. On page 275, there is an interesting example that should be included but not for the reasons the authors give. The example demonstrates that if one solves a system of differential equations using the D operator technique without sufficient prior thought one can introduce extraneous constants in the solution. The real flaw is on the preceding page—the algebraic technique given there is not the only method (or even a desirable method) for solving a system of differential equations. There are a few notational lapses in Chapter 7. For example, the elements of a matrix X should be denoted by x_{ij}. Using both an upper and lower index is confusing to students who have taken a course on linear algebra.

Overall, the book is well written. Perhaps in the fourth edition the authors will consider improving the section on series solutions to differential equations at regular singular points. Any other flaws in the book are minor. There are two chapters that are especially good—Chapter 3 on the solutions to second order linear equations and Chapter 6 on solutions to differential equations by using the Laplace transform. I am also enthusiastic about the solutions to systems of differential equations by matrix methods in Chapter 7.

This is the third edition of a book that has been very successful at our university. It is probably one of the best books available for an introductory course on ordinary differential equations.

D. H. TRAHAN, Naval Postgraduate School

FILMS

The Geometry Euclid Didn't Know. Produced by David Nulsen. Mathematical consultant: Paul Johnson. Sixteen and one-half minutes, 16mm, sound and color. Purchase from David Nulsen Enterprises, 3211 Pico Boulevard, Santa Monica, CA 90405. Price $235.

This film covers the highlights of the history of non-Euclidean geometry, from Euclid's formulation of the parallel postulate, through unsuccessful attempts to prove it, and culminating in the discovery of hyperbolic and elliptic geometry. While it is impossible to cover this field adequately in fifteen minutes, the film does achieve a nice blend of the historical and mathematical, and could be profitably viewed at both the beginning and the end of a course in non-Euclidean geometry. High school geometry students, too, could enjoy and benefit from this film.

The film's documentary style holds the viewer's attention, and the graphic segments are well done. But this raises the question of how the film medium can best be utilized in mathematics. It is this reviewer's opinion that the best use of film (at the college level) is to provide dynamic graphic illustration of phenomena which would be difficult to do by static means (blackboard, slides, handouts). However, the dynamic graphics in this film are not particularly illuminating, and much the same effect could probably be obtained with a narrated slide show.

The film's treatment of non-Euclidean geometry in the twentieth century could be improved. While connections with relativity are mentioned, too much time is spent on whether our three-dimensional space is Euclidean. It is suggested that fine enough measurements of angles from distant stars could settle this question, without mentioning the limitations of this approach or the meaningfulness of the question. The problem of giving a physical interpretation to "points" and "lines" is ignored, and the distinction between abstract geometry as a mathematical model, as opposed to a descriptive theory, is not made clear.

STANLEY WAGON, Smith College

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S***(14-16), P***, L***, *Gödel, Escher, Bach: An Eternal Golden Braid*. Douglas R. Hofstadter. Basic Books, 1979, xxi + 777 pp, $18.50. [ISBN: 0-465-02685-0] A *tour de force* of scholarly and playful metaphor, linking the diverse mirrors with which art and science reflect reality. Hofstadter's themes are paradox and self-reflection; his tools are dialogues (à la Lewis Carroll) and puns (visual as well as verbal). His subjects include--in addition to Gödel, Escher, and Bach--genetic codes, computer science, particle physics, proof theory, Zen Buddhism, and the nature of intelligence. A brilliant, creative and very personal synthesis, without precedent or peer in modern literature. LAS

GENERAL, T(13-16: 1), L, *Mathematics: Ideas and Uses*. David Russell. D. Van Nostrand, 1979, x + 412 pp, $13.95 [ISBN: 0-442-27202-2]; *Instructor's Manual*, iii + 43 pp, (P). [ISBN: 0-442-27203-0] Non-algebraic. For majors in education and liberal arts. Five interrelated chapters on sets, logic, and the computer; followed by an independent nine chapters on the metric system through calculus. Geometry slighted. Non-challenging exercises and problems. Topics for further investigation with over 100 references. Attractive appropriate photographs especially at chapter beginnings. Worth a look if your course fits. JK

GENERAL, S(13), *Math Equals: Biographies of Women Mathematicians and Related Activities*. Teri Perl. A-W, 1978, v + 250 pp, $8.95 (P). [ISBN: 0-201-05709-3] A book written to help change sex role stereotypes, raise questions about roles of women, and increase awareness of career choices. LLK

GENERAL, S(13-16), *Professional Opportunities in the Mathematical Sciences, Tenth Edition*. MAA, 1978, 35 pp, (P). A timely new edition of a very useful guidebook, covering teaching, industry and government, applied mathematical sciences (computer science, operations research, statistics, actuarial science) and new interdisciplinary areas. Describes job opportunities, working conditions and salary structure. Concludes with an extensive list of references. LAS

GENERAL, S, L, *Mathematical Challenge! One Hundred Problems for the Olympiad Enthusiast*. L.S.R. Mbili. Math Digest (Dept. of Math., U. of Cape Town, 7700 Rondebosch, South Africa), 1978, 50 pp, $2.50 (P). [ISBN: 0-7992-0222-3] Problems from elementary number theory, high school algebra, trigonometry, and geometry, with solutions. LCL

GENERAL, S(13-14), L, *Proof in Mathematics ('If', 'then' and 'perhaps')*. P.R. Baxandall, *et al.* U of Keele, 1978, v + 130 pp, (P). A diffuse collection of examples from elementary mathematics arranged to illustrate common proof techniques and strategies. Somewhat unfinished and sketchy, especially in the second half. LCL

GENERAL, P, L**, *Books About Mathematics: History, Philosophy, Education, Models, System Theory, and Works of Reference, etc.: A Bibliography*. Else Høyrup. Roskilde U Center, 1979, viii + 180 pp, (P). A well-organized, up-to-date bibliography of 1400 titles, mostly books, on history and biography; philosophy, psychology and sociology of mathematics; models and systems theory; bibliographies, dictionaries, encyclopedias and handbooks. The author is a research librarian in mathematics at Roskilde University, where the mathematics collection stresses the subjects covered by this volume. LAS

GENERAL, T(13), S? *Thinking Metric, Second Edition*. Thomas F. and Marilyn B. Gilbert. Wiley, 1978, xv + 140 pp, $3.95 (P). [ISBN: 0-471-03427-4] Revision of 1973 edition (First Edition, TR, November 1974) of a self-instructional booklet. Expanded conversion tables and emphasis on approximate conversions "because conversions will be necessary until we are fully metric." PJ

GENERAL, S?? *The Giza Pyramids: Final Decoding*. Rocky McCollum, Gerald J. Fraccaro, Elmer D. Robinson. Pyramid of Aquarius Pub (PO Box 163, Dearborn Hts., MI 48127), 1978, 230 pp, (P). Arm-chair treasure hunting by subdividing maps into golden rectangles and golden triangles and superimposing logarithmic spirals in an attempt to locate records of prehistoric civilization. (No luck to date.)LCL

GENERAL, P, *Lecture Notes in Mathematics-677: Séminaire Bourbaki vol. 1976/1977, Exposés 489-506*. Springer-Verlag, 1978, 264 pp, $14.30 (P). [ISBN: 0-387-08937-3; 3-540-08937-3]

BASIC, T(13-14: 1, 2), *Elementary Technical Mathematics*. James F. Connelly, Robert A. Fratangelo. Macmillan, 1978, xiv + 706 pp, $12.95. [ISBN: 0-02-324430-5] The mathematics deemed necessary for study of the applications of mathematics in technology programs of community colleges, junior colleges and technical institutions. Covers elementary algebra, advanced algebra and topics from college algebra and analytic geometry. Traditional content and methods, no applications to technology. 600 plus examples, 3000 plus exercises, 1000 plus review questions. PJ

BASIC, S(13), *Plotting Graphs*. G.E. Shilov. Trans: S. Sosinsky. MIR, 1978, 29 pp, $1.25 (P). Another in the Little Mathematics Library series. Brief, but clear, introduction to the methods of graphing algebraic equations by carrying out arithmetic operations geometrically. No calculus. Recommended supplementary reading for calculus students. JK

BASIC, T(13: 1), S. *Business Mathematics.* Esther H. Highland. Reston, 1978, x + 466 pp, $15.95. [ISBN: 0-87909-104-5] The goals of this textbook are to teach the mathematical procedures used in courses in accounting, finance, marketing and statistics, an understanding of business procedures and terms, and an approach to problem solving in the business context. The format consists primarily of lots of examples and exercises. CEC

BASIC, T(13: 1). *Business Mathematics, A Collegiate Approach, Third Edition.* Nelda W. Roueche. P-H, 1978, xii + 670 pp, $14.95. [ISBN: 0-13-105007-9] Arithmetic, simple algebra, descriptive statistics applied to accounting (e.g., taxes, insurance, financial statement analysis), retailing (e.g., discounts and markups), finance (e.g., simple and compound interest). Easy to read; problem sets include computational problems and "written" problems (none require a hand calculator). LCL

BASIC, T(13: 1), L. *Essential College Algebra.* Doris S. Stockton. HM, 1979, ix + 573 pp, $14.95. [ISBN: 0-395-26544-4] Fundamental algebraic techniques. Concise review of basic algebra. Concise development of functions, analytic geometry, matrices, induction, combinatorial operations, discrete probability and other useful algebra. Self-contained topic development. Numerous solved problems and self-scoring quizzes. Good for self-paced pre-calculus or algebra review. WLC

EDUCATION, S?(15). *A Quick Guide to the New Math.* Sumner B. Cotzin. U Pr of America, 1977, 171 pp, $8.60 (P). Brief explanations for parents of about forty concepts (with symbols) from grades 1 to 8 of the 1960's and early 1970's. Set notation, number systems and properties, arithmetic of rationals, functions, etc. No geometry. PJ

HISTORY, L**. *Abrégé d'histoire des mathématiques 1700-1900.* Jean Dieudonné. Hermann, 1978, 272F set [ISBN: 2-7056-5859-9]; *I: Algèbre, Analyse classique, Théorie des nombres,* 395 pp; *II: Fonctions elliptiques, Analyse fonctionnelle, Topologie, Géométrie différentielle, Probabilités, Logique mathématique,* vii + 472 pp. More an encyclopaedia than a monograph, each of the thirteen chapters is devoted to a different area of mathematics, and is written by a different individual working under Dieudonné's general editorship. Despite the title, the various essays frequently move well into the twentieth century. Extensive references are given at the end of each chapter; a skimpy general index is contained at the end of *Volume II,* as is a useful register of mathematicians with outline (3-4 line) biographies. LAS

HISTORY, P, L*. *History of Analysis.* Ed: R.J. Stanton, R.O. Wells, Jr. Rice U, 1978, iii + 228 pp, $11 (P). [ISBN: 0-89263-236-4] This 1978 combined spring-summer issue of *Rice University Studies* features a 150-page historical survey of harmonic analysis ("the exploitation of symmetry") by George Mackey, a brief look at the creation of the theory of group characters by Thomas Hawkins based on the Dedekind-Frobenius correspondence, and a survey of the history of analysis ("in the Renaissance and after") by Salomon Bochner. LAS

FOUNDATIONS, P. *Lecture Notes in Mathematics-689: Cabal Seminar 76-77.* Ed: A.S. Kechris, Y.N. Moschovakis. Springer-Verlag, 1978, 282 pp, $14.30 (P). [ISBN: 0-387-09086-X; 3-540-09086-X] Material presented at the Caltech-UCLA seminar together with two earlier papers: Kechris' "On spector classes," and Kechris and Moschovakis' "Notes on the theory of scales." JAS

FOUNDATIONS, P. *Lecture Notes in Mathematics-669: Higher Set Theory.* Ed: G.H. Müller, D.S. Scott. Springer-Verlag, 1978, 476 pp, $21.30 (P). [ISBN: 0-387-08926-8; 3-540-08926-8] Papers originating from Oberwolfach conference, April 1977, covering a wide range of research topics in set theory and related domains. Distinguished by a major expository paper by Kanamori and Magidor surveying the evolution of large cardinal axioms in set theory, with emphasis on main themes and unifying concepts. GHM

FOUNDATIONS, P. *Logic Colloquium '77.* Ed: Angus Macintyre, Leszek Pacholski, Jeff Paris. Stud. in Logic and Found. of Math., V. 96. North-Holland, 1978, x + 311 pp, $35.50. [ISBN: 0-444-85178-X] Proceedings of international conference in logic, Wroclaw, Poland, August 1977. Considerable variety in subject matter, with research reports in arithmetic, set theory (combinatorial, large cardinal, descriptive), model theory, generalized languages and quantifiers. Includes first-time publication of a classical study (Mostowski and Tarski, 1930's and 40's) of the elementary theory of well-ordering. GHM

FOUNDATIONS, P. *Generalized Recursion Theory II: Proceedings of the 1977 Oslo Symposium.* Ed: J.E. Fenstad, R.O. Gandy, G.E. Sacks. Stud. in Logic and Found. of Math., V. 94. North-Holland, 1978, vii + 417 pp, $42.25. [ISBN: 0-444-85163-1] Nineteen papers covering almost all areas of generalized recursion theory. LCL

FOUNDATIONS, T*(15-17), P, L*. *Sets: Naïve, Axiomatic and Applied.* D. van Dalen, H.C. Doets, H. de Swart. Pergamon Pr, 1978, xvii + 342 pp, $17.50 (P). [ISBN: 0-08-023047-4] Good coverage of naive as well as basic axiomatic set theory (without formal logic). Stresses material useful to the general mathematician, but includes topics considerably beyond the usual naive treatment, such as the cumulative hierarchy, models of set theory, measurable cardinals, the Borel hierarchy and the axiom of determinateness. Can function either as a beginner's text (many routine exercises) or as a useful reference book for working mathematicians. GHM

FOUNDATIONS, S(18), P. *Classification Theory and the Number of Non-Isomorphic Models.* S. Shelah. Stud. in Logic and Found. of Math., V. 92. North-Holland, 1978, xvi + 544 pp, $68.25 [ISBN: 0-7204-0757-5] Compilation of the author's incredible contributions to the model-theoretic classification of first-order theories. Key concepts of rank, stability, forking, prime models, saturativity, etc., are powerfully exploited to study the numbers of models of theories. Not a light book to read, "the right [and probably only] way is to put it on your desk in the day, below your pillow at night, devoting yourself to the reading." The many exercises should help one digest the overwhelming detail. GHM

COMBINATORICS, P, *Lecture Notes in Mathematics-686: Combinatorial Mathematics*. Ed: D.A. Holton, Jennifer Seberry. Springer-Verlag, 1978, ix + 353 pp, $17.80 (P). [ISBN: 0-387-08953-5; 3-540-08953-5] Twenty-eight papers and the invited lectures from the conference held at the Australian National University from August 16 to 27, 1977. Twelve other papers will be published elsewhere. JAS

COMBINATORICS, P, *Advances in Graph Theory*. Ed: B. Bollobás. Annals of Discrete Math., No. 3. North-Holland, 1978, vii + 295 pp, $53.75. [ISBN: 0-7204-0843-1] Collection of 23 papers on a broad range of topics, mostly within graph theory, some in combinatorics related to graph theory only peripherally. Authors are experts who dedicate this excellent volume to W.T. Tutte. SS

ALGEBRA, P, *Ring Theory: Proceedings of the 1977 Antwerp Conference*. Ed: F. Van Oystaeyen. Pure and Appl. Math., V. 40. Dekker, 1978, vii + 207 pp, $22.50 (P). From the preface: "The present volume is a ripened fruit of 'The Week of Ring Theory' organized at the University of Antwerp from April 4-8, 1977." JAS

ALGEBRA, T*(15-16: 1), S, P, L**, *Field Theory and Its Classical Problems*. Charles Robert Hadlock. Carus Math. Mono., No. 19. MAA, 1978, xvi + 323 pp, $16. [ISBN: 0-88385-020-6] "I wrote this book for myself...to piece together my own path through Galois Theory...[solving] simple, interesting questions...as quickly and directly as possible... . I approached...as an inquirer rather than as an expert...to share some of the sense of discovery and excitement I experienced. There is great mathematics here." Indeed there is: three Greek problems, constructibility, solution by radicals, polynomials with symmetric groups, all sensitively treated in a concrete manner suitable to the mathematical maturity of beginning students. Complete solutions (100 pp.) to all problems. LAS

ALGEBRA, P, *Quasi-Ideals in Rings and Semigroups*. Ottó Steinfeld. Akadémiai Kiadó, 1978, xi + 154 pp, $14.50. [ISBN: 963-05-1696-9] A non-empty subset (subring) Q of a semigroup (ring) R is a quasi-ideal if $QR \cap RQ \subset Q$. The author's intent is to give "a systematic survey of the most important results on quasi-ideals," and includes decomposition theorems, characterization of regular elements, etc. Bibliography, indices, and a list of unsolved problems. JS

ALGEBRA, P, *Representations of Finite Chevalley Groups*. G. Lusztig. CBMS Reg. Conf. in Math., No. 39. AMS, 1978, v + 48 pp, $7.60 (P). [ISBN: 0-8218-1689-6] Lecture notes given in a CBMS Regional Conference held at Madison, Wisconsin in 1977. Main purpose: to show how ℓ-adic cohomology of algebraic varieties can be used to get information on the representations of finite Chevalley groups. LCL

ALGEBRA, S(17-18), P, *Lecture Notes in Mathematics-682: The Representation Theory of the Symmetric Groups*. G.D. James. Springer-Verlag, 1978, v + 156 pp, $9.80 (P). [ISBN: 0-387-08948-9; 3-540-08948-9] Includes the basic theorems in the subject; the approach adopted is characteristic-free. Largely self-contained with instructive examples sprinkled throughout. LCL

ALGEBRA, T(16-18), S, P, L, *Combinatorial Group Theory: Presentations of Groups in Terms of Generators and Relations, Second Revised Edition*. Wilhelm Magnus, Abraham Karrass, Donald Solitar. Dover, 1976, x + 444 pp, $6 (P). [ISBN: 0-486-63281-4] Unabridged reproduction of the original (TR, April 1967; ER, May 1967) with corrections and references to developments since 1966. LCL

ALGEBRA, S(18), P, *Formal Groups and Applications*. Michiel Hazewinkel. Pure and Appl. Math., V. 78. Acad Pr, 1978, xxii + 573 pp, $49.50. [ISBN: 0-12-335150-2] A *tour de force* which takes the reader through most of the known results on formal groups, from the power series point of view, as well as easily accessible applications. LCL

ALGEBRA, P, *Lecture Notes in Mathematics-691: Algèbres de Lie Libres et Monoïdes Libres*. Gérard Viennot. Springer-Verlag, 1978, 124 pp, $9.90 (P). [ISBN: 0-387-09090-8; 3-540-09090-8] There are three chapters: I--Bases for free Lie algebras; II--Bisection of free Lie modules; III--Factorization of free modules and families of bases for free Lie algebras. Bibliography, index. JS

ALGEBRA, P, *Lecture Notes in Mathematics-654: Icosahedral Galois Representations*. Joe P. Buhler. Springer-Verlag, 1978, 143 pp, $8 (P). [ISBN: 0-387-08844-X; 3-540-08844-X] The main result is to show that there is an icosahedral form of level 800. The details of the proof are computational and require access to a computer to verify. This is the first known instance of the validity of a conjecture of Artin for the existence of a non-solvable Galois representation that is not a positive linear combination of monomial representations. CEC

ALGEBRA, P, *American Mathematical Society Translations, Series 2, V. 113: Twelve Papers in Logic and Algebra*. AMS, 1979, v + 250 pp, $24.40. [ISBN: 0-8218-3063-5]

ALGEBRA, P, *Lecture Notes in Mathematics-674: Homology of Classical Groups Over Finite Fields and Their Associated Infinite Loop Spaces*. Zbigniew Fierdorowicz, Stewart Priddy. Springer-Verlag, 1978, vi + 434 pp, $19.50 (P). [ISBN: 0-387-08932-2; 3-540-08932-2]

FINITE MATHEMATICS, T(13-14: 1), *Finite Mathematics with Applications to Life, Second Edition*. Lawrence G. Gilligan, Robert B. Nenno. Goodyear, 1979, x + 581 pp, $15.95. [ISBN: 0-87620-314-4] Standard material. Pitched at students with only one year of high school mathematics, majoring in the social sciences, business, economics, data processing or the liberal arts. Strengths include numerous worked-out examples and thoughtfully-prepared exercises. Useful chapter summaries with vocabulary and symbols. Interesting asides, notes and drawings are sprinkled throughout the text. JK

FINITE MATHEMATICS, T(13: 1), *Practical Finite Mathematics*. Gareth Williams. Allyn, 1979, viii + 408 pp, $14.95. [ISBN: 0-205-06525-2] Good choice of topics and applications of those topics: matrices, graph theory, linear programming, probability and statistics, Markov chains, and game theory. Also an introduction to computers. LLK

CALCULUS, T(13: 1, 2). *Much Ado About Calculus: A Modern Treatment with Applications Prepared for Use with the Computer.* Robert L. Wilson. Springer-Verlag, 1979, xvii + 788 pp, $16.80. [ISBN: 0-387-90347-X; 3-540-90347-X] An innovative two-term text featuring topics specially germane to computer applications: interpolation, regression ("converting data to functions"), numerical methods and, most notably, early introduction (p. 108) of Riemann-Stieltjes integration instead of the ordinary Riemann integral. Includes infinite series, differential equations and a brief introduction to partial derivatives and iterated integrals. Begins with a potentially deadly "Prologue" containing background material (real and complex numbers, functions and continuity, induction, notation) largely separated from its intrinsic motivation; concludes with appendices on trigonometry, analytic geometry, Fortran and Basic. LAS

CALCULUS, T?(13-14: 1, 2), S*. *Einführung in die Differential- und Integralrechnung.* Johann Cigler. Manzsche Verlag, 1978. *1,* 198 pp, DM 21 (P) [ISBN: 3-214-00050-0]; *2,* 205 pp, DM 23 (P). [ISBN: 3-214-00060-8] "Calculus for Graduate Students" would be a better title. This book offers a thorough theoretical treatise on calculus and its basic applications, including multidimensional calculus, with historical comments. The exercises are mostly theoretical and there is unfortunately no index. However, a fairly extensive table of contents makes it possible to use this as a reference work. JAS

COMPLEX ANALYSIS, P. *Lecture Notes in Mathematics-670: Fonctions de Plusieurs Variables Complexes III.* Ed: François Norguet. Springer-Verlag, 1978, xii + 394 pp, $19.50 (P). [ISBN: 0-387-08927-6; 3-540-08927-6] Proceedings of the Séminaire François Norguet from October 1975 to June 1977. JAS

COMPLEX ANALYSIS, S(18), P. *Lecture Notes in Mathematics-705: Konstruktion verseller Familien kompakter komplexer Räume.* Otto Forster, Knut Knorr. Springer-Verlag, 1979, vii + 141 pp, $9.90 (P). [ISBN: 0-387-09122-X; 3-540-09122-X]

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-660: Equations aux Dérivée Partielles.* Ed: Pham The Lai. Springer-Verlag, 1978, vi + 216 pp, $12.40 (P). [ISBN: 0-387-08913-6; 3-540-08913-6] Proceedings of the conference held at Saint-Jean-de-Monts, June 1-4, 1977. JAS

DIFFERENTIAL EQUATIONS, P. *Singular Perturbations of Hyperbolic Type.* R. Geel. Math. Centre Tracts, No. 98. Math Centrum, 1978, xxii + 184 pp, Dfl. 23 (P). [ISBN: 90-6196-166-1] Monographs on existence, a priori estimates and asymptotic expansions for initial value problems for linear ordinary, nonlinear ordinary, linear hyperbolic and nonlinear hyperbolic differential equations; also, a priori estimates and approximations for linear hyperbolic differential equations. JK

DIFFERENTIAL EQUATIONS, P. *Retarded Functional Differential Equations: A Global Point of View.* S.E.A. Mohammed. Research Notes in Math., No. 21. Fearon-Pitman, 1978, 147 pp, $13.25 (P). [ISBN: 0-273-08401-1]

DIFFERENTIAL EQUATIONS, S(17-18), P. *Methoden der Analytischen Störungsrechnung und ihre Anwendungen.* Urs Kirchgraber, Eduard Stiefel. Teubner, Stuttgart, 1978, viii + 294 pp. [ISBN: 3-519-02346-6] An account, for specialists, of the method of averaging and perturbation theory for systems of ordinary differential equations. JD-B

DIFFERENTIAL EQUATIONS, P. *Nonlinear Evolution Equations Solvable by the Spectral Transform.* Ed: F. Calogero. Research Notes in Math., No. 26. Fearon-Pitman, 1978, xvii + 257 pp, $19.50 (P). [ISBN: 0-273-08402-X] The text of the invited lectures presented at an international symposium that took place at the Accademia dei Lincei in Rome, June 1977. TRS

DIFFERENTIAL EQUATIONS, P. *Differential Equations and Applications.* Ed: Wiktor Eckhaus, Eduard M. de Jager. Math. Stud., V. 31. North-Holland, 1978, viii + 264 pp, $31 (P). [ISBN: 0-444-85185-2] Proceedings of a September 1977 conference at Scheveningen, the Netherlands, printed directly from author-supplied typewritten manuscript. 20 papers on diverse topics. LAS

DIFFERENTIAL EQUATIONS, P. *Alexander Weinstein Selecta.* Ed: J.B. Diaz. Fearon-Pitman, 1978, xxi + 629 pp, $36.75. [ISBN: 0-273-08411-9] About half of Weinstein's papers, published between 1920 and 1976, treating primarily eigenvalues, partial differential equations and singular equations. Includes a very brief professional and biographical sketch. LAS

DIFFERENTIAL EQUATIONS, P. *Applications des inéquations variationnelles en contrôle stochastique.* A. Bensoussan, J.-L. Lions. Dunod (US Distr: SMPF, 14 E. 60th St., NY 10022), 1978, 545 pp, 180 F. [ISBN: 2-04-010336-8] An advanced monograph with the purpose of devising "constructive methods to calculate, if necessary with the resources of numerical analysis, the solution to optimal control problems, particularly with stopping time." JAS

NUMERICAL ANALYSIS, S(18), P. *Numerical Methods for Solving Time-Dependent Problems for Partial Differential Equations.* Heinz-Otto Kreiss. Pr U Montreal, 1978, 114 pp, $8 (P). [ISBN: 0-8405-0430-6] Lecture notes. Properties of solutions to Cauchy problems including variable coefficients and inhomogeneities; corresponding properties of the approximate solutions obtained by several methods. RWN

NUMERICAL ANALYSIS, T(17: 1), P. *A Practical Guide to Splines.* Carl de Boor. Appl. Math. Sci., V. 27. Springer-Verlag, 1978, xxiv + 392 pp, $14.80 (P). [ISBN: 0-387-90356-9; 3-540-90356-9] The use of splines in interpolation, smoothing, least squares and collocation methods. Includes some of the essential theory based on B-splines. Lists and discusses several useful Fortran subroutines. Examples and problems. RWN

NUMERICAL ANALYSIS, T(16-17: 1), S, L*. *Numerical Solution of Partial Differential Equations: Finite Difference Methods, Second Edition.* G.D. Smith. Clarendon Pr, 1978, xii + 304 pp, $11.50 (P); $23.50. [ISBN: 0-19-859626-X] *Second Edition* of a rather useful, introductory reference. Changes include a chapter on iterative methods, use of matrix algebra and more exercises. (First Edition, TR, December 1976.) RWN

NUMERICAL ANALYSIS, T(15-17: 1), L*. *Introduction to Numerical Computations.* James S. Vandergraft. Comp. Sci. and App. Math. Acad Pr, 1978, xi + 344 pp, $21. [ISBN: 0-12-711350-9] Numerical methods for function evaluation, interpolation and approximation, differentiation and integration, linear systems, nonlinear equations and differential equations with an emphasis on practical considerations such as rounding errors, efficiency and reliability. RWN

NUMERICAL ANALYSIS, T(16-17: 1), S, L. *Einführung in die Numerische Mathematik.* Franz Locher. Wissenschaftliche Buchgesellschaft, 1978, 265 pp, DM 43 (P). [ISBN: 3-534-07284-7] An introduction to numerical analysis treating the usual topics except for the numerical solution of differential equations. Assumes knowledge of advanced calculus and linear algebra; proves some elementary theorems of functional analysis. JD-B

FUNCTIONAL ANALYSIS, S(17-18), P. *Non-Archimedean Functional Analysis.* A.C.M. van Rooij. Pure and Appl. Math., V. 51. Dekker, 1978, x + 404 pp, $29.50. [ISBN: 0-8247-6556-7] Textbook-type coverage of functional analysis over topological (complete valued) fields other than R and C. (Not to be confused with nonstandard analysis.) Not simply a generalization of classical analysis, many strong negations of classical theorems can be proved in this setting, thus helping to pinpoint the role of various properties of the scalar field. GHM

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-666: Espaces d'Interpolation Réels: Topologie et Géométrie.* Bernard Beauzamy. Springer-Verlag, 1978, x + 104 pp, $9 (P). [ISBN: 0-387-08923-3; 3-540-08923-3] An exposition of the current state of development of interpolation spaces for the real case. JAS

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-693: Hilbert Space Operators.* Ed: J.M. Bachar, Jr., D.W. Hadwin. Springer-Verlag, 1978, viii + 184 pp, $9.80 (P). [ISBN: 0-387-09097-5; 3-540-09097-5] Proceedings of the conference held at California University at Long Beach during the week of June 20-24, 1977. JAS

FUNCTIONAL ANALYSIS, P. *Topics in Functional Analysis: Essays Dedicated to M.G. Krein on the Occasion of His 70th Birthday.* Ed: I. Gohberg, M. Kac. Acad Pr, 1978, xxi + 395 pp, $41. [ISBN: 0-12-287150-2]

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Physics-74: A Renormalization Group Analysis of the Hierarchical Model in Statistical Mechanics.* Pierre Collet, Jean-Pierre Eckmann. Springer-Verlag, 1978, 199 pp, $8 (P). [ISBN: 0-387-08670-6; 3-540-08670-6] Introductory exposition based on statistical mechanics followed by mathematical derivations addressed to readers familiar with functional analysis. LCL

FUNCTIONAL ANALYSIS, P. *Generalized Inverses and Operator Theory.* S.R. Caradus. Pure and Appl. Math., No. 50. Queen's U, 1978, 206 pp, (P). Expansion, updating, and continuation of the author's earlier contribution to this series (No. 38, *Operator Theory of the Pseudo-Inverse,* TR, February 1975). LCL

FUNCTIONAL ANALYSIS, T(17: 1), S, P. *A First Look at Numerical Functional Analysis.* W.W. Sawyer. Clarendon Pr, 1978, xi + 186 pp, $22; $9.95 (P). [ISBN: 0-19-859628-6] Concepts of Banach and Hilbert space motivated by numerical analysis. Contraction maps, convergence, generalized derivatives, and useful techniques. Problems. RWN

OPTIMIZATION, P. *Stochastic Optimal Control, The Discrete Time Case.* Dimitri P. Bertsekas, Steven E. Shreve. Math. in Sci. and Eng., V. 139. Acad Pr, 1978, xiii + 323 pp, $32. [ISBN: 0-12-093260-1] In two logically independent parts. The first deals with a unified framework in which a large variety of dynamic programming and optimal control models can be analysed. The second settles the (until now) open question of the measurability associated with stochastic optimal control problems by expanding the class of admissible policies to include universally measurable policies. Contains an extensive and useful bibliography. TAV

OPTIMIZATION, T?(16), P. *Mathematical Programming and Control Theory.* B.D. Craven. Chapman & Hall, 1978, xi + 163 pp, $19.95. [ISBN: 0-470-26407-1] Using a setting of Banach spaces, the author presents a unified approach to nonlinear programming including Lagrangian and Pontryagin theory. A shortage of exercises makes this a questionable choice for a text, but is profitable reading for those with an elementary knowledge of functional analysis. TAV

OPTIMIZATION, T(17-18), P. *Convexity and Optimization in Banach Spaces.* V. Barbu, Th. Precupanu. Editura Academiei (Romania), 1978, xi + 316 pp, $25 (P). [ISBN: 90-286-0018-3] The basic results and methods of convex analysis which make up the natural framework for the treatment of infinite dimensional optimization problems. The theory is illustrated by problems involving partial differential equations. LCL

OPTIMIZATION, P. *Interfaces Between Computer Science and Operations Research.* Ed: J.K. Lenstra, A.H.F. Rinnooy Kan, P. van Emde Boas. Math. Centre Tracts, No. 99. Math Centrum, 1978, ix + 231 pp, Dfl. 28 (P). [ISBN: 90-6196-170-X] 8 papers from a symposium held in Amsterdam in September 1976. Survey papers on formal models of computers, choice of data structures, NP-completeness, deterministic scheduling and worst-case analysis of heuristic methods. More specialized papers on methods for solving the knapsack problem, the travelling salesman problem and a problem on the spread of diseases. RWN

ANALYSIS, T(17-18: 1), P. *Bifurcation of Maps and Applications.* G. Iooss. Math. Stud., V. 36. North-Holland, 1979, x + 232 pp, $31.75 (P). [ISBN: 0-444-85304-9] Lecture notes intended to provide explicit formulas for bifurcated objects, thus providing a means to compute in particular applications. Presentation is strictly local; no global results. Applications refer to particular differential equations studied in detail. No index, limited bibliography, some exercises. TLS

ANALYSIS, P. *Unitary Group Representations in Physics, Probability, and Number Theory.* George W. Mackey. Benjamin/Cummings, 1978, xiv + 402 pp, $19.50 (P); $31.50. [ISBN: 0-8053-6703-9; 0-8053-6702-0] A published version of the "Oxford Notes," lectures given at Oxford University in 1966-1967, plus several paragraphs of "Notes and References." The principal thesis is that "the extensive modern applications of the theory of unitary group representations to number theory, physics, and probability are natural generalizations and extensions of much older applications of classical Fourier analysis to the very same subjects." LCL

ANALYSIS, P. *Lecture Notes in Mathematics-663: Compact Right Topological Semigroups and Generalizations of Almost Periodicity.* J.F. Berglund, H.D. Junghenn, P. Milnes. Springer-Verlag, 1978, x + 243 pp, $12 (P). [ISBN: 0-387-08919-5; 3-540-08919-5] An exposition of compactifications from a universal mapping point of view. JAS

ANALYSIS, P. *Lecture Notes in Mathematics-681: Séminaire de Théorie du Potentiel Paris, No. 3.* F. Hirsch, G. Mikobodzki. Springer-Verlag, 1978, vi + 294 pp, $16 (P). [ISBN: 0-387-08947-0; 3-540-08947-0] The majority of the lectures given at the seminar during the years 1976 and 1977. JAS

ANALYSIS, P. *Lecture Notes in Mathematics-665: Journées d'Analyse Non Linéaire.* Ed: P. Bénilan, J. Robert. Springer-Verlag, 1978, viii + 256 pp, $14.30 (P). [ISBN: 0-387-08922-5; 3-540-08922-5] Proceedings of the conference held at Besançon, France in June 1977. JAS

ANALYSIS, T?(14), S*, L. *A First Course in Mathematical Analysis.* J.C. Burkill. Cambridge U Pr, 1978, vii + 186 pp, $8.95 (P). [ISBN: 0-521-29468-1] Topics are quite typical of a "first course:" sequences, series, differentiation, integration, continuity, functions of several variables, but the treatment is somewhat sketchy for a text. There are a few exercises with notes on them, not just answers. This could serve nicely as a reference book for students studying from a more traditional text. TAV

ANALYSIS, T(18: 1), S, P. *Uniform Algebras and Jensen Measures.* T.W. Gamelin. London Math. Soc. Lect. Note Ser., No. 32. Cambridge U Pr, 1978, 162 pp, $13.50 (P). [ISBN: 0-521-22280-X] Lecture notes on function algebras: Choquet theory, R(K), abstract Dirichlet problem, Cole's theory of conjugation operators. LAS

ALGEBRAIC GEOMETRY, P. *Lecture Notes in Mathematics-687: Algebraic Geometry.* Ed: Loren D. Olson. Springer-Verlag, 1978, 244 pp, $12.50 (P). [ISBN: 0-387-08954-3; 3-540-08954-3] Papers based on the talks given at the symposium held June 27 to July 8, 1977 at the University of Tromsø, Norway. Papers primarily focus on intersection theory and space curves. JAS

ALGEBRAIC GEOMETRY, P. *Lecture Notes in Mathematics-683: Variétés Analytiques Compactes.* Ed: Y. Hervier, A. Hirschowitz. Springer-Verlag, 1978, 248 pp, $14.30 (P). [ISBN: 0-387-08949-7; 3-540-08949-7] Proceedings of the conference held at Nice, France, September 19-23, 1977. JAS

DIFFERENTIAL GEOMETRY, T(16-18: 2-4), S*, P, L*. *A Comprehensive Introduction to Differential Geometry, Second Edition.* Michael Spivak. Publish or Perish, 1979. [ISBN: 0-914098-83-7 set] *V. I,* xiii + 668 pp; *V. II,* xiv + 423 pp, $31.25 set; *V. III,* xi + 466 pp; *V. IV,* vii + 561 pp; *V. V,* vii + 661 pp, $62.50 set. Hindsight is best! This *Second Edition* includes, especially in *Volumes I* and *II* which have been completely retyped, corrections and moderate revisions. In particular, the notation and contents have been revised to make the whole five-volume set better unified. This book is no drill text with exercises, but it's good to see this fine exposition of the core of differential geometry in its historical setting revised and available in hard cover at such a reasonable price. (First Edition, TR, *V. I & II,* April 1975; *V. III,* June-July 1975; *V. IV & V,* December 1975.) JAS

DIFFERENTIAL GEOMETRY, S(18), P. *Naturally Reductive Metrics and Einstein Metrics on Compact Lie Groups.* J.E. D'Atri, W. Ziller. Memoirs No. 215. AMS, 1979, iii + 72 pp, $6.40 (P). [ISBN: 0-8218-2215-2] A construction is given to obtain naturally reductive metrics for compact Lie groups and a classification theory is obtained for the simple groups. Which of these metrics are Einstein metrics is discussed and numerous examples are given. JS

GEOMETRY, P. *Computational Geometry for Design and Manufacture.* I.D. Faux, M.J. Pratt. Ellis Horwood, 1979, 329 pp, $21.50. [ISBN: 0-85312-114-1] For engineers, designers and industrial mathematicians concerned with computer-aided design or manufacture. Geometrical view of developments in the last thirty years of the mathematical principles and techniques in the computer representation, analysis and synthesis of "shape information." Intriguing applications of classical analytic and differential geometry. Numerous up-to-date references. JK

GEOMETRY, S(14-16), P, L. *Geometric Principles and Procedures for Computer Graphic Applications.* Sylvan H. Chasen. P-H, 1978, xiv + 241 pp, $14.95. [ISBN: 0-13-352559-7] Analytic geometry applied to construction of smooth curves (and surfaces): splines, Bezier curves, conic sections (e.g., circle tangent to two given circles, passing through a given point), and more. A spare treatment, giving results and strategies rather than full details. LAS

GEOMETRY, S(13-18), P. *Geometrische Perspektive.* F. Rehbock. Springer-Verlag, 1979, ix + 154 pp, $16.30. [ISBN: 0-387-09053-3; 3-540-09053-3] A series of brief practical lessons with historical but no theoretical remarks. This book would be useful for provoking discussion in a geometry course or for authors who wish to provide quality drawings with their published papers. JAS

TOPOLOGY, P. *Lecture Notes in Mathematics-685: Knot Theory.* Ed: J.C. Hausmann. Springer-Verlag, 1978, 311 pp, $16 (P). [ISBN: 0-387-08952-7; 3-540-08952-7] The texts of the lectures presented at the seminar held in Plans-sur-Bex, Switzerland, in 1977. Includes two survey lectures and a collection of open problems. JAS

TOPOLOGY, T(17-18: 1, 2), P, *Topological Uniform Structures.* Warren Page. Wiley, 1978, xv + 398 pp, $34.95. [ISBN: 0-471-02231-4] For the mathematician or student with a background in the basics of analysis, linear algebra, and point set topology, this book offers a broad scope of topological vector spaces and abstract analysis with the unifying theme of "topologies compatible with increasingly enriched algebraic structures." Lots of problems, references, and a sizeable index make this appropriate for an advanced text or a reference work. JAS

TOPOLOGY, T?(16-17: 2), S, *Dynamic Topology.* Gordon Whyburn, Edwin Duda. Springer-Verlag, 1979, xi + 152 pp, $12. [ISBN: 0-387-90358-5; 3-540-90358-5] "The intention of the book is to lead the student, through his own efforts, rather quickly to some important theorems concerning mappings on topological spaces..." To set this pace the topics have been trimmed to the essentials. A reprint of Whyburn's *Dynamic Topology* is provided as a reward for the student's effort. The only flaw in this otherwise excellent text is that solutions immediately follow the exercises, ruining the purpose of this Moore-type format. TLS

TOPOLOGY, T(17-18: 1), S, P, *Actions of Finite Abelian Groups.* C. Kosniowski. Research Notes in Math., No. 18. Fearon-Pitman, 1978, 230 pp, $15 (P). [ISBN: 0-273-08405-4] Study of smooth actions of finite abelian groups on differentiable manifolds from the perspectives of bordism groups and of cutting and pasting groups. First part of text contains discussion of such groups, followed by a chapter on classifying spaces. The latter part of the work is concerned with generators of equivariant bordism groups and of equivariant cutting and pasting groups. Places significant emphasis on the idea of families of slice types. Index. References. RJA

TOPOLOGY, P, *Isolated Invariant Sets and the Morse Index.* Charles Conley. CBMS Reg. Conf. in Math., No. 38. AMS, 1978, 89 pp, $9.60 (P). [ISBN: 0-8218-1688-8] The author reports a generalization of the Morse index of a non-degenerate critical point of a gradient flow for more general invariant sets of arbitrary flows. The first two chapters analyse the ideas heuristically and give basic properties of flows. The last two chapters then construct this Morse index and analyse its homotopy properties. TLS

PROBABILITY, P, *Lecture Notes in Mathematics-678: Ecole d'Eté de Probabilités de Saint-Flour VII-1977.* D. Dacunha-Castelle, H. Heyer, B. Roynette. Springer-Verlag, 1978, ix + 379 pp, $17.80 (P). [ISBN: 0-387-08938-1; 3-540-08938-1] Three major papers by D. Dacunha-Castelle, H. Heyer, and B. Roynette. Most of the rest of the papers appeared as number 65 of the Annales Scientifiques de l'Université de Clermont. JAS

PROBABILITY, S(18), P, L, *Studies in Probability Theory.* Ed: Murray Rosenblatt. Stud. in Math., V. 18. MAA, 1978, xi + 268 pp, $16. [ISBN: 0-88385-118-0] Six advanced expositions sampling contemporary probability research (sequential analysis, random processes, extreme value theory, stochastic equations, statistical mechanics, ergodic theory) by Kiefer, Rosenblatt, Leadbetter, Papanicolaou, Kac, Ornstein. Presumes at least a year's graduate course in probability theory. LAS

PROBABILITY, P, *Bayesian Control of Markov Chains.* K.M. van Hee. Math. Centre Tracts, No. 95. Math Centrum, 1978, vii + 190 pp, Dfl. 23 (P). [ISBN: 90-6196-163-7] The monograph is concerned with the problem of controlling a Markov process with incompletely known transition law. The methods of dynamic programming are employed along with a Bayes criterion. The author shows that the method of successive approximations may be applied to rather large models that have a suitable parameter structure. Extensive bibliography. TAV

PROBABILITY, P, *Proceedings of the International Symposium on Stochastic Differential Equations, Kyoto, 1976.* Ed: Kiyosi Itô. Wiley, 1978, xxx + 507 pp, $35. [ISBN: 0-471-05375-9] 27 invited lectures, preceded by a 30-page introduction to stochastic differential equations by Itô and S. Watanabe. LAS

STATISTICS, T*(15-17: 1, 2), P, *Applied Regression Analysis and Other Multivariable Methods.* David G. Kleinbaum, Lawrence L. Kupper. Duxbury Pr, 1978, xv + 556 pp, $19.95. [ISBN: 0-87872-139-8] First half is a brief introduction followed by a detailed coverage of regression procedures. Second half contains four chapters on one-way and two-way analysis of variance, and single chapters on factor analysis, two-group discriminant analysis, and categorical data analysis. Examples are primarily from health and related fields. Presumes an elementary statistics course, but no calculus or matrix algebra is used. Solutions to exercises are included. RSK

STATISTICS, S(15-18), P*, L*, *Studies in Statistics.* Ed: Robert V. Hogg. Stud. in Math., V. 19. MAA, 1978, xiii + 213 pp, $14. [ISBN: 0-88385-119-9] An effective exposition of major statistical themes (experimental design, nonparametric statistics, chi-square tests, sample surveys, decision theory), prefaced by thematic and mathematical introductions. An excellent primer for mathematicians and mathematics students with little prior background in statistics. LAS

STATISTICS, P, *COMPSTAT 1978: Proceedings in Computational Statistics.* Ed: L.C.A. Corsten, J. Hermans. Physica-Verlag, 1978, 540 pp, DM 59 (P). [ISBN: 3-7908-0196-8] Proceedings of the third Compstat Symposium held in Leiden, Netherlands, in 1978. Contains 69 short papers primarily in the following subject areas: linear and nonlinear regression (14), statistical software (11), simulation and optimization (9), discriminant analysis (8), exploratory techniques (7), cluster analysis (7). RSK

STATISTICS, P, *COMPSTAT Lectures I: Lectures in Computational Statistics.* Ed: H. Skarabis, P.P. Sint. Physica-Verlag, 1978, 132 pp, DM 30 (P). [ISBN: 3-7908-0197-6] Eight articles related to the three Compstat Symposia (*COMPSTAT 1974,* TR, March 1975; *COMPSTAT 1976,* TR, October 1977; *COMPSTAT 1978,* TR above). Highlights some important developments in the field of computational statistics. RSK

STATISTICS, T(13), *Elementary Computer-Assisted Statistics, Revised Edition.* Frank Scalzo, Rowland Hughes. D. Van Nostrand, 1978, xv + 345 pp, $14.95. [ISBN: 0-442-80316-8] Additional material on computer use (see TR, October 1977). LCL

STATISTICS, T*(13: 1), S. *Statistics: Concepts and Controversies.* David S. Moore. Freeman, 1979, xv + 313 pp, $15; $6.95 (P). [ISBN: 0-7167-1022-6; 0-7167-1021-8] Interesting and readable text, dealing with concepts and issues rather than techniques. Divided into three sections: Collecting Data, Organizing Data, and Drawing Conclusions from Data (approximately one-fourth of the text). Contains many examples and exercises using real data. RSK

STATISTICS, P. *Lecture Notes in Mathematics-690: Robust Statistical Methods.* William J.J. Rey. Springer-Verlag, 1978, vi + 128 pp, $9 (P). [ISBN: 0-387-09091-6; 3-540-09091-6] Primarily concerned with M-estimators and simultaneous M-estimators, with emphasis on applications in regression problems. Theoretical tools are mainly the jackknife and the influence function. Good set of references. RSK

STATISTICS, T(13-14: 1, 2). *Statistics: An Introduction to Numerical Reasoning, Pilot Edition.* Ray A. Waller. Holden-Day, 1979, xiii + 555 pp, $15 (P). [ISBN: 0-8162-9314-7] Presupposes only algebra. The usual topics. No Bayesian methods. FLW

STATISTICS, P. *Statistical Services in Ten Years' Time: The Operational Environment; Organization and Coordination; An 'Ideal' System; Use of Computers; Setting Priorities; Technical Developments.* Ed: Joseph W. Duncan. Pergamon Pr, 1978, vii + 189 pp, $30. [ISBN: 0-08-022416-4] Contains all the papers, along with discussion summaries, presented at a Seminar of the Conference of European Statisticians, a Joint Body of the United National Economic Commission for Europe, held in Washington, D.C., in March 1977. Primarily concerned with the role of the Central Statistical Office in the 1980's. RSK

STATISTICS, S(13), L. *Reasoning with Statistics, Second Edition.* Frederick Williams. HR&W, 1979, x + 204 pp, $7.95 (P). [ISBN: 0-03-019536-5] The main goal is to aid readers in understanding studies involving statistical methods rather than to prepare them to use such methods. Tests of significance, correlation and regression, multivariate analysis (including factor analysis and discriminant analysis). LCL

STATISTICS, T**(13: 1). *Modern Elementary Statistics, Fifth Edition.* John E. Freund. P-H, 1979, xv + 510 pp, $15.95. [ISBN: 0-13-593491-5] Extensive revision of the author's well-known 1973 *Fourth Edition* (TR, March 1974), in a new improved format. Basic coverage remains the same, but the material has been reorganized and rewritten, with new examples and exercises. A sound, well-written and readable text. RSK

STATISTICS, P. *The Analysis of Cross-tabulated Data.* Graham J.G. Upton. Wiley, 1978, xii + 148 pp, $26.50. [ISBN: 0-471-99659-9] Aimed primarily at the social science research worker, it deals mainly with the application of the log-linear model to multiway cross-classifications. Assumes access to a computer program for iterative fitting of the models, but gives little information about the algorithm itself. Mostly concerned with complete tables of dichotomous variables, with limited discussions of several other cases. RSK

STATISTICS, T(13-14: 1, 2), S. *Business Statistics, Basic Concepts and Methodology, Second Edition.* Wayne W. Daniel, James C. Terrell. HM, 1979, xv + 672 pp, $16.95. [ISBN: 0-395-26762-5] The usual topics plus time series, multiple regression, survey sampling, decision theory, quality control, and the use of computers. FLW

COMPUTER PROGRAMMING, T(15-17: 1), S, L. *TORRIX, A Programming System for Operations on Vectors and Matrices Over Arbitrary Fields and of Variable Size, Volume 1.* S.G. van der Meulen, M. Veldhorst. Math. Centre Tracts, No. 86. Math Centrum, 1978, vi + 231 pp, Dfl. 28 (P). [ISBN: 90-6196-152-1] Torrix, a descendent of Algol, is, as indicated in the sub-title, intended to deal with programming problems in linear algebra in as general a context as possible. Treatment is essentially self-contained, concerned only with basic matrix operations. A projected Volume 2 is to deal with applications to "complex Hermitian and sparse matrix systems." Index, bibliography. JS

COMPUTER PROGRAMMING, T(14: 1), S*, P, L. *Program Style, Design, Efficiency, Debugging, and Testing, Second Edition.* Dennie van Tassel. P-H, 1978, ix + 323 pp, $14.95. [ISBN: 0-13-729947-8] The design considerations use a top-down approach much more than in the first edition (TR, August/September 1975). Additional problems and suggested projects make it more useful a text. RWN

COMPUTER PROGRAMMING, T*(13-18: 1), S. *Machine and Assembly Language Programming of the PDP-11.* Arthur Gill. P-H, 1978, xv + 191 pp, $16.50. [ISBN: 0-13-541870-4] Intended for use with a stand along PDP-11. Hardware manuals and appropriate software manuals may be needed to accompany the present text when used in an actual course. Presentation is clear and omits unnecessary details. This approach makes the text ideal for a course in this area where students have no previous knowledge of machine level concepts. Good examples. Includes entire illustrative programming examples. Chapter exercises. Seven appendixes, one on programming style. Index. RJA

COMPUTER PROGRAMMING, S*(13). *BASIC for Home Computers.* Bob Albrecht, Leroy Finkel, Jerald R. Brown. Wiley, 1978, xi + 336 pp, $5.95 (P). [ISBN: 0-471-03204-2] This book uses a self-instructional format to teach "Microsoft" Basic. Access to a computer is not essential, but highly desirable. An excellent book to get started with. CEC

COMPUTER PROGRAMMING, T(13-16: 1), S. *Advanced Programming Techniques: A Second Course in Programming Using FORTRAN.* Charles E. Hughes, Charles P. Pfleeger, Lawrence L. Rose. Wiley, 1978, xiii + 287 pp, $15.95. [ISBN: 0-471-02611-5] Includes topics found in the second course of the revised ACM Curriculum '68 recommendations. Begins with review of the features of elementary Fortran. Next is a discussion on various aspects of good programming style. Contains chapters on subprograms, nonarithmetic programming, extended I/O, data structures, machine representation of data, computer organization, and operating system facilities. Two appendices. Index. Chapter exercises. Many varied examples. RJA

COMPUTER PROGRAMMING, S(13-16). *Compatible Fortran.* A. Colin Day. Cambridge U Pr, 1978, vii + 107 pp, $13.95. [ISBN: 0-521-22027-0] Compatible Fortran is, by definition, acceptable to more than one type of computer. Text describes a subset of 1966 ANSI Fortran that is compatible with versions running on most machines. Provides explanations for the inclusion or exclusion of features for a compatible Fortran. Appendix. Bibliography. Index. RJA

COMPUTER PROGRAMMING, T(13: 1), S. *Beginning BASIC.* Paul M. Chirlian. Dilithium Pr, 1978, x + 224 pp, $9.95 (P). [ISBN: 0-918398-06-1] Pocketbook size, introductory text covering almost all aspects of the Basic programming language. LCL

COMPUTER PROGRAMMING, S(13). *The BASIC Idea: An Introduction to Computer Programming.* Richard Forsyth. Chapman and Hall, 1978, vi + 154 pp, $4.95 (P). [ISBN: 0-470-99397-9] A brief introduction to Basic programming. It may be a bit difficult for self-study, but it is appropriate for a short course. Includes exercises and some solutions. CEC

COMPUTER SCIENCE, T(14: 1), S, L. *Information Representation and Manipulation in a Computer, Second Edition.* E.S. Page, L.B. Wilson. Comp. Sci. Texts, No. 2. Cambridge U Pr, 1978, ix + 271 pp, $7.95 (P); $27.50. [ISBN: 0-521-29357-X; 0-521-22088-2] External and internal representations of information. Information structures: arrays, stacks, linked lists, trees; and operations on them. Searching and sorting techniques. A good variety of exercises. RWN

COMPUTER SCIENCE, S, P, L. *Lecture Notes in Control and Information Sciences-10: The Modelling of Systems with Small Observation Sets.* Jan M. Maciejowski. Springer-Verlag, 1978, vi + 242 pp, $12.50 (P). [ISBN: 0-387-09004-5; 3-540-09004-5] Proposes a method of assessing the "trade-off" between the complexity of a model and its accuracy in terms of computational complexity. Should interest those concerned with philosophical questions about scientific methods. FLW

COMPUTER SCIENCE, T*(15-18: 1, 2), S, L. *Principles of Compiler Design, Second Printing.* Alfred V. Aho, Jeffrey D. Ullman. A-W, 1978, x + 604 pp, $19.95. [ISBN: 0-201-00022-9] Text can be used at beginning, intermediate, or advanced levels. Covers all main aspects of compilers. Most chapters written to proceed from general principles to specific details of special cases. Text format is very attractive with special care given to location of charts and diagrams. Chapter exercises and bibliographic notes. One appendix outlines three actual compilers and another appendix describes a student compiler project. Bibliography and index. RJA

COMPUTER SCIENCE, S. *Your Home Computer: An Introduction to Personal Computing in this Era of Microcomputers.* James White. Dymax, 1977, vii + 211 pp, $9.95 (P). [ISBN: 0-918138-05-1] For those who have fallen in love with micro-computers (but may still be scared of them) this provides detailed advice to the lovelorn. This book offers a great deal of practical information about the role and nature of the home (micro) computer as well as very solid information about commercial aspects of the booming market in home computers: addresses of clubs, stores, manufacturers, as well as discussion of essential hardware and software questions. This is not a mathematics book but rather the background to a (non-existent?) buyers' guide. JAS

COMPUTER SCIENCE, S(15-18), P. *The Architecture of Concurrent Programs.* Per Brinch Hansen. P-H, 1977, xvii + 317 pp, $21. [ISBN: 0-13-044628-9] Presents a systematic method of developing concurrent programs in the programming language Concurrent Pascal. Text is divided into three parts: (1) programming tools--design principles, programming concepts, sequential Pascal, concurrent Pascal; (2) concurrent programs--the Solo operating system, the Job Stream operating system, real-time scheduler for process control; (3) language--concurrent Pascal report and machine. References. Index. RJA

COMPUTER SCIENCE, S. *The Complete STAR SHIP: A Simulation Project.* Roger Garrett. Dilithium Pr, 1978, x + 122 pp, $6.95 (P). [ISBN: 0-918398-10-X] A detailed study of computer simulation built around the example originally published by the author in the August, September, and October 1977 issues of *Interface Age* magazine. The structure of the simulation alone is presented, rendering the approach essentially hardware and software independent. JAS

COMPUTER SCIENCE, S(16-18), P, L. *Current Trends in Programming Methodology, V. II: Program Validation.* Ed: Raymond T. Yeh. P-H, 1977, xi + 322 pp, $19.95. [ISBN: 0-13-195719-8] Software validation attempts to determine if a program actually performs its intended logical functions. Subject divides into two areas: testing and verification. This work contains five articles on program testing and five on verification. Concludes with an extensive annotated bibliography. Index. RJA

COMPUTER SCIENCE, S(13), L. *Home Computers: A Beginners Glossary and Guide.* Merl K. Miller, Charles J. Sippl. Dilithium Pr, 1978, viii + 147 pp, $6.95 (P). [ISBN: 0-918398-02-9] Terminology, advice on type and size of memory, arithmetic, logic and components. RWN

COMPUTER SCIENCE, P, L. *Advances in Computers, Volume 16.* Ed: Morris Rubinoff, Marshall C. Yovits. Acad Pr, 1977, xiii + 357 pp, $36.50. [ISBN: 0-12-012116-6] 5 survey papers, each with excellent references, on 3D graphics, automatic programming, clinical computing, networks, and privacy protection. LAS

COMPUTER SCIENCE, S(16-17), P. *Algorithmen in Zellularautomaten.* Roland Vollmar. B.G. Teubner, 1979, 192 pp, DM 21,80 (P). [ISBN: 3-519-02350-4] On parallel algorithms, most of them non-numerical, for use on cellular automata. JD-B

SYSTEMS THEORY, P. *Recent Theoretical Developments in Control.* Ed: M.J. Gregson. Acad Pr, 1978, xv + 672 pp, $31.50. [ISBN: 0-12-301650-9] Proceedings of a September 1976 conference at the University of Leicester: 37 papers on algebraic systems theory, stability, optimal control, numerical methods, stochastic control, and estimation theory. LAS

SYSTEMS THEORY, P. *Lecture Notes in Control and Information Sciences-8: Infinite Dimensional Linear Systems Theory.* Ruth F. Curtain, Anthony J. Pritchard. Springer-Verlag, 1978, 297 pp, $14.80 (P). [ISBN: 0-387-08961-6; 3-540-08961-6] An introduction to infinite dimensional systems theory geared for readers with a knowledge of finite dimensional theory and some functional analysis. Includes an excellent bibliography. CEC

SYSTEMS THEORY, T(17-18), P. *Introduction to System Sensitivity Theory.* Paul M. Frank. Acad Pr, 1978, xiv + 386 pp, $19. [ISBN: 0-12-265650-4] Sensitivity theory provides the basic methods for studying the sensitivity of a mathematical model to parameter variations. Fundamental principles are presented here in a form understandable to engineers. LCL

SYSTEMS THEORY, T(15: 1), S, L. *Principles of Dynamic Programming, Part 1: Basic Analytic and Computational Methods.* Robert E. Larson, John L. Casti. Control and Systems Theory, V. 7. Dekker, 1978, xi + 330 pp, $22.50. [ISBN: 0-8247-6589-3] A well-written introduction presuming linear algebra and differential equations. Describes multistage decision processes, derives the fundamental iterative equations, discusses the basic computational procedures and treats various extensions. Includes numerous exercises. RWN

SYSTEMS THEORY, S(13-17). *A Handbook of Systems Analysis, Second Edition.* John E. Bingham, Garth W.P. Davies. Halsted Pr, 1978, ix + 229 pp, $18.95. [ISBN: 0-470-99129-1] An overview of the "major aspects and techniques of systems analysis." FLW

APPLICATIONS, P. *Modelling and Simulation in Practice.* Ed: M. Cross, *et al.* Wiley, 1979, 358 pp, $39.95. [ISBN: 0-470-26645-7] Typescript proceedings (with some discussion) of the "Inaugural Conference" of the North East [England] Polytechnics Mathematical Modelling and Computer Simulation Group held at Sunderland Polytechnic, 24-25 May 1978. Diverse topics, from coke ovens to drug distribution. LAS

APPLICATIONS, P, L. *The Maximum Entropy Formalism.* Ed: Raphael D. Levine, Myron Tribus. MIT Pr, 1979, xii + 498 pp, $22.50. [ISBN: 0-262-12080-1] Proceedings (in typescript) of a conference held at M.I.T. in May 1978, approximately 100 years after Boltzmann suggested the concept of entropy, and 75 years after Gibbs and Planck showed how it could be used. Contains several extensive historical expositions and topical surveys on application of maximum entropy to the physical and biological sciences.LAS

APPLICATIONS, P. *Differential Games and Control Theory III.* Ed: Pan-Tai Liu, Emilio Roxin. Lect. Notes in Pure and Appl. Math., V. 44. Dekker, 1979, vii + 244 pp, $27 (P). [ISBN: 0-8247-6845-0] Contains eleven papers presented at the Third Kingston Conference, 1978 devoted either to deterministic or stochastic systems. Papers range from applied to theoretical. (TR, *I,* November 1975; TR, *II,* April 1978.) TLS

APPLICATIONS (AGRICULTURE), S(15), P. *Quadratic Programming Models Applied to Agricultural Policies.* Anton D. Meister, Carl C. Chen, Earl O. Heady. Iowa St U Pr, 1978, x + 110 pp, $6. [ISBN: 0-8138-1930-X] After very brief introductions to linear and quadratic programming techniques, the authors discuss the applicability of such techniques to problems of export capacity alternatives, price supports, and fertilizer restrictions. TAV

APPLICATIONS (ARTIFICIAL INTELLIGENCE), P. *Associative Networks: Representation and Use of Knowledge by Computers.* Ed: Nicholas V. Findler. Acad Pr, 1979, xvii + 462 pp, $39.50. [ISBN: 0-12-256380-8] Fourteen chapters written by some twenty-one experts in the field are divided into three parts: general systems; theoretical efforts; and application areas. Chapter reference lists. RJA

APPLICATIONS (ARTIFICIAL INTELLIGENCE), P. *Human and Artificial Intelligence.* Ed: Friedhart Klix. Fund. Stud. in Comp. Sci., V. 8. North-Holland, 1979, xv + 227 pp, $36.50. [ISBN: 0-444-85173-9] Series of papers on artificial intelligence that were given at a symposium which was held during the XXI International Conference of Psychology, Paris, July 18-25, 1976. RJA

APPLICATIONS (ARTIFICIAL INTELLIGENCE), P. *Pattern-Directed Inference Systems.* Ed: D.A. Waterman, Frederick Hayes-Roth. Acad Pr, 1978, xiv + 658 pp, $33. [ISBN: 0-12-737550-3] A selection of papers presented at the Workshop on Pattern-Directed Inference Systems, May 23-27, 1977, in Honolulu. Text includes an introductory overview and papers on architecture and design, deductive inference, learning, cognitive modeling, natural language understanding, multilevel systems, and complexity. Bibliography. Author and subject indices. RJA

APPLICATIONS (ASTRONOMY), T(16-18), P. *Theory of Rotating Stars.* Jean-Louis Tassoul. Princeton U Pr, 1978, xiii + 506 pp, $40; $15 (P). Synthesis of a large number of theoretical investigations with a concentration on those topics (structure, stability, evolution) accessible within the framework of Newtonian mechanics. LCL

APPLICATIONS (BIOLOGY), T(16-17: 1), L. *Mathematical Models for the Growth of Human Populations.* J.H. Pollard. Cambridge U Pr, 1973, xii + 186 pp, $7.50 (P). [ISBN: 0-521-20111-X] Paperback edition (hardcover, TR, May 1974) of a book that provides concise surveys of various population models from Malthus to the present. Contains exercises (with solutions) and extensive references. LAS

APPLICATIONS (BIOLOGY), T(14-16), S*, P, L*. *Models in Ecology.* J. Maynard Smith. Cambridge U Pr, 1978, xii + 146 pp, $6.50 (P). [ISBN: 0-521-29440-1] "Ecology is still a branch of science in which it is usually better to rely on the judgment of an experienced practitioner than on the predictions of a theorist. Nevertheless, ecology will not come of age until it has a sound theoretical basis." Smith provides a concise survey of mathematical (difference and differential equations, Markov chains, simulation) and laboratory models, including several published here for the first time. Written for ecologists, the mathematical prerequisites are quite modest--just basic calculus. LAS

APPLICATIONS (BUSINESS), S(13). *Programming Learning Aid for Quantitative Methods with Applications in Accounting and Business.* Gordon B. Harwood. Irwin, 1979, xi + 160 pp, $4.95 (P). [ISBN: 0-256-02084-1] A brief non-rigorous review of (or simplified introduction to) several topics: calculus (as slope), matrices, linear programming, and some business-related topics such as inventory control. LLK

APPLICATIONS (BUSINESS), T(13-14: 2), L. *Mathematics for Business and Social Science, An Applied Approach, Second Edition.* Abe Mizrahi, Michael Sullivan. Wiley, 1979, xvii + 797 pp, $18.95. [ISBN: 0-471-03334-0] Fundamental mathematical techniques required for modern study of economics, business, and social science. Technique oriented, minimum mathematical theory. Useful and correct definitions, numerous examples to motivate interest. Comprehensive, applied, interesting writing style, wide range of tables. (First Edition, TR, December 1976.) WLC

APPLICATIONS (CHEMISTRY), T(15-18), S, P, L. *Hückel Theory for Organic Chemists.* C.A. Coulson, Brian O'Leary, R.B. Mallion. Acad Pr, 1978, xvi + 182 pp, $19.65. [ISBN: 0-12-193250-8] Lectures on molecular-orbital theory given to third year undergraduate chemists at Oxford, 1967-1973, by the late C.A. Coulson, the man largely responsible for the development and widespread application of this theory. Mathematics used includes calculus, linear algebra and graph theory. LCL

APPLICATIONS (ENGINEERING), T(15-18: 1, 2), S, P, L. *Computer Arithmetic: Principles, Architecture, and Design.* Kai Hwang. Wiley, 1979, xiii + 423 pp, $21.95. [ISBN: 0-471-03496-7] Basic concepts of machine arithmetic, theory and design of arithmetic units that perform fixed-point and floating-point arithmetic. Includes evaluation of elementary functions, pipelined arithmetic design, and error control techniques. Case studies of arithmetic designs in specific large computers. Chapter bibliographic notes, references, and problems. Solutions manual available for problems. Author and subject indexes. RJA

APPLICATIONS (ENGINEERING), S(17), P. *The Boundary Element Method for Engineers.* C.A. Brebbia. Halsted Pr, 1978, 189 pp, $27.50. [ISBN: 0-470-26438-1] An introduction to a method which is competitive with the finite difference and finite element methods for large classes of problems. The general approach used is through weighted residual methods. Applied to potential and elasticity problems in one and two dimensions. Includes Fortran programs. RWN

APPLICATIONS (ENGINEERING), S?(16-17), P. *Applied Mathematics: An Intellectual Orientation.* Francis J. Murray. Math. Concepts and Methods in Sci. and Eng., V. 12. Plenum Pr, 1978, xiv + 225 pp, $29.50. [ISBN: 0-306-39252-6] An extreme, idiosyncratic expression of a contemporary natural philosophy, steeped in simulation, engineering and classical mathematics. Opinion and exhortation are enhanced by rather arbitrary (and often highly technical) examples from ancient history and mathematical physics. Concludes with a superficial argument supporting total mathematical determinism: "Scientific understanding encompasses all experience...Philosophical "principles"...must either be a consequence of science [or] illusory. ...The theory of evolution...has a precise mathematical description... This understanding precludes the existence of God." LAS

APPLICATIONS (ENGINEERING), T(15-18: 1, 2), S, L. *Digital Systems: Hardware Organization and Design, Second Edition.* Fredrick J. Hill, Gerald R. Peterson. Wiley, 1978, xiii + 701 pp, $23.95. [ISBN: 0-471-39608-7] Text involves the student in the design process through introduction and use of A Hardware Programming Language (AHPL). This provides the student with a meaningful description of the implementation of computer subsystems and with concrete design examples. Reorganized material on the control unit, memories, and I/O intersystem communications. New material on microprocessors, interface design, and digital communications logic design. Problems. Appendices. Index. Compilers and simulators for AHPL available. RJA

APPLICATIONS (ENGINEERING), S(17-18), P. *Pattern Recognition: Ideas in Practice.* Ed: Bruce G. Batchelor. Plenum Pr, 1978, xv + 485 pp, $39.50. [ISBN: 0-306-31020-1] Describes pragmatic approaches to pattern recognition. Articles in the first half of the text focus on various methods and types of machines developed for pattern recognition purposes. The second half is devoted to applications. The editor opens with a general presentation on the subject and its relationship to other disciplines; he closes the work with some of the social aspects of the field. References accompany each article. Volume index. RJA

APPLICATIONS (ENVIRONMENTAL SCIENCE), P. *Modeling, Identification and Control in Environmental Systems.* Ed: G.C. Vansteenkiste. North-Holland, 1978, xviii + 1028 pp, $89. [ISBN: 0-444-85180-1] Proceedings of an IFIP (International Federation of Information Processing) working conference held in Ghent, Belgium in September 1977. Features mathematical and computer models of air, land and water resources, often viewed as "gray box" problems. LAS

APPLICATIONS (FLUID DYNAMICS), P. *Numerical Methods in Laminar and Turbulent Flow.* Ed: C. Taylor, K. Morgan, C.A. Brebbia. Halsted Pr, 1978, 1006 pp, $55. [ISBN: 0-470-26462-4] The proceedings of the conference held at Swansea, July 17-21, 1978. JAS

APPLICATIONS (GENETICS), P. *A New Mathematical Framework for the Study of Linkage and Selection.* S. Shahshahani. Memoirs No. 211. AMS, 1979, ix + 34 pp, $6 (P). [ISBN: 0-8128-2211-X] A model for the evolution of a genetic system based on the tools and ideas of differential geometry and differential dynamical systems. LCL

APPLICATIONS (INFORMATION THEORY), T(16-18: 1, 2), S, P. *Coding Theorems of Information Theory, Third Edition.* Jacob Wolfowitz. Ergebnisse der Math., B. 31. Springer-Verlag, 1978, xi + 173 pp, $27. [ISBN: 0-387-08548-3; 3-540-08548-3] Presentation of the main theorems of probabilistic information theory. Written from the mathematician's point of view. Begins with a short heuristic introduction followed by a chapter on necessary combinatorial background information. References. RJA

APPLICATIONS (PATTERN RECOGNITION), S(16-18), P. *Pattern Analysis: Lectures in Pattern Theory, Volume II.* U. Grenander. Appl. Math. Sci., V. 24. Springer-Verlag, 1978, viii + 605 pp, $18.80 (P). [ISBN: 0-387-90310-0; 3-540-90310-0] Continues the author's work on pattern recognition. FLW

APPLICATIONS (PHYSICS), L*. *Collected Papers on Wave Mechanics, Second (Unaltered) Edition.* E. Schrödinger. Chelsea, 1978, xiii + 146 pp, $9.95 [ISBN: 0-8284-0302-3] A "long-life" edition, unaltered from the 1928 second English edition, containing nine papers from 1926-27. LAS

APPLICATIONS (PHYSICS), T(18: 2), S, P. *Theory of Group Representations and Applications.* Asim O. Barut, Ryszard Rączka. PWN, 1977, xix + 717 pp, $35. Intended primarily for theoretical physicists, this is a comprehensive survey of the present state of development of representation theory for topological groups, particularly Lie groups. Encyclopedic in breadth with all of the major theorems (but few of the proofs); numerous applications to quantum theory. Includes chapter exercises (difficult), bibliography, indices, appendices on Hilbert space operators. JS

APPLICATIONS (POLITICAL SCIENCE), T(13-15: 1), S. *Spatial Models of Election Competition.* Steven J. Brams. EDC/UMAP, 1979, v + 94 pp, $4 (P). Elementary (high school algebra) logical and mathematical modelling applied to an (American) election campaign. Uses simple one-dimensional (left-right) analysis to determine optimal positions in a campaign. Aimed primarily at students in quantitatively oriented courses in political science, to show that "by deducing consequences from models, one can see more clearly what is happening than one can by trying to deal with reality in all its unmanageable detail." A typescript preliminary edition sponsored by the Undergraduate Mathematics and its Applications Project (UMAP). LAS

APPLICATIONS (POLITICAL SCIENCE), S(16-18), P, L. *Game Theory and Political Science.* Ed: Peter C. Ordeshook. New York U Pr, 1978, xii + 627 pp, $28.50. [ISBN: 0-8147-6156-9] 19 papers, both experimental and theoretical, by political scientists, economists, and mathematicians on applications of game theory to political theory. Based on a July 1977 conference at Hyannis, Mass., sponsored by the Mathematical Social Science Board, these contributions, according to the editor, suggest forthcoming major theoretical advances and conceptual departures from the classical theory of games. LAS

APPLICATIONS (SIMULATION), T(15-16: 1), S, P*, L**. *Lecture Notes in Control and Information Sciences-4: An Introduction to the Regenerative Method for Simulation Analysis.* M.A. Crane, A.J. Lemoine. Springer-Verlag, 1977, vii + 111 pp, $8.30 (P). [ISBN: 0-387-08408-8; 3-540-08408-8] Simulation provides one of the most practical approaches for the study of complicated real-world problems that involve systems of a random nature--communication networks, queues, maintenance and repair operations, inventory systems, etc. The recently developed "regenerative method" provides a technique for producing valid statistical results (including answers to tactical problems concerning the simulation setup). This presentation provides an informal but precise tutorial for potential users; extensive use of examples to motivate and illustrate fundamental ideas and results. Prerequisite: basic introduction to probability and statistics, and some acquaintance with the notion of modeling for problem solving. LCL

APPLICATIONS (SOCIAL SCIENCE), S(15-18), P, L. *Modeling of Complex Systems, An Introduction.* V. Vemuri. Acad Pr, 1978, xvi + 448 pp, $19.50. [ISBN: 0-12-716550-9] Interesting, often provocative, approach to mathematical modeling for large scale systems. Text is a blend of philosophical commentary and general methodology interspersed with applications. Mathematics used is largely elementary; statistics, differential equations, linear algebra, graph theory, etc. Wide variety of examples and suggestions for individual projects, but no exercises. Index, bibliography. JS

APPLICATIONS (SOCIAL SCIENCE), T(15-16: 1), S, L. *Computer Simulation and Modeling: An Introduction.* Richard S. Lehman. Halsted Pr, 1977, xii + 411 pp, $19.95. [ISBN: 0-470-99296-4] Intended for students of social or behavioral science. Several good examples. Fortran programming techniques, elementary data structures, and random processes. Discusses modeling and validation. RWN

APPLICATIONS (SOCIAL SCIENCE), T(18: 1), P*. *Mathematics of Manpower Planning.* S. Vajda. Wiley, 1978, ix + 206 pp, $29. [ISBN: 0-471-99627-0] A reasonably complete treatment of deterministic models of manpower (population) planning. In two parts: long-term development and short term, step by step considerations. Appendices on matrix theory and linear programming, the major tools, and an extensive bibliography. TAV

APPLICATIONS (SOCIAL SCIENCE), P. *Transformations: Mathematical Approaches to Culture Change.* Ed: Colin Renfrew, Kenneth L. Cooke. Acad Pr, 1979, xxii + 515 pp, $39.50. [ISBN: 0-12-586050-1] 21 pioneering essays by anthropologists, archeologists, mathematicians, geneticists, physiologists, computer scientists and systems engineers on the general subject of morphogenesis--the emergence of form. Most essays touch on "soft" catastrophe theory; each has an extensive bibliography of diverse sources. LAS

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; William L. Carlson, St. Olaf; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Paul Jorgensen, Carleton; Lorraine L. Keller, St. Olaf; Richard S. Kleber, St. Olaf; Joseph Konhauser, Macalester; Loren C. Larson, St. Olaf; George H. Mills, St. Olaf; R.W. Nau, Carleton; Thomas R. Savage, St. Olaf; John Schue, Macalester; Seymour Schuster, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Timothy L. Strotman, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036

PERSONAL ITEMS

Millsaps College, Jackson, Mississippi: Dr. John Ramsey, Louisiana State University, has been appointed Assistant Professor. Associate Professor Arnold A. Ritchie has retired with the title of Professor Emeritus.

SUNY of Oneonta: Associate Professor Leo J. Alex has been promoted to Professor. Assistant Professor Theodore S. Bolis has been promoted to Associate Professor.

Indiana University Southeast: Associate Professor James W. Woeppel has been appointed Computer Services Director. Assistant Professor Lawrence Mand joined an actuarial firm in Louisville, Kentucky. Associate Professor and Chairman C.S. Kim will be on leave-of-absence during the 1979-80 academic year.

North Carolina State University: Associate Professor Robert E. Hartwig has been promoted to Professor. Assistant Professor John E. Franke has been promoted to Associate Professor. Dr. Elmor L. Peterson, Northwestern University, has been appointed Professor. Associate Professors Henry C. Cooke and Charles H. Little have retired.

Stanford University: Associate Professor Mary Sunseri has been promoted to Professor. Professor Hans Samelson has been appointed Chairman of the Mathematics Department for 1979-81.

Professor N.S. Mendelsohn, Head of the Department of Mathematics, University of Manitoba, has been awarded the Henry Marshall Tory Medal by the Royal Society of Canada at its meeting on May 28, 1979.

Associate Professor and Chairman Edward H. Brande, S.J., Fordham University, has been appointed Professor of Mathematics and Academic Vice President.

Associate Professor William Walker Smith, University of North Carolina, has been promoted to Professor.

Assistant Professors Patrick D. Cassady and Thomas W. Osgood, Indiana University East, have been promoted to Associate Professors.

Dr. Eric L. Wilson, Wittenberg University, has been appointed Chairman of the Mathematics Department for a three year term.

Professor A. Clifford Cohen retired from the University of Georgia on June 30, 1978 after thirty one years of service. He has been named Professor Emeritus of Statistics.

Professor James M. Horner, Vice President and Provost at Illinois State University, has been appointed President of Central Missouri State University.

Professor Jerry P. King, Lehigh University, has been named Associate Dean of the College of Arts and Sciences.

Dr. Edward T.H. Wang, Wilfrid Laurier University, Waterloo, Canada, is on sabbatical leave at the Institute of Mathematics, Academia Sinica, Nanking, Taipei, Taiwan 115, from July 1, 1979 to June 30, 1980.

Associate Professor Ray O. Hamel, Eastern Washington University, Cheney, has been promoted to Professor.

Professor Paul R. Halmos, Indiana University at Bloomington, has been elected an Honorary Member of the Hungarian Academy of Sciences.

Associate Professor James Uebelacker, University of New Haven, has been named Assistant Provost.

Professor Raymond L. Spencer, Dearborn, Michigan, died on March 19, 1978. He was a member of the Association for thirty years.

Dr. Robert C. Seber, Western Michigan University, died on September 6, 1978. He was a member of the Association for thirty years.

Dr. Hubert E. Bray, Houston, Texas, died on August 30, 1978. He was a member of the Association for fifty five years.

Dr. Joel S. Georges, Winston Salem, North Carolina, died on November 22, 1978. He was a member of the Association for fifty nine years.

Professor Robert C. Seber, Western Michigan University, died on September 6, 1978. He was a member of the Association for thirty years.

Professor John Salerno, Bayero University, Kano, Nigeria, died on September 7, 1978, at the age of sixty four. He was a member of the Association for thirty two years.

Professor H.D. Block, Cornell University, died on October 6, 1978, at the age of fifty eight. He was a member of the Association for twenty nine years.

Sister M. Vaudreuil, Notre Dame Health CC, died on October 26, 1978, at the age of eighty four. She was a member of the Association for forty six years.

Professor Robert F. Tidd, North Dakota State University, died on February 2, 1979. He was a member of the Association for twenty eight years.

Professor Billy J. Pettis, University of North Carolina, died on April 17, 1979, at the age of sixty six. He was a member of the Association for thirty years.

Dr. Sumner I. Vrooman, Professor Emeritus, Pratt Institute, died in 1979 in Norway at the age of seventy nine. He was a member of the Association for thirty years.

Professor Lino G. Gutiervez-Novoa, University of Alabama, died in 1979 at the age of sixty. He was a member of the Association for three years.

Dr. Edmund E. Ingalls, Greeley, Colorado, died in 1979. He was a member of the Association for fifty one years.

Professor Robert D. Klein, Northeastern University, died in 1979 at the age of forty five. He was a member of the Association for seventeen years.

Professor Emeritus Ralph D. James, University of British Columbia, died on May 28, 1979.

Dr. W.H. Fagerstrom, Executive Director Emeritus of the Annual High School Mathematics Contest, died on September 10, 1978, at the age of eighty seven. In 1949-50 he proposed that the Metropolitan New York Section sponsor a mathematics contest for high school students. He became Chairman of the Appropriate Section Committee and invited high schools far and wide to participate. Out of this effort grew the seed for the Annual High School Mathematics Contest.

THE 1979 WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

The 40th Annual William Lowell Putnam Mathematical Competition will be held at participating institutions on Saturday, December 1, 1979. This competition is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship and is administered by The Mathematical Association of America. All colleges and universities in Canada and the United States may register eligible undergraduates. Registration forms will be mailed to institutions that participated in the 39th competition by September 24, 1979. Other institutions that wish to enter undergraduates should request registration forms from Professor L. F. Klosinski, Director; The William Lowell Putnam Mathematical Competition; University of Santa Clara; Santa Clara, CA 95053. Completed registrations must be received by the Director no later than October 19, 1979.

Further details are given in the Announcement Brochure that is mailed with the registration material. Reports of previous competitions, including examination questions and outlines of solutions, are in past issues of this *MONTHLY;* the most recent of these reports were in the issues of March 1979, January 1978, November 1976, November 1975, and December 1974.

SYMPOSIUM ON ILL-POSED PROBLEMS

This symposium will be held from October 2 to October 6, 1979 at the University of Delaware under the joint sponsorship of the Army Research Office and the Air Force Office of Scientific Research. The program will give equal weight to theory and applications. There will be featured overview lectures with the keynote address to be given by Professor Fritz John of the Courant Institute. In addition, there will be invited talks, contributed talks, and workshops. For further information, write to Professor M. Z. Nashed, Symposium Director, Department of Mathematical Sciences, University of Delaware, Newark, Delaware 19711.

ANNOUNCEMENT OF ALLENDOERFER, FORD AND POLYA AWARDS

At its meeting on January 28, 1977, in St. Louis, Missouri, the Board of Governors authorized a number of awards to authors of expository articles published in the *MONTHLY,* to be named after Lester R. Ford, Sr., *MATHEMATICS MAGAZINE,* to be named after Carl B. Allendoerfer, and the *TWO-YEAR COLLEGE MATHEMATICS JOURNAL,* to be named after George Polya. A maximum of two Carl B. Allendoerfer Awards, five Lester R. Ford Awards, and two George Polya Awards will be made annually; each award is in the amount of $100. The articles are to be selected by committees appointed by the President of the Association for this purpose and the Chairman of the Committee on Publications is to be an *ex-officio* member of each of these committees.

The recipients of the Carl B. Allendoerfer Awards for 1978 were selected by a committee consisting of Roy Dubisch, Chairman; Edwin F. Beckenbach, *ex-officio;* and Thomas W. Tucker. The recipients for Allendoerfer Awards for article published in 1978 were the following:

Doris Schattschneider, *Tiling the Plane with Congruent Pentagons, MATHEMATICS MAGAZINE (1978),* page 29

Bruce Berndt, *Ramanujan's Notebooks, MATHEMATICS MAGAZINE (1978),* page 147

The recipients of the Lester R. Ford Awards for 1978 were selected by a committee consisting of Lida K. Barrett, Chairman; Edwin F. Beckenbach, *ex-officio*; and Branko Grünbaum. The recipients for Ford Awards for articles published in 1978 were the following:

Bradley Efron, *Controversies in the Foundation of Statistics, MONTHLY 85 (1978),* April, p. 231

L. A. Shepp and J. B. Kruskal, *Computerized Tomography: The New Medical X-Ray Technology, MONTHLY 85,* June-July, p. 420

Ned Glick, *Breaking Records and Breaking Boards, MONTHLY 85,* January, p. 2

Kenneth I. Gross, *On the Evolution of Noncommutative Harmonic Analysis, MONTHLY 85,* August-September, p. 525

The recipients of the George Polya Awards for 1978 were selected by a committee consisting of Joseph Hashisaki, Chairman; Edwin F. Beckenbach, *ex-officio;* and Kay W. Dundas. The recipients for Polya Awards for articles published in 1978 were the following:

Richard Plagge, *Fractions Without Quotients: Arithmetic of Repeating Decimals, TYCMJ, 9, #1,* (1978), pages 11-15

Richard L. Francis, *A Note on Angle Construction, TYCMJ, 9, #2,* (1978), pages 75-80

DAVID P. ROSELLE, *Secretary*

U.S. MATH OLYMPIAD TEAM

Eight of the nation's most brilliant high school mathematics students won positions on the U.S. team for the 21st International Mathematical Olympiad. Selection of the U.S. team began in March with the Annual High School Mathematics Contest. Over 360,000 students in the U.S. and Canada participated.

About one hundred students with top contest scores were invited to compete in the Eighth U.S.A. Mathematical Olympiad, the final round of team selection. The eight winners went on to represent the U.S. at the international olympiad July 2 and 3, 1979, in London, England.

The olympiads test a student's ingenuity as well as mathematical background. The U.S. has done well in the international competition, placing second last year and winning the year before.

The members of this year's team are:
Michael V. Finn of Annandale, Virginia,
Bruce K. Smith of San Rafael, California,
Michael J. Larsen of Lexington, Massachusetts,
Lawrence E. Penn of Great Neck, New York,
Ronald F. Kaminsky of Albany, New York,
Mark G. Pleszkoch of Manassas, Virginia,
Randy L. Ekl of North Huntington, Pennsylvania, and
Richard G. Agin of Chicago, Illinois.

The U.S. team was honored on June 5 in Washington, D.C., at the Eighth U.S. Olympiad awards ceremony, which was held at the National Academy of Sciences and the Diplomatic Reception Rooms of the U.S. Department of State.

After the awards ceremony the eight U.S. Olympiad winners prepared for the international contest by attending a three-week training seminar at the U.S. Military Academy, West Point, New York.

The U.S.A. Mathematical Olympiad and the Annual High School Mathematics Contests (HSC) are sponsored by five major organizations of mathematicians and actuaries: The Mathematical Association of America, The Society of Actuaries, The National Council of Teachers of Mathematics, Mu Alpha Theta, and the Casualty Actuarial Society. The awards ceremony, the training sessions, and team travel expenses are supported by grants from IBM, the Army Research Office, and the Office of Naval Research.

THE 1980-81 SABBATICAL EXCHANGE INFORMATION SERVICE

Do you feel the need for a change of scene for a semester or a year, but are not eligible for a sabbatical leave? The MAA Sabbatical Exchange Information Service (SEIS) may help you arrange for a "No-cost sabbatical" by exchanging positions with a faculty member with similar interests in another institution. The stimulation of a sabbatical exchange can benefit you and your institution and cost little or nothing. We invite your participation in SEIS in 1980-81 looking toward an exchange in 1980 or later. Here is how SEIS works.

Many MAA members are in institutions not offering faculty members a program of sabbatical leaves. Even in institutions with sabbatical leave programs, individual faculty members often find themselves ineligible for such leave at a time when the desire for one is strongest. We all recognize the rejuvenating effect of an occasional change of scene, even if it does not involve release from teaching duties. The Association therefore suggests that an occasional exchange between two faculty members of similar interests, training, and experience at different institutions could be of great benefit to the individuals and also to their institutions. The individuals and institutions all stand to gain from the refreshment of exchanged ideas and insights.

It is often possible for two such faculty members to trade identities, so to speak, for a year. Such an exchange might involve trading teaching responsibilities, living quarters, and some departmental responsibilities. The extent of the exchange would depend on the individual circumstances. It is suggested, however, that salaries should not be exchanged or even discussed. Each faculty member would remain on the payroll of his permanent institution and receive all of his normal fringe benefits. Financially, his institution would not recognize the exchange at all.

The MAA proposes to become involved only to the extent of assisting in bringing together like-minded mathematics faculty members who are interested in an exchange. The information exchange will be accomplished by the annual publication by the Association in December of a list containing the names, addresses, and other pertinent information about members of the Association interested in arranging a "Sabbatical Exchange" with a colleague in another institution. This list will be sent free of charge to all those on the list and to any other MAA member who requests it.

Members interested in being listed in December 1979 should write to "SEIS, The Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036," enclosing the following information about themselves: (Please type or print legibly your information to ensure the accuracy of the listing).

1. Name
2. Institution
3. Department
4. Address
5. Rank
6. Major field of interest
7. Highest earned degree
8. Names of from one to five courses recently taught
9. Normal teaching load

10. Section of country preferred for visit: Northeast, Southeast, Northcentral, Southcentral, Northwest, Southwest

11. Period for which exchange is desired, e.g., all of the academic year 1980-81, or the first two quarters of 1980-81 or the second semester of 1980-81, etc.

Communications must reach the Washington office by December 1, 1979 for inclusion in the December 1979 list.

CORRECTION: Professor Carlton E. Lemke is a member of the faculty of Rensselaer Polytechnic Institute, and not of Stanford University as stated in the April, 1979, News and Notices.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

OKLAHOMA-ARKANSAS MARCH MEETING

This meeting was held at Oklahoma State University on March 30-31, in Stillwater, Oklahoma. The attendance was approximately one hundred and twenty with thirty-eight papers given by faculty and students. The highlights of the meeting were the N. A. Court Lecture given by Dr. H. O. Pollak, Director, Mathematics and Statistic Research Center, Bell Laboratories, and the Invited Address given by Dr. Philip Davis, Professor of Applied Mathematics, Brown University. The meeting was coordinated with an Applied Mathematics Symposium that was held at Oklahoma State University on March 29-30.

The following talks completed the program:

Current and Past Roles of the Statistician in Space Applications, Patrick L. Odell, University of Texas at Dallas

Mathematics Used in Research in Sandia Laboratories, Charlotte C. Acken, Sandia Laboratories

Applications of the Bohman-Korovkin Theorem, R. B. Deal, Oklahoma University Health Sciences Center

Integer Programming Solutions of Two Problems from Recreational Mathematics, Richard Greenhaw, Oklahoma Christian College

Actuarial Science Profession, Al Litchenburg, American Fidelity

Shift Registers and Graph Embeddings, Morris L. Marx, University of Oklahoma

Asymptotic Distribution of Time to First Repeat, R. B. Deal, Oklahoma University Health Sciences Center

Comparison of Product Approximation Methods, Lawrence Huff, University of Central Arkansas

Computer Applications in Population Genetics, Cassandra Scrimshire, Hendrix College

The Development of an Equation for Sequences and Polynomial Curves, Ronald R. Spencer, University of Oklahoma

Solution to the Schrodinger Equation Involving the $V = V_o(x/a-a/x)^2$ *Potential Using the Factorization Method,* Barry R. Ashworth, Central State University

Elliptic Geometry, Randy Lee Ringeisen, Oral Roberts University

Higher Order Symmetric Derivatives, Gene Weber, Hendrix College

Uniform Algebras and Scattered Spaces, Robert C. Smith, University of Arkansas at Fayetteville

Mathematical Model and Solution for the Production of Photoacoustic Signals in Solids, Dean P. Neikirk, Oklahoma State University

The Kernel of the Laplace Transform, David Sutherland, Hendrix College

Semigroups and Groupoids With Involution Like Operator, M. K. Sen and Naoki Kimura, University of Arkansas at Fayetteville

Relative Homological Properties of Isotype Subgroups, Daniel Fitzgerald, Oklahoma State University

Groupoids Acting on Sets, Mridul Kanti Sen, University of Arkansas at Fayetteville

A Program of Undergraduate Mathematical Research, Robert C. Eslinger, Hendrix College

Classroom Applications for Microcomputers, Dennis Reimer, Southwestern Oklahoma State University

School Mathematics in Iran, William R. Orton, University of Arkansas at Fayetteville

A Convergence Structure of Continuous Extended Real-valued Functions, William Feldman, University of Arkansas at Fayetteville

Bi-Topological Spaces, Joe Wiley, Southeastern Oklahoma State University

Characterizations of Functionally Compact Spaces, Larry L. Herrington, University of Arkansas at Pine Bluff

Rarely-Continuous Functions, Paul E. Long, University of Arkansas at Fayetteville

Eigenvalue Problem of $M_n = (\omega_{jk})$, $(j, k=1, \omega \ldots, n$ *where*

$$\omega_{jk} = \frac{1}{\sqrt{n}}\left(e^{\frac{2\pi i}{n}}\right)^{(j-1)(k-1)}$$

Tetsundo Sekiguchi and Naoki Kimura, University of Arkansas at Fayetteville

Extending the Schwarz Reflection Principle, James R. Choike, Oklahoma State University

A Theorem Concerning the Boundary Behavior of Vector-Valued Harmonic Functions, James L. Meek, University of Arkansas at Fayetteville

A Summer Mathematics Program for Women, James F. Potter, University of Arkansas at Fayetteville

Two Models for Large Class Instruction, Harold Huneke, University of Oklahoma
The New Education Law: What Does It Mean?, Richard Thiessen, Oral Roberts University
Uniform Limits of Step Functions, Agnes L. Tulio, Hendrix College
A Queue for Waiting, Pacquitta Crim, Oral Roberts University
Characterizations of Convex Sets, Lisa M. Orton, Hendrix College
Gravitational Geometry, Tetsundo Sekiguchi, University of Arkansas at Fayetteville
Θ-convergence Space Characterizations of H-closed Spaces, T. R. Hamlett, Arkansas Tech University

MARCH MEETING OF THE MISSOURI SECTION

The Missouri Section of the MAA held its Spring meeting on the Kansas City Campus of the University of Missouri on March 30 and 31, 1979. There were 73 persons in attendance. Featured speakers were Professor Leonard Gillman, treasurer of the MAA, who spoke on *The Dog with the Cocked Head,* Professor Walter Mientka of the University of Nebraska who spoke on *Elementary Sieve Methods and Arithmetic Problems in the Theory of Numbers,* and Tam Lonigan, a Fellow of the Actuary Society associated with Meidinger and Associates of Kansas City who spoke on *Insights in the Actuarial Profession.*

Section Chairman Yudell L. Luke presided over the business meeting. The Section voted to start a Section Newsletter with Elizabeth Bermann, University of Missouri-Kansas City, as editor. The Putman Award *Studies in Real and Complex Analysis* edited by I. I. Hirschman, Jr. was awarded to Phillip Harrington of Washington University.

Section Officers for academic year 1979-80 are: Mike Z. Williams, Westminster College, Chairperson; Merry McDonald, Northwest Missouri State University, Chairperson Elect; Yudell L. Luke, University of Missouri-Kansas City, Past Chairperson; John D. Kubicek, Southwest Missouri State University, Secretary-Treasurer. Also, Leonard Palmer, Southeast Missouri State University, serves as chairman of the High School Lecture Program and Al Tinsley, Central Missouri State University, serves as chairman of the MAA High School Mathematics Competition.

The program for the meeting also included meetings for the MAA Departmental Representatives and for Mathematics Department chairperson. A tour of Linda Hall Library followed the meetings.

The following contributed papers completed the program:

The Circle of Apollonius and Some Packaging Problems, M. Ahuja, Southeast Missouri State University
How Children Learn Fractions; Part-group, Part-whole Models, G. Austin-Martin, Stephens College
Physical Exercises for Signed Numbers, E. Berman, University of Missouri-Kansas City
Analytic Interpretation for Finite Geometries, R. L. Boehning, Missouri Southern State College
Math Confidence and Performance as a Function of Individual Differences in Math Aptitude, M. Bowling, Stephens College
An Elementary Proof of a Theorem Concerning the Order p of a Linear k-step Method, C. Cooper, Central Missouri State University
Operators in a Partially Ordered Linear Algebra, E. Davenport, Central Missouri State University
Young Children's Spatial Concepts, T. Goodman, Central Missouri State University
Sex Differences in Mathematics Achievement; Implications for Careers for Women, M. Gulliver, Stephens College
A Banach Type Fixed Point Theorem, T. Hicks, University of Missouri-Rolla
Digital Sums, Niven Numbers and Natural Density, R. Kennedy, Central Missouri State University
A Math Lab for Developmental Students, F. Mangan, St. Louis Community College at Meramec
Some Observations on the Dragon Curve and Related Rations, J. Mathis, William Jewell College
Overcoming Math Avoidance, C. Stuth, Stephens College
On a Class of Saks Spaces, P. K. Subramanian, Missouri Southern State College

JOHN D. KUBICEK, *Secretary-Treasurer*

ANNUAL MEETING OF THE KANSAS SECTION

The sixty-fourth annual meeting of the Kansas Section of the MAA was held on April 7, 1979 at Johnson County Community College, Overland Park, Kansas. Section Chairman, John Hutchinson, presided. Approximately 75 people attended.

Invited addresses were *Arithmetic Problems in Theory of Numbers* and *The Number of Solutions of a Set of Divisibility Conditions* by Walter Mientka, University of Nebraska-Lincoln.

The following contributed papers were presented:

A Power Formula for Transition Matrices of Regular Ergodic Markov Chains, William H. Richardson, Wichita State University
Spiral Kernals, Lucio Arteaga, Wichita State University
Pursuit and Evasion in a Graph (or Cave), Elwyn H. Davis, Pittsburg State University
A Problem in Scheduling Round Robin Tournaments, Donald Quiring, Bethel College
Multiple Choice Tests: An Assessment of Guessing, Richard M. Rubison, Kansas State University
Ramanujan Minus Hardy Equals What?, Prem N. Bajaj, Wichita State University
Measuration Units in Solution, O. Terry Gilbert, Johnson County Community College
Classroom Examples from the World of Business, Margaret W. Maxfield, Kansas State University
What Hath Codd Wrought: (data base systems), Felix Dreher, Pittsburg State University
A Synthetic Look at the Sterographic Projection, William M. Self, Pittsburg State University
Some Observations on Certain Finite Configurations in Mathematics, D. V. Chopra, Wichita State U.

The four top performers from the Section in the 1978 Putnam Examination were awarded one-year

memberships in the MAA. They are: Arthur S. Parker and Bruce K. Holmer, University of Kansas; Daniel Grubb, Kansas State University, and Jonathan Jantz, Bethel College.

ROBERT H. THOMPSON, *Secretary-Treasurer*

SPRING MEETING OF THE TEXAS SECTION

The Annual Spring Meeting of the Texas Section was held at Texas Tech University in Lubbock, Texas on April 6-7, 1979. Professor Dalton Tarwater of Texas Tech University was in charge of arrangements. There were 169 registered persons in attendance.

Invited Speakers included Dr. Milton Holloway, Executive Director of the Texas Energy Advisory Council, who spoke on *The Use of Mathematical Models for a National Energy Policy;* Dr. Mark Wells of the Los Alamos Scientific Laboratory, whose title was *An Experimental Algorithm for N-dimensional Adaptive Quadrature;* Professor Fred Roberts of Rutgers University, whose topic was *Interval Graphs, Traffic Phasing, Ship Building, and Mobil Radio Frequencies;* and Dr. A. B. Willcox, Executive Director of the MAA, whose presentation was titled *Mathematics: Where Are We Going and What Can Educators Do About it?* Dr. Willcox was the official MAA representative to the meeting. He also presented a slide program on the new Washington Headquarters.

A panel discussion on *Trends in the Two-Year College Mathematics Curriculum* was presented by William Keils of San Antonio College, Joe McMillian of North Harris County Community College, Mary Parker of Austin Community College and Betty Travis of St. Phillips College. David Sanchez of San Antonio College was moderator.

A panel discussion of *Trends in the Undergraduate Mathematics Curriculum in Texas* was presented by George Berzsenyi of Lamar University. Larry Guseman of Texas A & M University and Ernest Ratcliff of Southwest Texas State University. The program was arranged by Margaret Hutchinson of St. Thomas University and moderated by Richard Alo of Lamar University.

E. Richard Heineman and Emmett Hazlewood were honored at the annual Section Dinner for outstanding service to the Texas Section. Entertainment at the banquet was provided by a Texas Tech University string ensemble.

The Spring Meeting incorporated special group meetings for Department Chairman, Institutional Representatives and the Executive Committee of the Texas Section.

Nominations for appropriate positions on the Executive Committee were presented by Howard Rolf of Baylor University, Chairman of the Nominations Committee, and elected by acclamation with a unanimous vote. The selection was made contingent upon approval of proposed revisions in the Texas Section Bylaws by the Board of Governors in its next meeting. The Executive Committee for 1979-80 is as follows:

Chairman, Dalton Tarwater, Texas Tech University, Lubbock, Texas 79409; Chairman-Elect, Bill Anderson, East Texas State University, Commerce, Texas 75428; Secretary/Treasurer, Glen Mattingly, Sam Houston State University, Huntsville, Texas 77341; Immediate Past Chairman, R. G. Dean, Stephen F. Austin State University, Nacogdoches, Texas 75962; Level-I Director, David Sanchez, San Antonio College, San Antonio, Texas 78284; Level-II Director, Margaret Hutchinson, University of St. Thomas, Houston, Texas 77006; Level-III Director, Bennie Williams, The University of Texas at Arlington, Arlington, Texas 76019; Director-at-Large, Robert M. Thrall, Rice University, Houston, Texas 77001; Arrangements Chair, Bill Anderson, East Texas State University, Commerce, Texas 75428; Arrangements Chair-Elect, Ray Tebbetts, San Antonio College, San Antonio, Texas 78284; State Contest Director, J. R. Boone, Texas A & M University, College Station, Texas 77843; Governor, James N. Younglove, University of Houston, Houston, Texas 770074.

Contributed papers were as follows:

Teaching the Reading of Mathematics; A Follow-Up, Shirley Tucker, Austin
An Applications Module in Spherical Trigonometry, Carol Congleton, Moody College
A Study of the Effect of Selected Mathematics Instruction Sequences on Retention and Transfer with Logical Reasoning Controlled, Vera R. King, Prairie View A & M
Mathematics and Esperanto, Michael Jones, ELA
The Diagnostic Capabilities of the Computer, Betty Travis, St. Philip's College
The Mexican Connection: A Division Algorithm from South of the Border, James A. Bell, Laredo State U
A Geometrical Approach to Determinants, John Huber, Pan American University
Monotone Operators and Convexity of Chebyshev Sets, Russell Bilyeu, North Texas State University
Minkowski-Farkas Operators Defined by an Infinite Matrix, Ronald Teemley, Texas Christian University
Summability Factors for Cesaro Methods, David F. Dawson, North Texas State University
Semi Open Sets and Semi Separation Axioms, Charles Dorsett, North Texas State University
Hyperspaces of Compactifications of the Half-Line, Ann Petrus, Our Lady of the Lake University
Dimensions of Projections of Cells in E^n, Carl P. Pixley, Southwest Texas State University
Generalized Omology Theories, Robert M. Nehs, Texas Southern University
An Observation Concerning Whitford's Binet Formula Generalized, Montie G. Monzingo, Southern Methodist University
The Factorization of Integers in Algebraic Number Fields, Roy G. Brooks, The University of Texas of the Permian Basin
Embedding a Class of Modular Lattices in Projective Geometries, Don E. Edmondsen, The University of Texas at Austin
On Positive Integers with a Fixed Number of Small Prime Factors (Preliminary Report), Donald G. Hazlewood, Southwest Texas State University
Construction of Three Dimensional Geometric Models with the Aid of Computer Drawn Nets, Norman W. Naugle, Texas A & M University
A Mathematical Model of General Equilibrium in Economic Theory, L. E. Broome and George Carter, Moody College

Mathematical Aspects of a Problem Concerning a Talk on a Lunar Shuttle Landing Simulation, John Lamb, Jr., East Texas State University
New Bases of Monodiffric Polynomials, Charles R. Deeter and Kiritkumar Talati, Texas Christian U
Normal = Binomial + Continuity Correction?, L. Dwane Snider, Tarleton State University
Euler's nth Difference Formula and Applications, Mark D. Bray, East Texas State University
A Note on the Summability of Infinite Series, Thomas A. Keagy, Texas Eastern University
Geometric Applications of Symmetric Functions, J. M. Stark, Lamar University
Texas Mathematical Olympiad, George Berzenyi, Lamar University
Progress Report on College Mathematics in Texas, C. P. Benner, The University of Houston

APRIL MEETING OF THE IOWA SECTION

The 66th regular meeting of the Iowa Section, MAA, was held on the campus of Cornell College, Mt. Vernon, Iowa on April 20 and 21, 1979. Chairperson Donald Meyer presided. There were 51 people in attendance, 32 of whom were members of the section.

The program, arranged by Frank Kosier, consisted of the contributed papers, invited addresses, and panel discussion listed below along with the Governor's report presented by James Cornette.

At the business meeting, Arnold Adelberg of Grinnell College was elected to the office of Chairperson-Elect. A discussion of the Section's relation to the Iowa Academy of Sciences was conducted. A committee consisting of Donald Meyer and the members of the executive committee was directed to explore these relations and also potential relations with the Iowa Sections of the Society of Industrial and Applied Mathematics and the American Statistical Association. The committee is to have a proposal ready for the next annual meeting. The membership voted to make the Iowa State University Library the official repository for the records of the Section. Thanks were extended to the Cornell College mathematics department for their services.

Papers, invited addresses, and panel discussion:
Modeling and Control of Childhood Diseases, David Tudor, Mt. Vernon
Determining Elementary Topological Properties Through an Explicit Construction, Donald Marxen, Dubuque
A Generalization of Dixmier's Center-Valued Trace Theory, An invited address, Istvan Kovacs, Dubuque
A Psychologist's Contributions to Mathematics, Elizabeth A. Blobaum, Dubuque
Mathematics in the Biological Sciences, Renata M. Korona, Dubuque
Linear Algebra and Its Tools, Janne M. Haupert, Dubuque
Mathematics: Where Are We Going and What Can Educators Do About It?, Invited address, A. B. Willcox, Executive Director of the Mathematical Association of America, Washington, D.C.
Infinitesimal Analysis - Then and Now, Invited address, K. D. Stroyan, Iowa City
Classroom Experiences in the Teaching of the Infinitesimal Calculus, Panel discussion, Eugene Madison, Iowa City; Dean Karns, Cedar Rapids; John Pais, St. Louis, Missouri

E. J. PEAKE, *Secretary-Treasurer*

APRIL MEETING OF THE INDIANA SECTION

The spring meeting of the Indiana Section of the MAA was held at Butler University in Indianapolis on Saturday, April 21, 1979 with 38 members and 58 non-members present.

The chairman of the Section, Meyer Jerison of Purdue University presided. The meeting was held in conjunction with the Indiana Small College Math Competition, in which 12 teams participated.

The following papers were presented:
Balancing chemical reactions: Matrix methods with computer assistance, R. P. Grimaldi, Rose-Hulman Institute of Technology
Digital simulation of turbulence of the Space Shuttle Orbiter during entry and landing, C. M. Murphy, Purdue University Calumet
Techniques for determining particular solutions to the Schwarz-Christoffel transformation, W. A. Marion, Valparaisor University
Almost arithmetic sequences, C. Kimberling, University of Evansville
Are fair elections possible? A look at Arrow's impossibility theorem, C. A. Cheney, Indiana State U
Pegboard Solitaire, H. L. Montgomery, University of Michigan
The dog with the cocked head, L. Gillman, University of Texas

L. Cote, Purdue University, recognized three students for solving problems appearing in the Indiana School Mathematics Journal: Mark Beitz, Highland High School; Emily Smith, Garrett High School; Mike Hollander, Lakeside Junior High School.

At the business meeting, the following were elected for 1979-1980: Chairman, U. Dudley, DePauw University; Vice-Chairman, D. E. Deal, Ball State University; Secretary-Treasurer, R. R. Patterson, Indiana University, Purdue University at Indianapolis. G. Sherman, Rose-Hulman Institute of Technology, continues as Governor. L. Gillman, national MAA Treasurer, reported on the new MAA headquarters building. Memberships in the MAA were awarded to Kendall S. Stanley, Purdue University and Kevin A. Fosso, Wabash College, in recognition of their performance on the Putnam examination.

The Section noted the death of Professor Arthur Hellerberg of Valparaiso University and requested the Secretary-Treasurer to send a message of condolence to his wife.

R. R. PATTERSON, *Secretary-Treasurer*

CORRECTION: An item on page 425 of the May, 1979, issue of the *MONTHLY* stated that Doris Schattschneider was elected editor of *MATHEMATICS MAGAZINE* for a five-year term beginning on January 1, 1980. It should read "a five-year term beginning on January 1, 1981".

CALENDAR OF FUTURE MEETINGS

Sixty-third Annual Meeting, San Antonio, Texas, January 5-7, 1980.

Sixtieth Summer Meeting, University of Michigan, Ann Arbor, August 18-20, 1980.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, last weekend in April or first weekend in May. Deadline for papers 6 weeks before meeting.

FLORIDA, early March. Deadline for paper titles 2 weeks before meeting.

ILLINOIS, first Friday/Saturday in May.

INDIANA, Wabash College, Crawfordsville, October 27, 1979.

INTERMOUNTAIN

IOWA, third weekend in April. Deadline for papers February 1.

KANSAS, March or April. Deadline for papers January 1.

KENTUCKY, early April. Deadline for papers 6 weeks before meeting.

LOUISIANA–MISSISSIPPI, Louisiana Tech University, Ruston, February 15-16, 1980.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Prince Georges Community College, Largo, Maryland, November 10, 1979.

METROPOLITAN NEW YORK, spring. Deadline for papers 2 weeks before meeting.

MICHIGAN, first Friday and Saturday in May. Deadline for papers 6 weeks before meeting.

MISSOURI, late March/early April. Deadline for papers January 31.

NEBRASKA, April.

NEW JERSEY, Essex County College, Newark, November 3, 1979.

NORTH CENTRAL, University of North Dakota, Grand Forks, October 26–27, 1979.

NORTHEASTERN, University of Hartford, Hartford, Connecticut, November 18, 1979.

NORTHERN CALIFORNIA, Naval Postgraduate School, Monterey, February 23, 1980.

OHIO, College of Wooster, Wooster, October 26–27, 1979.

OKLAHOMA–ARKANSAS, (approx.) Friday and Saturday of first weekend in April. Deadline for papers 3 weeks before meeting.

PACIFIC NORTHWEST, second Saturday in June. Deadline for papers 6 weeks before meeting.

PHILADELPHIA, Drexel University, Philadelphia, November 17, 1979.

ROCKY MOUNTAIN, last weekend in April or first in May. Deadline for papers 8 weeks before meeting.

SEAWAY, SUNY at Albany, October 26–27, 1979.

SOUTHEASTERN

SOUTHERN CALIFORNIA, University of California, Riverside, November 16–17, 1979.

SOUTHWESTERN, usually in April. Deadline for papers 2 weeks before meeting.

TEXAS, Friday and Saturday in early April. Deadline for papers March 1.

WISCONSIN, Friday and Saturday between mid-April and first week in May. Deadline for papers 6 weeks before meeting.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, San Francisco, January 3-8, 1980.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES, Town & Country Hotel, San Diego, California, October 17-20, 1979.

AMERICAN MATHEMATICAL SOCIETY, San Antonio, Texas, January 3-6, 1980.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Amherst, Massachusetts, June 23–26, 1980.

ASSOCIATION FOR COMPUTING MACHINERY, Plaza Hotel, Detroit, Michigan, October 29-31, 1979

ASSOCIATION FOR SYMBOLIC LOGIC, New York City, December 28-29,1979.

ASSOCIATION FOR WOMEN IN MATHEMATICS

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Seattle, Washington, April 16-19, 1980.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Marc Plaza Hotel, Milwaukee, Wisconsin, October 15-17, 1979.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Radisson Muehlbach, Kansas City, Missouri, November 1-3, 1979.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Denver Marriott, Denver, Colorado, November 12-14, 1979.

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OCTOBER

THE AMERICAN MATHEMATICAL MONTHLY

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1979

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NOTICE TO AUTHORS

Research papers per se are unsuitable; see statement of policy (Vol. 85, p. 1).

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DIGITAL TOPOLOGY

AZRIEL ROSENFELD

Digital pictures are rectangular arrays of nonnegative numbers. The analysis of a digital picture usually involves "segmenting" it into parts and measuring various properties of and relationships among the parts. In particular, one often wants to separate out the connected components of a picture subset to determine the adjacency relationships among those components, to track and encode their borders, or to "thin" them down to "skeletons" that have no interiors, without changing their connectedness properties. There are standard algorithms for doing all of these tasks; but to prove that they work, one needs to establish some basic topological properties of digital picture subsets. This paper provides an introduction to the study of such properties, which we call *digital topology*.

1. Introduction. *Digital image processing* or *picture processing* [1] is a rapidly growing discipline with broad applications in business (document reading), industry (automated assembly and inspection), medicine (radiology, hematology, etc.), and the environmental sciences (meteorology, geology, land-use management, etc.), among many other fields [2]. Most of this work involves picture analysis: given a picture, to construct a description of it in terms of the objects it contains or the regions of which it is composed and their properties and relationships. For example, a printed page is made up of characters on a background; a blood smear on a microscope slide contains blood cells on a background; a chest x-ray shows the heart, lungs, ribs, etc.; a satellite TV image of terrain is composed of terrain types; and so on. The process of decomposing a picture into regions, or into objects and background, is called *segmentation.*

A picture is input to the computer by *sampling* its brightness values at a discrete grid of points, and *digitizing* or *quantizing* these values to a finite number of binary places. The result of this process is called a *digital picture*; it is a rectangular array of discrete values. The elements of this array are called *pixels* (short for "picture elements"), or sometimes simply *points*, and the value of a pixel is called its *gray level*. Segmentation is basically a process of assigning the pixels to classes; one simple way of doing this, called "thresholding," classifies the pixels according to whether or not their gray levels exceed a given threshold value t. Methods of segmenting digital pictures will not be reviewed here; for an introduction to this subject see, e.g., [1, Chapter 8].

Once a picture has been segmented into subsets, it can be described in terms of properties of these subsets and relationships among them. Some of these properties depend on the gray levels of the points that belong to a subset, but others are "geometrical" properties which depend only on the positions of these points. Especially basic are *topological* properties of the subsets, involving such concepts as adjacency and connectedness, but not size or shape.

Topological properties of digital picture subsets are useful for a number of reasons. After a subset has been singled out, e.g., by thresholding, one usually wants to further segment it into connected regions, since these often correspond to distinct objects (characters, blood cells, etc.). One may also want to track the borders of these regions, since the sequences of moves around the borders provide a compact encoding of region shape. Alternatively, one may want to "thin" the regions into "skeletons," without changing their connectedness properties, since this too yields a compact representation (e.g., an elongated region is represented by a set of arcs or curves). The adjacency or surroundedness relations among the regions can be compactly

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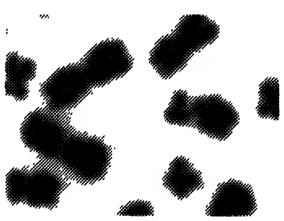
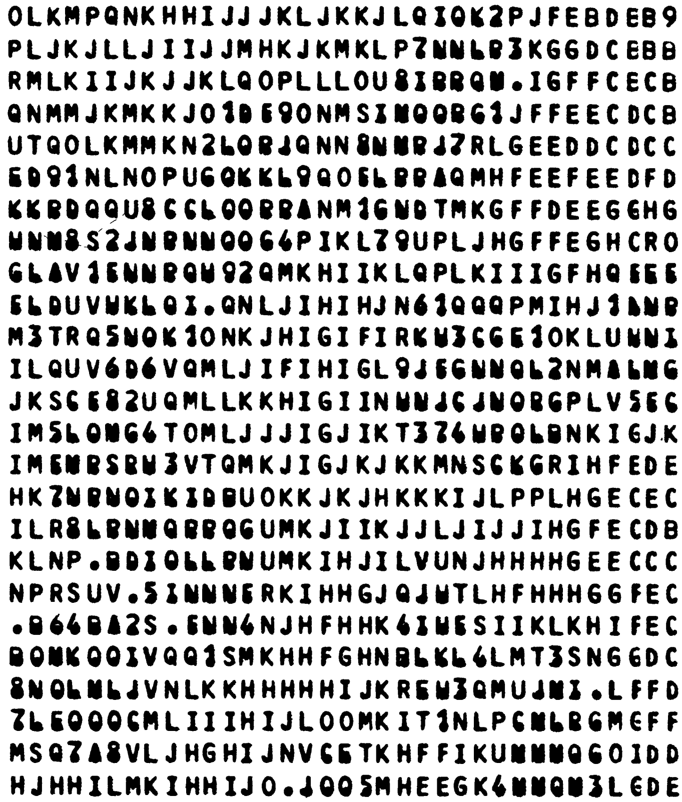
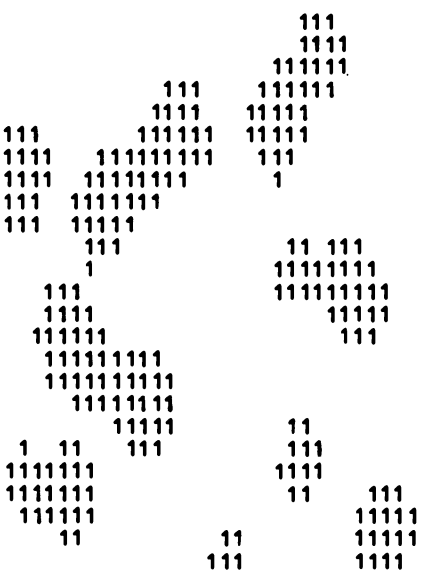


(a) Photomicrograph of some chromosomes.



(b) Array of numerical values obtained by coarsely sampling (a). Characters blank, 1, 2,...,9, A, B,...,V represent gray levels 0,1,...,31; the same characters overstruck with periods represent levels 32, 33,...,63.



(c) Result of thresholding (b) at gray level 40; values < 40 are displayed as blanks, values ⩾ 40 as 1's.

```
007654565533211012
0067645454554523211001111
007656442222
006666434122
071007765443344421
00777000766544334434231 2
07654312
701076654344321
00765444221
06441
```

(d) Move sequences representing the borders of the connected components of 1's in (c); see the end of Section 5 for the code that is used. The starting point for each border is at the leftmost of its uppermost points, and the borders are followed clockwise. Note that two of the components (near the upper left) are diagonally adjacent.

FIG. 1

represented by a graph whose nodes are the regions, and in which two nodes are joined by an arc iff those two regions are adjacent.

Many algorithms exist for segmenting a picture subset into its connected components, border following, thinning, and constructing the adjacency graph of a partition of a picture; see, e.g., [**1**, Chapter 9]. To prove that these algorithms work correctly, or even (in some cases) to state them precisely, it is necessary to establish some of the basic topological properties of digital picture subsets. This paper provides an introduction to the study of such properties, which we call *digital topology*. Of course, this is nothing more than the study of some simple properties of finite sets of lattice points; but it should be of interest because of the widespread practical use of these ideas in digital image processing.

Section 2 introduces the concepts of adjacency and connectedness for digital picture subsets. Section 3 defines digital arcs and curves, and develops some of their basic properties. Section 4 deals with thinning, Section 5 with border following, and Section 6 with adjacency and surroundedness. Only sketches of the proofs are given; a more detailed treatment can be found in [**3**, Chapter 2], which in turn is based on the material in [**4**]–[**7**].

Figure 1 shows a picture of some chromosomes, the corresponding array of coarsely sampled numerical values, a thresholded version of the picture containing ten connected objects, and a list of the move sequences representing the borders of these objects.

2. Connectedness. We begin by formulating the concept of connectedness for subsets of a digital picture Π. For concreteness, we assume that Π is an array of lattice points having positive integer coordinates (x,y), where $1 \leqslant x \leqslant M$, $1 \leqslant y \leqslant N$.

DEFINITION 2.1a. The *4-neighbors* of (x,y) are its four horizontal and vertical neighbors $(x \pm 1,y)$ and $(x,y \pm 1)$.

DEFINITION 2.1b. The *8-neighbors* of (x,y) consist of its 4-neighbors together with its four diagonal neighbors $(x+1,y \pm 1)$ and $(x-1,y \pm 1)$.

Note that if (x,y) is a border point of Π, i.e., if $x=1$ or $M, y=1$ or N, some of these neighbors do not exist. If the points P and Q of Π are neighbors, we call them *(4- or 8-) adjacent*.

DEFINITION 2.2. Let P, Q be points of Π. By a *path* from P to Q we mean a sequence of points $P = P_0, P_1, \ldots, P_n = Q$ such that P_i is a neighbor of P_{i-1}, $1 \leqslant i \leqslant n$.

Note that this is two definitions in one, depending on whether "neighbor" means 4-neighbor or 8-neighbor; we refer to these two types of paths as *4-path* and *8-path*, respectively.

Let S be a nonempty subset of Π. To avoid special cases, we assume that S does not meet the border of Π.

DEFINITION 2.3. We say that P and Q are *connected* in S if there exists a path from P to Q consisting entirely of points of S.

Again, this is two definitions in one; we speak of P and Q being *4-connected* or *8-connected*.

PROPOSITION 2.1. *"Connectedness in S" is an equivalence relation.*

DEFINITION 2.4. The equivalence classes defined by this relation are called the *(connected) components* of S. If S has only one component, it is called *connected*.

Let $\bar{S}$ be the complement of S.

DEFINITION 2.5. The unique component of $\bar{S}$ that contains the border of Π is called the *background* of S; all other components, if any, are called *holes* in S. If S has no holes, it is called *simply connected*.

We shall see shortly that when we study connectedness in digital pictures, both 4- and 8-

definitions must be used—in fact, whichever one we use for S, we must use the other one for $\bar{S}$. If Π were a hexagonal rather than a rectangular array, there would be only one type ("6") of neighbor, connectedness, etc., which we could use for both S and $\bar{S}$; but in practice, only rectangular arrays are used in digital image processing.

Our definition of connectedness is an "arcwise" definition, rather than a definition in terms of open and closed sets. We can, however, define a topology on Π in which the standard notion of connectedness reduces to our definition [8]. Let

$$\begin{aligned} U(P) \equiv U(x,y) &= \{P\}, \text{ if } x+y \text{ is odd} \\ &= \{P \text{ and its 4-neighbors}\}, \text{ if } x+y \text{ is even.} \end{aligned}$$

If we take the U's as a basis for the open sets, then a set is connected in the resulting topology iff it is 4-connected as defined above.

3. Arcs and curves. A commonly used method of shape analysis in digital picture processing involves reducing "thick" digital point sets to idealized "thin" forms—e.g., reducing elongated, simply connected objects to arcs, or objects that have a single hole to closed curves. We will discuss "thinning" processes of this sort in Section 4; but first we must introduce digital definitions of arcs and curves.

DEFINITION 3.1. $S \subseteq \Pi$ is called an *arc* if it is connected, and all but two of its points (its "endpoints") have exactly two neighbors in S, while those two have exactly one.

It is easily seen that an arc can be regarded as a path which neither crosses nor "touches" itself —i.e., its points can be numbered $Q_1, \ldots, Q_n$ so that Q_i is a neighbor of Q_j iff $i = j \pm 1$. To rule out degenerate cases, we shall assume that an arc always has at least two points. Readily, S cannot be both a 4-arc and an 8-arc unless it is a horizontal or vertical straight line segment.

PROPOSITION 3.1. *An arc is simply connected.*

REMARK. This proposition is not true if we use 4-connectedness for both the arc and its complement, since the 4-arc

```
PPP
P P
PP
```

has a 4-hole. The proposition can be proved by induction on the number of points in the arc, using the fact that, if we delete an endpoint from an arc, the result is still an arc (if it has more than one point); the details, which involve an enumeration of cases, will not be given here.

DEFINITION 3.2. $S \subseteq \Pi$ is called a *curve* if it is connected, and each of its points has exactly two neighbors in S.

Readily, we can number the points of a curve $Q_1, \ldots, Q_n$ so that Q_i is a neighbor of Q_j iff $i \equiv j \pm 1$ (modulo n). To rule out degenerate cases, we will assume that a 4-curve always has at least eight points; and an 8-curve, four points. Note that no S can be both a 4-curve and an 8-curve.

PROPOSITION 3.2. *A curve has at most one hole.*

This follows from Proposition 3.1 and the fact that deleting any point from a curve makes it an arc. Note that it, too, is false if we use 4-connectedness for both S and $\bar{S}$; for example the 4-curve

```
PPP
P PP
PP P
 PPP
```

has two 4-holes. Indeed, as we shall next see, if we use opposite types of connectedness for S and

$\bar{S}$, then a curve has exactly one hole; but if we use 8-connectedness for both, then the 8-curve

$$\begin{array}{ccc} & P & \\ P & & P \\ & P & \end{array}$$

has no 8-holes.

THEOREM 3.3. *A curve has exactly one hole.*

This is the Jordan Curve Theorem for digital curves. The proof is similar to a standard proof of the theorem for polygons. Let S be a curve, and $P \notin S$; we say that $P \equiv (x,y)$ is "inside" S if the half-line $H_p \equiv \{(z,y) | x \leqslant z \leqslant M\}$ crosses S an odd number of times, and "outside" S otherwise. ("Crosses" must be properly defined, since H_p may meet S in runs of consecutive points; such a run is a crossing if S enters the run from the row above H and exits to the row below H, or vice versa.) It can then be shown that neighboring points of $\bar{S}$ are either both inside or both outside S; hence points in the same component of $\bar{S}$ are either all inside or all outside. The theorem follows from this and the fact that the inside and outside of a curve are both nonempty.

We can also prove

PROPOSITION 3.4. *Every point of a curve S is adjacent (in the sense of $\bar{S}$'s connectedness) to both components of $\bar{S}$.*

This follows readily from the fact that if we delete any point from S, it becomes an arc, which is simply connected.

4. Thinning. The goal of *thinning* is to remove points from a set S without changing the connectedness properties of either S or $\bar{S}$. The class of points which can be safely removed is characterized by the following proposition, in which $N(P)$ denotes the set of 8-neighbors of P.

PROPOSITION 4.1. *The following properties of the point P of S are all equivalent:*
(a) *$S \cap N(P)$ has the same number of components (in the S sense) as $S \cap [N(P) \cup \{P\}]$*
(b) *$\bar{S} \cap N(P)$ has the same number of components (in the $\bar{S}$ sense) as $[\bar{S} \cap N(P)] \cup \{P\}$*
(c) *$S \cap N(P)$ has just one component adjacent to P ("component" and "adjacent" in the S sense)*
(d) *$\bar{S} \cap N(P)$ has just one component adjacent to P ("component" and "adjacent" in the $\bar{S}$ sense)*
(e) *$S - \{P\}$ has the same number of components (in the S sense) as S, and $\bar{S} \cup \{P\}$ has the same number of components (in the $\bar{S}$ sense) as $\bar{S}$.*

DEFINITION 4.1. A point having the properties of Proposition 4.1 is called *simple*.

Evidently, an *isolated* point of S (having no neighbors in S) and an *interior* point of S (having all eight neighbors in S) cannot be simple; while an *end* point of S (having exactly one neighbor in S) is always simple. The proof of the last part of the proposition is not trivial; it requires use of Proposition 3.4.

It follows from Proposition 4.1 that if S is simply connected, and $P \in S$ is not an isolated, interior, or simple point, then $S - \{P\}$ is not connected, but consists of components that are simply connected. Using this observation, we can show, using induction on the number of points in S, that if S is simply connected and has more than two points it must have at least two simple points. In fact, we can show that if S has only two simple points, they must both be ends, and that if S has an interior point it has a simple point that is not an end.

These remarks provide a basis for defining a crude thinning process for simply connected sets S. We repeatedly delete from S simple points that are not ends; each such deletion leaves S simply connected. When no further deletion is possible, S can no longer have interior points and so is (relatively) "thin." Note that the result depends on the order in which the points are deleted. We can also establish the following neat characterization of arcs:

THEOREM 4.2. *S is an arc iff it is simply connected and has exactly two simple points.*

Analogous remarks can be made about connected S's that have only one hole; by Proposition 4.1, if $P \in S$ is not isolated, interior, or simple, then $S-\{P\}$ is either simply connected or not connected. This allows us to prove

THEOREM 4.3. *S is a curve iff it is connected, has exactly one hole, and has no simple points.*

It follows that a connected S having just one hole can be thinned to a curve by repeatedly deleting its simple points. As a further corollary, we obtain a converse to the digital Jordan Curve Theorem:

COROLLARY 4.4. *Let S be connected, let $\bar{S}$ have exactly two components, and let every point of S be adjacent (in the $\bar{S}$ sense) to both of these components; then S is a curve.*

The hypothesis that S is connected is unnecessary, as we shall see in Section 6.

5. Border following. A set $S \subseteq \Pi$ can be represented by specifying its borders; each border can be specified by defining a starting point and a sequence of moves from neighbor to neighbor. This representation, which is often quite compact, is very commonly used in image processing. In this section we define the border representation and give an algorithm for constructing it. We sketch a proof that this algorithm is valid, based on the results of Section 4.

DEFINITION 5.1. The *border* of $S \subseteq \Pi$ is the set of points of S that have 4-neighbors in $\bar{S}$.

One could also define a "thicker" border consisting of points that have 8-neighbors in $\bar{S}$; but the 4-neighbor definition is the one usually used.

The border of S consists, in general, of many parts, since S may have many components, each of which has many holes. In order to define border following, we must single out one of these parts at a time:

DEFINITION 5.2. Let C be a component of S and D a component of $\bar{S}$. The *D-border* of C is the set of points of C that have 4-neighbors in D. We denote this border by C_D.

We now describe an algorithm that successively visits all the points of the D-border of C. We assume that C is 4-connected and D 8-connected; that C has more than one point; and that we are given an initial pair of 4-neighboring points $P_0 \in C$, $Q_0 \in D$, which we assume to be distinctively marked. The algorithm, which we call BF_4, specifies how to find a new point pair (P_{i+1}, Q_{i+1}), given the current pair (P_i, Q_i).

BF_4 operates as follows: Let the 8-neighbors of P_i, in clockwise order starting with Q_i, be $R_{i1} = Q_i, R_{i2}, \ldots, R_{i8}$. Let R_{ij} be the first of the R's that is in C and is a 4-neighbor of P_i (i.e., j is odd); such an R_{ij} must exist, since C is 4-connected and has more than one point. If $R_{i,j-1}$ is in D, take R_{ij} as P_{i+1} and $R_{i,j-1}$ as Q_{i+1}; otherwise, take $R_{i,j-1}$ as P_{i+1} and $R_{i,j-2}$ as Q_{i+1}. If, for some $i>0$, P_i is P_0 and one of $R_{i1}, \ldots, R_{ij}$ is Q_0, stop.

To illustrate the operation of BF_4, we give a simple example. Let C be the set of P's shown below; the blanks are in $\bar{S}$, while P^* is in S but not in C. Let P_0 be the P on the third row, and let Q_0 be the blank on its left. Then the successive steps of BF_4 are as follows:

Input:

	P^*
P	
P	
P	P

Step 1.

$$\begin{array}{ccc} & & P^* \\ R_{02} & P & \\ Q_0 & P_0 & \\ & P & P \end{array}$$

Here $R_{03}=P_1$, $R_{02}=Q_1$.

Step 2.

$$\begin{array}{ccc} R_{12} & R_{13} & P^* \\ Q_1 & P_1 & R_{15} \\ & P & R_{16} \\ & P & P \end{array}$$

Here $R_{17}=P_2$, $R_{16}=Q_2$. Note that $R_{14}=P^*$ is in S, but is ignored.

Step 3.

$$\begin{array}{cc} & P^* \\ P & \\ P_2 & Q_2 \\ P & P \end{array}$$

Here $R_{22}=P_3$, $R_{21}=Q_3$ $(=Q_2)$. Note that $P_2=P_0$, but the algorithm does not stop since Q_0 is not one of R_{21}, R_{22}, R_{23}.

Step 4.

$$\begin{array}{ccc} & P^* & \\ P & & \\ P & Q_3 & R_{32} \\ P & P_3 & R_{33} \\ R_{36} & R_{35} & R_{34} \end{array}$$

Here $R_{37}=P_4$, $R_{36}=Q_4$.

Step 5.

$$\begin{array}{ccc} & & P^* \\ & P & \\ R_{44} & P & \\ R_{43} & P_4 & P \\ R_{42} & Q_4 & \end{array}$$

Here $R_{45}=P_5$, $R_{44}=Q_5$

Step 6.

$$\begin{array}{ccc} & & P^* \\ & P & \\ Q_5 & P_5 & \\ & P & P \end{array}$$

Here $P_5=P_0$ and $Q_5=Q_0$, so the algorithm stops.

It is easily seen that the successive P_i's chosen by BF_4 are 4-connected to each other in S (though they may not be 4-neighbors); the successive Q_i's are 8-connected to each other in $\bar{S}$; and P_i is 4-adjacent to Q_i. Thus the P_i's are all in C, the Q_i's all in D, and the P_i's are all on the D-border of C. The proof that the P_i's constitute the entire D-border can be outlined as follows: Readily, the operation of BF_4 is unaffected if all points of $\bar{S}$ except those in D (and in the background component) are transferred from $\bar{S}$ to S; hence it suffices to prove the assertion for C's that have at most one hole. For simply connected C's, we can use induction on the number of points in C; by Section 4, C has simple points, and readily if BF_4 works when a simple point

is deleted, it still works when the point is present. For C's with one hole, we can first show that BF_4 works if C is a curve and then use a similar induction argument based on Theorem 4.3.

The algorithm ("BF_8") for the case where C is 8-connected and D 4-connected is very similar. Here we simply let R_{ij} be the first of the R's that is in C, and take $P_{i+1}=R_{ij}$, $Q_{i+1}=R_{i,j-1}$. Thus P_{i+1} is an 8-neighbor of P_i, and Q_{i+1} is 4-connected in $\bar{S}$ to Q. Incidentally, our choice of clockwise order for the R's implies that borders are followed keeping C on the right; thus the outer border of C is followed clockwise, and its hole borders counterclockwise. (On the meaning of "outer border" and "hole border," see Proposition 6.2.)

Since the successive P_i's chosen by BF_4 or BF_8 are 8-neighbors of each other, we can specify the D-border of C by giving the position of the starting point P_0 together with a string of 3-bit numbers $(0,\ldots,7)$ representing the moves from one P_i to the next. For example, we can use the code

$$\begin{matrix} 3 & 2 & 1 \\ 4 & P_1 & 0 \\ 5 & 6 & 7 \end{matrix}$$

to represent these moves (mnemonic: code i corresponds to a move in direction $45i°$). This representation is called a *chain code*.

To reconstruct C from its borders, we need to know the pair of points (P_0, Q_0) and the chain code for each border C_D. It is then straightforward to mark the points of C_D, as well as a band of points in D adjacent to C_D, for each D. When this has been done, it is easy to "color in" the interior of C. Note that if we had not marked the points in D that adjoin C, it would not be easy to decide which side of the D-border of C is interior to C.

6. The adjacency tree. Given $S\subseteq\Pi$, the components of S and $\bar{S}$ partition Π into connected regions. A useful way of (partially) describing a partition of Π is in terms of its *adjacency graph*, which specifies the regions and their adjacencies. When the partition consists of the components of a set and its complement, we can show that its adjacency graph is a tree. It can also be shown that if a component of S and a component of $\bar{S}$ are adjacent, one of them surrounds the other; thus, under the relationship "surrounds," the tree becomes a directed tree. In this section we define these concepts more precisely and sketch the proofs of these assertions (which, incidentally, are true only when we use opposite types of connectedness for S and $\bar{S}$).

DEFINITION 6.1. Let $\mathbb{S}\equiv\{S_1,\ldots,S_n\}$ be a partition of Π. The *adjacency graph* $\mathcal{G}$ of this partition is the graph whose node set is $\mathbb{S}$, and in which two nodes S_i,S_j are joined by an arc iff the sets S_i and S_j are adjacent (i.e., some point of S_i is a neighbor of some point of S_j).

When $\mathbb{S}$ consists of the connected components of S and $\bar{S}$, we shall denote its adjacency graph by $\mathcal{G}_s$. In this case it does not matter whether we use 4-neighbors or 8-neighbors to define the adjacency relationship, since if a component of S and a component of $\bar{S}$ are 8-adjacent, they must also be 4-adjacent.

THEOREM 6.1. *$\mathcal{G}_s$ is a tree.*

Sketch of proof: We must show that $\mathcal{G}_s$ does not contain a cycle. Let T be a component of S and U, V components of $\bar{S}$ that are adjacent to T (or vice versa); then any path from U to V, in the sense of the connectedness of $\bar{S}$, must meet T, since otherwise the regions encountered by the path, together with T, would constitute a cycle. If we knew that U and V had to be in different components of $\bar{T}$, then no path between them could lie entirely in $\bar{T}$. Suppose they were in the same component W of $\bar{T}$; since they are both adjacent to T, they would both have to meet W_T, the T-border of W. But since BF works, we know that W_T is connected (in the $\bar{S}$ sense); and evidently $W_T\subseteq\bar{S}$, since points of $\bar{T}$ that are adjacent to the component T of S cannot be in S. Thus U and V cannot both meet W_T, since they are different components of $\bar{S}$;

hence they cannot be in the same component of $\bar{T}$, so that T separates them, which proves that $\mathcal{G}_s$ has no cycles.

DEFINITION 6.2. Let A, B be any subsets of Π. We say that A *surrounds* B if any 4-path from B to the border of Π meets A.

PROPOSITION 6.2. *Let C, D be adjacent components of S, $\bar{S}$, respectively; then either C surrounds D or D surrounds C. Moreover, exactly one component of $\bar{S}$ surrounds each component of S (and vice versa, for non-background components of $\bar{S}$).*

Sketch of proof: As seen in the proof of Theorem 6.1, two D's cannot be in the same component of $\bar{C}$; hence at most one D can be in the background component, so that all others are in holes and so are surrounded by C. On the other hand, there does exist a D_0 not surrounded by C (e.g., the point just to the right of a rightmost point of C is in such a D_0). On any 4-path from C to the border of Π, let P_i be the last point of C; then P_{i+1} is in some D, but is not surrounded by C, hence is in D_0, so that D_0 surrounds C. These observations also give us

THEOREM 6.3. *Under the relation "surrounds," $\mathcal{G}_s$ can be regarded as a directed tree, with the background component of $\bar{S}$ as root.*

We can also prove the promised stronger version of Corollary 4.4. Let U, V be components of S and W, Z components of $\bar{S}$; then U, V cannot both be adjacent to both W and Z, since this would imply that $\mathcal{G}_s$ contained the cycle U, W, V, Z. Thus if $\bar{S}$ has two components, and every point of S is adjacent to both of them, S cannot have two components, and so is connected. This proves

PROPOSITION 6.4. *If $\bar{S}$ has two components, and every point of S is adjacent to both of them, S is a curve.*

7. Concluding remarks. Many other topics could have been included in this paper; the study of geometrical properties of digital picture subsets ("digital geometry") has a rapidly growing literature. Some additional references on digital topology are **[9]**, on homotopy; **[10]**, on dimension; **[11]**, on genus; **[12]**–**[14]**, on shrinking, and **[15]**–**[16]**, on thinning; as well as **[17]**, which provides an alternative approach to some of the basic results. Other areas of digital geometry deal with natural metrics on digital pictures **[18]**; with perimeter and diameter measurement **[19]**–**[21]** and the isoperimetric inequality **[22]**; and with geodesics **[23]**. There is also considerable literature on convexity (e.g., **[24]**–**[25]**: When can a digital object be the digitization of a convex object?) and straightness (e.g., **[26]**: When can a chain code be the digitization of a straight line?). It is hoped that this paper will help bring this work to the attention of mathematicians.

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FINDING HOW MANY ROOTS A POLYNOMIAL HAS IN (0, 1) OR (0, ∞)

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1. Introduction and Summary. A real polynomial is given, perhaps of high degree. We are interested in its roots either in (0, 1) or (0, ∞). We suspect there is either none or one. Is there an easier way to prove it than Sturm's theorem if Descartes's rule does not suffice? This paper gives one, involving little more than simple addition.

The question was motivated by "internal rates of return" or "equivalent annual interest rates." Let $c_0, c_1, \ldots, c_n$ be a given sequence of positive and negative cash flows, representing the anticipated after-tax returns at times $0, 1, \ldots, n$ of a project under consideration, or the differences between those of two projects to be compared. At interest rate r, the total accumulated value of the c_j at time n would be $\Sigma c_j(1+r)^{n-j}$. The corresponding value in present dollars, or "present value," is $\Sigma c_j x^j$, where $x = 1/(1+r)$ is the one-period "discount factor"; it has advantages of vividness, comparability across horizons n, and for our purposes, of coefficients which are directly at hand. As the appropriate rate r is often debatable, special interest attaches to rates at which the present or equivalently accumulated value is 0, called "internal rates of

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return." They correspond to roots between 0 and 1 of the polynomial $\Sigma c_j x^j$. The existence of more than one is unusual and complicates understanding of a business situation. By the method to be described, n additions usually suffice to eliminate this possibility, and business examples requiring more than $2n$ seem exceedingly rare except when artificially constructed. Sturm's method is far harder both to understand and to execute, ordinarily requiring about n^2 multiplications and divisions and $3n(n-1)/2$ additions and subtractions.

More generally, given any real polynomial

$$P(x)=\Sigma_{j=0}^{n} c_j x^j, \tag{1}$$

a family of eventually tight upper bounds on the number of roots in either (0, 1) or (0, ∞) can be obtained as follows. Start with

$$c_{0,j}=c_j, \qquad 0 \leqslant j \leqslant n, \text{ if the domain of interest is } (0, 1), \tag{2a}$$

$$c_{n+1-j,j}=c_j, \qquad 0 \leqslant j \leqslant n, \text{ if the domain of interest is } (0, \infty), \tag{2b}$$

and

$$c_{i,0}=c_0 \text{ for all } i, \qquad c_{0,j}=0 \text{ for } j>n. \tag{3}$$

By successive additions, calculate as many values

$$c_{i,j}=c_{i,j-1}+c_{i-1,j}, \qquad i \geqslant 1, j \geqslant 1, \tag{4}$$

as you wish. (In the tableau thus generated, each row consists of the partial sums of the previous one. The same holds for columns except for addition of a constant. Each rising diagonal consists of the pairwise sums of the previous one.) Take any path through the tableau of the $c_{i,j}$ going from any $c_{i,0}$ to any $c_{1,j}$ with $j \geqslant n$ by horizontal, vertical and diagonal steps. (Call such a path *allowed*.) Count the number of changes of sign along the path, ignoring 0's.

THEOREM 1. *The number of changes of sign along any allowed path through the tableau generated by the coefficients of $P(x)$ is an upper bound on the number of roots of $P(x)$ in the relevant domain, including multiplicities.*

Though backward and downward steps are allowed, they are not useful because: after any finite number of calculations of the form (4), the outer boundary of the tableau of calculated values is a *monotonic* path, meaning that each step on the path is upward, diagonally upward, or to the right, and

THEOREM 2. *The number of changes of sign along a monotonic path is no greater than the number along any allowed path in the tableau it bounds.*

For example, Table 1 is generated by $6-22x+29x^2-15x^3$ on (0, 1), or from the diagonal $i+j=4$ on, by $6-4x+3x^2-2x^3$ on (0, ∞). The outer boundary has just one sign change, implying at most one root. The diagonal $i+j=5$ is monotonic with three sign changes, so no path with $i+j \leqslant 5$ has fewer. (See also §3.3.)

TABLE 1

i \ j	0	1	2	3	4
0	6	−22	29	−15	
1	6	−16	13	−2	−2
2	6	−10	3	1	−1
3	6	−4	−1	0	−1
4	6	2	1	1	
5	6				

The following theorem, of interest in itself, also motivates some ensuing developments.

THEOREM 3. *The $c_{i,j}$ generated by $P(x)$ satisfy*

$$c_{i,j}=\Sigma_{t=0}^{n}\binom{m-t}{j-t}c_t=\binom{m}{j}\left[P\left(\frac{j}{m}\right)+O\left(\frac{1}{m}\right)\right] \tag{5a}$$

when the domain of interest is (0, 1), *and*

$$c_{i,j}=\Sigma_{t=0}^{n}\binom{m-n}{j-t}c_t=\binom{m}{j}\left[\left(\frac{i}{m}\right)^{n}P\left(\frac{j}{i}\right)+O\left(\frac{1}{m}\right)\right] \tag{5b}$$

when the domain of interest is $(0, \infty)$, *where in each case* $m=i+j-1$ *and* $O(1/m)$ *holds uniformly in* i, j *as* $m\to\infty$.

Thus, along a monotonic path with $i+j$ large, the $c_{i,j}/\binom{i+j-1}{j}$ sweep out approximately the values of $P(x)$ or $P(x)/(1+x)^n$. As this suggests,

THEOREM 4. *For M sufficiently large, if $i+j>M$ at every $c_{i,j}$ on a monotonic path, then the number of changes of sign along the path equals the number of roots of $P(x)$ in the relevant domain.*

Theorem 2 says that additional labor, to enlarge the tableau, cannot harm you. Theorem 4 says that it will ultimately be rewarded. But how will you know when you have received your ultimate reward? The following procedure will eventually tell you, provided all roots are simple.

On any monotonic path, let $(i_1,j_1), (i_2,j_2),\ldots,(i_s,j_s)$ index the points at which changes of sign occur. Then Theorem 3 suggests that there are roots of $P(x)$ near the values

$$x_k=j_k/(i_k+j_k) \quad \text{if the domain of interest is } (0, 1), \tag{6a}$$

$$x_k=j_k/i_k \quad \text{if the domain of interest is } (0, \infty). \tag{6b}$$

(These values could be used as starting points of a search.) Monotonicity of the path implies $x_1<x_2<\cdots<x_s$. Let

$$y_k=(x_k+x_{k+1})/2, \qquad 1\leqslant k\leqslant s-1. \tag{7}$$

Now $y_1<y_2<\cdots<y_{s-1}$; also $P(0)=c_0$, while $P(1)=c_{1,n}$ if the domain of interest is (0, 1) and $P(x)$ has the same sign at $x=\infty$ as $c_n=c_{1,n}$ if the domain of interest is $(0, \infty)$. Therefore,

THEOREM 5. *The number of changes of sign of the sequence*

$$c_0, P(y_1), P(y_2),\ldots,P(y_{s-1}), c_{1,n} \tag{8}$$

is a lower bound on the number of roots of $P(x)$ in the relevant domain. If all roots are simple and neither c_0 nor $c_{1,n}$ is 0, then this lower bound is eventually tight, and hence will eventually agree with the upper bound for the same path.

If $c_0=0$, then one may start with the first nonzero c_j; this amounts to dividing $P(x)$ by x^j, eliminating the root 0 with its multiplicity j and deleting j columns of 0's. If the domain of interest is $(0, \infty)$, then $c_{1,n}=c_n\neq 0$, provided n is defined as the degree of the polynomial, not larger. If the domain of interest is (0, 1), then $c_{1,n}=0$ if and only if 1 is a root. In this case, $P(x)/(1-x)=\Sigma_{j=0}^{n-1}c_{1,j}x^j$. Thus eliminating the root 1, with its multiplicity, is equivalent to treating the first row i for which $c_{i,n-i}\neq 0$ as the first row of the tableau.

The foregoing theorems are proved in the next section, in the order 3, 2, 1, 4, 5. Section 3 contains a variety of remarks.

2. Proofs. We have

$$\Sigma_{j=0}^{\infty}c_{i,j}x^j=P_0(x)/(1-x)^i \text{ for } -1<x<1, \tag{9}$$

where $P_0(x)=\Sigma_{j=0}^{n}c_{0,j}x^j$, because the $c_{i,j}$ in row i are the partial sums of those in row $i-1$, and because the partial sums of the coefficients of any power series $Q(x)$ are the coefficients of the power series for $Q(x)/(1-x)$. By similar but even more elementary reasoning, for rising diagonals $i+j=m+1$ with $m\geqslant n$,

$$\Sigma_{j=0}^{m}c_{m+1-j,j}y^j=(1+y)^{m-n}R_n(y) \tag{10}$$

where $R_n(y)=\Sigma_{j=0}^{n}c_{n+1-j,j}y^j$. Thus successive rows of the tableau correspond to successive divisions by $1-x$, successive rising diagonals to successive multiplications by $1+y$.

Clearly, $P_0(x)=P(x)$ when the domain of interest is (0, 1), and $R_n(y)=P(y)$ when the domain of interest is $(0, \infty)$. The two cases are related by

$$R_n(y)=(1+y)^n P_0(y/(1+y)), \qquad P_0(x)=(1-x)^n R_n(x/(1-x)); \tag{11}$$

that is, the same tableau is generated by $P_0(x)$ with domain (0, 1) as by $R_n(y)$ with domain $(0, \infty)$. One way to see this is to start with $P_0(x)=\Sigma_{j=0}^{n}c_jx^j$ and let

$$R_t(y)=(1+y)^t\Sigma_{j=0}^{t}c_j[y/(1+y)]^j=\Sigma_{j=0}^{t}c_jy^j(1+y)^{t-j}. \tag{12}$$

$R_t(y)$ is a polynomial of degree t whose coefficients are determinable from a recursion relation derived from

$$R_t(y)=(1+y)R_{t-1}(y)+c_ty^t. \tag{13}$$

It can be verified that $c_{t+1-j,j}$ satisfies this recursion relation, including its initial conditions. Therefore

$$R_t(y)=\Sigma_{j=0}^{t}c_{t+1-j,j}y^j. \tag{14}$$

For $t=n$ we find that the $c_{n+1-j,j}$ generated by $P_0(x)$ are indeed the coefficients of $R_n(y)$ as defined by (11). Similarly, starting with $R_n(y)$ and filling in the tableau from the bottom up by (4) leads to $P_0(x)$ satisfying (11). Thus, both generate the same tableau.

The summation formulas (5) in Theorem 3 can now be obtained by expanding the right-hand sides of (9) and (10) and comparing powers of x and y, respectively. (They can also be verified from (2)–(4) by induction and (9) and (10) can be proved from them. It would even be possible to take them as definitions and prove (2)–(4) from them.) The second equality of (5a) now follows from

$$\binom{m-t}{j-t}\Big/\binom{m}{j}=\frac{j(j-1)\cdots(j-t+1)}{m(m-1)\cdots(m-t+1)}=\left(\frac{j}{m}\right)^t+O\left(\frac{1}{m}\right), \tag{15}$$

where $O(1/m)$ holds uniformly in j as $m\to\infty$ with t fixed. The second equality of (5b) is equivalent by the preceding paragraph. This proves Theorem 3. It might seem natural to replace the leading term in the second line of (5b) by $\binom{m-n}{j}P(j/i)$, but this is 0 for $i<n+1$, and even for $i\geqslant n+1$ the remainder requires a further condition on i.

Theorem 2 follows from the simple observation that if an allowed path is contained in the tableau bounded by a monotonic path, then the former can be transformed into the latter by a series of alterations, each of which alters the sequence of values along the path by one of the following rules.

(i) Replace an element of the sequence by two or more repetitions of it.

(ii) Delete an element of the sequence.

(iii) Sum two consecutive elements of a sequence and insert the sum between them.

Since such alterations obviously cannot increase the number of sign changes, Theorem 2 follows.

In view of Theorem 2, it suffices to prove Theorem 1 for rising diagonal paths with $m\geqslant n$. The number of changes of sign along such a path is an upper bound on the number of positive roots of (10), and hence of $R_n(y)$, by Descartes's rule of signs. Since $P(y)=R_n(y)$ when the domain of interest is $(0, \infty)$, and $P(x)=(1-x)^nR_n(x/(1-x))$ by (11) when the domain of interest is (0, 1), Theorem 1 follows.

To prove Theorem 4, the asymptotic expressions in (5) are inadequate. Some kind of differencing is needed to bound the local oscillations of the $c_{i,j}$ when P on the right-hand side of (5) is near 0. It turns out to be convenient to use (5a), divide it by its leading factor, and then difference along rising diagonals. Specifically, let

$$d_{m,j}=c_{m+1-j,j}\Big/\binom{m}{j}, \tag{16}$$

which has the same sign as $c_{m+1-j,j}$. Then (5a) becomes

$$d_{m,j} = \Sigma_{t=0}^{n} c_t j_{(t)} / m_{(t)} = P(j/m) + O(1/m), \tag{17}$$

where $j_{(t)} = j(j-1)\cdots(j-t+1)$. Now let Δ be the backward difference operator on j. Then $\Delta j_{(t)} = t(j-1)_{(t-1)}$, whence by repetition, $\Delta^r j_{(t)} = t_{(r)}(j-r)_{(t-r)}$. Therefore

$$\Delta^r d_{m,j} = \Sigma_{t=r}^{n} c_t t_{(r)}(j-r)_{(t-r)} / m_{(t)} = \left[P^{(r)}(j/m) + O(1/m) \right] m^r, \tag{18}$$

where $P^{(r)}$ is the rth derivative of P. Under suitable conventions, the case $r=0$ is (17). The terms $O(1/m)$ in (17) and (18) hold uniformly in j as $m \to \infty$.

We can now prove Theorem 4 for rising diagonal paths as follows. Let h be the smallest distance between distinct values of x which are roots of $P(x)$ or of any $P^{(r)}(x), r<n$. By (18), one can choose M so large that $\Delta^r d_{m,j}$ has the same sign as $P^{(r)}(j/m)$ for all r, $0 \leqslant r \leqslant n$, all $m \geqslant M$, and all j such that j/m differs from the nearest root of $P^{(r)}(x)$ by at least $h/3$. Consider the sequence $d_{m,j}$, $j=0,1,\ldots,m$, for any fixed $m \geqslant M$. It has (case $r=0$) no changes of sign in the segments where j/m differs from the nearest root of $P(x)$ by at least $h/3$. Now let z be any root of $P(x)$. Suppose z has multiplicity k, so $P^{(k)}(z) \neq 0$. If $|j/m - z| \leqslant 2h/3$, then j/m differs from the nearest root of $P^{(k)}(x)$ by at least $h/3$ and hence $\Delta^k d_{m,j}$ has the same sign as $P^{(k)}(j/m)$, which has the same sign as $P^{(k)}(z)$. Therefore $d_{m,j}$ has at most k changes of sign as j/m varies in this region. Provided some j/m in this region also differs from the nearest root of $P(x)$ by at least $h/3$, which is guaranteed if $m \geqslant 3/h$, it follows that $d_{m,j}$ changes sign altogether no more often than the total number of roots of $P(x)$ with multiplicities. This proves Theorem 4 for rising diagonals. The general case follows by Theorem 2.

The first part of Theorem 5 was proved in introducing it. The rest is proved as follows. Assume the domain of interest is (0, 1), for definiteness. The discussion of the last two paragraphs implies that, for large $i+j$, the only changes of sign of $c_{i,j}$ occur for $j/(i+j)$ near a root of $P(x)$, and there is at least one change per root. The points x_k defined by (6a) therefore approximate all the roots of $P(x)$. The points y_k, therefore, include points approximately halfway between each pair of successive roots of $P(x)$. If all roots are simple, then the values of $P(x)$ at these points alternate in sign, and if $P(0)$ and $P(1)$ are added to the sequence, then there is (exactly) one such alternation for each root. Thus the lower bound is tight.

3. Remarks. 3.1. *Parity*. All allowed paths and $P(x)$ start with the same sign as the first nonzero c_j and end with the same sign as $c_{1,n}$. Hence all bounds and the actual number of roots are even if these two signs are the same, odd if they are different, and an upper bound of 1 is always tight, provided $c_{1,n} \neq 0$. (An upper bound of 0 is trivially tight. If $c_{1,n} = 0$, unless it is eliminated as at the end of Section 1, the parities may differ, whatever conventions are used for terminal zeros and for roots at 1. Example: $P(x) = a - (1+a)x + x^2$ has roots a and 1, while the path $(c_{1,0}, c_{1,1}, c_{1,2}) = (a, -1, 0)$ has the same signs whether $0<a<1$ or $a>1$.)

3.2. *Rectangular tableaux*. It is natural to calculate the tableau one row or column at a time, by accumulating the previous row or column. This produces a rectangular tableau, with outer corner $c_{i,j}$ say, whose outer boundary we shall call the $c_{i,j}$ *corner path*. The corresponding formula is

$$P_0(x) = (1-x)^i \Sigma_{t=0}^{j} c_{i,t} x^t + x^{j+1} \Sigma_{t=1}^{i} c_{t,j} (1-x)^{t-1} \text{ for } j \geqslant n. \tag{19}$$

The bound obtained from the $c_{i,j}$ corner path is nonincreasing in i and j, by Theorem 2. If the last row of a rectangular tableau has either no sign change, or one with an even number of terminal zeros, then adding rows only (increasing i but not j) cannot reduce the bound. Furthermore, eventually all added rows will have only one sign, that of the first nonzero c_j. Similar statements hold for columns.

If a $c_{i,j}$ corner path for $P(x)$ has been calculated and a root x_0 is then found, the $c_{i,j}$ corner path for $P(x)x_0/(x_0-x)$ can be obtained directly from the $c_{i,j}$ corner path for $P(x)$, without intermediate calculation, by a simple rule given in Pratt and Hammond **[5]**.

3.3. *First paths giving tight bounds.* Given any $i, j \geqslant 2$, there exist polynomials for which the $c_{i,j}$ corner path with the corner itself omitted is the first monotonic path on which the bound is tight, meaning that it lies wholly within the tableau bounded by any other monotonic path on which the bound is tight. (Quadratic examples can be constructed by choosing any c_0/c_2 and c_1/c_2 in the (nonempty) triangle in $(c_0/c_2, c_1/c_2)$ space determined by the inequalities $c_{i-1,j-1} < 0, c_{i,j-1} \geqslant 0, c_{i-1,j} \geqslant 0$ where $c_{i,j}$ is given by (5a) or (5b). All are semidefinite since the first allowed path already gives a tight bound for indefinite quadratics, except that $i=j=2$ can occur with or without semidefiniteness when the domain of interest is (0, 1).)

Is there always a first path or a first monotonic path along which the number of changes of sign equals the number of roots? No. In Table 1, the $c_{3,4}$ and $c_{4,3}$ corner paths have only one change of sign, but any path in the tableau bounded by both has at least three, being in the tableau bounded by $c_{3,3}$ corner path, which has three.

3.4. *Infinite paths and power series.* §3.2, paragraph 2, implies that infinite paths could be allowed in Theorems 1, 2, and 4 but are of little interest for polynomials. For infinite power series, however, nonvacuous results require paths with infinite final segments because no others are allowed (since $n=\infty$). Specifically,

THEOREM 1′. *Let $P(x)=\sum_{j=0}^{\infty} c_j x^j$ have radius of convergence $r>0$. Generate a tableau by (2a) and (3) with $n=\infty$, and (4). Call a path through this tableau allowed if it starts at some $c_{i,0}$ and proceeds by horizontal, vertical, and diagonal steps to $j=\infty$ with i bounded. The number of changes of sign along any allowed path is an upper bound on the number of roots of $P(x)$ in $(0, \min(r, 1))$, including multiplicities.*

With this definition of "allowed," the definition of "monotonic" needs no change, and Theorem 2 still holds. Theorem 1′ follows from this extended Theorem 2, equation (9), which now holds for $|x| < \min(r, 1)$, and the following lemma, which is easily proved by induction. The lemma is also useful in the continuous case (§3.5), and Descartes's rule of signs is a direct consequence of it.

LEMMA. *If f is k times continuously differentiable on $(0, t)$, then the sequence of different signs displayed by $f(x)$, $0<x<t$, is an initial subsequence of the sequence displayed by $f(0)$, $f'(0), \ldots, f^{(k)}(0)$, $f^{(k)}(x)$, $0<x<t$, where j changes of sign are included at roots of $f(x)$ or $f^{(k)}(x)$ of multiplicity j. One can replace $0<x$ by $0 \leqslant x$ in both sequences provided that, if $f(0)=0$, the second sequence is adjusted to reflect the multiplicity of 0 as a root.*

In Theorem 3, the first line of (5a) holds for infinite power series with j in place of n, but the second line requires additional conditions. The final term in brackets approaches 0 uniformly in j if either $\sum_j |c_j| < \infty$ or j/m is bounded away from 1. However, for $P(x) = 1/(1-x)^2$, for example, it becomes $-j(i+j)/i^2(i+1)$, which does not even approach 0 for $i = O(j^{2/3})$. Theorems 4 and 5 also require additional conditions. For example, if $P(x) = (1-x)^k \log(1+x/2)$, then $c_{k,j} = (-1/2)^j/j, j \geqslant 1$ and paths approaching $j=\infty$ with $i=k$ will have infinitely many sign changes no matter how large M is in Theorem 4. Requiring $i>k$, the multiplicity of the root 1, eliminates this difficulty, but others remain.

3.5. *Continuous methods.* The continuous counterpart of the problem considered here concerns the positive 0's of a Laplace transform and requires integration in place of summation. For functions on a finite interval, infinite paths can be avoided by means of the lemma above. The work reported here began when the continuous counterpart of the bounds obtained from infinite rows was discovered by John S. Hammond, III [1], who was investigating generalizations of stochastic dominance in the context of utility theory and noticed that a result he had obtained for exponential utility is equivalent to a result for present value. Although the discrete case can be treated by continuous methods, the results are somewhat inconvenient. This is what stimulated me to look into discrete methods for the discrete case.

3.6. *Replacing x by x^s.* If s is any positive integer, then the polynomial $P(x^s)$ has the same number of roots in (0, 1) or $(0, \infty)$ as $P(x)$, with the same multiplicities, but the extra coefficients 0 can affect the bounds. As $s \to \infty$, when the domain of interest is (0, 1), the $c_{i,j}$ approach the values obtained by i integrations, except for a possible multiple depending on i, but the bounds need not approach the corresponding integration bounds. For example, for the polynomial $3-10x^s+15x^{2s}+4x^{3s}$ on (0, 1), the row $i=3$ has 2 sign changes for s odd and 0 for s even. The corresponding integration bound is 0. For further discussion see Pratt and Hammond [4]. Note that the correspondence (11) between polynomials on (0, 1) and $(0, \infty)$ is not preserved by substitution of x^s for x.

3.7. *Related literature.* Hausdorff [3, pp. 98–99] shows that if a polynomial $P(x)>0$ on [0, 1] then it can be expressed in the form $\Sigma_{j=0}^{m} a_j x^j (1-x)^{m-j}$ with all $a_j > 0$. This is essentially Theorem 4 in the case that $P(x)$ has no roots. Specifically, Theorem 4, in conjunction with (10) and (11), implies that, if $P(x)>0$ on [0, 1], then for some t,

$$P(x)=(1-x)^n R_n(x/(1-x))=\Sigma_{j=0}^{m} c_{m+1-j,j} x^j (1-x)^{m-j} \tag{20}$$

with all $c_{m+1-j,j} \geqslant 0$. For some larger m, the same will hold with all $c_{m+1-j,j} > 0$.

Hardy, Littlewood, and Pólya [2, Theorem 57] generalize Hausdorff's result to positive forms in any number of variables. The proof of Section 2 has some resemblance to their proof, though they have no need for differencing. It has no resemblance to Hausdorff's.

Special cases of the results given here have appeared in the financial literature. Budan's Theorem and Sturm's Theorem have also been applied in the financial context. For discussion of the relationships to Theorem 1 and references leading to others, see Pratt and Hammond [4], [5].

3.8. *Miscellany.* The c_j may change sign frequently in the financial context if they represent the difference between two projects or if the accounting period for the cash flows is very short. A relatively small amount of addition (4) may well eliminate many sign changes arising in such ways.

Formulas (5a) and (5b) can easily be generalized to express $c_{i,j}$ in terms of any earlier row, column, or diagonal. A shortcut can then be derived to determine from the $c_{3,n}$ corner path whether any $c_{3,j}$ corner path with $j>n$ would have fewer sign changes [4]. However, a simpler and stronger result can be obtained by the continuous method [5]. If this still gives a bound exceeding 1 in a financial problem, it is perhaps time to seek a root rather than an improved bound.

The coefficients of various polynomials and power series related to $P(x)$ may be found in the tableau, and this is a far simpler way to obtain them than direct use of the binomial theorem. See, e.g., (9), (10), and (19) with $i=\infty$. Also, the nth diagonal from NW to SE gives the coefficients of the accumulated value $\Sigma c_j(1+x)^{n-j}$; if the coefficients of $P_0(x)$ are reversed, it gives those of $P_0(1+x)$, which are used in Budan's theorem.

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SUMMING AN ALTERNATING SERIES

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Introduction. Isaac Asimov in a recent book [**1**, pp. 78–80] contrasts the sixteenth-century approximation for π, 355/113, with Leibniz' series for π,

$$\pi=\sum_{1}^{\infty}4(-1)^{k+1}/(2k-1), \tag{*}$$

given in 1673. Asimov asks how many terms of (*) one must take in order to improve the estimate 355/113.

More generally, one can ask about the accuracy one obtains in approximating the sum of a convergent alternating series $\Sigma_1^\infty(-1)^{k+1}a_k$ by the partial sum $\Sigma_1^n(-1)^{k+1}a_k$. We shall derive elementary, but useful, estimates in Section 1. In Section 3 we will use more sophisticated methods to derive other estimates.

Obviously, Leibniz' series (*) converges very slowly. Thus it is impractical to compute the sum of this series by simply adding up the terms. In Section 2 we will develop methods of approximating accurately the sum of a convergent alternating series $\Sigma(-1)^{k+1}a_k$.

In [**4**] Boas investigated the questions raised above primarily for series of positive terms. Calabrese [**6**] and Pinsky [**9**] considered some of the above questions for the alternating series.

1. Elementary remainder estimates. We first fix some notation. Throughout this paper, if $\Sigma_1^\infty b_k$ is a convergent series, we shall let $R_n=\Sigma_{n+1}^\infty b_k$, the error or remainder in approximating the actual sum by $\Sigma_1^n b_k$.

If $\{b_k\}$ is a sequence, we will let $\Delta b_k=b_k-b_{k+1}$, the first difference of the sequence $\{b_k\}$. Successive differences are defined inductively: $\Delta^{i+1}b_k=\Delta(\Delta^i b_k)$. We also set $\Delta^0 b_k=b_k$.

Throughout this section we shall assume that $\{a_k\}$ is a positive, decreasing sequence with limit zero. The standard calculus results (see, for example, [**10**, p. 587]) tell us that in this case $\Sigma(-1)^{k+1}a_k$ converges, $\{|R_n|\}$ decreases, and that if the sum of the series is denoted by L, then

$$0<|R_n|=(-1)^nR_n=(-1)^{n+1}\left[\sum_{1}^{n}(-1)^{k+1}a_k-L\right].$$

We take as our starting point the easily verified identity

$$\sum_{1}^{n}(-1)^{k+1}a_k=\left[a_1/2+(1/2)\sum_{1}^{n-1}(-1)^{k+1}\Delta a_k\right]+(-1)^{n+1}a_n/2. \tag{1.1}$$

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This identity is of a type studied extensively by Berndt and Schoenfeld [3]. Taking the limit as $n \to \infty$, we see that the series in brackets [ ] converges to L also. Now subtracting L and multiplying by $(-1)^{n+1}$ we obtain

$$\begin{aligned} |R_n| &= (-1)^{n+1}\left[\sum_1^n (-1)^{k+1} a_k - L\right] \\ &= \left\{(-1)^{n+1}\left[a_1/2 + (1/2)\sum_1^{n-1} (-1)^{k+1} \Delta a_k - L\right]\right\} + a_n/2. \end{aligned} \tag{1.2}$$

If $\{\Delta a_k\}$ decreases, the term in braces {} is negative and so $|R_n| < a_n/2$. This is Calabrese's result.

THEOREM 1.1 (Calabrese [6]). *If $\{\Delta a_k\}$ decreases, then $|R_n| < a_n/2$.*

COROLLARY (Pinsky [9]). *If $\{\Delta a_k\}$ and $\{\Delta^2 a_k\}$ decrease, then*

$$|R_n'| < \Delta a_n/4,$$

where R_n' is the remainder for the series

$$a_1/2 + (1/2)\sum_1^\infty (-1)^{k+1} \Delta a_k. \tag{1.3}$$

Proof. Apply Theorem 1.1 to the series (1.3). ∎

Pinsky [9] noted that if s_n denotes the nth partial sum of the series $\Sigma(-1)^{k+1} a_k$, then the nth partial sum of the series (1.3) is $(s_n + s_{n-1})/2$. Thus the Corollary above gives an estimate for the accuracy obtained by approximating the sum of the original series by the averages of its partial sums.

Shohat [12] in 1933 obtained the series (1.3) for the special case $a_k = 1/(2k-1)$ (which is $1/4$ of the Leibniz series (*)) by a much different and more complicated method than that given above. Unfortunately, his method is not sufficiently general. (For example, it gives no useful information for the series $\Sigma(-1)^{k+1}/k$.)

An identity similar to (1.1) is

$$\sum_1^n (-1)^{k+1} a_k = \left[a_1/2 + (1/2)\sum_1^n (-1)^{k+1} \Delta a_k\right] + (-1)^{n+1} a_{n+1}/2. \tag{1.4}$$

Again taking the limit as $n \to \infty$, the series in brackets [ ]converges to L. Again subtracting L and multiplying by $(-1)^{n+1}$, we obtain

$$\begin{aligned} |R_n| &= (-1)^{n+1}\left[\sum_1^n (-1)^{k+1} a_k - L\right] \\ &= \left\{(-1)^{n+1}\left[a_1/2 + (1/2)\sum_1^n (-1)^{k+1} \Delta a_k - L\right]\right\} + a_{n+1}/2. \end{aligned} \tag{1.5}$$

This time if $\{\Delta a_k\}$ decreases, the term in braces {} is positive so $a_{n+1}/2 < |R_n|$. This result has been noted before ([5]). The corresponding corollary is that if $\{\Delta^2 a_k\}$ decreases, then $\Delta a_{n+1}/4 < |R_n'|$. Combining these above results we have theorems giving both upper and lower estimates for the original series and series (1.3).

THEOREM 1.2. *If $\{a_k\}$ and $\{\Delta a_k\}$ decrease, then*

$$a_{n+1}/2 < |R_n| < a_n/2.$$

COROLLARY. *If $\{\Delta a_k\}$ and $\{\Delta^2 a_k\}$ decrease, then*

$$\Delta a_{n+1}/4 < |R_n'| < \Delta a_n/4.$$

We may extend these arguments to obtain increasingly sharp estimates. We state our results as a theorem.

THEOREM 1.3. *If* $\Sigma(-1)^{k+1}a_k$ *converges and* $\{\Delta^j a_k\}$ *decreases, then*

$$\sum_{i=1}^{j}\Delta^{i-1}a_{n+1}/2^i<|R_n|<a_n/2-\sum_{i=2}^{j}\Delta^{i-1}a_n/2^i.$$

Proof. In identity (1.4) replace n by $n-1$ and a_k by Δa_k, then substitute into equation (1.2) to obtain

$$|R_n|=\left\{(-1)^{n+1}\left[a_1/2+\Delta a_1/4+(1/4)\sum_{1}^{n-1}(-1)^{k+1}\Delta^2 a_k-L\right]\right\}$$
$$+a_n/2-\Delta a_n/4.$$

Repeating this procedure $j-2$ times, we obtain

$$|R_n|=\left\{(-1)^{n+1}\left[\sum_{i=1}^{j}\Delta^{i-1}a_1/2^i+(1/2^j)\sum_{k=1}^{n-1}(-1)^{k+1}\Delta^j a_k-L\right]\right\}$$
$$+a_n/2-\sum_{i=2}^{j}\Delta^{i-1}a_n/2^i.$$

Since the term in braces {} is always negative, the right inequality follows.

The left inequality is derived similarly beginning with identity (1.5). ∎

Now let f be a real-valued, differentiable function defined on $[1,\infty)$. There is a close relation between the difference Δf and the derivative f'. Indeed, by the mean-value theorem, we have

$$\Delta f(k)=-[f(k+1)-f(k)]=-f'(\xi)$$

for some $\xi\in(k,k+1)$. It follows that if f' is increasing, then $\{\Delta f(k)\}$ is decreasing. Similarly, if f'' exists everywhere, $\Delta^2 f(k)=f''(\xi)$ for some $\xi\in(k,k+1)$. Thus $\{\Delta^2 f(k)\}$ is decreasing if f'' is decreasing. Therefore, in order to assure that $\{\Delta^j f(k)\}$ decreases for $j=0,1,2,\ldots$, it suffices that f is decreasing, f' is increasing, f'' is decreasing,.... This latter condition is fulfilled if f' is negative, f'' is positive, f''' is negative,.... These conditions are satisfied for many of the functions f defining alternating series, especially those involving terms like $1/x^p$ and $\ln x$. A sequence $\{a_k\}$ for which $\{\Delta^j a_k\}$ is decreasing for $j=0,1,2,\ldots$ is called *completely monotone* (see [13]). We see that if $f^{(j)}(t)f^{(j+1)}(t)<0$ for $j=0,1,2,\ldots$ and for $t\in[1,\infty)$, then $\{f(k)\}$ is completely monotone.

As an illustration of the use of these remainder estimates, let $a_k=1/(2k-1)$. Suppose we wish to find the smallest n for which $|R_n|<\frac{1}{2}\times 10^{-4}$. (This precision is often referred to as four-place decimal accuracy.)

By the criteria above, $\{a_k\}$ is completely monotone, so that Theorem 1.3 applies. Using the estimate $|R_n|<a_n/2$, we solve $a_n/2\leqslant\frac{1}{2}\times 10^{-4}$, to obtain

$$5000.5=(10^4+1)/2\leqslant n.$$

This inequality suggests taking $n=5001$. Using the estimate $a_{n+1}/2<|R_n|$ we find that

$$4.9995\times 10^{-5}=a_{5001}/2<|R_{5000}|$$

and

$$\tfrac{1}{2}\times 10^{-4}<5.0005\times 10^{-5}=a_{5000}/2<|R_{4999}|.$$

Recalling that $\{|R_n|\}$ decreases, we see that the desired n is either 5000 or 5001. To decide between these two values we must employ a sharper estimate.

We use Theorem 1.3, with $j=3$, to obtain

$$|R_{5000}|<a_{5000}/2-\Delta a_{5000}/4-\Delta^2 a_{5000}/8$$
$$=4.99999995\times 10^{-5}<\tfrac{1}{2}\times 10^{-4}.$$

Therefore $n=5000$ is the desired value.

TABLE 1

(1) $\Sigma_1^\infty(-1)^{k+1}/\ln(\ln(k+2))$ (2) $\Sigma_1^\infty(-1)^{k+1}/\ln(k+1)$ (3) $\Sigma_1^\infty(-1)^{k+1}/k^p$
(4) $\Sigma_1^\infty(-1)^{k+1}\ln k/k$ (5) $\Sigma_1^\infty(-1)^{k+1}/(2k-1)$ (6) $\Sigma_0^\infty(-.9)^k$ (7) $\Sigma_0^\infty(-1)^k/k!$
(8) $\Sigma_1^\infty(-1)^{k+1}/k^{k^k}$

Series	Sum	Number of terms required to calculate the sum with absolute error less than $(1/2)\times$			
		10^{-2}	10^{-10}	10^{-100}	10^{-1000}
1	8.74955124	$T(1.17\times 10^{43})$	$T_3(9.7)$	$T_3(99)$	$T_3(1000)$
2	.924299897	2.688×10^{43}	$T(4.4\times 10^9)$	$T_2(100)$	$T_2(1000)$
3($p=.01$)	.5022548581	10^{200}	10^{1000}	$10^{10,000}$	$10^{100,000}$
3($p=.5$)	.6048986434	10^4	10^{20}	10^{200}	10^{2000}
4	−.1598689037	648	2.7×10^{11}	2.4×10^{102}	2.4×10^{1003}
3($p=1$)	.6931471805	100	10^{10}	10^{100}	10^{1000}
5	.7853981633	50	5×10^9	5×10^{99}	5×10^{999}
6	10/19	45	220	2186	21,855
7	.3678794411	6	15	71	451
8	.9375	2	2	3	4

Note: $T(x)=T_1(x)=10^x$, $T_n(x)=T(T_{n-1}(x))$.

Table 1 gives the number of terms required to approximate various sums with particular degrees of precision. The series are arranged roughly in order of their rates of convergence.

The next section is devoted to a discussion of how to calculate the sum of an alternating series. Of course some of the sums in Table 1 are known in closed form. For example, the sum of series (3) ($p=1$) is ln 2. The sums of the rapidly converging series can be calculated by direct addition. The interesting sums to approximate are those given by series such as (1) or (2) where the sum is not known in closed form and where the series converges so slowly that it is not feasible to directly add the terms.

We close this section by answering Asimov's question quoted in the introduction, which is to find the least n for which

$$|R_n|<355/113-\pi,$$

where R_n is the remainder for the series $\Sigma_1^\infty(-1)^{k+1}a_k$ with $a_k=4/(2k-1)$.

Letting $\varepsilon=355/113-\pi$, we approximate ε by $2.667641890\times 10^{-7}$. Solving $a_n/2<\varepsilon$ we obtain $n>3{,}748{,}629.5$. Now for $n=3{,}748{,}630$, we have $|R_n|<a_n/2<\varepsilon$. For $n=3{,}748{,}629$, we have $\varepsilon<2.667641956\times 10^{-7}=a_{n+1}/2+\Delta a_{n+1}/4<|R_n|$. Thus we must take a minimum of 3,748,630 terms of $\Sigma(-1)^{k+1}a_k$ to improve 355/113 as an approximation to π.

Obviously summing the series $\Sigma 4(-1)^{k+1}/(2k-1)$ directly is an impractical means of approximating π. Recent approximations of π to several thousand decimal places have been obtained by using certain arctangent relations, such as Machin's formula,

$$\pi/4=4\arctan(1/5)-\arctan(1/239),$$

and the Gregory series,

$$\arctan x = \sum_{1}^{\infty} (-1)^{k+1} x^{2k-1}/(2k-1),$$

to evaluate the arctangents. (See [2] and [14] for details and a history of the approximation of π.) Apparently, the record approximation to π is held by J. Guilloud, who computed one million decimal digits. A new algorithm for computing π published in 1976 by Salamin [11] promises to surpass previous methods. Salamin's formula is based on Gauss' method for calculating elliptic integrals and an elliptic integral relation of Legendre. A proof of Salamin's main result can be given using only results from elementary calculus. Guibas and Simonyi are working on a program to compute π to 67 million binary digits using Salamin's algorithm. This is equivalent to approximately 20 million decimal digits.

2. Summing the alternating series. The usual method of summing a slowly converging series Σa_k is to replace it with a series Σb_k which has the property that its partial sums $s_n = \Sigma_1^n b_k$, for small values of n, give good approximations to the sum of the series Σa_k. The series Σb_k may be either convergent or divergent. Indeed, perhaps the most powerful method for summing a slowly convergent series involves a divergent series Σb_k. However, at first we shall follow what is perhaps the most obvious route, simply replacing Σa_k by a convergent series Σb_k which has a more rapid rate of convergence.

Let $\{a_k\}$ be a positive, decreasing sequence with limit zero. As usual let R_n denote the remainder for the series

$$\sum_{1}^{\infty} (-1)^{k+1} a_k. \tag{2.1}$$

In Section 1 we showed that the series

$$a_1/2 + (1/2) \sum_{1}^{\infty} (-1)^{k+1} \Delta a_k \tag{2.2}$$

has the same sum as $\Sigma_1^\infty (-1)^{k+1} a_k$. Let R_n' denote the remainder for this series. Theorem 1.1 and its Corollary show that if $\{\Delta a_k\}$ and $\{\Delta^2 a_k\}$ decrease, the remainders satisfy

$$|R_n| < a_n/2, \ |R_n'| < \Delta a_n/4.$$

Therefore, replacing series (2.1) with series (2.2) improves the rate of convergence by at least a factor of two. Table 2 gives a comparison of the actual rates of convergence of the series (2.1) and (2.2) for $a_k = 1/k$.

Of course this process may be repeated. That is, we may replace the alternating series $\Sigma_1^\infty (-1)^{k+1} \Delta a_k$ in (2.2) by $\Delta a_1/2 + (1/2)\Sigma_1^\infty (-1)^{k+1} \Delta^2 a_k$. Substituting back into (2.2), we obtain

$$a_1/2 + \Delta a_1/4 + (1/4) \sum_{1}^{\infty} (-1)^{k+1} \Delta^2 a_k. \tag{2.3}$$

Letting R_n'' denote the remainder for this series, we find that if $\{\Delta^2 a_k\}$ and $\{\Delta^3 a_k\}$ decrease, then

$$|R_n''| < \Delta^2 a_n/8.$$

We again achieve at least a twofold rate of increase in convergence in replacing series (2.2) with (2.3). This process obviously may be continued to obtain remainders $R'''n, \ldots$. Table 2 also compares R_n and R_n'' for $\Sigma(-1)^{k+1}/k$.

If we do continue this process, we obtain

$$\sum_{1}^{\infty} (-1)^{k+1} a_k = \sum_{0}^{n-1} \Delta^k a_1/2^{k+1} + (1/2^n) \sum_{k=1}^{\infty} (-1)^{k+1} \Delta^n a_k. \tag{2.4}$$

If $\Sigma(-1)^{k+1} a_k$ converges, the series $(1/2^n)\Sigma_1^\infty (-1)^{k+1} \Delta^n a_k$ converges and equation (2.4) holds. This result follows from equation (1.1). In fact one need not assume that $\{a_k\}$ is monotone or even that $a_k > 0$, although most of our series will possess these properties.

TABLE 2.

Methods of Summing $\Sigma_1^\infty(-1)^{k+1}a_k$ for $a_k=1/k$

(1) $\Sigma_1^n(-1)^{k+1}a_k$ (2) $a_1/2+(1/2)\Sigma_1^n(-1)^{k+1}\Delta a_k$ (3) $a_1/2+\Delta a_1/4+(1/4)\Sigma_1^n(-1)^{k+1}\Delta^2 a_k$
(4) $\Sigma_0^n\Delta^k a_1/2^{k+1}$ (5) $\Sigma_1^{100}(-1)^{k+1}a_k+\Sigma_0^n\Delta^k a_{101}/2^{k+1}$ (6) $\Sigma_1^{100}(-1)^{k+1}a_k-\Sigma_1^n C_{2k}f^{(2k-1)}(101)$ $(f(t)=1/t)$

x' denotes the absolute value of the remainder in using (x) as an approximation to the sum $\Sigma_1^\infty(-1)^{k+1}a_k$.

Series	$n=1$	$n=2$	$n=3$	$n=4$	$n=5$	$n=10$	$n=50$	$n=100$
1	1.000000	.500000	.833333	.583333	.783333	.645635	.683247	.688172
1′	.306853	.193147	.140186	.109814	.090186	.047512	.009900	.004975
2	.750000	.666667	.708333	.683333	.700000	.691080	.693051	.693123
2′	.056853	.026481	.015186	.009814	.006853	.002058	.000096	.000025
3	.708333	.687500	.695833	.691667	.694048	.692983	.693145	.693147
3′	.015186	.005647	.002686	.001481	.000900	.000164	.0000018	2.4×10^{-7}
4	.625000	.666667	.682292	.688542	.691146	.693109	.6931471805	.6931471805
4′	.068147	.026481	.010856	.004606	.002001	.000038	1.6×10^{-12}	4×10^{-33}
5	.6931469414	.6931471770	.6931471804	.6931471805	.6931471805	.6931471805	.6931471805	.6931471805
5′	2.4×10^{-7}	3.5×10^{-9}	6.7×10^{-11}	2.2×10^{-12}	7×10^{-13}	1.8×10^{-19}	2.9×10^{-58}	4.3×10^{-92}
6	.6931471817	.6931471805	.6931471805	.6931471805	.6931471805	.6931471805	.6931471805	.6931471805
6′	1.2×10^{-9}	2.4×10^{-13}	9.8×10^{-17}	7×10^{-20}	7.7×10^{-23}	9.5×10^{-36}	2.7×10^{-95}	3.2×10^{-128}

We should be tempted to let $n\to\infty$ in equation (2.4) and use

$$\sum_{0}^{\infty}\Delta^k a_1/2^{k+1} \tag{2.5}$$

as an approximation to the original series (2.1). This will be permissible if

$$\lim_{n\to\infty}(1/2^n)\sum_{k=1}^{\infty}(-1)^{k+1}\Delta^n a_k=0.$$

This is, in fact, true under the sole assumption that series (2.1) is convergent. This result is due to Euler and series (2.5) is called the Euler transformation of series (2.1).

THEOREM 2.1 (Euler [8, pp. 244–246]). *If* $\Sigma_1^\infty(-1)^{k+1}a_k$ *converges, then* $\Sigma_0^\infty\Delta^k a_1/2^{k+1}$ *converges also and has the same sum.*

In Table 2 we compare series (2.5) with series (2.2) and (2.3) and we note that series (2.5) converges most rapidly.

To obtain an estimate for the remainder of the Euler transformation (2.5), we return to equation (2.4). Letting R_n^E denote the nth remainder of the Euler transformation, we see that

$$R_n^E=(1/2^n)\sum_{k=1}^{\infty}(-1)^{k+1}\Delta^n a_k.$$

Therefore, if $\{\Delta^n a_k\}$ decreases,

$$R_n^E<\Delta^n a_1/2^n.$$

For example, using this estimate, we can show that only 22 terms of the Euler transformation of Leibniz' series for π are needed to improve 355/113 as an estimate for π.

One can approximate the alternating series $\Sigma(-1)^{k+1}a_k$ very accurately by directly summing N terms, then using the Euler transformation to approximate the remainder $\Sigma_{N+1}^\infty(-1)^{k+1}a_k$. In Table 2, we approximate $\Sigma(-1)^{k+1}/k$ by

$$\sum_{1}^{100}(-1)^{k+1}/k+\sum_{0}^{n}\Delta^k a_{101}/2^{k+1}$$

for various n. Great accuracy is obtained.

The Euler transformation is most useful when the differences $\Delta^k a_1$ are easily computed. In general the most successful method to sum a slowly converging alternating series is to obtain an integral representation for the series. This we can easily do by beginning with equation (1.1). I am indebted to Professor Ralph P. Boas, Jr., for pointing out the intimate connection between equation (1.1) and our integral representation.

Assuming that f and its derivatives are continuous on $[1,\infty)$, we may write

$$\begin{aligned}\sum_{1}^{n}(-1)^{k+1}f(k)&=\left[f(1)+(-1)^{n+1}f(n)\right]/2+(1/2)\sum_{1}^{n-1}(-1)^{k+1}\Delta f(k)\\&=\left[f(1)+(-1)^{n+1}f(n)\right]/2+\int_1^n f'(t)X_1(t)\,dt,\end{aligned} \tag{2.6}$$

where $X_1(t)=-\frac{1}{2}$ on $[1,2),[3,4),[5,6),\ldots$ and $X_1(t)=+\frac{1}{2}$ on $[2,3),[4,5),[6,7),\ldots$. (See Fig. 1.) Equation (2.6) is a special case of Boole's formula.

To improve equation (2.6) we use integration by parts. It is convenient to choose the indefinite integral $X_2(t)$ of $X_1(t)$ to be periodic of period 2. (See Fig. 2.) We obtain

$$\begin{aligned}\int_1^n f'(t)X_1(t)\,dt&=f'(t)X_2(t)\Big|_1^n-\int_1^n f''(t)X_2(t)\,dt\\&=f'(n)X_2(n)-f'(1)X_2(1)-\int_1^n f''(t)X_2(t)\,dt\end{aligned}$$

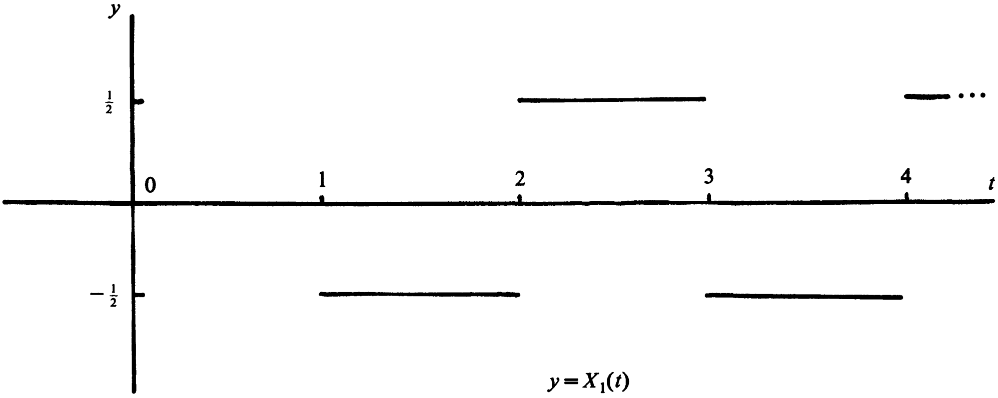

FIG. 1

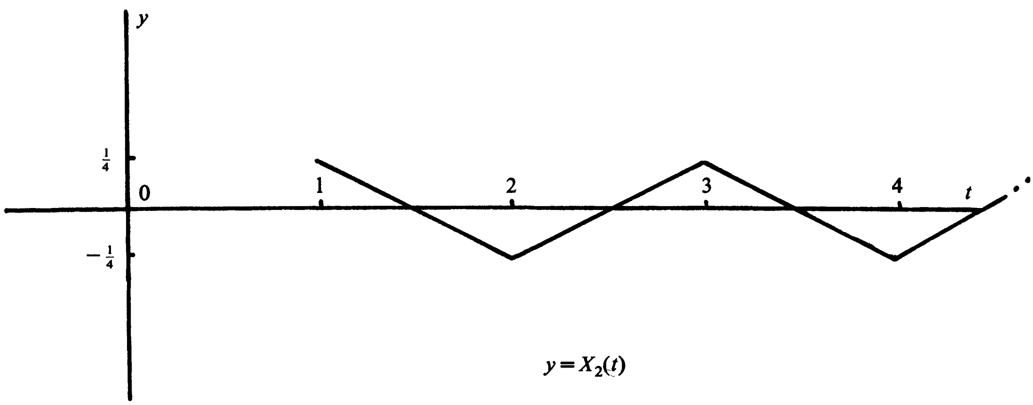

FIG. 2

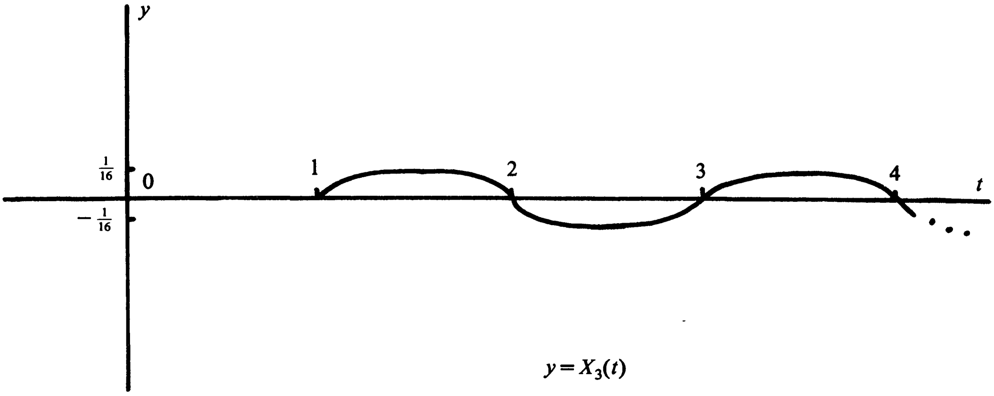

FIG. 3

$$=f'(n)(-1)^{n+1}C_2-f'(1)C_2-\int_1^n f''(t)X_2(t)dt,$$

where $C_2=(-1)^{n+1}X_2(n)=X_2(1)=1/4$. Substituting back into equation (2.6) we have

$$\sum_1^n(-1)^{k+1}f(k)=\left[f(1)+(-1)^{n+1}f(n)\right]/2+C_2\left[(-1)^{n+1}f'(n)-f'(1)\right]$$
$$-\int_1^n f''(t)X_2(t)dt.$$

If we again integrate by parts and again choose the periodic, indefinite integral $X_3(t)$ of $X_2(t)$ (see Fig. 3), we obtain

$$\sum_1^n(-1)^{k+1}f(k)=\left[f(1)+(-1)^{n+1}f(n)\right]/2+C_2\left[(-1)^{n+1}f'(n)-f'(1)\right]$$
$$+\int_1^n f'''(t)X_3(t)dt$$

since this time $X_3(1)=X_3(n)=0$.

Repeated integration by parts gives

$$\sum_1^n(-1)^{k+1}f(k)=\left[f(1)+(-1)^{n+1}f(n)\right]/2+\sum_1^m C_{2k}\left[(-1)^{n+1}f^{(2k-1)}(n)\right.$$
$$\left.-f^{(2k-1)}(1)\right]+\int_1^n f^{(2m+1)}(t)X_{2m+1}(t)dt, \tag{2.7}$$

where the constant C_{k+1} is chosen so that $X_{k+1}(t)$ is the periodic indefinite integral of $X_k(t)$. We shall always have $X_k(1)=X_k(n)=0$ for k odd and $k\geqslant 3$. (An empty sum, $\Sigma_{k=1}^m$ for $m<1$, is interpreted as zero.)

If we take $f(t)=t^{2m}$, the integral in equation (2.7) is zero. If we then take $n=2$, we can develop a formula from which the C_{2k} can be determined recursively. It is also possible (see [4]) to write the numbers C_{2k} in terms of the Bernoulli numbers B_{2k}. The relationship is given by the equation

$$C_{2k}=\frac{2^{2k}-1}{(2k)!}B_{2k}.$$

Some Bernoulli numbers are tabulated in [7]. The first few Bernoulli numbers are

$$B_1=-\frac{1}{2},\quad B_2=\frac{1}{6},\quad B_4=-\frac{1}{30},\quad B_6=\frac{1}{42},\quad B_8=-\frac{1}{30},$$
$$B_{10}=\frac{5}{66},\quad B_{12}=\frac{691}{2730}.$$

The functions $X_k(t)$ can be related to the periodic extensions of the Bernoulli polynomials. Details are given in [4].

It is tempting to let $m\to\infty$ in equation (2.7) to obtain an infinite series on the right to approximate the original series on the left. Unfortunately, the resulting series is usually divergent. Therefore, we proceed differently.

First, assume that $\Sigma(-1)^{k+1}f(k)$ converges and that the derivatives $f^{(k)}(t)$ converge to zero as $t\to\infty$. We may then let $n\to\infty$ in equation (2.7) to obtain

$$\sum_1^\infty(-1)^{k+1}f(k)=f(1)/2-\sum_1^m C_{2k}f^{(2k-1)}(1)+\int_1^\infty f^{(2m+1)}(t)X_{2m+1}(t)dt.$$

Replacing the lower index 1 by N we have

$$\sum_N^\infty(-1)^{k+1}f(k)=(-1)^{N+1}\left[f(N)/2-\sum_1^m C_{2k}f^{(2k-1)}(N)\right]+\int_N^\infty f^{(2m+1)}(t)X_{2m+1}(t)dt. \tag{2.8}$$

We shall show in Section 3 how to obtain accurate estimates of the integral $I_N = \int_N^\infty f^{(2m+1)}(t)X_{2m+1}(t)dt$. Thus, we can use equation (2.8) to approximate the sum $\Sigma_N^\infty(-1)^{k+1}f(k)$. This approach is similar to the situation we encountered in using the Euler transformation. We can sum a series to N terms, then use formula (2.8) to approximate the remainder.

For example, if $f(k)=1/k$, $m=1$, and $N=101$, we obtain

$$\sum_1^\infty(-1)^{k+1}/k=\sum_1^{101}(-1)^{k+1}/k+1/(2\cdot 101)+1/(4\cdot 101^2)+I_{101}.$$

Neglecting I_{101}, this sum is .6931471817, which differs from the true value by approximately 1.2×10^{-9}. For larger values of m, we obtain more accuracy. (See Table 2.)

3. Additional remainder estimates. In this section we will first derive estimates for the remainder $I_N = \int_N^\infty f^{(2m+1)}(t)X_{2m+1}(t)dt$ appearing in equation (2.8). We will then be able to justify the sums given in Table 1 and explain how they were derived. We will conclude by obtaining estimates for the remainder $\Sigma_{n+1}^\infty(-1)^{k+1}f(k)$ of the alternating series in terms of the derivatives of f. These formulas will complement the formulas given in Section 1 where the remainder is given in terms of the differences $\Delta^k f$.

Throughout this section we shall assume that f is positive and that the derivatives $f^{(k)}(t)$ tend to zero strictly monotonically as $t\to\infty$. It follows that $f^{(2k-1)}(t)$ is negative and increases to zero as $t\to\infty$.

We note that $X_k(t+1)=-X_k(t)$ for $k=1,2,3$ (see Figures 1, 2, and 3). Using induction we can show that $X_{2k+1}(t+1)=-X_{2k+1}(t)$ for $k=0,1,2,\ldots$. We also note that $X_1(t)\leqslant 0$ for $t\in[1,2)$ and $X_3(t)\geqslant 0$ for $t\in[1,2)$. Using induction we can show that $X_{2k+1}(t)\leqslant 0$ for k even and $t\in[1,2)$ and $X_{2k+1}(t)\geqslant 0$ for k odd and $t\in[1,2)$. Clearly, no function $X_k(t)$ is identically zero.

First, suppose that N is odd. If m is odd, in view of the above discussion

$$\begin{aligned}
&\int_N^{N+2} f^{(2m+1)}(t)X_{2m+1}(t)dt\\
&\qquad=\int_N^{N+1} f^{(2m+1)}(t)X_{2m+1}(t)dt+\int_{N+1}^{N+2} f^{(2m+1)}(t)X_{2m+1}(t)dt\\
&\qquad=\int_N^{N+1} f^{(2m+1)}(t)X_{2m+1}(t)dt-\int_N^{N+1} f^{(2m+1)}(t+1)X_{2m+1}(t)dt\\
&\qquad=\int_N^{N+1}\left[f^{(2m+1)}(t)-f^{(2m+1)}(t+1)\right]X_{2m+1}(t)dt<0
\end{aligned}$$

since $f^{(2m+1)}(t)-f^{(2m+1)}(t+1)<0$, $X_{2m+1}(t)\geqslant 0$ on $[N,N+1)$, and $X_{2m+1}(t)$ is not identically zero. Thus,

$$\int_i^{i+2} f^{(2m+1)}(t)X_{2m+1}(t)dt<0$$

for odd i. Therefore $\int_N^{N+2i} f^{(2m+1)}(t)X_{2m+1}(t)dt<0$, for $i=1,2,\ldots$. Taking the limit as $i\to\infty$ we obtain

$$I_N=\int_N^\infty f^{(2m+1)}(t)X_{2m+1}(t)dt<0.$$

If N is odd and m is even, the argument is the same except that $X_{2m+1}(t)\leqslant 0$ on $[N,N+1)$. In any case we have $(-1)^m I_N>0$. If N is even we get a change of sign so that, in general, $(-1)^{m+N+1}I_N>0$.

Fix N. Letting $J_m=\int_N^\infty f^{(2m+1)}(t)X_{2m+1}(t)dt$, J_m and J_{m+1} have opposite signs. Therefore J_m and J_m-J_{m+1} have the same sign. Thus $|J_m|<|J_m-J_{m+1}|$. However

$$J_m-J_{m+1}=-(-1)^{N+1}C_{2m+2}f^{(2m+1)}(N).$$

It follows that we may write

$$\int_N^\infty f^{(2m+1)}(t)X_{2m+1}(t)\,dt = J_m = -\eta(-1)^{N+1}C_{2m+2}f^{(2m+1)}(N),$$

for some η, $0<\eta<1$. We summarize our results as Theorem 3.1.

THEOREM 3.1. *Suppose f is positive and that f and all of its derivatives $f^{(k)}(t)$ tend to zero strictly monotonically as $t\to\infty$. Then*

$$\sum_N^\infty (-1)^{k+1}f(k) = (-1)^{N+1}\left[f(N)/2 - \sum_1^m C_{2k}f^{(2k-1)}(N) - \eta C_{2m+2}f^{(2m+1)}(N)\right]$$

for some η, $0<\eta<1$.

It can be shown that the numbers C_{2k} alternate in sign. Therefore, the series above on the right is an alternating series. Even though $\{|C_{2k}f^{(2k-1)}(N)|\}$ is rarely a decreasing sequence, the series behaves like an alternating series $\Sigma(-1)^{k+1}a_k$, where $\{a_k\}$ decreases, in that the remainder in stopping at any term is less than the absolute value of and has the same sign as the next term. This is the conclusion reached in introductory calculus texts for the alternating series $\Sigma(-1)^{k+1}a_k$, where $\{a_k\}$ decreases, (see, for example, [**10**, p. 587]).

In the previous section we estimated

$$\sum_{101}^\infty (-1)^{k+1}f(k) \approx f(101)/2 - f'(101)/4 = .6931471817,$$

where $f(k)=1/k$. We may now employ Theorem 3.1 to find that the error is at most the absolute value of and has the same sign as

$$-C_4f'''(101) = f'''(101)/48 = -1.201225430\times 10^{-9}.$$

This value is in very close agreement with the actual error (see Table 2).

We used Theorem 3.1 to sum the slowly converging series in Table 1 whose sums are not known in closed form. For example, to sum $\Sigma_1^\infty(-1)^{k+1}/k^{.01}$, we first summed 500 terms to obtain $\Sigma_1^{500}(-1)^{k+1}/k^{.01} = .03238675800$. We then estimated

$$\sum_{501}^\infty (-1)^{k+1}/k^{.01} \approx f(501)/2 - f'(501)/4 = .4698681001.$$

The error in this last approximation is less than the absolute value of and has the same sign as $f'''(501)/48 = -3.16\times 10^{-12}$. Therefore, the sum $\Sigma_1^\infty(-1)^{k+1}/k^{.01}$ is $.03238675800 + .4698681001 = .502254858$, correct to nine places. As Table 1 shows, it is impractical to sum this series by direct addition. The other sums in Table 1 were computed similarly.

To derive error estimates for the remainder $R_n = \Sigma_{n+1}^\infty(-1)^{k+1}f(k)$, we write

$$\begin{aligned} R_n &= R_{n-1} + (-1)^n f(n) \\ &= (-1)^n f(n)/2 + (-1)^{n+1}\left[-\sum_1^m C_{2k}f^{(2k-1)}(n) - \eta C_{2m+2}f^{(2m+1)}(n)\right], \end{aligned} \tag{3.1}$$

for some η, $0<\eta<1$. Taking $m=0$, we have

$$R_n = (-1)^n f(n)/2 + (-1)^n\eta f'(n)/4.$$

Thus

$$|R_n| = (-1)^n R_n = f(n)/2 + \eta f'(n)/4 < f(n)/2$$

since $f'(n)<0$. We again have Calabrese's result (Theorem 1.1). We can obtain further error estimates by taking more terms in equation (3.1).

THEOREM 3.2. *Suppose f is positive and that f and its derivatives $f^{(k)}(t)$ tend to zero strictly monotonically as $t\to\infty$. Let $R_n = \Sigma_{n+1}^\infty(-1)^{k+1}f(k)$ denote the nth remainder of the alternating*

series. Then

$$|R_n| = f(n)/2 + \sum_{1}^{m} C_{2k} f^{(2k-1)}(n) + \eta C_{2m+2} f^{(2m+1)}(n)$$

for some η, $0<\eta<1$. *Furthermore,*

$$f(n)/2 + \sum_{1}^{2i+1} C_{2k} f^{(2k-1)}(n) < |R_n| < f(n)/2 + \sum_{1}^{2i} C_{2k} f^{(2k-1)}(n)$$

for $i=0,1,2,\ldots$.

Proof. The equation is merely another way to write equation (3.1). Since $C_{2m}>0$ if m is odd and $C_{2m}<0$ if m is even, the inequality follows immediately. ■

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RECENT APPLICATIONS OF SOME OLD WORK OF LAGUERRE

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1. Introduction. Edmond Laguerre (1834–1886) is rightly considered as one of the foremost mathematicians of his time. He was a forerunner of Hadamard in the study of entire functions; the "Laguerre polynomials" are an important tool in several branches of pure and of applied mathematics, and Laguerre is also often quoted for his contributions to geometry ("theory of cycles"), algebraic equations, and continued fractions.

Nevertheless, he rates only four half-lines in the 1972 edition of the *Petit Larousse* **[15]**, and his name is not even mentioned in such excellent surveys as **[8]**, **[18]**, **[19]**, and **[14]**. To my surprise, not only does Laguerre not rate an entry, but he is not even mentioned under some other heading, in the *Encyclopaedia Britannica* (at least not in its 1954 edition) **[5]**. There is one brief mention of Laguerre in the four-volume *World of Mathematics* **[12]** in connection with the

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solution of a classical problem of tangent circles by the use of his "theory of cycles" and another brief mention in **[2]** in connection with three-dimensional analytical geometry.

Among Laguerre's numerous contributions to mathematics, is a theory of importance both to the theory of equations and to the study of the zeros of polynomials. It would be wrong to say that the theory is forgotten. In fact, G. Szegö **[20]** gives a sketchy proof of the main theorem, in view of some applications similar to those in the present paper; and a few years ago, Dočev used it to obtain an excellent bound for the absolute value of the zeros of Bessel Polynomials (see **[4]**). This theory, however, is not readily found in Laguerre's own papers. In order to obtain its more powerful results, the reader must combine several of Laguerre's papers (often very condensed notes published in the *Comptes Rendus* of the French Academy of Sciences), fortunately now collected in his two-volume *Oeuvres* **[10]**. Several very readable proofs of Laguerre's theorems can be found in the excellent monograph **[11]** by M. Marden. These are based mainly on considerations from mechanics (spherical and plane fields of forces, points of equilibrium, centers of mass, etc.). There exists also, however, a masterly presentation (quoted in **[11]**) of this material, together with some of its applications, in the unsurpassed work, *Aufgaben und Lehrsätze aus der Analysis* by Pólya and Szegö **[16**, vol. 2, Part 5, Chapter 2, Problems 101–120]. The reader who is willing to spend the time and effort needed to solve those 20 problems, and in this way to rediscover Laguerre's theory for himself under Pólya and Szegö's guidance, will profit greatly. His task has been made easier by the recent English translation **[16a]**. Such a reader is to be encouraged in his endeavor and need not continue to read the present article. Indeed, my purpose here is to make a leisurely, coherent presentation of Laguerre's theory, by following Pólya and Szegö's treatment to a large extent, for the benefit of those who choose a less arduous way to become acquainted with this beautiful work. Several applications, some of them recent, follow the theoretical part.

2. The center of mass. Let us consider n complex numbers $z_1, z_2, \ldots, z_n$, represent them in the complex plane, and assume that at each of these points there is a unit mass. Then the "center of mass" of the n unit masses is given by the formula

$$\zeta = \frac{1}{n}(z_1 + z_2 + \cdots + z_n).$$

It is an elementary exercise to verify that this definition has intrinsic meaning, i.e., that the position of ζ with respect to the given points does not depend on the location of the origin and the orientation of the axes. Indeed, if each $z_j (j = 1, 2, \ldots, n)$ is subject to the same translation, say $z_j \to z_j' = z_j + a$, then also $\zeta \to \zeta' = \zeta + a$. Similarly for a rotation, if $z_j \to z_j' = z_j e^{i\phi}$, then also $\zeta \to \zeta' = \zeta e^{i\phi}$. It also is clear that if $a \leqslant \operatorname{Re} z_j \leqslant b$, then also

$$a \leqslant \operatorname{Re} \zeta \leqslant b. \tag{1}$$

In fact, unless $\operatorname{Re} z_j = a$ or $\operatorname{Re} z_j = b$ for all z_j (which then would be collinear), one has strict inequalities in (1).

Let us now consider C, the smallest convex polygon that contains all the points z_j. By a rotation we may bring any side of C, say (see Fig. 1) $z_2 z_3$, into a vertical position, so that $\operatorname{Re} z_j \leqslant \operatorname{Re} z_2 = \operatorname{Re} z_3 = b$, say. Then, by (1), also $\operatorname{Re} \zeta \leqslant b$, with strict inequality, unless $\operatorname{Re} z_j = b$ for all $j = 1, 2, \ldots, n$. We may say that the line z_2, z_3 determines two half-planes, of which one does not contain any points z_j; then ζ belongs to the other half-plane. The same reasoning holds of course, for all sides of C, so that we conclude that ζ itself belongs to the intersection of those half-planes, i.e., to C. In fact, the strict inequalities in (1) show that ζ belongs to the interior of C, unless all z_j's are collinear (in which case no interior exists).

3. The generalized center of mass. We now proceed to generalize the concept of a "center of mass." The point (or complex number) ζ determined in section 2 will be said to be the "center of mass with respect to the point at infinity." When we want to stress this fact, we shall write ζ_∞. In

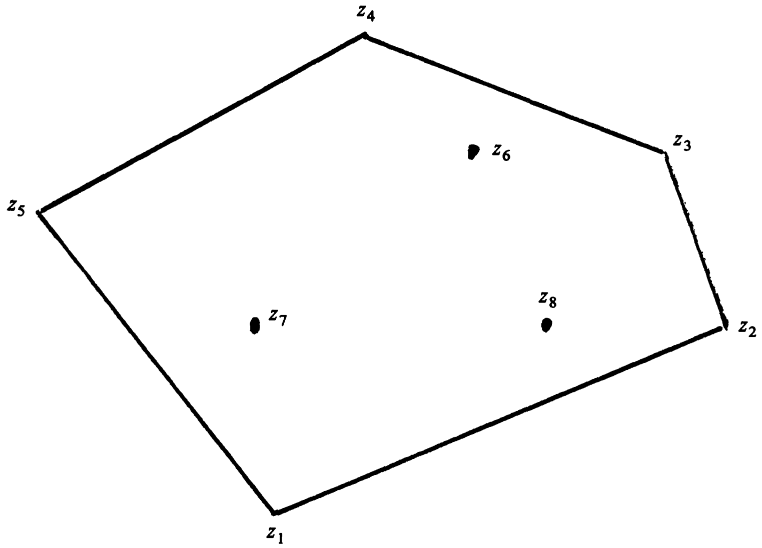

FIG. 1

other words, we define

$$\zeta = \zeta_\infty = \frac{1}{n}\sum_{j=1}^{n} z_j. \tag{2}$$

We now want to define a center of mass ζ of $z_1, z_2, \ldots, z_n$, with respect to an arbitrary point z_0 that we shall call (for want of a better name) the "pole." We do this by reducing the new problem to the old one. We first map the pole z_0 into the point at infinity by some appropriate linear fractional transformation, say

$$z \to z' = \frac{a}{z - z_0} + b. \tag{3}$$

Under (3), each z_j is mapped into

$$z_j' = \frac{a}{z_j - z_0} + b,$$

while $z_0 \to z_0' = \infty$. Next, we find the center of mass $\zeta' = \zeta_\infty'$ of the (z_j')'s with the pole $z_0' = \infty$ by (2), i.e.,

$$\zeta' = \frac{1}{n}\sum_{j=1}^{n} i_j'.$$

Finally, we map the whole configuration back, by the transformation inverse to (3). Under this inverse transformation, $z_j' \to z_j, z_0' = \infty \to z_0$ and ζ' is mapped into some point ζ_{z_0}, which we define as the center of mass of the points $z_j (j = 1, \ldots, n)$ with respect to the pole z_0.

While all the operations described are well defined, the question arises: does such a construction have any geometric meaning? Does ζ_{z_0} depend, as its name implies, only on the set $\{z_1, z_2, \ldots, z_n; z_0\}$? Indeed, we have used (3) in our construction, and (3) depends on the two arbitrary parameters a and b. Will not ζ_{z_0} also depend on our arbitrary choice of these parameters? Fortunately, ζ_{z_0} turns out to have an intrinsic meaning and is independent of the particular transformation (3) selected. Indeed, by (2) and (3),

$$\zeta' = \frac{1}{n}\sum_{j=1}^{n} z_j' = \frac{1}{n}\sum_{j=1}^{n}\left(\frac{a}{z_j - z_0} + b\right) = b + \frac{a}{n}\sum_{j=1}^{n}\frac{1}{z_j - z_0}.$$

On the other hand ζ' is the image of ζ_{z_0} under (3), so that

$$\zeta' = \frac{a}{\zeta_{z_0} - z_0} + b.$$

By equating the two expressions of ζ' we obtain

$$\frac{1}{\zeta_{z_0} - z_0} = \frac{1}{n} \sum_{j=1}^{n} \frac{1}{z_j - z_0},$$

or, solving for ζ_{z_0}:

$$\zeta_{z_0} = z_0 + n \left\{ \sum_{j=1}^{n} \frac{1}{z_j - z_0} \right\}^{-1}. \tag{4}$$

This explicit formula shows that ζ_{z_0} is indeed independent of a and b, as claimed.

4. A separation theorem. We consider now some properties of ζ_{z_0}. We recall that in section 2 $\zeta = \zeta_\infty$ was a point of C. We may think of C as follows: Consider all pairs of points $z_j, z_k (1 \leqslant j, k \leqslant n)$ and the straight line determined by them. This divides the plane into two half-planes. Sometimes both half-planes contain some of the given points (e.g., for z_5, z_7 in Fig. 1), but sometimes only one contains points (as in our first example, $z_2 z_3$, or, say, z_1, z_5 in Fig. 1). In the latter case, delete the half-plane without points (for z_1, z_5 one deletes the "southwesterly" half-plane below the infinite straight line through z_1 and z_5). The intersection of all remaining half-planes is precisely C. Finally, we observe that the straight line through z_j, z_k may also be considered as a generalized circle through z_j, z_k and the point at infinity (the "pole" in that construction).

Exactly the same considerations apply to $z_1', z_2', \ldots, z_n'$ with $\zeta' = \zeta_\infty'$, which lies inside the (ordinary) polygon C', the smallest convex polygon that contains all the (z_j')'s, while $z_0' = \infty$, the "pole," lies outside. When we map back, under the function inverse to (3), the straight lines that form the sides of C', say z_j', z_k', are transformed into circles through the pole z_0 (the image of $z_0' = \infty$) and the points z_j, z_k. The image of C' is, therefore, a curvilinear polygon, that we may also construct directly as follows: We take the points $z_1, z_2, \ldots, z_n$ two by two and construct a circle through the couple $z_j, z_k (1 \leqslant j, k \leqslant n)$ and the pole z_0. This circle divides the complex plane into two circular domains (its "inside" and its "outside"). It may happen that one of these two (open) domains contains no other points $z_j (j = 1, \ldots, n)$. In this case we "delete" it mentally. The intersection of the remaining circular domains is a closed curvilinear polygon, say $C = C_{z_0}$, the inverse image of C' under (3). Then C_{z_0} divides the complex plane into two regions (Jordan curve theorem!). In one of them we find z_0, the inverse image of $z_0' = \infty$; in the other, ζ_{z_0}, the inverse image of $\zeta' = \zeta_\infty'$. Indeed, C' separates $z_0' = \infty$ and $\zeta' = \zeta_\infty'$ and consequently its image C separates the corresponding images z_0 and ζ_{z_0}.

By dropping an unnecessary subscript, we may state the results obtained so far as a theorem.

THEOREM 1. *Let C_z be the curvilinear polygon defined above, corresponding to the points $z_1, z_2, \ldots, z_n$ and to the pole z. Then C_z separates z from the center of mass ζ_z of $z_1, z_2, \ldots, z_n$ relative to z.*

5. The Fundamental Theorem. Let $F(z) = \prod_{j=1}^{n} (z - z_j)$ be a polynomial and z an arbitrary point. Then the following theorem holds:

THEOREM 2. *The center of mass ζ_z of the zeros of $F(z)$, with respect to z is given by*

$$\zeta_z = z - nF(z)/F'(z). \tag{5}$$

Proof. Equation (5) follows immediately from (4) and

$$\frac{F'(z)}{F(z)} = \prod_{j=1}^{n} \frac{1}{z - z_j}.$$

THEOREM 3. *If the zeros* $z_j (j=1,2,\ldots,n)$ *of* $F(z)$ *belong to any circular domain* D (*i.e., either all are outside or all are inside some circle* Γ) *and if* z *is outside* D, *then* C_z *is in* D.

Proof. The smallest convex polygon C' containing all the (z_j')'s is inside any circle Γ' that contains all the (z_j')'s. Indeed, Γ' is convex and contains all (z_j')'s, while C' is the *smallest* convex set that contains all (z_j')'s. We also observe that $z'=\infty$ is outside Γ'. In particular, given the circle Γ that contains all or none of the z_j's, let Γ' be the image of Γ under (3). Then Γ' separates $z'=\infty$ from C'. Hence, when we map back, Γ separates C_z (the image of C') from z (the image of $z'=\infty$). Consequently, if z is not in the circular domain D, then C_z is in D and the proof is complete.

THEOREM 4. (Laguerre [**10**, vol. 1, pp. 161–166]). *Let* x *be a simple zero of the polynomial* $F(z)$ *of degree* n. *The center of mass of the remaining zeros with respect to* x *is*

$$X=X(x)=x-2(n-1)F'(x)/F''(x). \tag{6}$$

Proof. Let $F(z)=(z-x)f(z)$; then $F'(z)=f(z)+(z-x)f'(z)$ and $F''(z)=2f'(z)+(z-x)f''(z)$, so that $F'(x)=f(x)$ and $F''(x)=2f'(x)$. We now apply Theorem 2 to the polynomial $f(z)$ of degree $n-1$ and obtain (6), thus proving Theorem 4.

Now let Γ_1 be a circle through x that contains all the other zeros of $F(z)$. By assumption x is a simple zero; hence, one can deform Γ_1 slightly into a circle Γ that leaves x outside, but whose interior D still contains all other zeros. Let C_x be the curvilinear polygon of these other zeros of $F(z)$, with respect to x as a pole. By Theorem 3 we know that $C_x \subset D$, because x is outside D. Also, by Theorem 1, C_x separates x from $X(x)$; hence, $X(x)$ is inside C_x and, a fortiori, in D. This finishes the proof of the following fundamental theorem of Laguerre:

THEOREM 5. (Laguerre [**10**, vol. 1, pp. 161–166]). *Let* $f(z)$ *be a polynomial of degree* n *and define the function*

$$X(z)=z-2(n-1)f'(z)/f''(z). \tag{7}$$

Let z_1 be a simple zero of $f(z)$ and consider a circle C (possibly a straight line) through the point z_1 of the complex plane, and such that no zero of $f(z)$ belongs to one of the two open circular regions determined by C. Then $X(z_1)$ belongs to the other region.

6. The Corollaries. Theorem 5 may seem somewhat strange and rather complicated and one may wonder what kind of applications, if any, such a theorem could have. However, already the first corollary—an almost trivial inference from Theorem 5—may give a hint concerning the power of Theorem 5.

COROLLARY 1. *If* z_1 *is one of the zeros of largest modulus of a polynomial* $f(z)$, *then*

$$|X(z_1)| \leqslant |z_1|. \tag{8}$$

Proof. No zero belongs to the open circular domain $|z|>|z_1|$; hence $|X(z_1)|>|z_1|$ is ruled out by Theorem 5 and Corollary 1 follows.

We shall soon need

LEMMA 1. *Let* $f(z)$ *be a polynomial solution of the linear differential equation*

$$P(z)y''+Q(z)y'+R(z)y=0; \tag{9}$$

then, for each zero z_0 *of* $f(z)$,

$$\frac{f'(z_0)}{f''(z_0)}=-\frac{P(z_0)}{Q(z_0)}. \tag{10}$$

Proof. If we write $f(z)$ for y in (9) and then substitute z_0 for z so that $y(z_0)=0$, we obtain $P(z_0)f''+Q(z_0)f'(z_0)=0$; this proves (10). It follows that if the polynomial $f(z)$ satisfies a

differential equation (9), then we may replace the ratio $f'(z_1)/f''(z_1)$ in (7) by $-P(z_1)/Q(z_1)$. We formalize this conclusion as

COROLLARY 2. *Let us assume that the polynomial $f(z)$ of degree n is a solution of the differential equation* (9). *Let z_1 be any simple zero of $f(z)$ and let C be any circle* (*or straight line*) *through z_1, such that one of the two circular regions* (*or half planes*) *determined by C contains no zero of $f(z)$; then*

$$X(z_1)=z_1+2(n-1)P(z_1)/Q(z_1)$$

belongs to the other region.

7. Reality of roots of polynomials. As a first application of this theory, we shall find a condition that is both necessary and sufficient for the reality of all the zeros of a polynomial.

THEOREM 6. *Let $\zeta=\zeta_z$ be the center of mass of the zeros of the polynomial $f(z)$ with respect to the pole z. A necessary and sufficient condition for all the zeros of $f(z)$ to be real is that when z varies, z and ζ_z should have imaginary parts of opposite signs* (*unless z and ζ are both real; in the present context $z=\infty$ is to be considered real*).

In the proof of Theorem 6 we shall use the following lemma, interesting in its own right:

LEMMA 2. *Let the fixed points $z_1,z_2,\ldots,z_n$ be given and let z be a variable point. Then, as z approaches any of the points z_k, the point ζ_z also approaches z_k.*

Proof. The second term in (4) vanishes in the limit and the lemma follows.

Proof of Theorem 6. (a) Let all zeros of $f(z)$ be real, and assume that $\operatorname{Im} z=a>0$. Take the horizontal line $y=\varepsilon$, $0<\varepsilon<a$, as "circle" Γ in Theorem 3. Then ζ_z is inside D, i.e., $\operatorname{Im}\zeta_z<\varepsilon$. As ε is arbitrarily small, $\operatorname{Im}\zeta_z\leqslant 0$. However, the equality is not possible for $a>0$; otherwise, by a slight deformation of Γ we could obtain a circle Γ_1, such that all zeros z_j of $f(z)$ are in one circular domain, while both z and ζ_z are in the other (see Fig. 2), thus contradicting Theorem 3. It follows that if $\operatorname{Im} z>0$, then $\operatorname{Im}\zeta_z<0$. The proof for $\operatorname{Im} z<0$ is analogous. This proves the necessity of the condition.

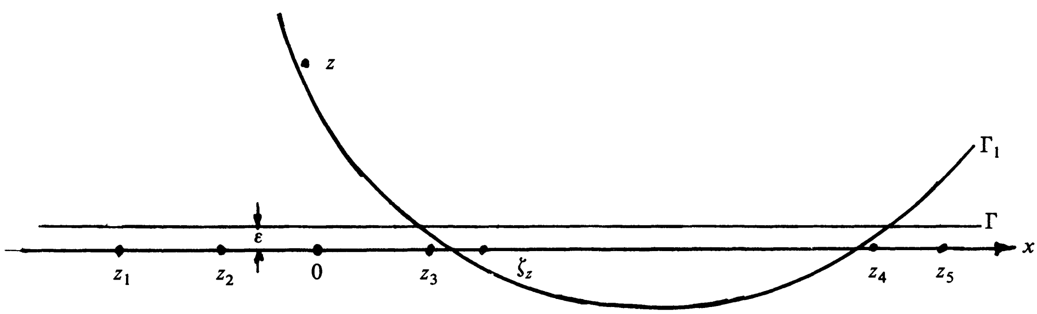

FIG. 2

(b) Assume that $\operatorname{Im} z\cdot\operatorname{Im}\zeta_z<0$ and that one of the zeros of $f(z)$, say z_1, has $\operatorname{Im} z_1\neq 0$. Then, for $z\to z_1$, it follows from Lemma 2 that also $\zeta_z\to z_1$; hence, the imaginary parts of z and of ζ_z are of the same sign, contrary to the assumption. This proves the sufficiency of the condition.

8. Zeros of classical orthogonal polynomials. Important particular cases of Theorem 6 are the classical orthogonal polynomials of Hermite, Laguerre and Legendre. They all satisfy differential equations of the form (9) and we recall here the corresponding values of $P(z), Q(z)$, and $R(z)$ (normalizations as in [17]).

	$P(z)$	$Q(z)$	$R(z)$
Hermite	1	$-2z$	$2n$
Laguerre	z	$1+\alpha-z$	n
Legendre	$1-z^2$	$-2z$	$n(n+1)$

(11)

In all these cases the function $R(z)$ is actually independent of z and depends only on an integral parameter n, which turns out to be also the exact degree of the polynomial solution of (9). We also recall the well-known

LEMMA 3 (see [**10**, vol. 1, pp. 133–143]). *All polynomial solutions of* (9) *have only simple zeros.*

Proof. If z_0 is a multiple zero of $y(z)$, then $y(z_0)=y'(z_0)=0$. Unless $P(z_0)=0$, this implies that $y''(z_0)=0$. By successive differentiations of (9) and induction on the order m of differentiation we obtain that $y^{(m)}(z_0)=0$ for all $m \geqslant 0$, so that $y(z)$ vanishes identically, which is not the case. If $P(z_0)=0$, then the equation obtained from (9) by differentiation, namely $P'y''+Py'''+Q'y'+Qy''+Ry'=0$ (recall: $R'\equiv 0$), reduces for $z=z_0$ to $P'(z_0)y''(z_0)$, whence again $y''(z_0)$ because $P(z)$ has no multiple zeros. The proof is again completed by induction on m.

THEOREM 7 (see [17]). *The classical orthogonal polynomials of Hermite, of Laguerre* (*of parameter* $\alpha > -1$) *and of Legendre have only simple, real zeros.*

Proof. (a) Let us assume that the Hermite polynomial $H_n(z)$ of degree n has complex zeros with, say, positive imaginary parts; then there is one of them, say $z_1=\gamma+i\delta$, with maximal imaginary part $\delta>0$. By Lemma 3, z_1 is a simple zero. It now follows from Theorem 3 that the center of mass ζ_{z_1} of the set of the other zeros of $H_n(z)$, with respect to the pole z_1, belongs to the "circular domain" $\operatorname{Im} z \leqslant \delta$. However, by (6), Lemma 1 and (11),

$$\zeta=z_1-2(n-1)H_n'(z_1)/H_n''(z_1)=z_1-\frac{n-1}{z_1}=z_1-\frac{(n-1)\overline{z_1}}{|z_1|^2}$$

and $\operatorname{Im}\zeta=\delta+\dfrac{(n-1)\delta}{\gamma^2+\delta^2}>\delta$, a contradiction. The case of $\delta<0$ is treated similarly.

(b) In the case of the Laguerre polynomials of real parameter $\alpha>-1$, the corresponding assumption of a zero $z_1=\gamma+i\delta$ with $\delta>0$ maximal, also leads, on account of Theorem 3, to $\operatorname{Im}\zeta_{z_1}\leqslant\delta$; however, by (6) Lemma 1, (10) and (11),

$$\begin{aligned}\zeta_{z_1}&=z_1-2(n-1)L_n(z_1,\alpha)/L_n''(z_1,\alpha)=z_1+2(n-1)z_1/\ (1+\alpha-z_1)\\&=z_1+2(n-1)z_1\big(1+\alpha-\overline{z_1}\big)|1+\alpha-z_1|^{-2}\\&=\gamma+i\delta+2(n-1)(\gamma+i\delta)(1+\alpha-\gamma+i\delta)|1+\alpha-z_1|^2\end{aligned}$$

and $\operatorname{Im}\zeta_{z_1}=\delta+2(n-1)\delta(1+\alpha)|1+\alpha-z_1|^2>\delta$, a contradiction.

(c) By the same reasoning we obtain for the Legendre polynomial $P_n(z)$, by assuming the existence of a zero $z_1=\gamma+i\delta$ with $\delta>0$, maximal, that on the one hand $\operatorname{Im}\zeta_{z_1}\leqslant 0$, while on the other hand

$$\begin{aligned}\zeta_{z_1}&=z_1-2(n-1)P_n'(z_1)/P_n''(z_1)=z_1-(n-1)(1-z_1^2)/z_1=nz_1-(n-1)/z_1\\&=nz_1-(n-1)\overline{z_1}\,|z_1|^{-2}=n(\gamma+i\delta)-(n-1)(\gamma-i\delta)(\gamma^2+\delta^2)^{-1}\end{aligned}$$

and

$$\operatorname{Im}\zeta_{z_1}=\delta\big(n+(n-1)(\gamma^2+\delta^2)^{-1}\big)>\delta,$$

a contradiction.

The reader is now invited to give the proof that, more generally, the Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$ with α,β real have only real zeros by using the fact that these generalizations of Legendre's polynomials (to which they reduce for $\alpha=\beta=0$) satisfy the differential equation (9)

with

$$P(z)=1-z^2,\quad Q(z)=\beta-\alpha-(2+\alpha+\beta)z,\quad R(z)=n(1+\alpha+\beta+n). \tag{12}$$

9. Bounds for the zeros of polynomials. In the present section we want to use Laguerre's theory in order to find bounds for the zeros of certain polynomials, including the classical orthogonal polynomials. G. Szegö (see [**20**, Chapter 6]) considers this problem in depth and studies it by several methods. One of them is precisely Laguerre's theorem, but Szegö's way of applying it is somewhat different from the present one. Next, he uses much more sophisticated methods and obtains correspondingly more precise results. It is instructive to compare the bounds obtained in the present section with those of, say, Theorem 6.32 of [**20**].

For our present purpose we recall that if z_1 is one of the zeros of largest modulus of a polynomial then, by (8), $|X(z_1)|\leqslant|z_1|$.

(a) As a first example we consider again Hermite's polynomials $H_n(z)$ and observe that, by Corollary 2

$$|z_1|>|z_1|\left|1+\frac{2(n-1)}{-2z_1^2}\right|,\ \text{or}\ |1-(n-1)z_1^{-2}|\leqslant 1,$$

where z_1 stands for one of the zeros of $H_n(z)$ of largest modulus. By Theorem 7, z_1 is real, so that $-1\leqslant 1-(n-1)z_1^{-2}\leqslant 1$, or $0\leqslant(n-1)z_1^{-2}\leqslant 2$, and, if $z_1>0$ (i.e., for the largest zero) we find that $z_1\geqslant\{(n+1)/2\}^{\frac{1}{2}}$. Similarly, for $z_1<0$ (i.e., for the smallest zero), we find that $z_1\leqslant-\{(n-1)/2\}^{\frac{1}{2}}$.

(b) For the largest zero z_1 of the Laguerre polynomial $L_n(z;\alpha)$ with $\alpha>-1$, Corollary 2 yields $|1+2(n-1)(1+\alpha-z_1)^{-1}|\leqslant 1$. By Theorem 7, z_1 is real, so that $-2\leqslant 2(n-1)(1+\alpha-z_1)^{-1}\leqslant 0$ and, successively, $0\leqslant(n-1)(z_1-\alpha-1)^{-1}\leqslant 1$, $z_1-\alpha-1\geqslant 0$, $z_1-\alpha-1\geqslant n-1$, and finally $z_1\geqslant n+\alpha$. If we also use the well-known fact (see [**20**], [**17**], [**7**], or [**1**]) that all zeros of $L_n(z;\alpha)$ are positive, then this result can be improved. We already know from Theorem 7 that z_1 is real; this with (7) and (10) shows that $X(z_1)$ is real. Let us assume that $X(z_1)\leqslant 0$; then for $\varepsilon>0$ sufficiently small a circle with diameter from $X(z_1)+\varepsilon$ to z_1 would contain all zeros of $L_n(z;\alpha)$, but $X(z_1)$ would be outside, thus contradicting Theorem 5. It follows that $X(z_1)>0$, so that $1+2(n-1)(1+\alpha-z_1)^{-1}>0$, whence $z_1>2n+\alpha-1$. In particular, for the "simple" Laguerre polynomial with $\alpha=0$, the largest zero z_1 satisfies the inequality $z_1>2n-1$.

(c) We now consider the case of the Jacobi polynomials $P_n^{(\alpha,\beta)}(z)$. We use (12) and, for simplicity of exposition, restrict ourselves to real values of α and β. Again z_1 will stand for a zero of largest modulus and we recall that, by Theorem 7 (as generalized by you), z_1 is real. By Corollary 2 we obtain that

$$\left|1+2(n-1)\frac{1-z_1^2}{z_1(\beta-\alpha)-(2+\alpha+\beta)z_1^2}\right|\leqslant 1.$$

By using the reality of the quantities involved, we obtain successively

$$-1\leqslant 1+2(n-1)\frac{1-z_1^2}{z_1(\beta-\alpha)-z_1^2(2+\alpha+\beta)}\leqslant 1,$$

$$-2\leqslant 2(n-1)\frac{1-z_1^2}{z_1(\beta-\alpha)-z_1^2(2+\alpha+\beta)}\leqslant 0,$$

and

$$0\leqslant(n-1)\frac{1-z_1^2}{z_1(\alpha-\beta)-z_1^2(2+\alpha+\beta)}\leqslant 1.$$

At this point we invoke the well-known fact (see, e.g., [**17**], [**20**], [**7**], or [**1**]) that all zeros of $P_n^{(\alpha,\beta)}(z)$ are not only simple and real, but also belong to the interval $(-1,+1)$, so that $1-z_1^2>0$.

We conclude that the last denominator is positive, so that z_1 satisfies the inequalities $0 \leqslant (n-1)(1-z_1^2) \leqslant (\alpha-\beta)z_1+(2+\alpha+\beta)z_1^2$, i.e., $n-1 \leqslant (n+1+\alpha+\beta)z_1^2+(\alpha-\beta)z_1$. If we solve the quadratic in the case $z_1>0$, we get

$$z_1 \geqslant \frac{\beta-\alpha+\sqrt{(\beta-\alpha)^2+4(n-1)(n+1+\alpha+\beta)}}{2(n+1+\alpha+\beta)}.$$

By an elementary computation this inequality can be written in the more revealing form

$$z_1 \geqslant \sqrt{\frac{n-1}{n+1+\alpha+\beta}}\sqrt{1+\frac{(\beta-\alpha)^2}{n^2-1+(n-1)(\alpha+\beta)}}+\frac{\beta-\alpha}{2(n+1+\alpha+\beta)}. \tag{13}$$

If we combine (13) with the already stated inequality $z_1<1$, we obtain a fairly sharp localization of the largest zero of the Jacobi polynomial $P_n^{(\alpha,\beta)}(z)$, especially if n is large.

(d) The particular case $\alpha=\beta$ is of special importance; the corresponding polynomials are known as ultraspherical polynomials and $Q(z)$ becomes $-2(1+\alpha)z$, while $P(z)=1-z^2$ remains unchanged. Formula (13) now simplifies drastically and the largest zero of $P_n^{(\alpha,\beta)}(z)$ satisfies

$$\sqrt{\frac{n-1}{n+1+2\alpha}} \leqslant z_1<1, \tag{14}$$

with

$$-1<z_n \leqslant -\sqrt{\frac{n-1}{n+1+2\alpha}}$$

for the smallest zero.

(e) By performing a change of parameter (a different normalization is also customary) we obtain the Gegenbauer polynomials $C_n^{(\nu)}(z)$. They are solutions of (9), with $P(z)=1-z^2$ and $Q(z)=-(2\nu+1)z$. For our purpose it is sufficient to observe that if α is the parameter of the ultraspherical polynomial, then the parameter ν of the Gegenbauer polynomial is $\nu=\alpha+1/2$. Hence (14) becomes

$$\sqrt{\frac{n-1}{n+2\nu}} \leqslant z_1<1, \tag{15}$$

as bounds for the largest zero of $C_n^{(\nu)}(z)$; similarly, the smallest zero satisfies $-1<z_n \leqslant -\sqrt{(n-1)/(n+2\nu)}$. The reader is encouraged to obtain this result directly from (8), without use of (13), or (14).

(f) The Jacobi polynomials with $\alpha=\beta=0$ are precisely the Legendre polynomials $P_n(z)$. From either (13), or (14), or from (15) with $\nu=1/2$, we obtain for the largest zero of $P_n(z)$ that

$$\sqrt{\frac{n-1}{n+1}} \leqslant z_1<1. \tag{16}$$

Direct use of Corollary 2 yields the same result via the inequality $|1-(n-1)(1-z_1^2)z_1^{-2}|<1$, which reduces to $|n-(n-1)z_1^{-2}| \leqslant 1$, whence (16) follows. The smallest zero z_n then satisfies $-1<z_n \leqslant -\sqrt{(n-1)/(n+1)}$.

(g) Also of interest are the Tchebycheff polynomials. We consider here only the polynomials $T_n(z)$, solutions of (9) with $P(z)=1-z^2$ and $Q(z)=-z$. They may be considered as Jacobi (or ultraspherical) polynomials with $\alpha=\beta=-1/2$. Either directly from Corollary 2, or from (14), we obtain for the largest zero z_1 the inequalities

$$\sqrt{1-1/n} \leqslant z_1<1,$$

while the smallest zero z_n satisfies $-1<z_n \leqslant -\sqrt{1-1/n}$.

Only little more care is required (see the end of Section 2) in order to show that in all these inequalities the equal sign may be omitted.

10. In this last section we shall consider the zeros of the Bessel polynomials $y_n(z;a,b)$, solutions of (9) with $P(z)=z^2, Q(z)=az+b$ and $R(z)=-n(n+a-1)$ (see, e.g., **[9]**, **[6]**, **[3]**, and **[17]**). The best available results on bounds valid for all Bessel polynomials were obtained by Dočev **[4]** by the present methods. Stronger results, but valid only asymptotically, for $n\to\infty$, were obtained by an entirely different method by Olver **[13]**.

It is not a serious restriction to assume, as we shall see, that b is real and positive; in fact, one may set $b=2$ (or any other positive constant) without much loss of generality.

By Corollary 2 we obtain for a zero z_1 of $y_n(z;a,b)$, of largest modulus, the inequality

$$\left|1+2(n-1)\frac{z_1}{az_1+a}\right|\leqslant 1. \tag{17}$$

We can no longer handle (17) by assuming that z_1 is real, because most of the zeros of $y_n(z;a,b)$ are not real. We, therefore, proceed as follows. Let $z_1^{-1}=v$; then (17) becomes

$$\left|1+\frac{2(n-1)}{a+bv}\right|\leqslant 1. \tag{18}$$

We first determine the locus of points v for which (18) holds with the equal sign. It is given by $1+2(n-1)/(a+bv)=e^{i\phi}(0\leqslant\phi<2\pi)$, so that, solving for v, an elementary computation leads to

$$v=-b^{-1}\left(a+n-1+i(n-1)\operatorname{cotg}\frac{\phi}{2}\right)=-b^{-1}\left\{n-1+\operatorname{Re}a+i\left(\operatorname{Im}a+(n-1)\operatorname{cotg}\frac{\phi}{2}\right)\right\}.$$

As ϕ varies from 0 to 2π, v describes the straight line L, parallel to the imaginary axis, of abscissa $-(n-1+\operatorname{Re}a)/b$. This permits us to determine the locus of $z=v^{-1}$, i.e., of the values of z_1 for which (17) holds with the equal sign; it is the inverse of L in the unit circle. This is a circle through the origin, with center on the real axis and passing through the (real) point $-b/(n-1+\operatorname{Re}a)$. Its equation is therefore

$$\left|z+\frac{b}{2(n-1+\operatorname{Re}a)}\right|=\left|\frac{b}{2(n-1)+\operatorname{Re}a}\right|. \tag{19}$$

For sufficiently small negative z and $b>0$, (17) obviously holds; hence the points that satisfy (17) are the points inside and on the circle (19). In particular, a zero z_1 of $y_n(z;a,b)$ of largest modulus satisfies

$$|z_1|\leqslant\frac{b}{n-1+\operatorname{Re}a}. \tag{20}$$

For $b=2$, in particular, (20) becomes $|z_1|\leqslant 2/(n-1+\operatorname{Re}a)$. The most interesting particular case (that of the "simple" Bessel polynomials, see **[9]**) is $a=b=2$, when (20) yields

$$|z_1|\leqslant\frac{2}{n+1}. \tag{21}$$

The readers who compare this result with **[4]** should keep in mind that Dočev considers only the case $b=-1$ and denotes $a-2$ by m; hence (19) leads him to

$$|z_1|\leqslant\frac{1}{n+1+\operatorname{Re}m}.$$

As already mentioned, Olver **[13]** obtains better bounds for the location of the zeros of $y_n^{(z)}=y_n(z;2,2)$, but his results hold only asymptotically, for large n. Specifically, he shows that, for $n\to\infty$,

$$|z_1|\cong v/n, \tag{22}$$

where $v \cong 1.50888\ldots$. However, for $n=1, y_1(z;2,2)=z+1$, so that $z_1=-1$. We see that Dočev's (21) yields the exact result, while Olver's bound (22) is too large. Even for $n=3$, (21) still yields $|z_1| \leqslant .5$, which is closer to the true value $z_1 \cong -.4305$ than the bound (22), which gives $.5029\ldots$. For larger n, however, Olver's result is the better one.

The number of applications of Laguerre's method can be increased almost indefinitely. It does not always lead to best possible results, but its power is considerable and it provides a unified approach to a variety of problems. I hope that my examples have illustrated both the power and the versatility of the method. The interested reader will find further applications by turning for inspiration to the original "Oeuvres" of Laguerre.

Theorems as powerful and aesthetically appealing as those of Laguerre were bound to excite the imaginations of many mathematicians. They obtained many generalizations and found numerous new applications of these theorems. Most of them can be found in **[11]** and this is not the place to quote all of them. It may be worthwhile, however, to call the attention of the reader to one of the most often quoted generalizations, known as Grace's theorem; it can be found in **[11]** as well as in **[20]**.

Note: After the present paper was completed, the article **[21]** appeared. In it Dočev's result is improved as follows: it is shown that, for $n+a-1>0$, the zeros of $y_n(z,a)$ are located not only inside the semi-circle $r=2/(n+a-1), |\theta-\pi|<\pi/2$, but in fact inside the cardioid $r=(1-\cos\theta)/(n+a-1), |\theta-\pi|<\pi/2$. The maximal absolute value of the zeros is not improved.

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NONSTANDARD SET THEORY

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Infinitely small and infinitely large quantities were systematically introduced into mathematics with the invention of calculus by Newton and Leibniz. The use of such quantities, however, was accompanied by logical contradictions, which mathematicians of the seventeenth and eighteenth centuries were unable to resolve. Although the method of infinitesimals generally yielded correct results, no one ever succeeded in formulating a precise, noncontradictory set of rules governing these objects; and infinitesimal quantities were gradually displaced (at least, in pure mathematics) by the familiar ε-δ calculus. A mathematically sound model of infinitely small and infinitely large objects became possible only after advances in mathematical logic in the twentieth century. Nonstandard Analysis, developed by A. Robinson in 1960, not only provided foundations for the calculus of infinitesimals in the classical spirit but also enabled mathematicians to use "nonstandard" objects in ways that could not be attempted on the basis of vague, intuitive understanding alone. Since then, interesting applications were found in various branches of mathematics, mathematical physics, and economics.

Robinson's exposition in **[10]** and its subsequent simplifications unfortunately involve the cumbersome apparatus of mathematical logic. Our aim here is to present methods of Nonstandard Analysis at a level of formalism customary in other branches of mathematics. We view nonstandard objects as ideal, imaginary elements adjoined to the universe of the standard mathematics and formulate a few simple and reasonably intuitive principles governing their behavior. We then show, on examples selected to illustrate a variety of nonstandard constructions, how nonstandard mathematics can be developed from these principles.

The basic framework for Nonstandard Analysis is presented in §§1–3; this system was introduced in [3], where its relative consistency with respect to the Zermelo-Fraenkel set theory is shown. We examine the real line and some concepts of general topology from our point of view in §§4–5; these results can be found in Robinson **[10]** and Luxemburg [7], **[8]**. Section 6 is devoted to nonstandard measure theory; our approach is basically that of Loeb **[6]** (except that we construct Loeb's extension in Theorem 3 of §6 directly, rather than using Carathéodory's Theorem), with some ideas coming from Anderson **[1]**. The final §7, part I, contains a more formal description of the logical foundations, and parts II and III discuss the relationship between our approach and the classical one based on higher-order nonstandard models, as well as some other axiomatizations of Nonstandard Analysis.

1. The standard universe. Our starting point is the universe of objects ordinarily studied by "standard" mathematicians—numbers, sets, geometric figures—together with the usual relations between such objects: order, set membership, etc. In order to distinguish these objects from other entities by which we enrich the mathematical universe, we qualify them by using the adjective *standard*. For example, 0, 1 and 17 are standard natural numbers, $\sqrt{2}$ and π are standard real numbers, R is the standard set of all real numbers, $\in$ and $<$ are standard relations, etc. Notation $\mathbb{S}(A)$ is sometimes used to express the fact that A is a standard object. A property or a statement is called *standard* if all quantified variables in it range over standard objects. Standard concepts (relations, operations, constants) are the ones defined by standard properties. They can then be used in other standard statements. (Here, as well as in the rest of the paper, we do not distinguish between properties and their descriptions in some formal

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language. So, "properties" are the same things as "formulas." Statements are properties (formulas) with no parameters (free variables). The reader desiring a more exact explanation of these matters might at this point begin to read part I of §7, and keep referring to it in §§2 and 3.)

We illustrate these ideas on some typical examples.

1. The relation of inclusion, $\subseteq$, is a standard concept defined for all standard sets A and B by the standard property,

For every standard x, $x \in A$ implies $x \in B$.

2. Continuity of a standard function $f: R \to R$ at a standard point $a \in R$ is a standard property of f and a; namely:

> For every standard $\varepsilon > 0$, $\varepsilon \in R$, there is a standard $\delta > 0$, $\delta \in R$, such that, for all standard $x \in R$, $|x-a| < \delta$ implies $|f(x)-f(a)| < \varepsilon$.

Indeed, we see that all quantified variables range over standard objects, and all concepts mentioned by this property (such as $>$, $\in$, 0, R, $|\cdot|$, $-$) are standard; either primitive ($\in$) or else understood to be defined, in their turn, by their usual standard properties.

We conclude with an observation that all theorems of standard mathematics are (presumably true) standard statements in our sense.

2. The internal universe. We now adopt a point of view that, besides the standard, "real" mathematical objects, there are also nonstandard, "ideal" objects, possessing formally the same properties as the standard ones. For a picturesque example, consider the set B of all birds; B is a standard set having eagles, storks, and sparrows among its standard elements. From the nonstandard standpoint, however, B has also fictitious elements, such as phoenixes. The nonstandard elements of B have all the properties of the standard ones: they are bipeds, females lay eggs, etc.

In concurrence with the established practice, the objects that are either standard or nonstandard are called *internal*. Notation $\mathcal{I}(A)$ means that A is an object from the universe of internal sets; so, of course, $\mathbb{S}(A)$ implies $\mathcal{I}(A)$, but not vice versa. Our first principle is intended to make precise the idea that nonstandard objects have formally the same properties as the standard ones.

We say that a property $\Phi^{\mathcal{I}}$ is an *internalization* of the standard property Φ if $\Phi^{\mathcal{I}}$ is obtained from Φ by replacing all quantified variables ranging (as they have to) over standard objects by variables ranging over internal objects.

PRINCIPLE OF EMBEDDING. *Standard objects $A, B, \ldots$ have a standard property Φ if and only if they have the internalized property $\Phi^{\mathcal{I}}$.*

To clarify this principle and its use, let us consider a few examples.

1. The standard relation of inclusion is defined by the statement:

> For all standard A and B, $A \subseteq B$ if and only if for all standard x, $x \in A$ implies $x \in B$. (1)

By the Principle of Embedding, this statement is equivalent to its internalization:

> For all internal A and B, $A \subseteq B$ if and only if for all internal x, $x \in A$ implies $x \in B$. (2)

We see from (2) that the standard relation $\subseteq$, originally defined only for pairs of standard sets, becomes, from the nonstandard viewpoint, applicable to all pairs of internal sets, and has the intended meaning. If, in particular, A and B are standard sets, then it follows from (1) and (2) that all standard elements of A belong to B if and only if all internal elements of A belong to B. In other words, the meaning of $A \subseteq B$ for standard A, B is the same, whether we look at A and B from the standard or from the nonstandard point of view; and we could use either (1) or (2) as a definition of $\subseteq$.

2. Let N be the standard set of natural numbers; we define N by the following property:

$$\begin{aligned}&\text{(i) } 0=\varnothing\in N;\\ &\text{(ii) for all standard } n, \text{ if } n\in N, \text{ then } n+1=n\cup\{n\}\in N;\\ &\text{(iii) if } N' \text{ is any standard set having properties (i) and (ii), then } N\subseteq N'.\end{aligned}\tag{3}$$

The internalization of (3) then asserts:

$$\begin{aligned}&\text{(i) } 0=\varnothing\in N;\\ &\text{(ii) for all internal } n, \text{ if } n\in N, \text{ then } n+1=n\cup\{n\}\in N;\\ &\text{(iii) if } N' \text{ is any internal set having properties (i) and (ii), then } N\subseteq N'.\end{aligned}\tag{4}$$

Briefly, the standard mathematician defines N as the smallest standard set containing zero and, with each standard element, also its successor. The Principle of Embedding shows that this same set N can also be defined internally as the smallest internal set containing zero and, with each internal element, also its successor. While (3) is a basis for proofs by standard induction, (4) can be used to justify proofs by internal induction (see the beginning of §3). Notice also that the operation of successor is well defined for all internal sets; again due to the Principle of Embedding: Since for every standard set n there is a unique standard set $n+1=n\cup\{n\}$, we can conclude that for every internal set n there is a unique internal set $n+1=n\cup\{n\}$. We also see that the standard arithmetic properties of natural numbers carry over into the internal universe. For example,

$$\text{For all standard natural numbers } n \text{ and } m,\ n+m=m+n.\tag{5}$$

is a standard theorem; therefore

$$\text{for all internal natural numbers } n \text{ and } m,\ n+m=m+n.\tag{6}$$

is an (internal) theorem. It is in this sense that the internal natural numbers (and internal objects in general) have the same properties as the corresponding standard ones.

3. Let us consider the standard definition of continuity from §1, Example 2. Continuity is a standard concept; by the Principle of Embedding, it is automatically applicable to internal functions and internal points:

> An internal function $f: R\to R$ is continuous at an internal point $a\in R$ if and only if for every internal $\varepsilon>0$, $\varepsilon\in R$, there is an internal $\delta>0$, $\delta\in R$, such that, for all internal $x\in R$, $|x-a|<\delta$ implies $|f(x)-f(a)|<\varepsilon$.

Moreover, for standard f and a the internal definition is equivalent to the standard one. Here, of course, the standard operations of absolute value and subtraction, the standard relations $\in$ and $<$, and other standard concepts, are applicable to internal elements, as in analogous instances in Examples 1 and 2. Similarly, the expression "an internal function f" is merely a shorthand for "f is some internal object which is a function," where "being a function" is a standard property, whose internalized definition reads: "An internal set of ordered pairs f is a function if, for all internal a, b_1, b_2, $(a,b_1)\in f$ and $(a,b_2)\in f$ imply $b_1=b_2$."

4. Let $\Phi(x,y)$ be a standard property. For any standard A and y, $\{\text{standard } x\in A\,|\,\Phi(x,y)\}$ denotes the standard set of all standard elements of A with the property Φ; that is, for all standard z,

$$z\in\{\text{standard } x\in A\,|\,\Phi(x,y)\} \text{ if and only if } z\in A \text{ and } \Phi(z,y).$$

The Principle of Embedding makes the standard operation $\{\text{standard } x\in A\,|\,\Phi(x,y)\}$ applicable to all internal A and y, with the intended result: for all internal z,

$$z\in\{\text{standard } x\in A\,|\,\Phi(x,y)\} \text{ if and only if } z\in A \text{ and } \Phi^{\mathfrak{I}}(z,y).$$

From now on, we always employ the more suggestive notation {internal $x \in A | \Phi^{\mathfrak{I}}(x,y)$} for this set, whenever either A or y is nonstandard.

These examples show how the Principle of Embedding automatically extends the scope of each standard concept into the internal universe, in such a way that the property which defines the concept over the standard universe (i.e., with variables ranging over the standard objects) also defines it over the internal universe (i.e., with variables ranging over the internal objects after internalization). There is also a useful reformulation of the Principle of Embedding from a somewhat different angle:

Let Φ be a standard property. If there is an internal object having the property $\Phi^{\mathfrak{I}}$, then there is a standard object having the property Φ.

Proof is simple. "There is an internal x such that $\Phi^{\mathfrak{I}}(x)$" is an internal statement, and is therefore equivalent to the standard statement of which it is an internalization: "There is a standard x such that $\Phi(x)$."■

For example, if we prove that a standard function f is continuous at some internal point z, we can conclude that f is continuous also at some standard point x.

Next we have to address the question of *existence* of nonstandard objects; it is easy to see that the Principle of Embedding alone does not imply existence of any. Our professed aim is to enrich the standard universe by all possible sorts of imaginary elements. As an example, we would like to add infinitesimals, that is, nonstandard real numbers ι such that $\iota \neq 0$, but $|\iota| < 1/n$ for every standard $n \in N$. This is a typical way of describing ideal elements; one formulates a list of properties, say, $\Phi(0,\iota)$, $\Phi(1,\iota)$, $\Phi(2,\iota), \ldots$, and requires the ideal element to satisfy all of them simultaneously. (In our example, $\Phi(0,\iota)$ is "$\iota \in R$ and $\iota \neq 0$" and $\Phi(n,\iota)$ is "$|\iota| < 1/n$," for standard $n \neq 0$.) Obviously, this will not always work; if we assume existence of ι satisfying simultaneously $\Phi(0,\iota)$ and $\Phi(1,\iota)$ where $\Phi(0,\iota)$ is "$\iota \in N$" and $\Phi(1,\iota)$ is "$\iota \notin N$", we get a contradiction. We can postulate existence of an object satisfying simultaneously a given list of properties only if these properties are mutually consistent. Guided by the principle that whatever can be consistently imagined exists (as an idea, though not necessarily as a "real," standard object), and, more pragmatically, by a desire to have as many ideal elements available as possible, we formulate

THE WEAK PRINCIPLE OF SATURATION. *Let Φ be a standard property. If A is a standard set, and if for every standard finite $a \subseteq A$ there is a standard y such that $\Phi(x,y)$ holds simultaneously for all standard $x \in a$, then there is an internal y such that $\Phi^{\mathfrak{I}}(x,y)$ holds simultaneously for all standard $x \in A$.*

As an example, we prove existence of infinitesimals. Let $\Phi(x,y)$ be the property "($x=0$ and $y \in R$ and $y \neq 0$) or ($x \neq 0$ and $|y| < 1/x$)," and let $A = N$ be the standard set of natural numbers. Then for every standard finite $a \subseteq N$ there is a standard y, namely, $1/(\max(a)+1)$, which satisfies Φ simultaneously for all standard $x \in a$. If y is the internal object provided by the Principle of Saturation, then $\Phi^{\mathfrak{I}}(x,y)$ holds for all standard $x \in N$, that is, $y \in R$, $y \neq 0$ and $|y| < 1/x$ for all standard $x \in N (x \neq 0)$. This is precisely what we require of infinitesimals.

The intuitive rationale for our formulation of the Principle of Saturation goes along the lines of the previous discussion. If we assume that, for any standard finite $a \subseteq A$, there is a standard y such that $\Phi(x,y)$ holds for all standard $x \in a$, then this y satisfies $\Phi^{\mathfrak{I}}(x,y)$ simultaneously for all standard $x \in a$ ($\Phi(x,y)$ holds if and only if $\Phi^{\mathfrak{I}}(x,y)$ holds, by the Principle of Embedding), and each finite subcollection of the properties $\Phi(x,y)$, $x \in A$, x standard, is consistent. Thus, intuitively, the whole collection is consistent (any proof of a contradiction can use only finitely many of the properties).

We now further illustrate the use of the Principle of Saturation by proving two simple theorems of Nonstandard Analysis.

THEOREM 1. *Every standard infinite set has nonstandard elements.*

Proof. Let A be standard and infinite. For every $a \subseteq A$, standard and finite, there is a standard $y \in A \setminus a$. Then "$y \in A$ and $y \neq x$" holds simultaneously for all standard $x \in a$. We conclude, on the basis of the Principle of Saturation, that there is an internal y such that "$y \in A$ and $y \neq x$" holds simultaneously for all standard $x \in A$. But this means that y is a nonstandard element of A.■

THEOREM 2. *For every standard set A there is an internal finite set $\alpha \subseteq A$ containing all standard elements of A.*

Proof. If $a \subseteq A$ is standard and finite, then there is a standard set y such that "$y \subseteq A$, y is finite, and $x \in y$" holds simultaneously for all standard $x \in a$ (let $y = a$). By the Principle of Saturation, there is an internal set α such that $\alpha \subseteq A$, α is finite, and $x \in \alpha$ holds for all standard $x \in A$.■

Theorem 2 may appear at first almost contradictory, as it seemingly asserts that a finite set can contain an infinite set. Let us examine it in some detail. Since α is an internal finite set, it has, from the point of view of an internal mathematician, all properties of finite sets. In particular, every internal subset of α is finite. Therefore, if A is infinite, then A is not a subset of α. All standard elements of A belong to α, but there must be nonstandard elements of A which do not belong to α. Also, α itself is then nonstandard (because standard sets having the same standard elements are equal). So far so good, but let us look now at the set $^\circ A$, whose elements are precisely the standard elements of A: $^\circ A = \{x \in A | \mathbb{S}(x)\}$. Obviously $^\circ A \subseteq \alpha$ and $^\circ A$ is infinite (if A is). The threat of a contradiction is resolved by realizing that existence of $^\circ A$ does not follow from the principles we accepted so far. (Notice that the property used to define $^\circ A$ is not internal—see §7, part I, for details.) In other words, *$^\circ A$ is not an internal set*! The above argument simply shows existence of entities outside of the internal universe. It should not be surprising that, for some *external* point of view, the nonstandard finite sets contain infinite subsets and, indeed, are themselves infinite.

Noninternal entities are useful for better understanding of internal objects, as a notational convenience, and as an essential tool in some constructions. We therefore devote the next paragraph to the principles of work with them.

3. The external universe. We begin with an example. The standard set of natural numbers N contains the standard elements $0, 1, 2, \ldots$, but, besides them, also some nonstandard elements (see Theorem 1 in §2). Let us now consider a collection $^\circ N$, whose elements are precisely the standard natural numbers; clearly $^\circ N \subseteq N$, but $^\circ N \neq N$. We see immediately that $^\circ N$ is not standard (if two standard sets have the same standard elements, they are equal). Moreover, $^\circ N$ is not even internal. This can be proved by internal induction, as presented in (4) of Example 2 in §2. It is clear that $0 \in {}^\circ N$. If an internal $n \in {}^\circ N$, then n is standard, and therefore $n+1 \in {}^\circ N$. If $^\circ N$ were internal, we would have $N \subseteq {}^\circ N$, a contradiction.

Another proof of this result may provide additional insight. Since $<$ is a standard, therefore internal, well-ordering of N, every nonempty internal subset of N has a $<$-least element. If $^\circ N$ were internal, $N \setminus {}^\circ N$ would be a nonempty internal subset of N; but $N \setminus {}^\circ N$ is the collection of all nonstandard natural numbers, and does not have a least element (if $n \in N \setminus {}^\circ N$, then $(n-1) \in N \setminus {}^\circ N$).

It is useful to be able to work with objects such as $^\circ N$. We therefore extend our universe of discourse further by adding to it also some noninternal objects. We call objects which are either internal or noninternal, *external* (here we differ from Robinson **[10]**, who uses "external" in the sense of our "noninternal"). One should visualize the external objects as subsets of the universe of internal objects (of course, some of them are already internal), sets of such subsets, sets of sets of such subsets, etc. (a cumulative process well known to set theorists). Thus all noninternal objects are sets, and the scope of the set membership relation naturally extends to them. Internal objects may be members of external sets, but noninternal objects cannot belong to internal sets

(the internal universe was formed prior to its extension by noninternal objects). The familiar intuitive arguments, used to motivate the axioms of the standard set theory, can be used here, too, to justify validity of

THE PRINCIPLES OF ZERMELO SET THEORY WITH THE AXIOM OF CHOICE

in the external universe. We do not state these axioms here (see [3]); for practical purposes, this means merely that all elementary properties of sets used in topology, measure theory, functional analysis, etc., are valid in our universe of external sets. As we point out in [3, Theorem 3], Fraenkel's Axiom Schema of Replacement is false in our universe, and one has to avoid constructions which depend on it. The only such constructions of possible interest to nonlogicians occur in the theory of ordinal and cardinal numbers, and extra care is needed there (for example, not every external well-ordering is isomorphic to an external ordinal). However, we know of no use for external infinite ordinals and cardinals in Nonstandard Analysis; since we can assume that all of the axioms of Zermelo-Fraenkel set theory with Choice are valid in the standard universe, there are no obstacles to the study of standard and internal ordinals and cardinals in the usual way.

An important question is the relationship between internal concepts and the corresponding external ones. It is no longer true, as in the case of standard vs. internal, that each internal concept automatically agrees with its external counterpart. Nevertheless, with little effort one can prove that it is so for most of the elementary set-theoretic relations and operations, and we give a few examples below. We use the usual notation for the internal concepts, and a superscript $\mathfrak{E}$ for the corresponding concepts defined in the external universe.

1. For any external sets A and B, $A \subseteq^{\mathfrak{E}} B$ means that for all external x, $x \in A$ implies $x \in B$. If now A and B are internal, then all elements of A and B are also internal, and $A \subseteq^{\mathfrak{E}} B$ is equivalent to $A \subseteq B$ (i.e., to the statement: for all internal x, $x \in A$ implies $x \in B$). The relation of inclusion defined in the external universe therefore agrees, on internal sets, with the previously defined relation of inclusion in the internal universe, and we can (and do) drop the superscript $\mathfrak{E}$ and write simply $A \subseteq B$, whether A and B are internal or not.

2. The same sort of argument shows that, for example,

$$A \,{}^{\mathfrak{E}}\!\cup B = A \cup B \qquad A \,{}^{\mathfrak{E}}\!\cap B = A \cap B$$

$$^{\mathfrak{E}}\{A,B\} = \{A,B\} \qquad A \,{}^{\mathfrak{E}}\!\times B = A \times B$$

f is a $\mathfrak{E}$-function if and only if f is a function

$<$ is a $\mathfrak{E}$-ordering if and only if $<$ is an ordering; $^{\mathfrak{E}}\bigcup_{\iota \in I} A_\iota = \bigcup_{\iota \in I} A_\iota$, etc.,

whenever A, B, f, $<$ and $\langle A_\iota | \iota \in I \rangle$ are internal sets. In all these and similar cases, we drop the unnecessary superscript $\mathfrak{E}$.

3. An example of a concept for which the two definitions diverge is the operation of power-set. If we define

$$X \in {}^{\mathfrak{E}}\mathcal{P}(A) \text{ if and only if } X \subseteq A \text{ and}$$

$$X \in \mathcal{P}(A) \text{ if and only if } X \text{ is internal and } X \subseteq A,$$

then obviously $\mathcal{P}(A) \subseteq {}^{\mathfrak{E}}\mathcal{P}(A)$; however, ${}^{\circ}N \in {}^{\mathfrak{E}}\mathcal{P}(N) \setminus \mathcal{P}(N)$, and so the two concepts do not agree even on standard sets.

4. If Φ is any property (not necessarily standard, or even internal) and A, y are sets, $\{x \in A | \Phi(x,y)\} = {}^{\mathfrak{E}}\{x \in A | \Phi(x,y)\}$ denotes the external set of all external $x \in A$ with the property Φ. We have to distinguish carefully between this notion and that of $\{\text{standard } x \in A | \Phi(x,y)\}$

defined (so far) only for standard Φ, A and y. For example, if $\Phi(x)$ is the standard property "for some standard n, $x \leqslant n$," $\{\text{standard } x \in N | \Phi(x)\} = \{\text{standard } x \in N | x \leqslant n \text{ for some standard } n\} = N$, while $\{x \in N | \Phi(x)\} = ^{\mathfrak{S}}\{x \in N | x \leqslant n \text{ for some standard } n\} = {}^{\circ}N$. The point here is that $\{\text{standard } \ldots\}$ is being evaluated in the standard (or, equivalently, internal) universe, where we do not separate the standard elements from the nonstandard ones, while $\{\ldots\}$ is evaluated in the external universe, where we do.

We further clarify and extend conventions dealing with the set abstraction symbol later in this paragraph. First, however, we need to introduce one other important principle (our last):

THE PRINCIPLE OF STANDARDIZATION. *For every external set A there is a standard set $^*A$ such that A and $^*A$ have the same standard elements.*

We first note that use of the notation $^*A$ is justified, because the set in question is uniquely determined by A: If $^{**}A$ is another standard set having the same standard elements as A, then $^*A$ and $^{**}A$ have the same standard elements, and so $^*A = {}^{**}A$.

The Principle of Standardization does not assert anything about the nonstandard elements of A; they may or may not belong to $^*A$, and $^*A$ may have nonstandard elements which do not belong to A. (However, no noninternal elements of A may belong to $^*A$, since $^*A$ is internal.) Obviously, $^*A = A$ if A is standard; on the other hand, $^*({}^{\circ}N) = N \neq {}^{\circ}N$. For any A, we define ${}^{\circ}A$ to be the external *set of all standard elements of* A; that is, ${}^{\circ}A = \{x \in A | x \text{ is standard}\}$. If A is standard, we have ${}^{\circ}A \subseteq A$ and $^*({}^{\circ}A) = A$.

Returning now to our discussion of the abstraction symbol, consider a set A and a property Φ. Then $B = \{x \in A | \Phi(x)\}$ is an external set, and $^*B$ is a standard set with the property that a *standard* x belongs to $^*B$ if and only if $x \in A$ and $\Phi(x)$. We denote $^*B$ by $\{\text{standard } x \in A | \Phi(x)\}$, rather than $^*\{x \in A | \Phi(x)\}$, and notice that for standard Φ and A, this agrees with our previous conventions.

In summary: Given any property Φ and any set A, $\{\text{standard } x \in A | \Phi(x)\}$ denotes a standard set whose standard elements are precisely those standard $x \in A$ having the property Φ. On the other hand, $\{x \in A | \Phi(x)\}$ is an external set, whose elements are precisely those external $x \in A$ having the property Φ. If Φ is an internalization of a standard property and A is internal, then $\{\text{internal } x \in A | \Phi(x)\}$ is the internal set of all internal $x \in A$ having the property Φ.

As a first example of the role played by the Principle of Standardization, we prove a theorem which can be used to justify proofs by induction for any statement about natural numbers (not only a standard one).

THEOREM 1. *If A is a set such that*
(i) $0 \in A$, *and*
(ii) *for all standard natural numbers n, if $n \in A$ then $(n+1) \in A$,*
then A contains all standard natural numbers (*i.e.*, ${}^{\circ}N \subseteq A$).

Notice that we cannot conclude that $N \subseteq A$, unless A is standard.

Proof. Let us assume that ${}^{\circ}N \setminus A \neq \varnothing$. Since all elements of ${}^{\circ}N \setminus A$ are standard, this implies that the standard set $^*({}^{\circ}N \setminus A) \neq \varnothing$. Every nonempty standard set of natural numbers has a standard least element. Since ${}^{\circ}N \setminus A$ and $^*({}^{\circ}N \setminus A)$ have the same standard elements, the standard least element, k, of $^*({}^{\circ}N \setminus A)$ is also the least element of ${}^{\circ}N \setminus A$. If $k = 0$, we have a contradiction with (i), and if $k = l + 1$, then $l \in A$ and we have a contradiction with (ii).■

The next theorem establishes a simple relationship between standard and external concepts of finiteness and natural number. In particular, it implies that all elements of a standard finite set are standard.

THEOREM 2. *An external set is standard finite if and only if it is external finite and all of its elements are standard. The standard natural numbers coincide with the external natural numbers. Standard algebraic operations on standard natural numbers coincide with the corresponding external operations.*

Proof. We proceed to prove that every standard finite set X is external finite and has only standard elements, by induction on the standard number of elements of X, $\|X\|$. The statement is clearly true if $\|X\|=0$; so assume that $\|X\|=n+1$ where n is standard, and the statement is true for all standard sets with n elements. We write $X=(X\setminus\{x\})\cup\{x\}$ where x is some standard element of $X(X\neq\varnothing)$. Since $\|X\setminus\{x\}\|=n$, $X\setminus\{x\}$ is external finite and all its elements are standard, by the inductive assumption. The set $\{x\}$ is also external finite, and its single element is standard. We conclude that X is external finite (the union of two finite sets is finite) and all elements of X are standard, thus proving the statement for all standard sets with $n+1$ elements. Theorem 1 then shows that it holds for all standard finite sets.

Conversely, let X be an external finite set, all elements of which are standard; we prove that X is a standard finite set by external induction on the external number of elements of X, ${}^{\mathfrak{E}}\|X\|$ (the Principle of Induction is of course a consequence of the axioms of Zermelo set theory, and thus holds in the external universe). The case ${}^{\mathfrak{E}}\|X\|={}^{\mathfrak{E}}0$ is again clear, so we proceed with the inductive step. If ${}^{\mathfrak{E}}\|X\|={}^{\mathfrak{E}}n\,{}^{\mathfrak{E}}+{}^{\mathfrak{E}}1$, we write $X=(X\,{}^{\mathfrak{E}}\setminus{}^{\mathfrak{E}}\{x\})\,{}^{\mathfrak{E}}\cup{}^{\mathfrak{E}}\{x\}=(X\setminus\{x\})\cup\{x\}$ (for some $x\in X$) and conclude that $X\setminus\{x\}$ is standard finite by the inductive assumption, and $\{x\}$ is standard finite, because x is standard. Therefore X is standard finite and we are done.

The proofs of the remaining claims are similar, and we omit them. Induction based on Theorem 1 should be used in one direction, and external induction in the other direction.■

Integers (rationals, respectively) are usually defined as certain ordered pairs of natural numbers (integers, respectively). Since external ordered pairs coincide with the standard ones for standard objects, it is not surprising that Theorem 2 holds for integers and rational numbers as well; the proof is trivial but tedious, and we again omit it. The relationship between standard and external real numbers is examined in §4.

We conclude with a restatement of the Principle of Saturation in a stronger form. In place of a standard set A of standard parameters, we allow any external set of internal parameters, as long as it is not too large. We say that an external set A has a *standard size*, if its elements can be enumerated by standard elements of some standard set; more precisely, if there is a standard set B and an external one-to-one mapping of ${}^\circ B$ onto A($\|B\|$ is then called the *size* of A).

THE STRONG PRINCIPLE OF SATURATION. *Let Φ be a standard property. If A is a set of standard size, all elements of which are internal, and if for every external finite $a\subseteq A$ there is an internal y such that $\Phi^{\mathfrak{s}}(x,y)$ holds simultaneously for all $x\in a$, then there is an internal y such that $\Phi^{\mathfrak{s}}(x,y)$ holds simultaneously for all $x\in A$.*

In most applications, one uses the following corollary:

THEOREM 3 (The Principle of Extension). *Let A be a standard set. If f is an external function defined on ${}^\circ A$ and with internal values, then there is an internal function $F\supseteq f$.*

Proof. (The graph of) f is a set of internal objects having a standard size, as witnessed by the mapping which assigns to each $x\in{}^\circ A$ the pair $(x,f(x))$. If $a\subseteq f$ is external finite, then a is (a graph of) a function, whose domain is an external finite set of standard elements, and all values of which are internal. It follows by Theorem 2, that dom(a) is a standard finite set, and then, by induction based on Theorem 1, that a is internal. Therefore, $y=a$ is an internal set such that "y is a function and $x\in y$" holds simultaneously for all $x\in a$. We conclude that there is an internal F for which "F is a function and $x\in F$" holds simultaneously for all $x\in f$. But then F is an internal function and $F\supseteq f$.■

This completes description of our universe of discourse. The next three paragraphs present samples of nonstandard arguments in our system. From now on, we adopt the following notational convention: Unless explicitly stated otherwise, lightface letters denote standard objects, Greek letters denote internal objects, and boldface letters stand for arbitrary (external) objects.

4. The real numbers. Let R be the standard set of real numbers; we know already that it has nonstandard elements (since it is infinite). Our first goal is to classify elements of R. We say that an internal real number ρ is *infinitely large* (or just *infinite*) if $|\rho| \geqslant n$ for every standard natural number n. Otherwise, it is called *finitely large* (or just *finite*). We say that ρ is *infinitely small* (or *infinitesimal*) if $|\rho| \leqslant 1/n$ for every standard natural number $n \neq 0$.

THEOREM 1. *Every nonstandard natural number is infinitely large.*

Proof. Let $\nu \in N \setminus {}^\circ N$ and $n \in {}^\circ N$; since $n = \{0, 1, \ldots, n-1\}$ is a standard finite set, all of its elements are standard (Theorem 2 in §3) and $\nu \notin \{0, 1, \ldots, n-1\}$. By linearity of $<$, $\nu \geqslant n$.■

We see that infinitely large real numbers exist. If ρ is infinitely large, i.e., $|\rho| \geqslant n$ for all $n \in {}^\circ N$, then $|1/\rho| < 1/n$ for all $n \in {}^\circ N$, and $1/\rho$ is infinitesimal. The only standard infinitesimal is 0; there are no standard infinitely large reals (Archimedean property). 1 and $1 + (1/\nu)$, for ν infinitely large, are examples of finite, but not infinitely small, real numbers.

For ρ, $\sigma \in R$, we write $\rho \approx \sigma$, and say that ρ is *infinitely close* to σ, if $\rho - \sigma$ is infinitesimal. Clearly, $\approx$ is an equivalence relation on R; it is good to know that $\approx$ is not internal.

Proof. Otherwise, $\mathbf{I} = \{\text{internal } \sigma \in R \mid \sigma \approx 0\}$ (the set of all infinitesimals) and then also $N \cap \{\text{internal } \rho \in R \mid \rho = 1/\sigma \text{ for some } \sigma \in \mathbf{I},\ \sigma \neq 0\}$ would be internal; the latter set, however, is exactly $N \setminus {}^\circ N$, which we proved not to be internal.■

The *monad* of $\rho \in R$, $\mathbf{M}(\rho)$, is defined as the set $\{\sigma \in R \mid \sigma \approx \rho\}$; again, $\mathbf{M}(\rho)$ is not internal for any $\rho \in R$ (otherwise, $\mathbf{I} = \mathbf{M}(0) = \{\sigma - \rho \mid \sigma \in \mathbf{M}(\rho)\}$ would be internal).

A very important tool in Nonstandard Analysis is the mapping st, which assigns to each finite real number a standard real number infinitely close to it.

THEOREM 2. *For each finitely large $\rho \in R$ there is a unique standard $r \in R$ such that $\rho \approx r$ (i.e., $\rho \in \mathbf{M}(r)$); we denote this r by* st (ρ) *and call it the standard part of ρ.*

Notice that dom(st) is the noninternal set of all finite reals, ran(st) is the noninternal set ${}^\circ R$ of all standard reals, and $\mathrm{st}^{-1}(r) = \mathbf{M}(r)$ for all $r \in {}^\circ R$. In particular, st is not internal.

Proof. Since ρ is finite, $\rho \leqslant n$ for some $n \in {}^\circ N$. The standard set $X = \{\text{standard } r \in R \mid r \leqslant \rho\}$ is bounded from above by n, and therefore has a supremum. Let $r = \sup X$; we prove that $r \approx \rho$. But if not, then there is $n \in {}^\circ N$ such that $|r - \rho| \geqslant 1/n$, i.e., either $r - (1/n) \geqslant \rho$, or $r + (1/n) \leqslant \rho$. Both possibilities contradict the definition of r. To prove uniqueness, we note that $r \approx \rho$ and $s \approx \rho$ imply $r \approx s$, that is, $(r - s)$ infinitesimal. Since 0 is the only standard infinitesimal, we have $r - s = 0$.■

Some simple properties of the function st are needed later.

THEOREM 3. *If A, B and $\langle A_i \mid i \in I \rangle$ are standard, then*

$$\mathrm{st}^{-1}\Big(\bigcup_{i = I} A_i\Big) = \bigcup_{i \in {}^\circ I} \mathrm{st}^{-1}(A_i);$$

$$\mathrm{st}^{-1}\Big(\bigcap_{i \in I} A_i\Big) = \bigcap_{i \in {}^\circ I} \mathrm{st}^{-1}(A_i);$$

$$\mathrm{st}^{-1}(A \setminus B) = \mathrm{st}^{-1}(A) \setminus \mathrm{st}^{-1}(B).$$

Proof. We show only $\bigcap_{i \in {}^\circ I} \mathrm{st}^{-1}(A_i) \subseteq \mathrm{st}^{-1}(\bigcap_{i \in I} A_i)$; the other cases are similar. If $\rho \in \bigcap_{i \in {}^\circ I} \mathrm{st}^{-1}(A_i)$, then $r = \mathrm{st}(\rho) \in A_i$ for all standard $i \in I$. Since r is standard, it follows that $r \in \bigcap_{i \in I} A_i$ and therefore $\rho \in \mathrm{st}^{-1}(\bigcap_{i \in I} A_i)$.■

The relationship between standard and external rational numbers is discussed in §3; we see there that x is a standard rational if and only if it is an external rational. However, from the standard point of view, the set of all rationals is $Q=\{\text{standard } x \mid x \text{ is a rational}\}$, while externally it is ${}^{\mathfrak{E}}Q={}^{\circ}Q=\{x \mid x \text{ is a standard rational}\}$. Because of this, the standard and the external real numbers do not exactly coincide; nevertheless, the relationship between the two concepts is very simple. To be specific, we identify real numbers with the lower classes of the corresponding Dedekind cuts; if then r is a standard real number, $r=\{\text{standard } q\in Q \mid q\leqslant r\}$, then ${}^{\circ}r=\{q\in r \mid q \text{ is standard}\}\subseteq {}^{\circ}Q$ is an external real number such that, for all standard rational q, $q\leqslant r$ if and only if $q^{\mathfrak{E}}\leqslant {}^{\circ}r$. Conversely, if $\mathbf{r}=\{\mathbf{q}\in {}^{\mathfrak{E}}Q \mid \mathbf{q}^{\mathfrak{E}}\leqslant \mathbf{r}\}$ is an external real number, then ${}^{*}\mathbf{r}\subseteq Q$ is a standard real number such that, for all standard rational q, $q^{\mathfrak{E}}\leqslant \mathbf{r}$ if and only if $q\leqslant {}^{*}\mathbf{r}$. The correspondence via $^{\circ}$ and * is easily seen to preserve algebraic operations and other algebraic and analytic properties of real numbers; for example, if A is a standard set of real numbers and $\mathbf{A}=\{{}^{\circ}r \mid r\in A\}$ is the corresponding set of external real numbers, ${}^{\circ}(\sup(A)) = {}^{\mathfrak{E}}\sup(\mathbf{A})$. Since we are not going to be concerned with properties of reals as sets (Dedekind cuts), but merely with their algebraic properties, we identify the standard reals (i.e., the elements of ${}^{\circ}R$) with the external reals (i.e., the elements of ${}^{\mathfrak{E}}R$), and ${}^{\circ}R$ (but, of course, not R) with ${}^{\mathfrak{E}}R$. This convention is notationally convenient in §6.

5. Some general topological theorems. In this paragraph, we look at several basic topological notions from the nonstandard point of view. Let T be a standard Hausdorff topological space. For each standard $t\in T$, we define the *monad* of t, $\mathbf{M}(t)$, as the set of all $\tau\in T$ which belong into every standard neighborhood of t; i.e., $\mathbf{M}(t)=\bigcap\{U \mid U \text{ is a standard neighborhood of } t\}$. We say that elements of $\mathbf{M}(t)$ are *infinitely close* to t, and write $\tau\approx t$, when $\tau\in\mathbf{M}(t)$. Notice that we do not define monads of, and infinite closeness for, nonstandard elements of T; this can be done only if the topology of T is determined by a uniformity. A point $\tau\in T$ is called *near-standard* if $\tau\in\mathbf{M}(t)$ for some standard $t\in T$. We note that t is uniquely determined in Hausdorff spaces ($t\neq t'$ implies existence of neighborhoods U, U' such that $t\in U$, $t'\in U'$ and $U\cap U'=\varnothing$; but $\mathbf{M}(t)\subseteq U$, $\mathbf{M}(t')\subseteq U'$); we denote this t by $\mathrm{st}(\tau)$. In this way, st is an external mapping of near-standard points of T onto the standard points of T; it is easy to check that it has all properties proved in §4 for the special case $T=R$.

We now prove several typical results.

THEOREM 1. *A standard sequence $\langle t_n \mid n\in N\rangle$ of elements of T converges to $t\in T$ if and only if $t_\nu\approx t$ for all infinitely large $\nu\in N$.*

Proof. (1) Assume that $\lim_{n\to\infty}t_n=t$. If U is any standard neighborhood of t, then $t_n\in U$ for all $n\geqslant n_U$ where $n_U\in {}^{\circ}N$; in particular, $t_\nu\in U$ for all infinitely large $\nu\in N$. Thus $t_\nu\approx t$ for any such ν.

(2) Assume, conversely, that $t_\nu\approx t$ for all infinitely large $\nu\in N$. Let U be a standard neighborhood of t; then there is an internal $\nu_U\in N$ such that $t_\nu\in U$ for all internal $\nu\geqslant\nu_U$ (let ν_U be any infinitely large integer). By the Principle of Embedding, there is a standard $n_U\in N$ such that $t_n\in U$ for all standard $n\geqslant n_U$, and we have $\lim_{n\to\infty}t_n=t$. ∎

THEOREM 2. *Let $f: T\to R$ be a standard function, and let $t\in T$ be standard. Then f is continuous at t if and only if for all $\tau\in T$, $\tau\approx t$ implies $f(\tau)\approx f(t)$ (in other words, $f[\mathbf{M}(t)]\subseteq\mathbf{M}(f(t))$).*

Proof. (1) Assume that for any standard $\varepsilon>0$ there is a standard neighborhood U of t such that, for all $x\in T$, $x\in U$ implies $|f(x)-f(t)|<\varepsilon$. If $\tau\approx t$, then $\tau\in U$ for all standard U, and hence $|f(\tau)-f(t)|<\varepsilon$ for all standard $\varepsilon>0$. We see that $f(\tau)\approx f(t)$.

(2) Assume, conversely, that $\tau\approx t$ implies $f(\tau)\approx f(t)$ for all $\tau\in T$. Working toward a contradiction, let $\varepsilon>0$ be standard and such that for any standard neighborhood U of t there is $x\in U$ for which $|f(x)-f(t)|\geqslant\varepsilon$. Then the standard property "$x\in U$ and $|f(x)-f(t)|\geqslant\varepsilon$" can be satisfied simultaneously for any standard finite set $\{U_1,\dots,U_n\}$ of neighborhoods of t (because $\bigcap_{1\leqslant i\leqslant n}U_i$

is also a neighborhood of t), and the Weak Principle of Saturation provides $\tau \in T$ satisfying "$\tau \in U$ and $|f(\tau)-f(t)| \geqslant \varepsilon$" simultaneously for all such standard U. But then $\tau \approx t$ and not $f(\tau) \approx f(t)$, contradicting the assumption.■

Yet another argument in a similar vein establishes

THEOREM 3. *A standard set $A \subseteq T$ is open if and only if $\mathbf{M}(t) \subseteq A$ for all standard $t \in A$. A is closed if and only if, for any standard $t \in T$, $\mathbf{M}(t) \cap A \neq \varnothing$ implies $t \in A$,*

It is a little harder to obtain a characterization of compactness.

THEOREM 4. *A standard closed set $A \subseteq T$ is compact if and only if all elements of A are near-standard.*

Proof. (1) Assume that A is compact. Let $\tau \in A$; we show that $\tau \approx t$ for some $t \in A$. If not, then each standard $t \in A$ has a standard neighborhood U_t such that $\tau \notin U_t$. The system $\{\text{standard } U_t | t \in A \text{ and } t \text{ is standard}\}$ is obviously a standard cover of A, and therefore has a standard finite subcover, say $\{U_{t_1}, \ldots, U_{t_n}\}$. This means that $A \subseteq \bigcup_{1 \leqslant i \leqslant n} U_{t_i}$ and implies $\tau \in \bigcup_{1 \leqslant i \leqslant n} U_{t_i}$. Since all elements of the standard finite set $\{1,2,\ldots,n\}$ are standard, we conclude that $\tau \in U_{t_i}$ for some standard $t_i \in A$, contradicting the choice of U_{t_i}.

(2) Assume, conversely, that for each $\tau \in A$ there is a standard $t \in T$ such that $t \approx \tau$; since A is closed, $t = \mathrm{st}(\tau) \in A$ by Theorem 3. Working towards a contradiction, we assume that $\langle U_i | i \in I \rangle$ is a standard cover of A by open sets such that, for each standard finite $I_0 \subseteq I$, there is $x \in A \setminus U_i$ for all $i \in I_0$. By the Weak Principle of Saturation, we then can find $\xi \in T$ such that $\xi \in A \setminus U_i$ for all standard $i \in I$. But $\xi \in A$ is near-standard and $A \setminus U_i$ is a closed set; so Theorem 3 implies $x = \mathrm{st}(\xi) \in A \setminus U_i$, for all standard $i \in I$. This is a contradiction with the assumption that $\langle U_i | i \in I \rangle$ is a standard cover of A.■

As a last example, we give a nonstandard proof of a standard theorem.

THEOREM 5. *An image of a compact set by a continuous function is compact.*

Proof. Let $f: T \to R$ be a standard continuous function, and let $A \subseteq T$ be a standard compact set. According to Theorems 3 and 4, it suffices to show that, for all $\tau \in A$, $f(\tau)$ is near-standard and $\mathrm{st}(f(\tau)) \in f[A]$. However, since A is compact, $\tau \in A$ implies $\tau \approx t \in A$. By the continuity of f, we then have $f(\tau) \approx f(t) \in f[A]$ (see Theorem 2), so $f(\tau)$ is indeed near-standard and $\mathrm{st}(f(\tau)) = f(t) \in f[A]$.■

6. Topics in nonstandard analysis. Good expositions of the nonstandard approach to Calculus abound in the published literature; we recommend, in particular, Keisler [5]. Also, no methods essentially different from those illustrated by the previous examples are called for. For these reasons we merely hint at the flavor of the subject by formulating a definition of the derivative, and then devote the rest of this section to nonstandard measure theory. This topic is technically much more interesting, because of its heavy use of external concepts.

Let $\alpha, \beta, \iota \in R$, $\iota \neq 0$. We say that α is *infinitely close to β relative to ι* (notation: $\alpha \approx \beta(\iota)$) if $(\alpha - \beta)/\iota$ is infinitesimal. In particular then $\alpha \approx \beta$ if and only if $\alpha \approx \beta$ (1). Let f be a function from an interval I into R, and let $a \in I$. (All lightface sets are standard.) We say that $A \in R$ is the *derivative* of f at a if $\Delta f = f(a+\iota) - f(a)$ is infinitely close to $df = A \cdot \iota$ relative to ι, for all infinitesimal $\iota \neq 0$. We then set

$$f'(a) = A = \mathrm{st}\left(\frac{f(a+\iota)-f(a)}{\iota}\right).$$

A simple proof in the spirit of §5 shows that this definition agrees with the usual notion of derivative.

We now turn to nonstandard measure theory. The basic idea of the nonstandard approach, implicit already in Robinson **[11]**, and developed explicitly in various different ways by Bernstein, Wattenberg, Henson, Loeb, Anderson, and others (see **[4]** for references), is to approximate standard measures "infinitely closely" by internal measures on internal *finite* sets. Measures on finite sets are very simple objects: Each internal measure μ on an internal finite set Ω is completely determined by its point-mass function $\phi_\mu : \Omega \to R$ where $\phi_\mu(\omega) = \mu(\{\omega\})$ for all $\omega \in \Omega$. Conversely, every internal $\phi : \Omega \to R$ determines a unique measure μ_ϕ on (the algebra of all internal subsets of) Ω by $\mu_\phi(X) = \Sigma_{\omega \in X} \phi(\omega)$, for all internal $X \subseteq \Omega$. The simplest and most important example of a measure on a finite set Ω is the *counting measure* μ_c defined by $\mu_c(X) = \|X\|/\|\Omega\|$ for all internal $X \subseteq \Omega$; then of course $\phi_{\mu_c}(\omega) = 1/\|\Omega\|$ for all $\omega \in \Omega$. (Here, as in §3, $\|X\|$ denotes the internal number of elements of the internal finite set X.) The approximation procedure then makes it possible to substitute, for example, the counting measure on an internal finite set for the standard Lebesgue measure.

We say that (X, S, m) is a *measure space* if S is an algebra of subsets of X and m is a finitely additive nonnegative real-valued measure on S (we do not assume σ-completeness of S and σ-additivity of m unless explicitly specified). The following two theorems describe one way the previously mentioned approximation procedure can be realized.

THEOREM 1. *For every standard measure space (X, S, m) there is an internal measure space $(\Omega, \mathcal{P}(\Omega), \mu)$ such that $^\circ X \subseteq \Omega \subseteq X$, Ω is finite, and $m(A) = \mu(A \cap \Omega)$ for all standard $A \in S$.*

Proof. For any standard finite subsets, $\{x_1, \ldots, x_m\}$ and $\{A_1, \ldots, A_n\}$, of X and S, respectively, there are a standard finite $\boldsymbol{\sigma} \subseteq A$ and a standard function $f : \boldsymbol{\sigma} \to R$ such that "$x_i \in \boldsymbol{\sigma}$ and $m(A_j) = \Sigma_{x \in A_j \cap \boldsymbol{\sigma}} f(x)$" holds for all $i = 1, \ldots, m$ and $j = 1, \ldots, n$.

To see this, we consider the partition of X into 2^n sets of the form $\tilde{A}_1 \cap \cdots \cap \tilde{A}_n$, where each $\tilde{A}_j$ is either A_j or $X \setminus A_j$. Whenever $\tilde{A}_1 \cap \cdots \cap \tilde{A}_n \neq \varnothing$, we choose one standard $y \in \tilde{A}_1 \cap \cdots \cap \tilde{A}_n$, put y into $\boldsymbol{\sigma}$, and set $f(y) = m(\tilde{A}_1 \cap \cdots \cap \tilde{A}_n)$. If $x_i (i = 1, \ldots, m)$ is not one of the y's chosen, we put it into $\boldsymbol{\sigma}$ also, and then set $f(x_i) = 0$. It is easy to check that $\boldsymbol{\sigma}$ and f work.

An application of the Weak Principle of Saturation now yields an internal finite $\Omega \subseteq A$ and an internal function $\phi : \Omega \to R$ such that $x \in \Omega$ holds for all $x \in {}^\circ A$ and $m(A) = \Sigma_{\xi \in A \cap \Omega} \phi(\xi)$ holds for all standard $A \in S$. The measure μ_ϕ determined on $\mathcal{P}(\Omega)$ by the point-mass function ϕ has the required properties.■

THEOREM 2. *For every internal measure space $(\Omega, \mathcal{P}(\Omega), \mu)$ where $\mu(\Omega)$ is finitely large, and $^\circ X \subseteq \Omega \subseteq X$, there is a standard measure space $(X, \mathcal{P}(X), m)$ such that $m(A) \approx \mu(A \cap \Omega)$ for all standard $A \subseteq X$.*

Proof. The function $\mathbf{m}$ defined for standard $A \subseteq X$ by $\mathbf{m}(A) = \mathrm{st}(\mu(A \cap \Omega))$ is external. However, $m = {}^*\mathbf{m}$ is a standard set having exactly the same standard elements as $\mathbf{m}$; in particular, it is a function, and $m(A) = \mathbf{m}(A) \approx \mu(A \cap \Omega)$ for all $A \in \mathcal{P}(X)$. Finite additivity follows easily from the fact that the sum of a standard finite set of infinitesimals is infinitesimal (induction based on Theorem 1 in §3).■

It is easy to check, using approximations by simple functions, that Theorem 1 (Theorem 2, respectively) implies, for all standard bounded m-measurable (μ-measurable, respectively) functions $f : X \to R$ that $\int f\,dm \approx \int f \restriction \Omega\, d\mu$ where, for Ω internally finite, $\int f \restriction \Omega\, d\mu = \Sigma_{\omega \in \Omega} f(\omega)\phi_\mu(\omega)$. (As usual, a standard function $f : X \to R$ is *bounded* if $|f(x)| \leqslant K$ for some $K \in R$ and all $x \in X$.)

Another immediate corollary of the preceding theorems is the fact that every standard finitely additive measure on some algebra of subsets of A can be extended to a standard finitely additive measure on the standard algebra of all subsets of A. For example, starting with the measure m_1, defined on the algebra generated by left-closed, right-open subintervals of $[0, 1)$, and assigning to each interval its length, we use Theorem 1 to find its internal approximation μ, and then

Theorem 2 to find a standard approximation m_2 of μ, defined on all subsets of [0, 1); clearly, $m_1 \subseteq m_2$. Unfortunately, this approach does not seem to give us any clue as to which subsets of [0, 1) are Lebesgue measurable, or guarantee that m_2 is equal to the Lebesgue measure on such sets (or, for that matter, is even σ-additive).

The crucial new idea, introduced by Loeb in **[6]**, is to try to recover $m(A)$ from μ using $\mathrm{st}^{-1}(A) \cap \Omega$, rather than $A \cap \Omega$, as we do in Theorem 2. (To see the difference, note, e.g., that all nonzero infinitesimals belong to $\mathrm{st}^{-1}(N) \setminus N$, while all infinitely large natural numbers belong to $N \setminus \mathrm{st}^{-1}(N)$.) However, the set $\mathrm{st}^{-1}(A)$ is usually not internal, and we are thus faced with a problem of extending the measure μ to external sets. Loeb at this point relies on Carathéodory's Extension Theorem (see **[12]**); but it is instructive and not too difficult to proceed directly. This is what we now begin to do.

From now on, let (Ω, Σ, μ) be a fixed internal measure space where $\mu(\Omega)$ is finite. Ideally, we might want to extend μ itself to external $\mathbf{X} \subseteq \Omega$. A necessary condition on μ is that

$$\text{for all internal } A_1, A_2 \in \Sigma, \text{ if } A_1 \subseteq \mathbf{X} \subseteq A_2, \text{ then } \mu(A_1) \leqslant \mu(\mathbf{X}) \leqslant \mu(A_2). \qquad (*)$$

We could then define $\mu_-(\mathbf{X}) = {}^{\mathfrak{E}}\sup\{\mu(A) | A \in \Sigma \text{ and } A \subseteq \mathbf{X}\}$, $\mu^+(\mathbf{X}) = {}^{\mathfrak{E}}\inf\{\mu(A) | A \in \Sigma$ and $A \supseteq \mathbf{X}\}$, and say that $\mathbf{X}$ is measurable if $\mu_-(\mathbf{X}) = \mu^+(\mathbf{X})$. But, since not every external bounded set of internal real numbers has a supremum (consider ${}^\circ N$), the indicated ${}^{\mathfrak{E}}\sup$ and ${}^{\mathfrak{E}}\inf$ need not exist, and it would be necessary to externally complete the internal reals first. It appears that completions with reasonable algebraic properties automatically identify each finite $\alpha \in R$ with $\mathrm{st}(\alpha)$. Since we are ultimately interested in measures whose values are standard real numbers (infinitely close to the values of μ) anyway, it is simpler to take the standard part right away, and look for an external measure $\mathbf{m}$ satisfying

$$\text{for all internal } A_1, A_2 \in \Sigma, \text{ if } A_1 \subseteq \mathbf{X} \subseteq A_2, \text{ then } \mathrm{st}(\mu(A_1)) \leqslant \mathbf{m}(\mathbf{X}) \leqslant \mathrm{st}(\mu(A_2)). \qquad (**)$$

(We remind the reader that we identify external real numbers with the standard ones as in §4; so external measures have standard reals as values.)

These considerations motivate the following definition: For all external sets $\mathbf{X} \subseteq \Omega$,

$$\mathbf{m}_-(\mathbf{X}) = \sup\{\text{standard } r \in R \,|\, r = \mathrm{st}(\mu(A)) \text{ for some } A \in \Sigma \text{ such that } A \subseteq \mathbf{X}\},$$
$$\mathbf{m}^+(\mathbf{X}) = \inf\{\text{standard } r \in R \,|\, r = \mathrm{st}(\mu(A)) \text{ for some } A \in \Sigma \text{ such that } A \supseteq \mathbf{X}\}.$$

We let $\Sigma^* = \{\mathbf{X} \subseteq \Omega | \mathbf{m}_-(\mathbf{X}) = \mathbf{m}^+(\mathbf{X})\}$ and set $\mathbf{m}(\mathbf{X}) = \mathbf{m}_-(\mathbf{X}) = \mathbf{m}^+(\mathbf{X})$ for $\mathbf{X} \in \Sigma^*$.

THEOREM 3. *Σ^* is an external σ-algebra of subsets of Ω, $\Sigma^* \supseteq \Sigma$, $\mathbf{m}(A) \approx \mu(A)$ for $A \in \Sigma$, $\mathbf{m}$ is an external σ-additive measure on Σ^*, and Σ^* is complete with respect to $\mathbf{m}$. Moreover, Σ^* is the smallest external set with those properties, and $\mathbf{m}$ is uniquely determined on Σ^*.*

Proof. We first notice that $\mathbf{X} \in \Sigma^*$ if and only if for every standard $\varepsilon > 0$ there exist internal $A_1, A_2 \in \Sigma$ such that $A_1 \subseteq \mathbf{X} \subseteq A_2$ and $\mu(A_2 \setminus A_1) < \varepsilon$. It is then obvious that $\Sigma \subseteq \Sigma^*$, $\mathbf{m}(A) = \mathrm{st}(\mu(A))$ for $A \in \Sigma$, and Σ^* is closed under complements, and almost equally easy to show that it is closed under unions: If $\mathbf{X}_1, \mathbf{X}_2 \in \Sigma^*$ and $\varepsilon > 0$ is standard, choose $A_{11}, A_{12}, A_{21}, A_{22} \in \Sigma$ so that

$$A_{11} \subseteq \mathbf{X}_1 \subseteq A_{12} \quad \text{and} \quad \mu(A_{12} \setminus A_{11}) < \frac{\varepsilon}{2},$$
$$A_{21} \subseteq \mathbf{X}_2 \subseteq A_{22} \quad \text{and} \quad \mu(A_{22} \setminus A_{21}) < \frac{\varepsilon}{2}.$$

Then $A_{11} \cup A_{21}$ and $A_{12} \cup A_{22}$ belong to Σ, $A_{11} \cup A_{21} \subseteq \mathbf{X}_1 \cup \mathbf{X}_2 \subseteq A_{12} \cup A_{22}$, and

$$\mu[(A_{12} \cup A_{22}) \setminus (A_{11} \cup A_{21})] \leqslant \mu(A_{12} \setminus A_{11}) + \mu(A_{22} \setminus A_{21}) < \varepsilon.$$

We next show that, for each external sequence $\langle \mathbf{X}_n | n \in {}^{\mathfrak{E}}N = {}^\circ N \rangle$ of pairwise disjoint elements of Σ^*, $\mathbf{X} = \bigcup_{n \in {}^\circ N} \mathbf{X}_n \in \Sigma^*$ and $\mathbf{m}(\mathbf{X}) = \sum_{n \in {}^\circ N} \mathbf{m}(\mathbf{X}_n)$.

For each $n\in{}^\circ N$, choose $A_{n1}, A_{n2}\in\Sigma$ such that $A_{n1}\subseteq \mathbf{X}_n\subseteq A_{n2}$ and $\mu(A_{n2}\setminus A_{n1})<\varepsilon/2^n$. The naive attempt of letting $A_1=\bigcup_{n\in{}^\circ N}A_{n1}$ and $A_2=\bigcup_{n\in{}^\circ N}A_{n2}$ fails, because A_1 and A_2 need not be internal. We need a more complicated argument, using the Strong Principle of Saturation. To begin with, sets $\langle A_{n1}|n\in{}^\circ N\rangle$ are pairwise disjoint, and therefore, for any $k\in{}^\circ N$, $\Sigma_{0<n<k}\mathbf{m}(A_{n1})=\mathrm{st}(\mu(\bigcup_{0<n<k}A_{n1}))\leqslant \mathrm{st}(\mu(\Omega))$. Since $\mu(\Omega)$ is finitely large, this implies that the external series $\Sigma_{n\in{}^\circ N}\mathbf{m}(A_{n1})$ converges. The inequalities

$$\begin{aligned}\sum_{n\in{}^\circ N}\mathbf{m}(\mathbf{X}_n)&\leqslant\sum_{n\in{}^\circ N}\mathrm{st}(\mu(A_{n2}))\\&=\sum_{n\in{}^\circ N}[\mathrm{st}(\mu(A_{n1}))+\mathrm{st}(\mu(A_{n2}\setminus A_{n1}))]\\&\leqslant\sum_{n\in{}^\circ N}\left[\mathbf{m}(A_{n1})+\frac{\varepsilon}{2^n}\right]\leqslant\mathbf{m}(\Omega)+2\varepsilon\end{aligned}$$

then show that $\Sigma_{n\in{}^\circ N}\mathbf{m}(\mathbf{X}_n)$ and $\Sigma_{n\in{}^\circ N}\mathbf{m}(A_{n2})$ also converge; we let $M=\Sigma_{n\in{}^\circ N}\mathbf{m}(\mathbf{X}_n)$. It then also follows from these inequalities that one can choose $k\in{}^\circ N$ so that

$$M=\sum_{n\in{}^\circ N}\mathbf{m}(\mathbf{X}_n)<\sum_{0<n<k}\mathbf{m}(A_{n1})+3\varepsilon\quad\text{and}\quad\sum_{k<n\in{}^\circ N}\mathbf{m}(A_{n2})<\varepsilon.$$

We let $A_1=\bigcup_{0<n<k}A_{n1}$ be the desired lower approximation to $\mathbf{X}=\bigcup_{n\in{}^\circ N}\mathbf{X}_n$; clearly, $A_1\subseteq\mathbf{X}$ and $A_1\in\Sigma$. To get the upper approximation, we prove the following result:

CLAIM. *There is an internal finite sequence* $\langle A_{\nu 2}|\nu\leqslant\nu_0\rangle$ $(\nu_0\in N\setminus{}^\circ N)$ *of elements of* Σ *extending* $\langle A_{n2}|n\in{}^\circ N\rangle$ *and such that* $\Sigma_{k<\nu\leqslant\nu_0}\mu(A_{\nu 2})\leqslant\varepsilon$.

We then let $A_2=\bigcup_{\nu<\nu_0}A_{\nu 2}$; $A_2\in\Sigma$, because Σ is an internal algebra and is thus closed under internal finite unions, and $\mathbf{X}_n\subseteq A_{n2}$ for all $n\in{}^\circ N$, so that $\mathbf{X}=\bigcup_{n\in{}^\circ N}\mathbf{X}_n\subseteq\bigcup_{n\in{}^\circ N}A_{n2}\subseteq\bigcup_{\nu<\nu_0}A_{\nu 2}=A_2$. Moreover, $\mu(A_2\setminus A_1)\leqslant\mu[\bigcup_{0<n\leqslant k}\mu(A_{n2}\setminus A_{n1})\cup\bigcup_{k<\nu<\nu_0}A_{\nu 2}]\leqslant\Sigma_{0<n\leqslant k}\mu(A_{n2}\setminus A_{n1})+\Sigma_{k<\nu\leqslant\nu_0}\mu(A_{\nu 2})\leqslant(\Sigma_{0<n\leqslant k}(\varepsilon/2^n))+\varepsilon<3\varepsilon$. Since ε is an arbitrary standard positive number, this shows $\mathbf{X}\in\Sigma^*$. Furthermore,

$$\begin{aligned}\sum_{n\in{}^\circ N}\mathbf{m}(\mathbf{X}_n)-3\varepsilon&<\sum_{0<n\leqslant k}\mathbf{m}(A_{n1})\leqslant\mathbf{m}(A_1)\leqslant\mathbf{m}(\mathbf{X})\leqslant\mathbf{m}(A_2)\\&\leqslant\mathbf{m}(A_1)+\mathbf{m}(A_2\setminus A_1)\leqslant\sum_{n\in{}^\circ N}\mathbf{m}(\mathbf{X}_n)+3\varepsilon,\end{aligned}$$

being true for all standard $\varepsilon>0$, implies

$$\mathbf{m}\left(\bigcup_{n\in{}^\circ N}\mathbf{X}_n\right)=\sum_{n\in{}^\circ N}\mathbf{m}(\mathbf{X}_n).$$

To prove the Claim, we first use the Principle of Extension (Theorem 3 of §3) to find an internal function $\langle A_{\nu 2}|\nu\in\Delta\rangle$ such that ${}^\circ N\subseteq\Delta$ and $A_{\nu 2}=A_{n2}$ whenever $\nu=n\in{}^\circ N$. Let then $\Delta'=\{\text{internal }\nu\in\Delta|\nu\in N, A_{\nu' 2}\in\Sigma\text{ for all internal natural numbers }\nu'\leqslant\nu,\text{ and }\Sigma_{k<\nu'\leqslant\nu}\mu(A_{\nu' 2})\leqslant\varepsilon\}$. We note that Δ' is an internal set and that ${}^\circ N\subseteq\Delta'\subseteq N$. Since ${}^\circ N$ is not internal, there exists $\nu_0\in\Delta'\setminus{}^\circ N$; $\langle A_{\nu 2}|\nu\leqslant\nu_0\rangle$ then has all properties required by the Claim.

We now know that $\mathbf{m}$ is an external σ-additive measure on Σ^*. If $\mathbf{X}\in\Sigma^*$ and $\mathbf{m}(\mathbf{X})=0$, then $\inf\{\text{standard }r\in R|r=\mathrm{st}(\mu(A))\text{ for some }A\in\Sigma\text{ such that }A\supseteq\mathbf{X}\}=0$, so clearly $\inf\{\text{standard }r\in R|r=\mathrm{st}(\mu(A))\text{ for some }A\in\Sigma\text{ such that }A\supseteq\mathbf{Y}\}=0$ for any $\mathbf{Y}\subseteq\mathbf{X}$, showing $\mathbf{Y}\in\Sigma^*$ and $\mathbf{m}(\mathbf{Y})=0$. This means that Σ^* is complete with respect to $\mathbf{m}$.

To prove minimality of Σ^*, let Σ^{**} be another algebra with the properties stated in Theorem 3 (for Σ^*). If $\mathbf{X}\in\Sigma^*$, then, for each $n\in{}^\circ N$, there exist $A_{n1}, A_{n2}\in\Sigma$ such that $A_{n1}\subseteq\mathbf{X}\subseteq A_{n2}$ and $\mathbf{m}(A_{n2}\setminus A_{n1})<1/n$. Since $\Sigma\subseteq\Sigma^{**}$ and Σ^{**} is an external σ-algebra, $A_1=\bigcup_{n\in{}^\circ N}A_{n1}$ and $A_2=\bigcap_{n\in{}^\circ N}A_{n2}$ both belong to Σ^{**}; moreover, $A_1\subseteq\mathbf{X}\subseteq A_2$ and $\mathbf{m}(A_2\setminus A_1)\leqslant\mathbf{m}(A_{n2}\setminus A_{n1})<1/n$ for all standard n. But $\mathbf{m}(A_2\setminus A_1)$ is standard, so $\mathbf{m}(A_2\setminus A_1)=0$ and, by completeness of Σ^{**},

$\mathbf{X}\setminus A_1 \subseteq A_2 \setminus A_1$ belongs to Σ^{**}. Finally, $\mathbf{X}=(\mathbf{X}\setminus A_1)\cup A_1 \in \Sigma^{**}$. We see that $\Sigma^* \subseteq \Sigma^{**}$. In the process, we also showed that $\mathbf{m}(\mathbf{X})=\mathbf{m}(A_1)=\sup\{\text{standard } r\in R \mid r=\mathrm{st}(\mu(A_{n1})) \text{ for some } n\in {}^\circ N\}$ is uniquely determined.■

Again, the usual procedure of approximating by simple functions gives the following characterization of measurability and integral (which can be used as a definition of these concepts even prior to introduction of the notion of measure).

THEOREM 4. *An external bounded function* $\mathbf{f}:\Omega\to R$ *is* $\mathbf{m}$-*measurable if and only if*

$$\sup\left\{\text{standard } r\in R \mid r=\mathrm{st}\left(\sum_{\omega\in\Omega}\phi(\omega)\right) \textit{for some internal } \phi \textit{ such that } \phi(\omega)\leqslant \mathbf{f}(\omega) \textit{ for all } \omega\in\Omega\right\}=$$

$$\inf\left\{\text{standard } r\in R \mid r=\mathrm{st}\left(\sum_{\omega\in\Omega}\phi(\omega)\right) \textit{for some internal } \phi \textit{ such that } \phi(\omega)\geqslant \mathbf{f}(\omega) \textit{ for all } \omega\in\Omega\right\}.$$

If $\mathbf{f}$ *is* $\mathbf{m}$-*measurable, the common value is equal to* $\int \mathbf{f}\,d\mathbf{m}$. *All internal bounded functions* $\mathbf{f}$ *are* $\mathbf{m}$-*measurable and* $\int \mathbf{f}\,d\mathbf{m} - \mathrm{st}(\int \mathbf{f}\,d\mu)$ *for such* $\mathbf{f}$.

Although we restrict ourselves to finite measures for the sake of simplicity, conventional methods can be used to extend the construction of $\mathbf{m}$ to measure spaces where $\mu(\Omega)$ is not necessarily finite. One defines first a set Σ' by letting $\mathbf{X}\in\Sigma'$ if and only if $\mathbf{X}\subseteq A$ for some internal $A\in\Sigma$ such that $\mu(A)$ is finite, and $\sup\{\text{standard } r\in R \mid r=\mathrm{st}(\mu(A)) \text{ for } A\in\Sigma \text{ such that } A\subseteq\mathbf{X}\}=\inf\{\text{standard } r\in R\mid r=\mathrm{st}(\mu(A)) \text{ for } A\in\Sigma \text{ such that } A\supseteq\mathbf{X}\}$. One then lets $\mathbf{X}\in\Sigma^*$ if and only if $\mathbf{X}\cap A\in\Sigma'$ for all $A\in\Sigma$ such that $\mu(A)$ is finite, and sets

$$\mathbf{m}(\mathbf{X})=\sup\{\text{standard } r\in R\mid r=\mathrm{st}(\mu(\mathbf{X}\cap A)) \text{ for } A\in\Sigma \text{ such that } \mu(A) \text{ is finite}\}, \tag{+}$$

where it is understood that $\mathbf{m}(\mathbf{X})=+\infty$ if the set in (+) is unbounded. Arguments similar to those used in proof of Theorem 3 then show that Σ^* has all properties listed in Theorem 3, except perhaps minimality.

Our stated purpose is to obtain simple nonstandard "infinitely close" approximations of standard measures. We now return to this question and show how it can be done, using Theorem 3, in case of the Lebesgue measure on the interval [0, 1).

Let ν be an infinitely large natural number, and let $\Omega=\{\text{internal } \rho\in R\mid \rho=\alpha/\nu \text{ for some } \alpha\in N \text{ such that } 0\leqslant\alpha<\nu\}$. Σ is the algebra of all internal subsets of Ω, and μ is the counting measure on Ω, that is, $\mu(A)=\|A\cap\Omega\|/\nu$ for all $A\in\Sigma$. Theorem 3 supplies an external σ-algebra $\Sigma^*\supseteq\Sigma$ and an external σ-additive measure $\mathbf{m}$ on Σ^*. We then let $S=\{\text{standard } X\subseteq[0,1)\mid \mathrm{st}^{-1}(X)\cap\Omega\in\Sigma^*\}$ and define a standard function n on S by the requirement that $n(X)=\mathbf{m}(\mathrm{st}^{-1}(X)\cap\Omega)$ for all $X\in S$ (existence of such n follows from the Principle of Standardization). The simple properties of st listed in Theorem 3 of §4 immediately show that S is a standard σ-algebra of subsets of [0, 1), n is a standard σ-additive measure on S, and S is complete with respect to n. We also have $n([0,x))=x$ for all standard x, $0<x<1$. Indeed, we first see that

$$\mathbf{m}([0,x)\cap\Omega)=\mathrm{st}(\mu([0,x)\cap\Omega))=\mathrm{st}\left(\frac{[x\nu]-1}{\nu}\right)=x,$$

because $([x\nu]-1)/\nu<x\leqslant[x\nu]/\nu$ and $1/\nu$ is infinitesimal. (Here $[x\nu]$ is the largest integer $\leqslant x\nu$.) Next, for any $x'<x$, $[0,x')\cap\Omega\subseteq\mathrm{st}^{-1}([0,x))\cap\Omega\subseteq[0,x)\cap\Omega$, so that $x'\leqslant n([0,x))\leqslant x$ for any $x'<x$,

and thus $n([0,x))=x$. Therefore, S contains all Lebesgue measurable sets and n agrees with the Lebesgue measure on such sets. It is also evident that n is translation-invariant. Actually, it is possible to prove that S contains exactly all Lebesgue measurable sets (this result is due to Henson). Furthermore, a standard function $f:[0,1)\to R$ is n-measurable or n-integrable if and only if $(f\circ \mathrm{st})\restriction\Omega$ is $\mathbf{m}$-measurable or $\mathbf{m}$-integrable, and $\int f dn = \int (f\circ \mathrm{st})\restriction\Omega d\mathbf{m}$ whenever either side exists. (This is again proved using approximations by simple functions.) It is also possible to view the intervals $[\alpha/\nu,(\alpha+1)/\nu)$, rather than the points α/ν, as the atoms on which the counting measure μ operates (this is the approach taken by Anderson in **[1]**). μ-measurable functions are then the internal step-functions defined on the partition of $[0,1)$ determined by α/ν for $0\leqslant\alpha<\nu$, $\alpha\in N$, and, for n-integrable $f:[0,1)\to R$, $\int f dn=\sup\{\text{standard } r\in R\,|\,r=\mathrm{st}(\int \phi d\mu)\}=\inf\{\text{standard } r\in R\,|\,r=\mathrm{st}(\int \psi d\mu)\}$ where ϕ and ψ are internal step-functions approximating $f\circ\mathrm{st}$ from below and above, respectively.

We refer the reader to **[6]** and **[1]** for further results, as well as for applications of nonstandard measure theory to the theory of probability.

7. Final remarks. I. *Logical foundations for Nonstandard Set Theory*. In this section we describe the logical foundations for Nonstandard Set Theory somewhat more rigorously. We select Zermelo-Fraenkel set theory with the Axiom of Choice (ZFC) as a foundation for ordinary, "standard" mathematics. The only objects considered in ZFC are sets (i.e., variables $x,y,\ldots,$ $A,B,\ldots$ of ZFC range over sets) and the only primitive concept is the membership relation denoted by the symbol $\in$. In addition, the language of set theory contains symbols for equality $(=)$, logical connectives "and" $(\wedge)$, "or" $(\vee)$, "not" $(\sim)$, "implies" $(\Rightarrow)$, and "if and only if" $(\Leftrightarrow)$, and quantifiers "for all" $(\forall)$ and "there exists" $(\exists)$. Formulas of ZFC are constructed from these symbols according to inductive syntactic rules:

(i) $(X\in Y)$ and $(X=Y)$ are formulas if X and Y are variables;

(ii) $(\Phi\wedge\Psi)$, $(\Phi\vee\Psi)$, $(\sim\Phi)$, $(\Phi\Rightarrow\Psi)$ and $(\Phi\Leftrightarrow\Psi)$ are formulas if Φ and Ψ are formulas;

(iii) $(\forall X)\Phi$ and $(\exists X)\Phi$ are formulas if Φ is a formula and X is a variable.

Only those statements about sets, and those descriptions of properties of sets, that are expressible by formulas, are allowed. Of course, for convenience one introduces other symbols for specific relations, constants, and operations, for example, the symbol $\subseteq$ for set inclusion. However, such symbols can be viewed as merely a shorthand for their defining formulas; e.g., $A\subseteq B$ is a shorthand for

$$(\forall x)(x\in A\Rightarrow x\in B). \tag{1}$$

One concludes the development of ZFC by specifying certain statements about sets as axioms. (Since the actual axioms are of little interest to us here, we will not state them.) All theorems about sets have to be provable from these axioms.

In Nonstandard Set Theory (NZFC) we have to deal with three sorts of objects: standard sets, internal sets, and external sets (the most encompassing sort). Variables of NZFC therefore range over the external sets; but, in addition to $\in$, we need two other primitive concepts: the relations of being standard $(\mathbb{S})$ and internal $(\mathcal{I})$. Formulas of NZFC are then constructed according to the rules (i′), (ii), and (iii), with

(i′): $(X\in Y),(X=Y),\mathbb{S}(X),\mathcal{I}(X)$ are formulas if X and Y are variables and (ii), (iii) as before.

Although any formula Φ of ZFC is also a formula of NZFC, its meaning in the two theories is, in general, different. The reason is that quantified variables range over all external sets in NZFC, and over all standard sets in ZFC. If we want to write down a formula $\Phi^{\mathbb{S}}$ of NZFC having the same meaning as a given formula Φ of ZFC, we have to *relativize* all quantifiers in Φ to $\mathbb{S}$, i.e., replace each $(\forall X)\cdots$ by $(\forall X)(\mathbb{S}(X)\Rightarrow\cdots)$ and each $(\exists X)\cdots$ by $(\exists X)\,(\mathbb{S}(X)\wedge\cdots)$. In full detail, the inductive rules for construction of the *standardization* $\Phi^{\mathbb{S}}$ of Φ are as follows:

(i*) $(X \in Y)^{\mathbb{S}}$ is $(X \in Y)$; $(X = Y)^{\mathbb{S}}$ is $(X = Y)$;
(ii*) $(\Phi \wedge \Psi)^{\mathbb{S}}$ is $(\Phi^{\mathbb{S}} \wedge \Psi^{\mathbb{S}})$, $(\sim\Phi)^{\mathbb{S}}$ is $(\sim\Phi^{\mathbb{S}})$, and similarly for other connectives;
(iii*) $((\forall X)\Phi)^{\mathbb{S}}$ is $(\forall X)\ (\mathbb{S}(X) \Rightarrow \Phi^{\mathbb{S}})$, more conveniently denoted $(\forall \text{ standard } X)\Phi^{\mathbb{S}}$, and $((\exists X)\Phi)^{\mathbb{S}}$ is $(\exists X)\ (\mathbb{S}(X) \wedge \Phi^{\mathbb{S}})$, denoted $(\exists \text{ standard } X)\Phi^{\mathbb{S}}$.

In an entirely analogous way we can construct, for any formula Φ of ZFC, its *internalization* $\Phi^{\mathcal{I}}$; the appropriate rules are obtained from (i*)–(iii*) by replacing $\mathbb{S}$ with $\mathcal{I}$ and "standard" with "internal."

Formulas of the form $\Phi^{\mathbb{S}}$ [$\Phi^{\mathcal{I}}$, respectively] where Φ is a formula of ZFC, are called *standard* [*internal*, respectively]. Notice that $\mathcal{I}$ [$\mathbb{S}$, respectively] does not occur in them at all, and $\mathbb{S}$ [$\mathcal{I}$, respectively] can occur only in the form specified by (iii*) (i.e., as a part of a relativized quantifier). If $\Psi = \Phi^{\mathbb{S}}$ is a standard formula, then $\Phi^{\mathcal{I}}$ is also called the internalization of Ψ, and is denoted $\Psi^{\mathcal{I}}$.

Any concept (relation, constant, operation) defined in ZFC by a formula Φ therefore splits into three, generally distinct, concepts in NZFC: the standard analog defined by $\Phi^{\mathbb{S}}$, the internal analog defined by $\Phi^{\mathcal{I}}$, and the external analog defined by Φ. For example, the relation $\subseteq$ defined by the formula from (1) has three analogs in NZFC:

$$A \subseteq^{\mathbb{S}} B \text{ defined by } (\forall \text{ standard } x)\ (x \in A \Rightarrow x \in B);$$

$$A \subseteq^{\mathcal{I}} B \text{ defined by } (\forall \text{ internal } x)\ (x \in A \Rightarrow x \in B); \text{ and}$$

$$A \subseteq^{\mathfrak{E}} B \text{ defined by } (\forall x)\ (x \in A \Rightarrow x \in B).$$

For standard A and B, $\subseteq^{\mathbb{S}}$ has the same meaning as $\subseteq$ in ZFC, and we therefore write simply $\subseteq$ in place of $\subseteq^{\mathbb{S}}$. Whether or not the other two concepts, $\subseteq^{\mathcal{I}}$ and $\subseteq^{\mathfrak{E}}$, coincide with $\subseteq$ is a matter requiring some analysis, as explained in §§2 and 3. When constructing $\Phi^{\mathbb{S}}$ [$\Phi^{\mathcal{I}}$, respectively], we have to replace all previously defined concepts mentioned in Φ by their standard [internal, respectively] analogs.

The axioms for NZFC then consist of standardizations of the axioms of ZFC, together with the principles introduced in §§2 and 3 (see [3] for a rigorous complete presentation).

Although we attempted to motivate our principles intuitively, that of course does not guarantee their consistency. However, we proved in [3] that *NZFC is a conservative extension of ZFC*; that is, every standard theorem which can be proved in NZFC can also be proved in ZFC alone. This means, in particular, that NZFC is consistent, assuming that the standard set theory is consistent. Moreover, the proof of this result provides a procedure which automatically translates a nonstandard proof of a standard theorem into its standard proof. In most cases, the standard proofs obtained in this manner are not very enlightening; but then one reason for using nonstandard methods lies precisely in the fact that nonstandard proofs may be more transparent and easier to find. (In [3], the theory NZFC is called $\mathfrak{N}\mathfrak{S}_2$, and the result discussed presently is stated as Theorem 2. Its proof there does not require material from the very technical §2.)

II. *Nonstandard Analysis according to Robinson.* The usual framework for Nonstandard Analysis differs somewhat from the viewpoint described in this paper. We want to make a few comments on the relationship between the two approaches in order to facilitate transition to the published literature. One conspicuous feature of the usual approach, e.g., that of Robinson [**10**], is the use of higher-order nonstandard models containing all entities pertinent to the particular investigation (natural numbers, reals, etc.). From our point of view, the universe of external sets plays the role of such models; it contains all entities pertinent to any investigation, namely, all standard sets. However, the main difference is that, from the usual viewpoint, the standard sets are considered as having standard elements only. Generally speaking, this makes the usual standard concepts correspond to our external concepts. We give several examples. Our standard natural numbers have only standard elements, and this makes the usual standard natural numbers correspond to our standard (i.e., external) natural numbers. However, the usual

standard set of natural numbers corresponds to our external set of natural numbers ${}^{\mathfrak{S}}N = {}^{\circ}N$, rather that our N (which corresponds to what is usually denoted by N^*). Similarly, the usual standard set of rationals corresponds to our ${}^{\circ}Q$. The usual standard set of reals corresponds to our external set of external reals, ${}^{\mathfrak{S}}R$.

More formally, we can define *external standard sets* and a mapping * inductively as follows: $\mathbf{A}$ is an external standard set, $\mathbb{S}^{\mathfrak{S}}(\mathbf{A})$, if all elements of $\mathbf{A}$ are external standard sets, and then $\mathbf{A}^* = {}^*\{\mathbf{a}^* | \mathbf{a} \in \mathbf{A}\}$. Note that the induction starts with the empty set. For each external standard $\mathbf{A}$, $\mathbf{A}^*$ is a standard set; for example, $({}^{\mathfrak{S}}N)^* = N$, $({}^{\mathfrak{S}}Q)^* = Q$, $({}^{\mathfrak{S}}R)^* = R$. (Note also that this mapping * is not the same as the one defined in §3.) It is easy to prove inductively (and using the Principle of Standardization) that * is an isomorphism, with respect to $\in$, between the universe of external standard sets and the universe of standard sets. It is thus possible to adopt the usual viewpoint and consider the universe of (external) standard sets as embedded into the universe of internal sets by *. The Principle of Embedding then says that, for any set-theoretic property Φ, and any (external) standard $\mathbf{A}, \mathbf{B}, \ldots$, $\Phi^{\mathbb{S}^{\mathfrak{S}}}(\mathbf{A}, \mathbf{B}, \ldots)$ if and only if $\Phi^{\mathfrak{I}}(\mathbf{A}^*, \mathbf{B}^*, \ldots)$, i.e., * is an elementary embedding of the universe of (external) standard sets into the universe of internal sets. Similarly, the Principle of Saturation can be reformulated from this viewpoint. With a little practice, the reader should have no difficulties using either approach.

III. *Other Axiomatic Systems for Nonstandard Analysis.* My work on an axiomatic approach to Nonstandard Analysis began in the spring of 1972, when, stimulated by interest taken in the subject by students in my model theory course, I formulated the axioms of NZFC ($\mathfrak{N}\mathfrak{S}_2$) and proved that it is a conservative extension of ZFC. An analogous result for a related system ($\mathfrak{N}\mathfrak{S}_1$) turned out to be much harder, and the paper [3] was completed only in late 1974 (see also an abstract [2]). Since then, I have learned about two other attempts to develop nonstandard mathematics axiomatically.

Starting in 1973, *P. Vopěnka* and his students devised an *Alternative Set Theory* and developed some areas of mathematics (in particular, calculus and general topology) in this framework. We refer the interested reader to [13] for a detailed exposition and other references. It might be helpful to note that, by taking all external subsets of the standard set HF of all hereditarily finite sets, one obtains a model of the Alternative Set Theory (except for one axiom, essentially the Continuum Hypothesis).

E. Nelson developed an axiomatic theory called the *Internal Set Theory* in [9], and showed how a variety of nonstandard constructions can be performed in it. The Internal Set Theory deals only with standard and internal sets; its axioms correspond roughly to our principles of Embedding, Weak Saturation (there are differences here: saturation with respect to a proper class of standard parameters is allowed, the property Φ may have internal parameters (as in our Strong Saturation), and the principle is stated as an equivalence, rather than an implication) and Standardization (for external sets definable in the language of the Internal Set Theory). External sets can be introduced for the price of working with a model of set theory; this is satisfactory when external sets are used for bookkeeping purposes only, but becomes less wieldy when an interplay between external and internal concepts is important, such as in constructions of §6. (There, for example, a countable model of set theory would not do, because the external reals could not be identified with the standard ones.) The paper [9] contains many detailed explanations of pitfalls in nonstandard reasoning, and would be good reading for those interested in learning more about the developing field of Nonstandard Mathematics.

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FIFTY YEARS AGO

A fairly random selection from the topics of the 46 articles in volume 36 of this MONTHLY includes: "How can interest in calculus be increased?"; "The fundamental mathematical requirements of biology" (by a botanist); differential equations in electrical circuit theory, by T. C. Fry, of Bell Telephone Laboratories; Cantor's singular function, by Hille and Tamarkin; trigonometry in hyperspace; kinematics; polar conics and osculant conics of a nodal conic; the motion of a satellite of a spheroidal planet.

The winter meeting was in New York in December 1928; the summer meeting, at Boulder ("first time in the Far West"). At the former, E. R. Hedrick called for, among other things, the establishment of a mathematics abstracting journal. At the latter, there were addresses on the undergraduate mathematics curriculum in a liberal arts college and on factorization of large numbers (up to 1.5×10^9). The College Entrance Examination Board was being urged to modify its requirements so as to get more solid geometry into the high school curriculum.

In short, the character of the MONTHLY and the concerns of the Association seem to have changed rather little in half a century, except that algebra has largely replaced geometry and the Problems are now drawn from more different fields. There is little evidence of any awareness in 1929 that major changes were soon to occur in Mathematics; for example, there were no articles on topology or abstract algebra or (what was to become) functional analysis. The problems of teaching appear to have remained much the same for fifty years, but I hope that the MONTHLY may now be bringing its readers more awareness of how Mathematics itself is changing.

Volume 36 contained 560 pages (in larger type and with wider margins than the one you are reading) and a 67-page Register of officers and members. There were 1,964 individual members, whose dues were $4 a year. The winter meeting was attended by 340 people; the summer meeting, by 239. In 1928, American universities awarded 49 doctorates in Mathematics (they included A. A. Albert, L. W. Cohen, B. W. Jones, Morris Marden and Morgan Ward). Fine Hall, at Princeton, was about to be built at a cost of $400,000.

Notice the curious effects of various kinds of inflation: in 1979 we have about 10 times as many members, each paying more than 5 times as high dues, but the MONTHLY is only slightly more than twice as large and actually publishes fewer (although longer) main articles than in 1929. The number of short notes has approximately doubled; the number of extended book

reviews has greatly decreased. The total number of authors per year is now only about twice what it was in 1929. About 20 times as many Ph.D.'s are being produced; putting up a building costs something like 10 times as much; and we can routinely factor numbers with around twice as many digits.

R.P.B.

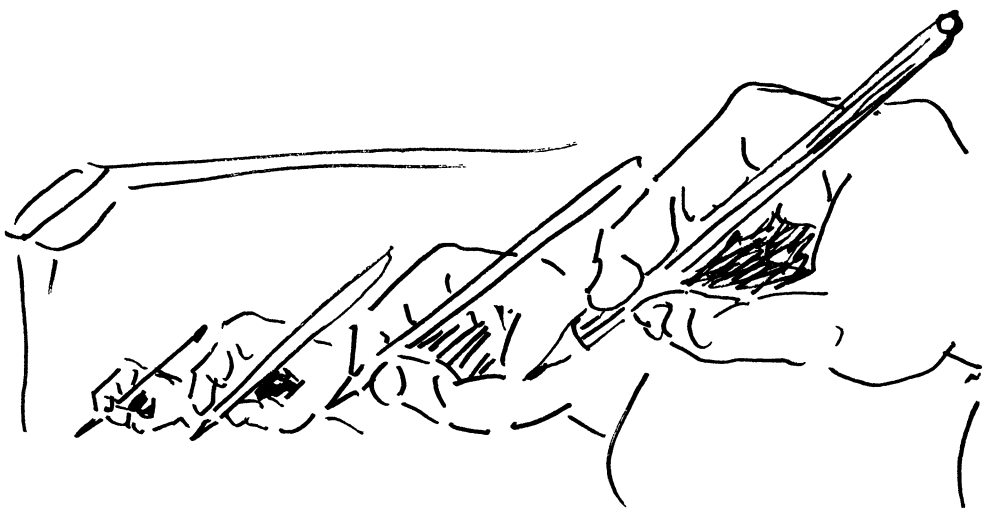
MISCELLANEA

28.

A DRAWING NO ONE CAN DRAW

M. STOJAKOVIĆ

Suppose you start to draw a picture and choose as your subject the sheet of paper on which you are drawing together with your hand holding the pencil with which you are drawing. Then you have to draw the sheet of paper and your hand holding the pencil and the sheet of paper on which you are drawing and your hand and the pencil and the sheet of paper . . . You could not finish the picture even if you could draw a million hands a second. Again, suppose you want to draw everything, say, twice life-size?



Drawing of a hand which is drawing the hand which is drawing the hand (· · ·).

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MATHEMATICAL NOTES

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

Material for this department should be sent to Professor Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis, MO 63121.

Advice to prospective authors: The editors have recently been receiving about **ten times** as many Mathematical Notes as can be used. It will simplify our work if authors will submit only especially interesting manuscripts. Mathematical Notes should be short papers of one to four printed pages which give new insights, new and improved proofs of old theorems, brief bits of mathematical folklore that have not found a home in the literature, or (occasionally!) new results that are not too technical. The topics should be of wide current interest. Papers that have already been rejected by a research journal are only very rarely suitable as Mathematical Notes.

R. P. B.

A SEQUENCE OF CONVERGENCE TESTS

J. R. NURCOMBE

1. Introduction. In recent years, various notes (e.g., [3] and [4]) have appeared purporting to contain some new extension of Cauchy's root test, although the material involved is essentially well known, being found in different forms in [1, p. 44] and [2, p. 282]. One possible reason for this is that the treatments given in [1] and [2] fail to establish any clear formal similarity between Cauchy's test and succeeding tests of the logarithmic scale of convergence. It is therefore perhaps worthwhile to remedy this defect. An attempt to do this is made in Theorem 1, and the remainder of this note endeavors to clarify some of the points arising in the references.

2. It is familiar that the convergence or divergence of a series of positive terms Σa_n may be deduced by comparing it with the standard series of the logarithmic scale, which converge for $p>1$ and diverge for $p\leqslant 1$:

$$\sum_{n=1}^{\infty} 1/p^n, \qquad \sum_{n=1}^{\infty} 1/n^p, \qquad \sum_{n=2}^{\infty} 1/n(\log n)^p, \qquad \sum_{n=3}^{\infty} 1/n\log n(\log\log n)^p,$$

etc. This can be expressed in a form similar to the root test.

THEOREM 1. *Let Σa_r be a series of positive terms. If* (i) $\overline{\lim}\, a_n^{1/n}<1$ *or* (ii) $\overline{\lim}\, a_n^{1/\log n}<1/e$, *or* (iii) $\overline{\lim}(na_n)^{1/\log_2 n}<1/e$, *or* (iv) $\overline{\lim}(n\log n\cdot a_n)^{1/\log_3 n}<1/e$, *then in each case, Σa_r converges, where $\log_2 n=\log\log n$, etc.*

Proof. The argument is the same in all cases and (iii) is chosen as typical. For all $n\geqslant N$, $(na_n)^{1/\log_2 n}\leqslant 1/e^p$, $p>1$. Thus $na_n\leqslant 1/e^{p\log_2 n}$ or $a_n\leqslant 1/n(\log n)^p$, implying Σa_r is convergent.

REMARKS. (1) The tests for divergence can be given in a similar manner, but since the limit forms are weaker than the non-limit forms in this case, they are omitted.

(2) (i) is Cauchy's test. (ii) and (iii) are, respectively, the comparison test equivalents of Raabe's and Gauss's tests. (iv) is the next comparison test of the logarithmic scale, and succeeding tests can be formulated in an analogous and obvious way. It is (ii) and (iii) which are given in [3].

(3) Substantially the same relationship exists between these tests and the corresponding ratio tests as exists between Cauchy's test and d'Alembert's test.

(4) The above method of formulating the comparison tests could be called the "exponential

form." The ratio tests can also be given in this form. D'Alembert's test is unchanged. Raabe's test becomes: if $\lim(a_{n+1}/a_n)^n < 1/e$, then Σa_r converges, and Gauss's test is:

$$\text{if } \lim n(a_{n+1}/a_n)^{n\log n} < 1/e, \text{ then } \Sigma a_r \text{ converges.}$$

3. Although the comparison tests are more powerful than the ratio tests, the latter can be strengthened and generalized in the following way.

THEOREM 2. *Let Σa_r be a series of positive terms, and k, a fixed positive integer. If* $\lim a_{n+k}/a_n \lessgtr 1$, *then* $\Sigma a_r \begin{cases} \textit{converges} \\ \textit{diverges}. \end{cases}$

Proof. Consider the k subseries,

$$\sum_{r=0}^{\infty} a_{1+rk}, \sum_{r=0}^{\infty} a_{2+rk}, \ldots, \sum_{r=1}^{\infty} a_{rk},$$

which collectively comprise all the terms of $\Sigma_{r=1}^{\infty} a_r$. Every subseries converges for the first hypothesis by d'Alembert's test and, since the sum of a series of positive terms is invariant under alterations in the order of the terms, Σa_r converges. For the second hypothesis, every subseries diverges by d'Alembert's test, so Σa_r cannot converge.

REMARKS. (1) Series can be readily constructed to show that $\underline{\lim}\, a_{n+k}/a_n \leqslant \underline{\lim}\, a_n^{1/n} < \overline{\lim}\, a_n^{1/n} \leqslant \overline{\lim}\, a_{n+k}/a_n$, which is known to be true if $k=1$, is false if $k>1$. For example, if $\Sigma a_r = u+v^2+u^3+v^4+\cdots$, take $u=1/2$, $v=2/3$. Then $\underline{\lim}\, a_{n+2}/a_n = 1/4 < \overline{\lim}\, a_{n+2}/a_n = 4/9 < \underline{\lim}\, a_n^{1/n} = 1/2 < \overline{\lim}\, a_n^{1/n} = 2/3$. Thus it is no longer necessarily true that Cauchy's test includes the extended forms of d'Alembert's test. Furthermore, for this example, the whole scale of ratio tests will fail when $k=1$, since the upper and lower limits of (a_{n+1}/a_n) include unity, although when $k=2$, the convergence of the series is easily deduced from theorem 2.

(2) Naturally, if there are infinitely many zero terms, the ratio tests are inapplicable.

(3) The same extensions apply to Raabe's, Gauss's, and the other tests of the logarithmic scale. For example, the extended form of Raabe's test when $k=2$ shows that the example in remark (1) of [3] converges. To prove this, we must show that $\lim n(a_{n+2}/a_n - 1) < -2$, when $a_{n-1} = a_n = (1-3/n)^{n\log n}$ for n even.

Now

$$\frac{a_{n+2}}{a_n} = \frac{\left(1-\dfrac{3}{n+2}\right)^{(n+2)\log(n+2)}}{\left(1-\dfrac{3}{n}\right)^{n\log n}}.$$

Also

$$(n+2)\log(n+2) = (n+2)\left(\log n + \log\left(1+\frac{2}{n}\right)\right)$$

$$= n\log n + 2\log n + 2 + O\left(\frac{1}{n}\right),$$

so

$$\frac{a_{n+2}}{a_n} = \left(1+\frac{3}{n-3}\right)^{n\log n} \cdot \left(1-\frac{3}{n+2}\right)^{n\log n + 2\log n + 2 + O\left(\frac{1}{n}\right)}$$

$$= \left\{\left(1-\frac{3}{n+2}\right)\left(1+\frac{3}{n-3}\right)\right\}^{n\log n} \cdot \left(1-\frac{3}{n+2}\right)^{2\log n + 2 + O\left(\frac{1}{n}\right)}$$

$$=\left(1+\frac{6}{(n+2)(n-3)}\right)^{n\log n}\cdot\left(1-\frac{6\log n}{n+2}-\frac{6}{n+2}+O\left(\frac{\log n}{n}\right)^2\right)$$

$$=\left(1+\frac{6n\log n}{(n+2)(n-3)}+O\left(\frac{\log n}{n}\right)^2\right)\left(1-\frac{6(1+\log n)}{n+2}+O\left(\frac{\log n}{n}\right)^2\right)$$

$$=1-\frac{6\left(1+\log n-\dfrac{n\log n}{n-3}\right)}{n+2}+O\left(\frac{\log n}{n}\right)^2.$$

Thus

$$n\left(\frac{a_{n+2}}{a_n}-1\right)=\frac{-6n\left(1-\dfrac{3\log n}{n-3}\right)}{n+2}+O\left(\frac{\log n}{n}\right)^2$$

and the required limit is -6, which implies convergence.

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TAYLOR POLYNOMIALS AND DIFFERENCE QUOTIENTS

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An nth order Taylor polynomial for a function f at x_0 is a polynomial $p(x)=a_0+a_1(x-x_0)+\cdots+a_n(x-x_0)^n$ such that $f(x)-p(x)=o(|x-x_0|^n)$. Many delicate estimates of differential calculus, such as tests for relative extrema and the evaluation of indeterminate forms, can be performed by replacing functions by their Taylor polynomials. Moreover, rules for computing higher derivatives of compound functions have simple interpretations as algebraic rules for combining Taylor polynomials; it is particularly helpful to view the chain rule in this light. In view of the fundamental importance of Taylor polynomials, it would seem to be of considerable interest to characterize those functions with Taylor polynomials and to give a rule for finding them. Here we present a complete solution to this problem and give examples to show that some apparently more natural characterizations are, in fact, not quite correct.

For orders 0 and 1 the conditions are precisely continuity and differentiability at x_0, but for higher orders the situation is more complicated. Taylor's theorem shows that f has an nth order Taylor polynomial at x_0 if $f^{(n)}$ is continuous in a neighborhood of x_0, but this condition is far from necessary. For example, suppose $f(x)=\exp(-x^{-2})$ for irrational x and $f(x)=0$ for rational x. Then for $p(x)\equiv 0$ we have $f(x)-p(x)=o(|x|^n)$ for every choice of n, although $f'(0)$ is the only derivative of f which exists.

A reasonable alternative is to replace the higher derivatives by limits of higher difference quotients. Milne-Thompson [1] gives an elementary presentation of difference schemes in general. The simplest and most common scheme is to define

$$\Delta_h f(x)=f(x+h)-f(x)$$

and

$$\Delta_h^{n+1}f(x)=(\Delta_h^n f)(x+h)-\Delta_h^n f(x)$$

for $n \geqslant 1$. One can then prove the following well-known result:

THEOREM 1. *Suppose $a_0 + a_1(x - x_0) + \cdots + a_n(x - x_0)^n$ is an nth order Taylor polynomial for f at x_0. Then $f(x_0) = a_0$ and*

$$\lim_{h \to 0} \Delta_h^k f(x_0)/h^k = k!\,a_k$$

for $k = 1, \ldots, n$

When $n = 2$ the converse is also true. We prove this converse below; the proof is the starting point of our investigation.

Assume

(i) $f(x_0) = a_0$

(ii) $\Delta_h f(x_0) = a_1 h + o(h)$

(iii) $\Delta_h^2 f(x_0) = 2a_2 h^2 + o(h^2)$.

Iterating the identity $\Delta_h f(x_0) = \Delta_{h/2}^2 f(x_0) + 2\Delta_{h/2} f(x_0)$ yields

$$\Delta_h f(x_0) = \Delta_{h/2}^2 f(x_0) + 2\Delta_{h/2^2}^2 f(x_0) + \cdots + 2^{m-1}\Delta_{h/2^m}^2 f(x_0) + 2^m \Delta_{h/2^m} f(x_0).$$

By (ii), $\lim_{m\to\infty} 2^m \Delta_{h/2^m} f(x_0) = a_1 h$. Hence

$$\Delta_h f(x_0) = a_1 h + \sum_{k=1}^{\infty} 2^{k-1} \Delta_{h/2^k}^2 f(x_0). \tag{1}$$

If we set $g(h) = \Delta_h^2 f(x_0) - 2a_2 h^2$, then (iii) implies

$$\sum_{k=1}^{\infty} 2^{k-1} g(h/2^k) = o(h^2).$$

Thus (1) yields

$$\Delta_h f(x_0) = a_1 h + \sum_{k=1}^{\infty} 2^{k-1} \times 2a_2 (h/2^k)^2 + o(h^2) = a_1 h + a_2 h^2 + o(h^2).$$

Putting $h = x - x_0$ and $a_0 = f(x_0)$ completes the proof.

The converse fails unless we explicitly assume that $\Delta_h f(x_0)/h$ has a limit. For example, let g be any non-zero function vanishing outside the interval $(1,2)$ and set

$$f(x) = \sum_{m=-\infty}^{\infty} 2^m g(2^{-m}x)$$

Since $f(0) = 0$ and $f(2x) \equiv 2f(x)$, we see $\Delta_h^2 f(0) = f(2h) - 2f(h) + f(0) = 0$ for all h, whereas in every neighborhood of 0 the function $\Delta_h f(0)/h$ achieves every value of $g(x)/x$.

Somewhat more surprisingly, the converse fails for $n = 3$. Let us define $f(x) = (-1)^{j+k} x^3$ if $x = \pm 2^j 3^k$ and $f(x) = 0$ otherwise. Then f has no third order Taylor polynomial at 0, although $\Delta_h f(0)$ and $\Delta_h^2 f(0)$ are $O(|h|^3)$ and

$$\begin{aligned}\Delta_h^3 f(0) &= f(3h) - 3f(2h) + 3f(h) - f(0)\\ &= -27f(h) + 24f(h) + 3f(h) = 0.\end{aligned}$$

In order to obtain the converse for $n \geqslant 3$, we need to look at another type of higher difference which retains the properties of Δ_h^n that imply Theorem 1 and which also allows us to prove an analogue of (1). We base our higher differences on binary increments.

DEFINITION. 1.

$$B_h^{(1)} f(x) = f(x+h) - f(x)$$

and

$$B_h^{(n+1)} f(x) = B_{2h}^{(n)} f(x) - 2^n B_h^{(n)} f(x)$$

for $n \geqslant 1$. We see $B_h^{(n)} f(x) = \Delta_h^n f(x)$ for $n = 1$ or 2 but not for $n \geqslant 3$.

DEFINITION. 2. For positive integers m and n, we define $T(m,n)$ by

$$T(1,n)=1 \quad \text{and} \quad T(m+1,n)=(2^n-2^m)T(m,n).$$

Let us look at some consequences of our definitions. For $p_n(x)=x^n$, $n\geqslant 1$, we see easily that

$$B_h^{(m)}p_n(0)=T(m,n)h^n.$$

In particular, $B_h^{(m)}p_n(0)=0$ if $m>n$.

We also have an important analogue of equation (1). Let us assume

$$A_m=\lim_{h\to 0} B_h^{(m)}f(x_0)/h^m.$$

Since

$$B_h^{(m)}f(x_0)=B_{h/2}^{(m+1)}f(x_0)+2^mB_{h/2}^{(m)}f(x_0),$$

iterating this equation and taking a limit yields

$$B_h^{(m)}f(x_0)=A_mh^m+\sum_{k=1}^{\infty}2^{m(k-1)}B_{h/2^k}^{(m+1)}f(x_0). \tag{2}$$

THEOREM 2. *For $n\geqslant 1$, a necessary and sufficient condition for the polynomial $p(x)=a_0+a_1(x-x_0)+\cdots+a_n(x-x_0)^n$ to be an nth order Taylor polynomial for f at x_0 is that*

$$a_0=f(x_0) \text{ and } \lim_{h\to 0}B_h^{(m)}f(x_0)/h^m=T(m,m)a_m$$

for $m=1,\ldots,n$.

Proof. First assume $f(x)=p(x)+o(|x-x_0|^n)$. Then

$$f(x_0)=p(x_0)=a_0 \quad \text{and} \quad B_h^{(m)}f(x_0)=B_h^{(m)}p(x_0)+o(|h|^n).$$

Since $B_h^{(m)}p(x_0)=\sum_{j=1}^{n}T(m,j)a_jh^j$ and $T(m,j)=0$ for $m>j$, it follows immediately that

$$\lim_{h\to 0}B_h^{(m)}f(x_0)/h^m=T(m,m)a_m$$

for $m=1,\ldots,n$.

For the sufficiency, let us observe that if $g(h)=o(|h|^n)$ and $1\leqslant m<n$, then

$$\sum_{k=0}^{\infty}2^{m(k-1)}g(h/2^k)=o(|h|^n). \tag{3}$$

By hypothesis we have

$$B_h^{(n)}f(x_0)=T(n,n)a_nh^n+o(|h|^n).$$

Successive calculations using (2), (3), and induction show

$$B_h^{(m)}f(x_0)=\sum_{j=m}^{n}T(m,j)a_jh^j+o(|h|^n) \tag{4}$$

for $m=n-1,n-2,\ldots,1$ as well as $m=n$. Assuming that (4) has been proved for $m+1$, we have

$$\begin{aligned}B_h^{(m)}f(x_0)&=T(m,m)a_mh^m+\sum_{k=1}^{\infty}2^{m(k-1)}B_{h/2^k}^{(m+1)}f(x_0)\\&=T(m,m)a_mh^m+\sum_{k=1}^{\infty}2^{m(k-1)}\sum_{j=m+1}^{n}T(m+1,j)a_j(h/2^k)^j+o(|h|^n).\end{aligned}$$

We simply compute

$$\begin{aligned}\sum_{k=1}^{\infty}2^{m(k-1)}T(m+1,j)a_j(h/2^k)^j&=T(m+1,j)a_jh^j\sum_{k=1}^{\infty}2^{m(k-1)-kj}\\&=T(m+1,j)a_jh^j/(2^j-2^m)\end{aligned}$$

$$= T(m,j)a_j h^j \quad \text{for} \quad m < j \leqslant n.$$

This establishes (4). Putting $m=1$ and $h=x-x_0$ in (4) gives

$$f(x) = a_0 + a_1(x-x_0) + \cdots + a_n(x-x_0)^n + o(|x-x_0|^n)$$

as desired.

REMARK. If we also have $B_h^{(n+1)}f(x_0) = O(|h|^{n+\varepsilon})$ or $o(|h|^{n+\varepsilon})$, $\varepsilon > 0$, then our methods also show that $f(x) - p(x)$ satisfies a bound of the same order.

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CONFORMAL AUTOMORPHISMS OF CIRCULAR REGIONS

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In studying conformal mappings of regions in the complex plane $\mathbb{C}$ or the extended complex plane $\mathbb{P}$, Möbius transformations (linear fractional transformations) play an important role. Recall that a Möbius transformation has the form $T(z) = (az+b)/(cz+d)$, where $a,b,c,d \in \mathbb{C}$ and $ad-bc \neq 0$. Each Möbius transformation is a conformal homeomorphism of $\mathbb{P}$. The set Möb of Möbius transformations forms a group under composition of mappings. Möbius transformations possess the important property that they map circles into circles. Throughout this paper the word *circle* will mean a point in $\mathbb{P}$, a circle in $\mathbb{C}$ or a straight line in $\mathbb{C}$ together with the point ∞. A *circular region* in $\mathbb{P}$ is an open connected subset of $\mathbb{P}$ having the property that every boundary component is a circle. The circle-preserving property of Möbius transformations is very useful in the construction of explicit conformal mappings of circular regions. Given any region Ω in $\mathbb{P}$, let $\mathcal{Q}(\Omega)$ denote the group of conformal automorphisms of Ω.

We begin by considering some particular circular regions and their conformal automorphism group. Let $\mathbb{B} = \{z \in \mathbb{C} : |z| < 1\}$, $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$ and $A(r) = \{z \in \mathbb{C} : r < |z| < 1/r\}$, where $r \in (0,1)$. The following conformal automorphism groups are well-known.

$$\begin{aligned}
\mathcal{Q}(\mathbb{B}) \quad &= \left\{ z \mapsto \lambda \frac{z-a}{1-\bar{a}z} : \lambda \in \Gamma, a \in \mathbb{B} \right\} \\
\mathcal{Q}(\mathbb{C}) \quad &= \{ z \mapsto az+b : a \in \mathbb{C}\backslash\{0\}, b \in \mathbb{C} \} \\
\mathcal{Q}(\mathbb{P}) \quad &= \text{Möb} \\
\mathcal{Q}(\mathbb{B}\backslash\{0\}) &= \{ z \mapsto \lambda z : \lambda \in \Gamma \} \\
\mathcal{Q}(\mathbb{C}\backslash\{0\}) &= \{ z \mapsto az : a \in \mathbb{C}\backslash\{0\} \} \cup \{ z \mapsto a/z : a \in \mathbb{C}\backslash\{0\} \} \\
\mathcal{Q}(A(r)) \quad &= \{ z \mapsto \lambda z : \lambda \in \Gamma \} \cup \{ z \mapsto \lambda/z : \lambda \in \Gamma \}
\end{aligned}$$

Observe that for each of these circular regions the group of conformal automorphisms is a subgroup of Möb; they will be called our standard (simply or doubly connected) circular regions. The letter D will be used to represent any one of the standard circular regions.

Because of these important examples, it is natural to ask which regions Ω in $\mathbb{P}$ have the property that $\mathcal{Q}(\Omega)$ is a subgroup of Möb. The standard circular regions can be used to generate many other examples. It is elementary to verify that if $f : \Omega_1 \to \Omega_2$ is a conformal mapping of Ω_1 onto Ω_2, then $\mathcal{Q}(\Omega_2) = f \circ \mathcal{Q}(\Omega_1) \circ f^{-1} = \{ f \circ g \circ f^{-1} : g \in \mathcal{Q}(\Omega_1) \}$. Consequently, if D is a standard circular region and $T \in$ Möb, then $T|D$ is a conformal mapping of D onto a circular region $T(D)$ and $\mathcal{Q}(T(D)) = T \circ \mathcal{Q}(D) \circ T^{-1}$ is a subgroup of Möb conjugate to $\mathcal{Q}(D)$. On the other hand, if Ω is a simply or doubly connected circular region, then there is a $T \in$ Möb and a unique

standard circular region D such that $T(D)=\Omega$. Thus, $\mathcal{Q}(\Omega)$ is a subgroup of Möb if Ω is any simply or doubly connected circular region. Do any other simply or doubly connected regions Ω have the property that $\mathcal{Q}(\Omega)\subset$ Möb? After the following lemma, we shall see that the expected answer is correct.

LEMMA. (a) *If $T\in$ Möb, $T\neq I$ (the identity) is an involution ($T^2=I$), then T has two distinct fixed points.*

(b) *Suppose $S,T\in$ Möb and neither is the identity. S and T commute if and only if S and T have the same fixed points or else S and T are involutions with fixed points a,b and c,d, respectively, whose cross-ratio $(a,b,c,d)=-1$.*

These results are elementary facts about Möbius transformations ([3, p. 72], [4, pp. 10–12]).

PROPOSITION. *Let Ω be a simply or doubly connected region in $\mathbb{P}$ such that $\mathcal{Q}(\Omega)\subset$ Möb; then Ω is a circular region.*

Proof. Any simply or doubly connected region in $\mathbb{P}$ is conformally equivalent to one of the standard circular regions. Let $f: D\to\Omega$ be a conformal mapping of a standard circular region onto Ω. Now, $\mathcal{Q}(D)\subset$ Möb and f induces a group isomorphism $f^*: \mathcal{Q}(D)\to\mathcal{Q}(\Omega)$ defined by $f^*(T)=f\circ T\circ f^{-1}$. Precisely, $T\in$ Möb and $f\circ T\circ f^{-1}$ is defined only on Ω. But by hypothesis each conformal automorphism of Ω is the restriction of a Möbius transformation, so we may regard $f^*(T)$ as an element of Möb. The isomorphism f^* is continuous in the sense that a sequence $(T_n)_{n=1}^{\infty}$ in $\mathcal{Q}(D)$ converges uniformly on compact subsets of D to $T\in\mathcal{Q}(D)$ if and only if $(S_n)_{n=1}^{\infty}$, $S_n=f^*(T_n)$, converges uniformly on compact subsets of Ω to $S=f^*(T)$. (The uniform convergence is relative to the chordal metric in case ∞ belongs to D or Ω.)

For $\lambda\in\Gamma$ define $T_\lambda(z)=\lambda z$. Since each of the standard circular regions is invariant under any rotation about the origin, $T_\lambda\in\mathcal{Q}(D)$. In fact, $\mathfrak{T}=\{T_\lambda:\lambda\in\Gamma\}$ is an abelian subgroup of $\mathcal{Q}(D)\subset$ Möb. Consequently, $\mathfrak{S}=\{S_\lambda=f^*(T_\lambda):\lambda\in\Gamma\}$ is an abelian subgroup of $\mathcal{Q}(\Omega)\subset$ Möb and $f^*:\mathfrak{T}\to\mathfrak{S}$ is an isomorphism. The function T_{-1} is an idempotent, so $S_{-1}=f^*(T_{-1})$ is also an idempotent and must have two distinct fixed points, say a and b. We may assume that 0 and ∞ are the fixed points. (If not, then let R be a Möbius transformation mapping a to 0 and b to ∞. Replace f by $R\circ f$ and Ω by $R(\Omega)$. Then $R(\Omega)$ is a circular region if and only if Ω is.) Because S_{-1} fixes both 0 and ∞, $S_{-1}(z)=\omega z$ for some $\omega\in\mathbb{C}\setminus\{0\}$. From $S_{-1}^2=I$ and $S_{-1}\neq I$, we conclude that $S_{-1}=-I$.

Now, take any $\lambda\in\Gamma$, $\lambda\neq\pm1$. T_λ is not an idempotent so neither is S_λ. Also, S_λ and S_{-1} commute. From the preceding lemma we conclude that S_λ and S_{-1} have the same fixed points, so $S_\lambda=\omega(\lambda)I$ for some $\omega(\lambda)\in\mathbb{C}\setminus\{0\}$. We want to show that $\{\omega(\lambda):\lambda\in\Gamma\}=\Gamma$. As a first step we show that $\omega(\lambda)$ is a root of unity if λ is. If $\lambda^n=1$, then $T_\lambda^n=T_{\lambda^n}=I$, so $I=S_{\omega(\lambda)}^n=\omega(\lambda)^nI$. This implies that $\omega(\lambda)$ is an nth root of unity if λ is. Suppose ω is any nth root of unity. Take λ to be a primitive nth root of unity. Then $\omega(\lambda),\omega(\lambda^2),\ldots,\omega(\lambda^n)$ are nth roots of unity which are distinct since f^* is an isomorphism. Hence, $\omega=\omega(\lambda^k)$ for some k, $1\leqslant k\leqslant n$.

Next, we prove that $\mathfrak{S}=\mathfrak{T}$. Let $\lambda\in\Gamma$. Since the set of roots of unity is dense in Γ, there is a sequence $(\lambda_n)_{n=1}^{\infty}$ of roots of unity with $\lambda_n\to\lambda$. Then T_{λ_n} converges to T_λ uniformly on compact subsets of D, so S_{λ_n} converges to S_λ uniformly on compact subsets of Ω. In particular, $\omega(\lambda_n)\to\omega(\lambda)$, so $\omega(\lambda)\in\Gamma$. Take any $\omega\in\Gamma$ and let $(\omega_n)_{n=1}^{\infty}$ be a sequence of roots of unity converging to ω. There exists $\lambda_n\in\Gamma$ with $\omega_n=\omega(\lambda_n)$. By passing to a subsequence if necessary, we may assume $\lambda_n\to\lambda\in\Gamma$. As before $\omega_n=\omega(\lambda_n)\to\omega(\lambda)$ so that $\omega=\omega(\lambda)$. This proves that $\mathfrak{S}=\mathfrak{T}$.

Since $\mathfrak{T}\subset\mathcal{Q}(\Omega)$, the region Ω is invariant under any rotation about the origin. Therefore, Ω is a circular region.

REMARK. Actually it is possible to show that either f^* is the identity or $f^*(T_\lambda)=T_{\bar\lambda}$, but this is not needed. Also, since Ω is a circular region, $f=T|D$ for some $T\in$ Möb.

So far we have considered only simply and doubly connected regions. What is the situation

for regions of higher connectivity? If Ω is a circular region of finite connectivity, then $\mathcal{A}(\Omega) \subset$ Möb [1, pp. 234–237]. The proof of this fact is not as elementary as the cases when Ω is simply or doubly connected. Also, any region of finite connectivity is conformally equivalent to a circular region [1, pp. 237–241]. Whether the same result holds for regions of infinite connectivity is a famous unsolved problem. Does the proposition extend to regions of finite connectivity? The proof was based on the fact that, for circular regions Ω of connectivity one or two, the group $\mathcal{A}(\Omega)$ contains an isomorphic copy of the circle group Γ. For regions Ω of finite connectivity $n \geqslant 3$ this no longer holds since $\mathcal{A}(\Omega)$ is finite. Actually, if Ω has connectivity $n \geqslant 3$, then the order of $\mathcal{A}(\Omega)$ is at most $2n$ unless $n = 4, 6, 8, 12, 20$ [2]. It is easy to produce a counterexample to the proposition for regions of connectivity $n \geqslant 3$. Suppose $0 < \rho < 1$ and $0 < \phi < \pi/3$. Define $\Sigma = \{z = re^{i\theta} : \rho \leqslant r \leqslant 1/\rho, -\phi \leqslant \theta \leqslant \phi\}$. If $\omega = \exp(2\pi i/3)$, then $\Omega = \mathbb{P} \setminus (\Sigma \cup \omega\Sigma \cup \omega^2\Sigma)$ is a triply connected region in $\mathbb{P}$. Also, $S(z) = 1/z$ and $T(z) = \omega z$ belong to $\mathcal{A}(\Omega)$ and generate a subgroup of order 6. Since the order of $\mathcal{A}(\Omega)$ is at most 6, we conclude that S and T generate $\mathcal{A}(\Omega)$. In particular, $\mathcal{A}(\Omega) \subset$ Möb but Ω is not a circular region. It is easy to modify this example to obtain a noncircular region of any finite connectivity $n \geqslant 3$ with automorphism group a subgroup of Möb.

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TRIANGLES WITH INTEGER SIDES

J. H. Jordan, Ray Walch, and R. J. Wisner

(Dedicated to the memory of Paul Clement, 1916–1976)

1. Introduction. Recall that the Pythagorean Triangles are right triangles with integer sides. Some Pythagorean Triangles have the same perimeter; for example, those with sides 24 : 45 : 51, 20 : 48 : 52, and 30 : 40 : 50 all have perimeter 120. Other triangles with integer sides have perimeter 120; for example, 40 : 40 : 40 and 36 : 38 : 46. A question that naturally arises is, "How many incongruent triangles with integer sides, not necessarily Pythagorean, have the same perimeter?" Some rather surprising results turned up in our attempt to answer this question. For example, a total of 300 incongruent triangles with integer sides have perimeter 120, while 310 incongruent triangles with integer sides have perimeter 119. In this paper, we answer the question that is posed above.

2. The Theorems. For notation, let $T(N)$ denote the number of incongruent triangles with integer sides that have perimeter N. The purpose of this paper is to completely characterize $T(N)$ by means of the following three theorems:

THEOREM 1. *$T(4) = 0$; $T(6) = T(8) = 1$; $T(10) = 2$; $T(12) = 3$; and $T(14) = 4$.*

THEOREM 2. *If $N \geqslant 4$ is even and R is defined by $N = 12K + R$ with $4 \leqslant R \leqslant 14$, then*

$$T(N) = \frac{N^2 - R^2}{48} + T(R).$$

THEOREM 3. *If* $N \geqslant 6$ *is even, then* $T(N) = T(N-3)$.

Notice that Theorem 2 tells us that $T(N)$ is a quadratic function of N, related recursively to the starting points given by Theorem 1, while Theorem 3 gives the number of incongruent triangles with odd perimeter.

The sequence $\{T(N): N \geqslant 3\}$ begins

$$1,0,1,1,2,1,3,2,4,3,5,4,7,5,8,7,10,$$
$$8,12,10,14,12,16,14,19,16,21,19,24,21,\ldots$$

and seems confusing if not intractible. However, by invitation of Theorem 3, an examination of the subsequence of its odd-numbered terms displays a clearer pattern and runs

$$1,1,2,3,4,5,7,8,10,12,14,16,19,21,24$$
$$27,30,33,37,40,44,48,52,56,61,65,70,75,\ldots$$

as does (after the first term) the subsequence of even-numbered terms. This explains the restriction to even arguments of Theorem 1 and part of the hypotheses of Theorem 2.

3. The Proofs. The proof of Theorem 1 requires only the enumeration of the possible triangles. The proof for $N=12$ consists of the list $2:5:5$, $3:4:5$, and $4:4:4$, so $T(12)=3$. The proofs of the other parts of this theorem are similar and are omitted.

For the proof of Theorem 2, let $N \geqslant 4$ be even with K and R defined by $N = 12K + R$, where $4 \leqslant R \leqslant 14$. We denote by $[x]$ the greatest integer $\leqslant x$.

Let a be the length of one side of the triangle. We may choose it so that $a \leqslant [N/3]$. The remaining perimeter, $N-a$, is the sum of the lengths of the other two sides. Let us begin counting these triangles with a given side length a by first counting the triangle with the other two sides being as near equal in length as possible. This triangle would be

$$a:\left[\frac{N-a}{2}\right]:N-a-\left[\frac{N-a}{2}\right].$$

The second triangle counted would be

$$a:\left[\frac{N-a}{2}\right]-1:N-a-\left[\frac{N-a}{2}\right]+1.$$

We can now continue this count by shortening the second side by one and lengthening the third side by one until the difference in length between the third side and the second side is $\geqslant a$. Hence we must only count the list:

$$\begin{array}{l}
a:\ \left[\frac{N-a}{2}\right]:\ N-a\left[\frac{N-a}{2}\right]\\
a:\ \left[\frac{N-a}{2}\right]-1:\ N-a-\left[\frac{N-a}{2}\right]+1\\
\quad\vdots\\
a:\ \left[\frac{N-a}{2}\right]-j:\ N-a-\left[\frac{N-a}{2}\right]+j\\
\quad\vdots\\
a:\ \left[\frac{N-a}{2}\right]-\left[\frac{a}{2}\right]+1:\ N-a-\left[\frac{N-a}{2}\right]+\left[\frac{a}{2}\right]-1.
\end{array}$$

The number of the triangular triples in this list is $[a/2]$, and of course each has perimeter N.

Since each triangle of perimeter N must have one side that is at most $[N/3]$, it follows that

$$T(N) \leqslant \sum_{a=1}^{\left[\frac{N}{3}\right]} \left[\frac{a}{2}\right].$$

If we order the sides $a:b:c$ so that $a \leqslant b \leqslant c$, then in the array above, there is duplication when $[(N-a)/2]-j<a$. This means that $[(N-a)/2]-a<j$, but $j \leqslant [a/2]-1$. There are thus no j's satisfying this last inequality when $[(N-a)/2]-a \geqslant [a/2]-1$. The exact number of duplications to be culled from the array is the number of integers in the half-open interval $((N-a)/2-a, a/2-1]$, which is

$$\left[\frac{a}{2}\right]-1-\left[\frac{N-a}{2}\right]+a=a-1-\frac{N}{2}+\left[\frac{a}{2}\right]-\left[\frac{-a}{2}\right]=2a-1-\frac{N}{2}.$$

When N is even, $[(N-a)/2]=N/2+[-a/2]$, so the inequality becomes $N/2+1 \geqslant [a/2]-[-a/2]+a=2a$, or $(N+2)/4 \geqslant a$. Therefore, duplications cannot occur until a satisfies $[(N+2)/4]+1 \leqslant a \leqslant [N/3]$. Hence, using the bounds found for a above, we obtain

$$T(N)=\sum_{a=1}^{\left[\frac{N}{3}\right]}\left[\frac{a}{2}\right]-\sum_{a=\left[\frac{N+2}{4}\right]+1}^{\left[\frac{N}{3}\right]}\left(2a-1-\frac{N}{2}\right).$$

The task is now to simplify this formula for $T(N)$. To do this, we consider separately the cases $R=4$, 6, 8, 10, 12, and 14. To illustrate the evaluations, we detail only the case $R=10$ and leave the other cases to the interested reader.

If $N=12K+10$, then $[N/3]=4K+3$ and $[(N+2)/4]=3K+3$. So for this case,

$$\begin{aligned}
T(N)&=\sum_{a=1}^{4K+3}\left[\frac{a}{2}\right]-\sum_{a=3K+4}^{4K+3}\left(2a-1-\frac{N}{2}\right)\\
&=0+1+1+2+2+\cdots+(2K+1)+(2K+1)\\
&+\left(1+\frac{N}{2}\right)(K)-2\sum_{a=3K+4}^{4K+3} a\\
&=(2K+1)(2K+2)+(6K+6)K-K(7K+7)\\
&=4K^2+6K+2+6K^2+6K-7K^2-7K\\
&=3K^2+5K+2.
\end{aligned}$$

Rearranging we have

$$\begin{aligned}
T(N)&=\frac{144K^2+240K+100-100}{48}+2\\
&=\frac{(12K+10)^2-10^2}{48}+T(10)\\
&=\frac{N^2-R^2}{48}+T(R).
\end{aligned}$$

The other five cases are handled similarly, proving Theorem 2.

For the proof of Theorem 3, let $\mathfrak{E}(N)$ be the set of incongruent triangles with perimeter N for N even, and let $\mathfrak{O}(M)$ be the set of incongruent triangles with perimeter M for M odd. Note that $1:b:c$ cannot be a member of $\mathfrak{E}(N)$ since $1+b+c$ is not even unless b and c have opposite parity implying $c-b \geqslant 1$, which violates the triangle inequality.

Now if $a:b:c$ is a triangle in $\mathfrak{E}(N)$, then $a \geqslant 2$ and it follows that $a-1:b-1:c-1$ is a triangle in $\mathfrak{O}(N-3)$; provided the triangular inequality holds. Since N is even, $a+b$ and c have the same parity and $a+b>c$, therefore $a+b \geqslant c+2$. Hence $a-1+b-1 \geqslant c > c-1$, verifying the triangular inequality. Since $a-1:b-1:c-1$ is in $\mathfrak{O}(N-3)$, it follows that $T(N) \leqslant T(N-3)$. On the other hand, if $a:b:c$ is a triangle in $\mathfrak{O}(M)$, then $a+1:b+1:c+1$ is clearly a triangle in $\mathfrak{E}(M+3)$; therefore, for M odd, $T(M) \leqslant T(M+3)$. It follows for N even that $T(N-3) \leqslant T(N)$ and $T(N) \leqslant T(N-3)$; therefore $T(N)=T(N-3)$, proving Theorem 3.

4. Corollaries. Some useful corollaries can be derived:

COROLLARY 1. *If M is odd and R is defined by $M=12K+R$ with $1 \leqslant R \leqslant 11$, then*

$$T(M)=\frac{M(M+6)-R(R+6)}{48}+T(R).$$

This follows by substituting $M+3$ for N in Theorem 2 and noticing that $T(R)=T(R+3)$ for R odd.

COROLLARY 2. *If N is even, then*

$$T(N+12)=T(N)+\frac{N}{2}+3.$$

This follows from the fact that $(N+12)^2=N^2+48(N/2+3)$.

COROLLARY 3. *If M is odd, then*

$$T(M+12)=T(M)+\frac{M+9}{2}.$$

This follows from the fact that $(M+12)(M+18)=M(M+6)+48((M+9)/2)$.

5. Remarks. A natural continuation of this topic is the problem of determining the number of incongruent *quadrilaterals* with a prescribed perimeter. The extra degree of freedom gives rise to many difficulties however, since there are for $3:5:6:9$ the configurations (see Figure 1) and many others.

The configurations.

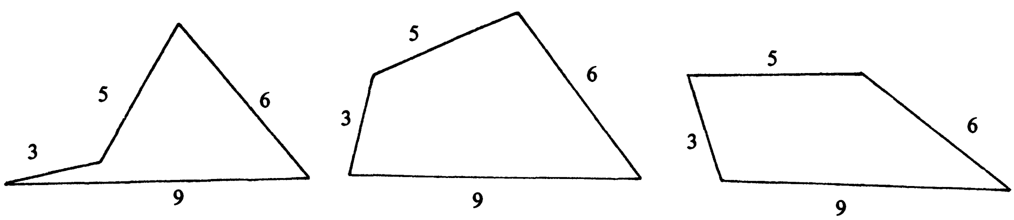

FIG. 1

One could choose to count only the convex quadrilaterals, making the problem easier. Yet even then, the side lengths do not uniquely determine a quadrilateral. Perhaps imposing maximal area would be a condition that brings about uniqueness. At any rate, the problem seems less interesting than does the one for triangles.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada T2N 1N4.

SOLUTIONS OF AN EQUATION IN ABELIAN GROUPS

RAPHAEL M. ROBINSON

Let $\mathfrak{G}$ be a finite Abelian group which is written multiplicatively. Let the elements of $\mathfrak{G}$ be $G_1, G_2, \ldots, G_m$, and consider all the linear combinations $a_1G_1 + a_2G_2 + \cdots + a_mG_m$, where the coefficients $a_1, a_2, \ldots, a_m$ are integers. If addition and multiplication of these expressions are defined in the obvious way, then they form a commutative ring $Z[\mathfrak{G}]$, the group ring of $\mathfrak{G}$ over Z.

We shall be concerned here with the equation

$$(B_1 - 1)(B_2 - 1)\cdots(B_n - 1) = 0,$$

where the unknowns $B_1, B_2, \ldots, B_n$ are to be chosen from some finite Abelian group $\mathfrak{G}$. The equality is to hold in $Z[\mathfrak{G}]$. This equation arose in Hajós [1], in connection with his proof of Minkowski's conjecture. I studied the equation in [2, §§3–6]. I found all the solutions for $n \leqslant 5$, and also proved some theorems about the solutions in general, and made some conjectures.

We shall consider just one problem here. What is the smallest order of any B_i? It turned out that, for $n = 1$ or 2, there is an element B_i of order 1; for $n = 3$, an element B_i of order 1 or 2; and for $n = 4$ or 5, an element B_i of order 1, 2, or 3. Is it true in general that, for any $n > 1$, there is some B_i of order less than n?

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CLASSROOM NOTES

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Material for this department should be sent to Professor Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis, MO 63121.

Advice to prospective authors: The editors have been receiving about **ten times** as many Classroom Notes as can be used. It will simplify our work if authors will submit only especially interesting manuscripts.

R. P. B.

other way is to write it as "Demoivre," like Descartes, etc. As this means a jump in the alphabetical order, I have taken the liberty of suggesting "Moivre." It may ease what always appears to be difficult, namely, to change a tradition.

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A USEFUL LEMMA IN THE THEORY OF SECOND-ORDER LINEAR DIFFERENTIAL EQUATIONS

WALTER LEIGHTON

The purpose of this note is to direct attention to a result that is of importance in the elementary theory of the second-order linear differential equation

$$a(x)y'' + b(x)y' + c(x)y = 0. \tag{1}$$

We suppose that $a(x) > 0$, that $a(x)$, $b(x)$, and $c(x)$ are continuous on some interval I, and that $u(x)$ and $v(x)$ are any two linearly independent solutions of this equation. It will be recalled that every solution of this differential equation may then be written in the form

$$c_1u(x) + c_2v(x),$$

where c_1 and c_2 are suitably chosen constants. Conversely, all such linear combinations of solutions are also solutions.

The result in question, which we shall call a lemma because it is primarily useful in establishing other results, is this.

LEMMA. *A necessary and sufficient condition that $x = b$ be conjugate to $x = a$ is that the determinant*

$$D \equiv \begin{vmatrix} u(a) & v(a) \\ u(b) & v(b) \end{vmatrix} = 0.$$

It will be recalled that a point $x = b$ is called *conjugate* to $x = a(b \neq a)$, if there exists a non-null solution of the differential equation that vanishes at both these points.

The proof of the lemma is easy. Suppose first that $D = 0$. There then exist constants c_1 and c_2, not both zero, such that

$$c_1u(a) + c_2v(a) = 0,$$
$$c_1u(b) + c_2v(b) = 0.$$

The solution $c_1u(x) + c_2v(x)$, with this choice of constants, vanishes at both $x = a$ and $x = b$ and is not identically zero. The proof of the sufficiency is complete.

To prove the necessity of the condition, suppose that $x = b$ is conjugate to $x = a$. There then exists a non-null solution of the differential equation that may be written in the form $c_1u(x) + c_2v(x)$ with the property that

$$c_1u(a) + c_2v(a) = 0,$$
$$c_1u(b) + c_2v(b) = 0.$$

Inasmuch as c_1 and c_2 are not both zero, it follows that D must be zero, and the proof of the lemma is complete.

A couple of examples will illustrate the use of the lemma.

If $x=b$ is not conjugate to $x=a$, the points (a,c) and (b,d) can be joined by a unique solution curve $y=y(x)$ of the differential equation.

Here c and d are arbitrary numbers. To prove this result it is sufficient to show that there is one and only one pair of constants c_1 and c_2 such that

$$c_1u(a)+c_2v(a)=c,$$
$$c_1u(b)+c_2v(b)=d.$$

But this follows at once from the fact that $D\neq 0$.

A second illustration is the following. Suppose that $q(x)\geqslant 0$ and that $q(x)$ and $f(x)$ are continuous on an interval $a\leqslant x\leqslant b$. To show that a system

$$\begin{aligned} y''-q(x)y&=f(x),\\ y(a)=y(b)&=0 \end{aligned} \tag{2}$$

always has a unique solution, recall that the general solution of (2) can be written in the form

$$c_1u(x)+c_2v(x)+y_0(x),$$

where $y_0(x)$ is any particular solution of (2), and $u(x)$ and $v(x)$ are linearly independent solutions of the corresponding homogeneous differential equation

$$y''-q(x)y=0.$$

It is sufficient then to show that there exists a unique pair of constants c_1 and c_2 such that

$$c_1u(a)+c_2v(a)+y_0(a)=0,$$
$$c_1u(b)+c_2v(b)+y_0(b)=0.$$

But this is a consequence of the fact that the determinant

$$D\equiv\begin{vmatrix} u(a) & v(a)\\ u(b) & v(b)\end{vmatrix}\neq 0,$$

for if D were zero, $x=b$ would be conjugate to $x=a$; however, by the Sturm comparison theorem, no non-null solution of the homogeneous differential equation

$$y''-q(x)y=0$$

can have more than one zero.

The proof is complete.

It is clear from the foregoing that if $u(x)$ and $v(x)$ are any two linearly independent solutions of (1), the zeros of the solution $v(a)u(x)-u(a)v(x)$ yield the conjugate points of $x=a$.

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AN ALGORITHM FOR THE CALCULATION OF π

GEORGE MIEL

The algorithm in question, which is readily programmed for a digital computer, may be shown in an elementary course in numerical analysis. The user chooses an arbitrary number $t\in(-1,1]$. The algorithm tries to find a series expansion which depends on t and which converges to π. If the attempt is successful, it uses the series to calculate π within a prescribed tolerance. Series generated from different choices of t are different. The algorithm thus defines a class of series expansions, which includes classical expansions due to Leibniz, Euler, and others. The algorithm can be used as the starting point of a discussion of the pertinent procedures and historical background leading to the recent computation of π to a million decimals.

To approximate π with an error bounded by a given $\epsilon>0$, proceed as follows:
Step 1. Choose $t\in(-1,1]$.
Step 2. Let $T_0=1$ and $T_i=(T_{i-1}-t)/(tT_{i-1}+1)$, $i\geqslant 1$.
Step 3. Iterate Step 2 with $i=1,2,\dots,N$ provided that the following conditions hold:

$$tT_i>-1, \qquad 1\leqslant i\leqslant N-1, \tag{1}$$

$$|T_N|\leqslant 1. \tag{2}$$

Step 4. Compute the sums

$$s_p=\sum_{i=0}^{p} a_i, \qquad a_i=\frac{(-1)^i t^{2i+1}}{2i+1},$$

$$\bar{s}_q=\sum_{j=0}^{q} \bar{a}_j, \qquad \bar{a}_j=\frac{(-1)^j T_N^{2j+1}}{2j+1},$$

where the numbers of terms p and q are chosen so that

$$Nt^2|a_p|+T_N^2|\bar{a}_q|\leqslant\frac{\epsilon}{4}.$$

Step 5. Approximate π by the number $\hat{\pi}=4Ns_p+4\bar{s}_q$.
The above algorithm does not consider rounding errors. Note that proper action must be taken to guarantee that the iterates $T_1,\dots,T_N$ computed in Step 3 satisfy *both* conditions (1) and (2): either continue the iterations until they do or else start over with a new t. The following simple lemma is needed to show that $|\pi-\hat{\pi}|\leqslant\epsilon$.

LEMMA. *If* $tT>-1$, *then* $\tan^{-1}(\frac{T-t}{tT+1})=\tan^{-1}T-\tan^{-1}t$.

Proof. Let $\theta=\tan^{-1}T$, $\phi=\tan^{-1}t$, and $I=(-\pi/2,\pi/2)$. Then

$$\theta,\phi\in I, \tag{3}$$

$$\tan(\theta-\phi)=\frac{\tan\theta-\tan\phi}{\tan\phi\tan\theta+1}=\frac{T-t}{Tt+1}.$$

The desired conclusion holds provided that $(\theta-\phi)\in I$. It follows from (3) that

$$\cos\theta>0, \qquad \cos\phi>0, \qquad (\theta-\phi)\in(-\pi,\pi).$$

Moreover,

$$\cos(\theta-\phi)=(1+tT)\cos\theta\cos\phi.$$

Consequently, $\cos(\theta-\phi)>0$ and $(\theta-\phi)\in I$.

Example: The hypothesis of the lemma is needed. If $T=-t=\sqrt{3}$, then $\tan^{-1}(\frac{T-t}{tT+1})=-\pi/3$ and $\tan^{-1}T-\tan^{-1}t=2\pi/3$.

THEOREM. *The number* $\hat{\pi}=4Ns_p+4\bar{s}_q$ *approximates* π *with an error bounded by* $4Nt^2|a_p|+4T_N^2|\bar{a}_q|$.

Proof. We first show that

$$\pi=4N\sum_{i=0}^{\infty} a_i+4\sum_{j=0}^{\infty}\bar{a}_j. \tag{4}$$

Use (1) and the lemma to get

$$\begin{aligned}\tan^{-1}T_N&=\tan^{-1}\left(\frac{T_{N-1}-t}{tT_{N-1}+1}\right)=\tan^{-1}T_{N-1}-\tan^{-1}t\\&=\tan^{-1}T_{N-2}-2\tan^{-1}t=\tan^{-1}T_0-N\tan^{-1}t.\end{aligned}$$

Since $T_0=1=\tan\frac{\pi}{4}$,

$$\pi=4N\tan^{-1}t+4\tan^{-1}T_N.$$

Since $|t|\leqslant 1$ and $|T_N|\leqslant 1$, both arctangents can be expanded into series to obtain (4).

Next, if $s=\lim_{p\to\infty}s_p$ and $\bar{s}=\lim_{q\to\infty}\bar{s}_q$ then

$$|\pi-\hat{\pi}|\leqslant 4N|s-s_p|+4|\bar{s}-\bar{s}_q|.$$

Use Leibniz's estimate of the remainder of alternating series to get

$$|s-s_p|\leqslant|a_{p+1}|\leqslant t^2|a_p|,$$

$$|\bar{s}-\bar{s}_q|\leqslant|\bar{a}_{q+1}|\leqslant T_N^2|\bar{a}_q|.$$

The desired error bound follows.

The identity (4), which may be denoted by the triple (t,N,T_N), includes classical expansions given on pages 382–384, 394, in Johnson [8]: the familiar series $(0,0,1)$ due to Leibniz; $(\frac{1}{2},1,\frac{1}{3})$ due to Euler; $(\frac{1}{2},2,\frac{-1}{7})$, $(\frac{1}{3},2,\frac{1}{7})$, $(\frac{1}{4},3,\frac{5}{99})$ due to Hutton; and $(\frac{1}{5},4,\frac{-1}{239})$ due to Machin. Observe that fast convergence requires small values for $|t|$ and $|T_N|$. Some advantageous combinations are:

$$\left(\tfrac{1}{6},5,\tfrac{-475}{11767}\right),\left(\tfrac{1}{7},5,\tfrac{237}{3116}\right),\left(\tfrac{1}{7},6,\tfrac{-1457}{22049}\right),\left(\tfrac{1}{8},6,\tfrac{15247}{388079}\right).$$

We note that most of the formulas generated by the algorithm are not suitable for high accuracy computations, as they can run the storage and time up drastically.

However, the algorithm serves well as a pedagogical device for initiating a description of the methods and history leading to the recent evaluation of π to a million decimals. The instructor may start such a description by pointing out that the computation of (t,N,T_N) is simplified when t and T_N are reciprocals of integers, and that Størmer [11] proved in 1895 that there exist exactly four such triples. The most celebrated of these is $(\frac{1}{5},4,\frac{-1}{239})$, originally used in 1706 by Machin to calculate π to 100 decimals, and programmed as recently as 1959 to carry π to over 16,000 decimals on an IBM 704 computer. These and other historical events, which occurred prior to 1960, are conveniently listed in [13]. For an account of more recent calculations, consult [2, Chap. 18]. It should be noted, however, that the latter reference contains inaccuracies; see [6]. Old notes in this MONTHLY [1], [3], [4], [9], [12] may also be used to discuss the historical development of high accuracy computation of π.

In our algorithm, as t is chosen smaller than $\frac{1}{5}$, T_N becomes progressively unwieldy. However, one can decompose the corresponding expansion (t,N,T_N) into a relation which contains three arctangents instead of two. For example, using the fact that if $tT<1$ then

$$\tan^{-1}\left(\frac{t+T}{1-tT}\right)=\tan^{-1}t+\tan^{-1}T,$$

we get

$$\tan^{-1}\frac{15247}{388079}=\tan^{-1}\frac{57}{1624}+\tan^{-1}\frac{1}{239},$$

$$\tan^{-1}\frac{57}{1624}=2\tan^{-1}\frac{1}{57}.$$

Consequently, the triple

$$\left(\frac{1}{8},6,\frac{15247}{388079}\right)$$

is equivalent to the more convenient relation,

$$\pi=24\tan^{-1}\frac{1}{8}+8\tan^{-1}\frac{1}{57}+4\tan^{-1}\frac{1}{239},\tag{5}$$

used by Shanks and Wrench [10] in 1961 to calculate π to over 100,000 decimals. The computation, which required 8 hours 43 minutes on an IBM 7090 system, was ingeniously verified [10, p. 77] by using the relation,

$$\pi = 48\tan^{-1}\frac{1}{18} + 32\tan^{-1}\frac{1}{57} - 20\tan^{-1}\frac{1}{239}, \tag{6}$$

which can be shown to be equivalent to the triple $(\frac{1}{18}, 12, T)$, where approximately $T = 0.1199869898$.

The two three-term arctangent relations (5) and (6) were subsequently used by Guilloud and associates in Paris to ultimately calculate π to 1,000,000 decimals; the final value appeared in 1974 in [7]. The computer used was a CDC 7600 and the running time, not including the verification, was 23 hours and 18 minutes, of which 1 hour 7 minutes was used to convert the final result from binary to decimal. An amusing anecdote, very effective in the classroom, is to point out that in 1966, the infamous Dr. Matrix [5, p. 100] predicted that the millionth digit of π would be 5, since in the King James Bible, third book, chapter 14, verse 16, the magical number seven appears and the seventh word has five letters. Surprisingly, the millionth digit of π does turn out to be 5. (The millionth decimal place, excluding the initial 3, is 1.)

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12. J. W. Wrench, Jr., On the derivation of arctangent equalities, this MONTHLY, 45 (1938) 108–109.

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MATHEMATICAL EDUCATION

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MATHEMATICS AS BACKGROUND FOR THE STUDY OF LAW

MICHAEL GEMIGNANI

About two years ago I was admitted to Indiana University–Indianapolis Law School. My primary interest was law as it relates to university administration. Although as a mathematician

I am happy to say that I have done quite well in legal studies, at least one law instructor expressed serious doubts about mathematics as a preparation for law. This instructor reasoned that mathematicians are trained to travel the strict path of logic from hypotheses to conclusion. In a given mathematical system, a given theorem is either true or false. "Facts" are determined either by what is accepted as axiomatic or by what is logically derivable from the initial assumptions.

Law, of course, is rather different. Lawyers should be able to argue both sides of a case, both to a conclusion and to its negation. It is this indeterminacy and apparent "illogic" that the instructor felt might cause the organized mind of the mathematician some confusion. Moreover, "laws" do not hold the same position for lawyers that axioms do for mathematicians. Even if we restrict our attention solely to the "Rules of Law" stated in our nation's Constitution, we observe that these rules have been interpreted by a multiplicity of courts in a variety of jurisdictions in different ways at different times. For example, one cannot use the Constitution as the basis for a mathematical theory of Constitutional law. Faced with a Constitutional question, a lawyer must look not merely to the Constitution—indeed, perhaps least of all to the Constitution—but also to how the courts have interpreted the Constitution with regard to the point in issue. The resulting search, as often as not, leads not in a straight line to a logical conclusion but to any one of a number of defensible positions by a tortuous maze of twisting and branching paths. And the real proof of any proposition is not the irrefutable logic of the argument used to derive it but how effectively that argument persuades a court to adopt that conclusion as its own.

All this being so, can we properly recruit students who wish to enter law school into the mathematical sciences with the promise that what we have to offer is one of the most appropriate backgrounds for what they will encounter later? My personal experience thus far leads me to believe the answer is an unqualified yes.

In the first place, although decisions of different courts with regard to seemingly the same question of law and identical factual situations sometimes differ, any court when it issues a written opinion tries to justify its decision by a more or less logical argument which begins with certain premises. These premises may be statutes, provisions of the State or Federal Constitutions, previous court decisions, or assumptions about society and what is best for the public welfare. Court opinions often differ not so much by their logic but, rather, in the premises which underlie their arguments. A mathematician is not only trained in logical skills to aid in following the argument, or finding the fallacies in it, but is more aware than most of the "axiomatic method" and hence is looking for the underlying hypotheses of the system.

Second, like many concepts in mathematics, most of the key concepts in law are "terms of art"; that is, they are fairly abstract and defy any simple definition. Their meaning becomes clear only to someone who has studied them at some length. In mathematics, for example, a group is a set which has certain properties—properties which, in turn, imply other properties. Groups can be categorized in various ways, such as abelian, nonabelian, simple, finite, infinite, etc., with respect to properties they possess above and beyond their merely being groups.

In a somewhat analogous way in law, a fundamental legal concept may be defined formally in a relatively brief manner. But that definition and its full range of implications can only be understood through a thorough study of what courts and legal scholars have said about the concept. We go from the general to the more specific, distinguishing ever more finely to arrive eventually at some real understanding of what the initial concept means in practice. In mathematics, merely knowing the definition of a group is of little value. We do not really know what a group is until we have studied that definition from the viewpoint of where it leads us and what it implies. Likewise, in law a basic definition is merely the starting point for a prolonged study that eventually leads to some comprehension of what lies hidden behind the deceptive simplicity of the initial definition.

Third, it is important in law to be able to make close and often subtle distinctions. One factual situation may appear much like another, but the law with respect to each may be quite different. If, for example, the Supreme Court has ruled that in certain circumstances evidence can be seized without a search warrant, then a lawyer for a defendant who is faced with incriminating evidence seized without a search warrant will argue that there is a critical distinction to be made between the circumstances involving his client and those of the previous rulings, while the prosecutor will try equally hard to show that the circumstances fit within those which justify seizure of evidence without a warrant.

Mathematicians often find themselves searching for some distinctive property that will give a desired result. It took years to develop a "good" definition of compactness. It takes many decisions of many courts before rules basic to a field of law are framed in a widely accepted manner. This search for elegance and appropriate distinctions should make a mathematician feel very much at home.

Fourth, as those who have taken law examinations know, these examinations are more often than not concerned with spotting issues in some given factual situation and then applying rules of law to those issues to argue toward a conclusion. This is not unlike what the mathematician does when he looks at a situation and seeks to produce order from it by finding the basic issues —i.e., questions to ask about the situation and the rules which govern the system—and then argues to a conclusion.

Fifth, there is much that is quantitative about law studies. Quite apart from those areas, such as taxation and estate planning, in which quantitative and computational skills are obviously important, there are other areas in which mathematics enters in more subtle ways, such as the geometry that is inherent in the description of land through surveys or by metes and bounds or the interpretation of statutes involving interrelated variables. Obviously too, with the growth of technology, there is both the use of that technology in the practice of law and the law of the technology itself. The law relating to computers is still evolving rapidly, as are computers themselves, and a lawyer with a solid grounding in computer science probably has a very bright future indeed. Lawyers themselves and the entire legal system are becoming more and more dependent on computers to keep track of pending and settled litigation, accounting and billing, and legal publications.

The Law School Admission Test (LSAT), which, together with grade-point average, is the indicator most used to determine admission to law school, contains virtually no questions that courses in the humanities would be especially useful in answering, other than what those courses might bring the applicant in the way of skills in reading comprehension and correct English usage. On the other hand, mathematical skills are extremely useful in the sections on data interpretation, quantitative comparison, and logical reasoning. They apply also to the practical judgment and case-principle sections. In practical judgment questions, persons are asked to identify major objectives, major and minor factors, major assumptions, and unimportant issues relating to a decision in a hypothetical business situation. In many of the case-principle questions, persons must find the narrowest rule which justifies a given court decision and which is also consistent with all of the rules chosen in previous questions in that same section. This is not at all unlike finding the defining characteristics of a given set and of some of its proper subsets.

In sum, for all of the reasons listed above and for others omitted in the interest of space, I believe that a prospective law student could do far worse, and not much better, than have mathematics as a major field.

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COMMENT ON THE DIVISION OF THE PLANE BY LINES

HASSLER WHITNEY

The purpose of this note is to illustrate what I consider a basic principle in mathematical research: If a problem is difficult, don't be afraid to get your hands dirty. That is, make up various examples, simple and complex, and test out what happens, in a rather complete manner.

The example I consider is: How many regions are formed by a set of straight lines in the plane? I refer to the recent paper of John E. Wetzel [1].

To follow the suggested principle, draw a number of lines in the plane, and actually look at the counting process. The question arises at once: In what order shall I count the regions? There are plenty of choices; so look for some simple choice.

If the "plane" is a large sheet of paper on a table, you might first count the ones furthest up, then move downwards. You could stretch a stick across and move it downwards, counting from left to right if several regions are touched at the same time. Or to avoid hiding parts of the figure, move a stretched string downwards. Call this a "sweep line"; we have come naturally to the "ingenious notion" in Wetzel's paper (p. 654). If the diagram is really large, and you are in the middle, you could first count those regions furthest away; the sweep line is replaced by a shrinking circle. A difficulty here (counting some regions twice) is apt to lead you to flatten the circles, thus bringing you to the sweep line.

My most important message here is toward the educational process: You learn math best by *doing* it; especially, by *needing* it in some situation. Especially in early grades, "dirtying your hands" becomes "act out the story." In this way, there is a much better chance that schoolchildren will regain some of the extraordinary rate of learning that they exhibited before entering school.

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Some of these remarks were presented in an address to the American Mathematical Society at Brown University, August 10, 1978.

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COMMENT ON "GRADING ANSWER-UNTIL-CORRECT TESTS"

ROBERT B. ISRAEL

In [1], J. D. Austin proposes a grading system for a multiple-choice test in which the student makes successive choices from the n possible answers until hitting on the correct answer. However, this novel feature is effectively nullified by an oversimplified model of the student's behavior: either he knows the answer or he makes completely random guesses. In this situation, whenever there is at least one error we know that the student is just guessing, and any further information on the number of errors is irrelevant. Thus it is not surprising that Austin's grading system is the same as for the ordinary multiple-choice test: all that counts is the first choice. Under this system, moreover, a student motivated by grades and knowing the grading system will not waste further mental effort after making one error.

A slightly more plausible model of the student's behavior is that with probability α_k he can (correctly) eliminate k of the possible answers ($k=0,1,\ldots,n-1$), and then guess randomly among the remaining $n-k$ until successful. If X is the number of wrong answers he makes, then

$$P(X=x)=\sum_{k=0}^{n-1-x}(n-k)^{-1}\alpha_k.$$

For a grading system that awards $f(x)$ points for x wrong answers, we have

$$Ef(X)=\sum_{k=0}^{n-1}(n-k)^{-1}\alpha_k\sum_{x=0}^{n-1-k}f(x).$$

Now we must choose just what we want to estimate by our grading system. One reasonable choice is $\lambda\equiv(n-1)^{-1}Ek=(n-1)^{-1}\Sigma k\alpha_k$. The unique unbiased estimator for this λ is $\hat{\lambda}(x)=1-2x/(n-1)$. On the other hand, we might wish to estimate α_{n-1}, the probability that the student knows the correct answer outright. The unique unbiased estimator for this is

$$\hat{\alpha}_{n-1}(x)=\begin{cases}1 & \text{if } x=0\\ -1 & \text{if } x=1\\ 0 & \text{otherwise}\end{cases}.$$

This would not be a workable system because $\hat{\alpha}_{n-1}(x)$ is not a decreasing function of x.

The above choice of λ is only one of several reasonable possibilities. It assumes that if the student operates according to the model, then the instructor rewards him equally for each answer he correctly eliminates. Depending on the nature of the question and answers, he might make another choice. The only constraint is that the student's score be a decreasing function of the number of wrong answers.

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PROBLEMS AND SOLUTIONS

EDITED BY A. P. HILLMAN

The editors receive more problems than they can use, but they are always on the lookout for interesting and novel problems. The task of selection will be much easier if proposers will keep the following guidelines in mind:

Send all proposed problems, in duplicate if possible, to American Mathematical Monthly, Department of Mathematics, Northwestern University, Evanston, IL 60201. Please include solutions and any information that will help the editors, including reasons that the problem is interesting. Problems in well-known textbooks and results in generally accessible sources are not acceptable.

Solutions should be sent to the addresses given at the head of each problem set.

An asterisk (*) indicates that the proposer did not supply a solution. If you submit a problem without a solution, you should tell the editors whether you know (or somebody else knows) how to solve the problem. If you are uncertain, the problem might be more suitable for the Research Problems section; see the instructions given there.

Proposers are asked to aim for the same audience as for the rest of the MONTHLY; *a rule of thumb is to think of people who have had at least a year of graduate work in Mathematics, very likely some years ago. Proposals that can be both understood and solved by high school students, or by those in the first two years of college, are usually more appropriate for the Mathematics Magazine or the Two-Year College Mathematics Journal. No precise line can be drawn between elementary and advanced problems, but in general an elementary problem will involve notions no more difficult than those encountered in the first year of graduate work and should be solvable by methods common at the same level. An advanced problem may deal with more advanced concepts or require more sophisticated techniques for its solution.*

A problem that takes half a page or more to state, or that involves several ad hoc definitions, is usually unsuitable. The statement of a problem should avoid specialized jargon and symbols whenever the English language is more readily understood: for example, "f is a continuous function" is preferable to "$f \in C$."

Before submitting a problem, be sure that it says exactly what you intend it to say. Solvers are not telepathic and will usually solve what was proposed, not what you had in mind.

PROBLEMS DEDICATED TO EMORY P. STARKE

Solutions of the problems in this issue dedicated to Professor Emory P. Starke should be mailed to Prof. A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, N.M. 87131 (USA) before February 29, 1980. To facilitate consideration, solutions should be typed (with double spacing).

S 19. *Proposed by Anon, Erewhon-upon-Spanish River.*

Let C be a smooth simple arc inside the unit disk, except for its endpoints, which are on the boundary. How long must C be if it cuts off one-third of the disk's area? Generalize.

S 20. *Proposed by A. P. Hillman, University of New Mexico, Albuquerque.*

Let n be a nonnegative integer and let S consist of all ordered quintuples $Q=(x_1,x_2,x_3,x_4,x_5)$ of nonnegative integers x_i with $x_1+x_2+x_3+x_4+x_5=n$. Prove or disprove that there are exactly the same number of Q in S with $x_2\leqslant x_3\leqslant x_4\leqslant x_5$ as there are satisfying the simultaneous conditions

$$x_1\leqslant x_2\leqslant x_4,\qquad x_1\leqslant x_3\leqslant x_4,\qquad \text{and}\quad x_3\leqslant x_5.$$

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be sent to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA (USA) 94303. To facilitate their consideration, solutions of Elementary Problems in this issue should be typed (in duplicate, with double spacing) and should be mailed before February 29, 1980. Please enclose a self-addressed label or card (for acknowledgment).

E 2791. *Proposed by John W. Vogel, Grinnell College.*

If the series of real numbers $\Sigma_{n=1}^{\infty}a_n$ converges, does $\Sigma_{n=1}^{\infty}a_n^3$ converge?

E 2792. *Proposed by Robert Patenaude, California State College, Bakersfield.*

Let U be a finite set. Characterize those collections C of subsets of U with the following property: there is a unique subset R of U such that the number of sets in C which R intersects is odd.

E 2793. *Proposed by E. D. Camier, Merseyside, England.*

P and Q are two points isogonally conjugate with respect to a triangle ABC of which the circumcenter, orthocenter, and nine-points center are O, H, and N, respectively. If $\overrightarrow{OR}=\overrightarrow{OP}+\overrightarrow{OQ}$, and U is the point symmetric to R with respect to N, show that the isogonal conjugate of U in the triangle ABC is the intersection V of the lines P_1Q and PQ_1 where P_1 and Q_1 are the inverses of P and Q in the circle ABC. (Assume that neither of P, Q is on the circle ABC.)

E 2794*. *Proposed by Robert A. Leslie, Agnes Scott College, Decatur, Ga.*

Let m, n, r, and c be positive integers with $rm=cn$. How many $m\times n$ matrices are there with each entry either 0 or 1 and where every row sum is r and every column sum is c?

E 2795. *Proposed by Doug Wiedemann, Institute for Defense Analyses, Princeton, N.J.*

Let S be a non-empty subset of $\{0,1\}^n=\{0,1\}\times\cdots\times\{0,1\}$ such that each member of S is adjacent to exactly k other members of S, where "adjacent" means differing in one coordinate position. Show that the size of S is even and at least 2^k. Furthermore, if the graph of the adjacency relation of S is connected, show that it will still be connected after removal of any point.

E 2796. *Proposed by P. Henrici, Eidgenössische Technische Hochschule, Zürich, Switzerland.*

Prove that the polynomial p with degree less than or equal to n that agrees with a given function $f(x)$ at the Chebyshev points

$$x_k=\cos\phi_k,\text{ where }\phi_k=(2k+1)\pi/(2n+2)\qquad(k=0,1,\ldots,n),$$

is, for x not in $\{x_0,\ldots,x_n\}$, given by $p(x)=N(x)/D(x)$ with

$$N(x)=\sum_{k=0}^{n}\frac{(-1)^k f(x_k)\sin\phi_k}{x-x_k},\qquad D(x)=\sum_{k=0}^{n}\frac{(-1)^k\sin\phi_k}{x-x_k}.$$

SOLUTIONS OF ELEMENTARY PROBLEMS

Hankel Determinant

E 2709 [1978, 276]. *Proposed by R. M. Norton, The College of Charleston, South Carolina.*

Let $A=(a_{ij})$, $0\leqslant i,j\leqslant n$ be a Hankel matrix defined by

$$a_{ij}=\begin{cases}0 & \text{if } i+j \text{ is odd}\\ \dbinom{2i+2j}{i+j} & \text{if } i+j \text{ is even.}\end{cases}$$

Compute $\det A$.

Two solvers decoded a typographical error in the statement of the problem: when $i+j=2k$ is even, the definition of a_{ij} should have been $\binom{2k}{k}$. Both solutions are sketched.

I. *Solution by A. A. Jagers, Twente University of Technology, Enschede, Netherlands.* Let F be the weight function of the associated Hankel form. Then $\int_{-\infty}^{\infty}x^hF(x)dx=\binom{2m}{m}$ or 0, according as $h=2m$ is even or h is odd. This classical moment problem may be solved by use of Fourier

transforms. Set $\phi(t)=\int_{-\infty}^{\infty}\exp(ixt)F(x)\,dx$. Then

$$\phi(t)=\sum_{n=0}^{\infty} t^n\phi^{(n)}(0)/n!,\qquad \sum_{n=0}^{\infty}(-1)^n\binom{2n}{n}t^{2n}/(2n)!=J_0(2t),$$

where J_0 is the Bessel function of the first kind and order 0. From a table of inverse transforms, $F(x)=\pi^{-1}(4-x^2)^{-1/2}$ if $|x|<2$, and $F(x)=0$ if $|x|>2$. Let $\{p_i\}$ be the associated sequence of orthonormal polynomials: $\deg p_i=i$, $\int_{-\infty}^{\infty}p_i(x)p_j(x)F(x)\,dx=\delta_{ij}$. Let k_i be the leading coefficient of p_i. Then $\det A=(k_0k_1\cdots k_n)^{-2}$. Now $p_0\equiv 1$, and $p_m(x)=\sqrt{2}\,T_m(x/2)$, $m\geqslant 1$, where T_m is the Chebyshev polynomial of the first kind: $T_m(\cos\theta)=\cos m\theta$. Then $k_0=1$, $k_m=1/\sqrt{2}$ $(m\geqslant 1)$. Thus $\det A=2^n$.

II. *Solution by Otto G. Ruehr, Michigan Technological University, Houghton, Michigan.* Since A is a Hankel matrix, we consider the associated power series and its continued fraction development. Define

$$f(x)=\sum_{k=0}^{\infty}\binom{2k}{k}x^{-(k+1)}=(x^2-4x)^{-1/2}$$
$$=\frac{b_0}{x}-\frac{b_1}{1}-\frac{b_2}{x}-\frac{b_3}{1}\cdots.$$

Let $\Delta_n=\det A_n$. The relation between Δ_n and the elements b_n of the above S-fraction for $f(x)$ is quoted on page 197 of H. S. Wall, *Analytic Theory of Continued Fractions* (Van Nostrand, 1948): $\Delta_{-1}=1$, $\Delta_n=\Delta_{n-1}\prod_{p=0}^{n}b_p$. Define $g(x)=(x/2)-\sqrt{(x^2/4-x)}$. Then $f(x)=[x-2g(x)]^{-1}$ and $g(x)$ has an S-fraction development, all of whose elements are unity. Therefore, $b_0=1$, $b_1=2$, $b_p=1, p\leqslant 2$. Solving the above recurrence relation then yields $\Delta_n=2^n$.

Solving the more general problem for which $a_{ij}=0$ if $i+j$ is odd, $a_{ij}=\binom{2rk}{rk}$ if $i+j=2k$, is tantamount to getting the continued fraction expansion for

$$G_r(x)=\sum_{k=0}^{\infty}(-1)^k\binom{2rk}{rk}x^k$$
$$=r^{-1}\sum_{j=1}^{r}\{1-4x^{1/r}\exp[i\pi(2j+1)/r]\}^{-1/2}.$$

Even though $G_2(x)$ is a relatively simple function, its continued fraction expansion is apparently not known. Extensive computation for the case $r=2$ (the problem as printed) shows that it is unlikely that a simple formula for $\det A$ exists in that case.

Also solved by Theodore S. Bolis.

A Multiplicative Map

E 2712 [1978, 277]. *Proposed by A. Wilansky, Lehigh University.*

Let A be a linear map from real bounded sequences to the real numbers such that for each sequence x some sub-sequence of x converges to $A(x)$. Must $A(xy)=A(x)A(y)$?

Solution I by P. R. Chernoff, University of California, Berkeley. It is convenient to regard sequences as functions on the set N of natural numbers. The given linear functional $A:l^{\infty}(N)\to R$ is obviously positive ($x\geqslant 0$ implies $A(x)\geqslant 0$), hence by a standard easy argument A is continuous relative to the sup norm. Accordingly, to show that A is multiplicative it is enough to show that it is multiplicative on the dense sub-algebra of simple functions—finite linear combinations of characteristic functions. By linearity, it suffices to show that $A(xy)=A(x)A(y)$ for characteristic functions x and y. Note that $A(x)=0$ or 1 for a characteristic function.

So consider two subsets E, F of N. First suppose that $A(\chi_E)=0$. Then $0\leqslant\chi_E\chi_F\leqslant\chi_E$ so, since A is order-preserving, $A(\chi_E\chi_F)=0=A(\chi_E)A(\chi_F)$. Next, suppose that $A(\chi_E)=1=A(\chi_F)$. If we denote complements by primes, we have the relation

$$\begin{aligned}1=A(1)&=A(\chi_E\chi_F+\chi_{E'}\chi_F+\chi_{F'})\\&=A(\chi_E\chi_F)+A(\chi_{E'}\chi_F)+A(\chi_{F'}).\end{aligned}$$

But $A(\chi_{E'})=0$ so $A(\chi_{E'}\chi_F)=0$. Likewise $A(\chi_{F'})=0$. Accordingly, $A(\chi_E\chi_F)=1=A(\chi_E)A(\chi_F)$. So in all cases A is multiplicative.

Solution II by J. W. Lark III, graduate student, Ohio State University. If x and $1/x\in l^\infty$, no subsequence of x may converge to 0. So $A(x)\neq 0$. The result follows from the theorem of Gleason, Kahane, and Zelazko given in Rudin, *Functional Analysis* (McGraw-Hill, 1973), p. 233.

Also solved by G. Gripenberg (Finland), J. Hagler, T. Jager, R. Johnsonbaugh, D. Langwitz, P. W. Lindstrom, S. P. Lloyd, N. Passell, W. A. Reagle, W. H. Ruckle, R. W. W. Taylor, K. Schilling, A. Stephanides, University of South Alabama Problem Group, and the proposer.

Editorial comment. Many interesting methods were proposed, e.g.: (a) A is a probability measure on βN and one shows that its support is a singleton. (b) A is a cluster point of $\{P_n\}$ in the weak* topology, hence $A\in\sigma(l^\infty)$, $l^\infty=C(\beta N)$. (c) If the ultrapower of $\hat{R}$ is taken with respect to the ultrafilter $\{E\subset N: A(\chi_E)=1\}$ then, for a standard sequence, x, $A(x)$ is the standard part of $x(\text{id})$, where id is the identity map on N.

Triangle Centroid

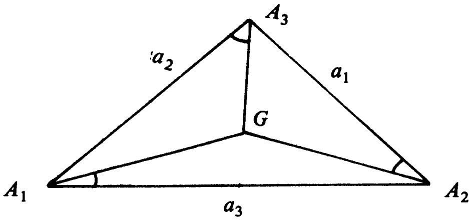
E 2715 [1978, 384]. *Proposed by Jack Garfunkel, Flushing, N.Y.*

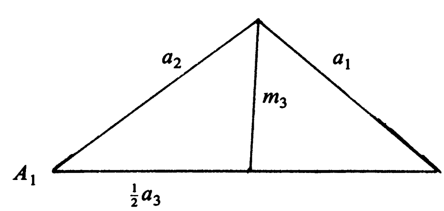
Let G be the centroid of the triangle $A_1A_2A_3$ and let

$$\theta_i=\measuredangle(\overrightarrow{A_iA_{i+1}},\overrightarrow{A_iG}),\qquad (i=1,2,3).$$

Prove or disprove that $\Sigma\sin\theta_i\leqslant 3/2$.

Solution by B. J. Venkatachala & C. R. Pranesachar, Indian Institute of Science, Bangalore, India.





The inequality is true. Let Δ denote the area, a_i the length of the side opposite A_i, and m_i the length of the median from A_i $(i=1,2,3)$. Then $\sin\theta_i=\Delta/(m_ia_{i-1})$, so that $\Sigma\sin\theta_i=\Delta[(m_1a_3)^{-1}+(m_2a_1)^{-1}+(m_3a_2)^{-1}]$. Hence by Cauchy's inequality,

$$\begin{aligned}(\Sigma\sin\theta_i)^2&\leqslant\Delta^2\left[a_3^{-2}+a_1^{-2}+a_2^{-2}\right]\left[m_1^{-2}+m_2^{-2}+m_3^{-2}\right]\\&=\Delta^2\left[a_2^2a_3^2+a_3^2a_1^2+a_1^2a_2^2\right]\left[m_2^2m_3^2+m_3^2m_1^2+m_1^2m_2^2\right]/(a_1a_2a_3m_1m_2m_3)^2.\end{aligned}$$

Now $16\Delta^2=2(a_2^2a_3^2+a_3^2a_1^2+a_1^2a_2^2)-a_1^4-a_2^4-a_3^4$; and from the law of cosines, $4m_1^2=2a_2^2+2a_3^2-a_1^2$, $4m_2^2=2a_3^2+2a_1^2-a_2^2$, $4m_3^2=2a_1^2+2a_2^2-a_3^2$. Thus $(4/9)\ (\Sigma\sin\theta_i)^2\geqslant N/D$, where

$$N=\left[2(a_2^2a_3^2+a_3^2a_1^2+a_1^2a_2^2)-a_1^4-a_2^4-a_3^4\right]\left[a_2^2a_3^2+a_3^2a_1^2+a_1^2a_2^2\right]^2,$$
$$D=a_1^2a_2^2a_3^2(2a_2^2+2a_3^2-a_1^2)(2a_3^2+2a_1^2-a_2^2)(2a_1^2+2a_2^2-a_3^2).$$

To see that $N/D\leqslant 1$, it is only necessary to check that $D=N+R^2$, where $R=(a_2^2-a_3^2)(a_3^2-a_1^2)(a_1^2-a_2^2)$. Set $D-N=f(x,y,z)$, $x=a_1^2$, $y=a_2^2$, $z=a_3^2$. From $f(y,y,z)=f_x(y,y,z)=0$, and from symmetry, it follows that $f(x,y,z)=kR^2$; clearly, $k=1$. Cauchy's inequality is strict unless $a_3/m_1=a_1/m_2=a_2/m_3$, in which case the triangle would be equilateral.

The above computations also show that the quantities $\pm R$, $4\Delta(a_2^2a_3^2+a_3^2a_1^2+a_1^2a_2^2)$, $8a_1a_2a_3m_1m_2m_3$ form the sides of a (possibly degenerate) right triangle.

It can be seen that if $e\leqslant 1$, the further inequality $\Sigma(\sin\theta_i)^e\leqslant 3\cdot 2^{-e}$ holds, with equality if and only if the triangle is equilateral.

A Functional Inequality

E 2720 [1978, 495]. *Proposed by Ralph P. Boas, Northwestern University.*

Show that $\sin^2 x<\sin(x^2)$ for $0<x\leqslant(\pi/2)^{1/2}$.

Solution I by P. Henrici, E. T. H., Zürich, and Stanford University. This is the special case $f(x)=\sin x$, $g(x)=x^2$, $a=\pi/2$, $b=(\pi/2)^{1/2}$ of the following theorem.

Let the functions f and g be differentiable and increasing on the intervals $[0,a]$ *and* $[0,b]$, *respectively, subject to the conditions*

$$\begin{array}{ll} \mathrm{H}_1: & f(0)=g(0)=0, \qquad f(a)\leqslant a=g(b);\\ \mathrm{H}_2: & g(f(a))=f(a);\\ \mathrm{H}_3: & f(x)<x \quad \textit{for } 0<x<a,\\ & g(x)<x \quad \textit{for } 0<x<f(a);\\ \mathrm{H}_4: & f' \textit{ is decreasing, } g' \textit{ is increasing.}\end{array}$$

Then

$$g\circ f(x)<f\circ g(x) \quad \textit{for } 0<x\leqslant b.$$

Proof. By H_1 and H_2, the equation

$$g\circ f(y)=f\circ g(x)$$

for each $x\in[0,b]$ has a uniquely determined solution $y=y(x)$. Denoting by $f^{[-1]}$ the inverse function of f,

$$y=f^{[-1]}\circ g^{[-1]}\circ f\circ g.$$

By the chain rule,

$$y'=f^{[-1]\prime}\circ g^{[-1]}\circ f\circ g\cdot g^{[-1]\prime}\circ f\circ g\cdot f'\circ g\cdot g'$$

or in view of $f^{[-1]\prime}=(f'\circ f^{[-1]})^{-1}$,

$$y'=\frac{f'\circ g}{f'\circ f^{[-1]}\circ g^{[-1]}\circ f\circ g}\,\frac{g'}{g'\circ g^{[-1]}\circ f\circ g}. \tag{1}$$

We wish to show that $y'(x)>1$ for $0<x<b$ and to this end show that each of the fractions on the right is >1. By H_4, the first fraction is >1 provided

$$f^{[-1]}\circ g^{[-1]}\circ f\circ g(x)>g(x), \qquad 0<x<b, \qquad \text{or}$$
$$f^{[-1]}\circ g^{[-1]}\circ f(t)>t, \qquad 0<t<a, \qquad \text{or}$$

$$g^{[-1]}\circ f(t) > f(t), \qquad 0<t<a, \qquad \text{or}$$

$$g^{[-1]}(s) > s, \qquad 0<s<f(a), \qquad \text{or}$$

$$s > g(s), \qquad 0<s<f(a).$$

But this is true by H_3. In a similar manner the second fraction in (1) is seen to be >1. From $y(0)=0, y'(x)>1, 0<x<b$, there follows

$$y(x)>x, \qquad 0<x\leqslant b.$$

Hence for the same values of x, using monotonicity,

$$g\circ f(x) < g\circ f(y) = f\circ g(x),$$

as was to be shown.

Solution II by A. A. Jagers, Technische Hogeschool Twente, Enschede, Netherlands, Mou-Liang Kung, Norfolk State College, and Leon Gerber, Saint John's University, New York (independently). The result is the special case $f\equiv\sin$, $a=\pi/2$, of the following idea. Let $a>0$, $f(0)=0$, $f'(0)=1$. On $(0,a]$ let f be increasing, $0<f(x)\leqslant 1$; let f' be decreasing. Then (*) $f(x)f(y)<f(xy)$ if $0<x,y,xy\leqslant a$. *Proof.* Let $0\leqslant t\leqslant 1$, $0\leqslant y\leqslant a$. Then $tf(y)=t\int_0^y f'(u)du\leqslant t\int_0^y f'(tu)du=f(ty)$. With $t=f(x)$, it follows that $f(x)f(y)\leqslant f(f(x)y)$. But $f(x)=\int_0^x f'(s)ds<x$. Therefore, (*) holds.

Corollary (Irving A. Dodes, Fairfield at St. James, Long Island City, N.Y.; C. Powder & R. Steinlage, University of Dayton): $\sin(x^n)<\sin^n x$ for $0<x^n\leqslant\pi/2$ (n integer >1).

Solution III by Stephen V. Noltie, California Polytechnic State University, San Luis Obispo, and K. F. Andersen, University of Alberta (independently). For $1\leqslant x\leqslant\pi/2$, the inequality is clear. For $0<t^2<t<1$, since $0<\cos t<\cos t^2$, it follows that $2\sin t\cos t<2t\cos t^2$. Integrate this from 0 to $x(<1)$.

Solution IV by H. Kestelman, University College, London, Otto G. Ruehr, Michigan Technological University, Michael Vowe, Rodersdorf, Switzerland, Robert E. Shafer, Berkeley, CA, Ignacy Kotlarski, Oklahoma State University, and C. C. Rousseau, Memphis State University (independently). Using the infinite product $\sin x = x\prod_{k=1}^{\infty}[1-x^2(\pi^2n^2)^{-1}]$, one sees that for $0<x^2<\pi/2$, the ratio $\sin(x^2)/\sin^2 x$ is $\prod_1^\infty a_n(x)$, with $a_n=b_n/c_n$, $b_n=1-x^4/n^2\pi^2$, $c_n=1-2x^2/n^2\pi^2+x^4/n^4\pi^4$. Since $b_n>0$, $c_n>0$, it is enough to prove that $1-b_n<1-c_n$, i.e., that $x^2(1+1/n^2\pi^2)<2$. But $\pi^2+1<12<4\pi$, so that $x^2(1+1/n^2\pi^2)<\frac{1}{2}\pi(1+1/\pi^2)<2$.

Several readers noted that the inequality remains valid to $x=x_0\doteq 1.3644$, and is reversed on $[x_0,\pi/2]$.

There were 45 solvers.

Editorial note. In "Inequalities for a Collection," *Math. Mag.*, 52 (1979) 28–31, the proposer proves the following general theorem.

Let f be continuous with domain $0\leqslant x<1$ *or* $0\leqslant x\leqslant 1$, $f(0)=0$, $f(1)>1$ *(including the possibility that* $f(1)=+\infty$*); let g be continuous with domain the range of f, and* $g(1)\leqslant 1$. *Let* $f(x)/x$ *and* $g(x)/x$ *be strictly increasing on their domains. Finally let* $f(x)\neq x$ *for* $0<x<1$. *Then* $f(g(x))\leqslant g(f(x))$ *for* $0<x<1$.

He derives (*) as a corollary by taking $f=\arcsin x$, $g(x)=x^b$.

Further note. Let $f(0)=0$; let f be increasing, f' decreasing, $0<f(t)\leqslant 1$, $f(y)/y<f'(y)$ on $(0,a]$. The sine function does not satisfy all these hypotheses. But with them, $yf(t)\leqslant y$, so that $f'(y)\leqslant f'(yf(t))$, $f(y)f'(t)<yf'(yf(t))f'(t)$. Integrating from 0 to x, the inequality $f(y)f(x)<f(yf(x))$ follows, and hence (*) holds. Thus (*) does not characterize the sine function.

An Urn with Balls of Two Colors

E 2722* [1978, 496]. *Proposed by Clark Kimberling, University of Evansville, Indiana.*

A ball is drawn from an urn containing one red ball and one green ball. If it is red it is

returned to the urn with one additional red ball and one additional green ball, but if it is green no balls are put into the urn. After the first drawing, subsequent drawings take place following the same rules. Find the probability that the urn contains all the time at least one green ball.

Remark by the editors. The required probability is unchanged if one stops when the urn contains no green balls. In this process let $P(r,g)$ be the probability that the urn ever contains r red and g green balls. (It can do so at most once.) We wish to calculate $1-\Sigma_{r=1}^{\infty}P(r,0)$. Enumeration shows $P(1,0)=\frac{1}{2}$ and $P(2,0)=\frac{1}{12}$. For $r\geqslant 2$ we have

$$P(r,0)=P(r,2)\frac{2}{r+2}\cdot\frac{1}{r+1}.$$

For $r>2$, $P(r,2)$ does not exceed $1-7/12=5/12$; for if there turn out to be only 1 or 2 red balls remaining when the urn is emptied of green balls, it must be true that the urn never did contain r red balls, and certainly not r red and 2 green balls. Thus

$$\sum_{r=1}^{\infty}P(r,0)\leqslant 1/2+1/12+(5/12)\sum_{r=3}^{\infty}\frac{2}{r+2}\cdot\frac{1}{r+1}=19/24.$$

Hence the probability the urn always contains at least one green ball is at least $5/24$. Three computer calculations (furnished by Ralph Jones, Joseph O'Rourke, and D. E. Penney) suggest that the answer is approximately 0.36.

An Urn With Balls of Three Colors

E 2724 [1978, 496]. *Proposed by Harry Lass, Jet Propulsion Laboratory, Pasadena, California.*

An urn contains k_1 white balls, k_2 red balls, and k_3 blue balls. The balls are withdrawn one at a time at random without replacement until all balls of one color (either red or white or blue) have been removed. (i) Determine the probability that all white balls are removed first. (ii) Determine the mean number of trials until all balls of some one color have been removed.

Solution adapted from those submitted independently by Michael Brozinsky, Queensborough Community College, and Michael Skalsky, Southern Illinois University. (i) Let $P(k_1,k_2,k_3)$ be the desired probability. If R denotes the event that the red balls are exhausted before the white balls and B denotes the event that the blue balls are exhausted before the white balls, $P(R)=k_1/(k_1+k_2)$, reflecting the fact that R occurs if and only if, among the red and white balls, a white ball is drawn last. Similarly, $P(B)=k_1/(k_1+k_3)$, and $P(R\cap B)=k_1/(k_1+k_2+k_3)$; thus

$$\begin{aligned}P(k_1,k_2,k_3)&=1-P(R\cup B)\\&=1-k_1/(k_1+k_2)-k_1/(k_1+k_3)+k_1/(k_1+k_2+k_3).\end{aligned}\tag{1}$$

Alternatively, the probability that exactly m red balls and n blue balls have been removed when the last white ball is removed is $N(m,n)/D(m,n)$, where

$$N(m,n)=\binom{k_1}{k_1-1}\binom{k_2}{m}\binom{k_3}{n};\ D(m,n)=\binom{k_1+k_2+k_3}{k_1-1+m+n}(k_2+k_3+1-m-n).\tag{2}$$

Summing over possible values of m,n gives

$$P(k_1,k_2,k_3)=\sum_{m=0}\sum_{n=0}N(m,n)/D(m,n),\tag{3}$$

the upper limits of summation being $m=k_2-1$, $n=k_3-1$. The identity obtained by equating right sides of (3) and (1) will be used in part (ii).

(ii) If X is the trial on which for the first time the last ball of some color is removed, $E(X)=S_W+S_R+S_B$ where S_W (respectively, S_R,S_B) is the contribution to $E(X)$

when the white balls (respectively, red, blue) are first removed. Using (2) we have $S_W = \Sigma_{m=0}\Sigma_{n=0}(k_1+m+n)N(m,n)/D(m,n)$. But using (3) easy manipulation shows this to be

$$(k_1+k_2+k_3+1)k_1(k_1+1)^{-1}P(k_1+1,k_2,k_3);$$

and substitution from (1) gives

$$(k_1+k_2+k_3+1)[k_1/(k_1+1)-k_1/(k_1+k_2+1)-k_1/(k_1+k_3+1) + k_1/(k_1+k_2+k_3+1)].$$

By permuting indices we evaluate S_R and S_B without computation. Combining the results gives

$$E(X)=(k_1+k_2+k_3+1)\cdot\{1-1/(k_1+1)-1/(k_2+1)-1/(k_3+1)+1/(k_1+k_2+1) + 1/(k_1+k_3+1)+1/(k_2+k_3+1)-1/(k_1+k_2+k_3+1)\}.$$

Also solved by Norman L. Johnson, M. S. Klamkin & A. Liu (Canada), Peter W. Lindstrom, L. E. Mattics, Victor K. Wu, Matthew F. Wyneken, and the proposer. Part (i) solved by Eli L. Isaacson and Paul B. Massell.

Johnson points out that the problem appears, with solution and generalization due to E. El-Neweihi, F. Proschan, and J. Sethuraman in N. L. Johnson and S. Kotz, *Urn Models and Their Application*, Wiley, New York, 1977. Klamkin credits Newman and Weissblum, *SIAM Review*, 9 (1967), with a generalization of Part (ii).

In case there are $r+1$ colors, Klamkin and Liu obtain an interesting compact formula for part (i), i.e., for $P(k_1,k_2,\ldots,k_{r+1})$.

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to Professor Roger C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109. To facilitate their consideration, solutions of Advanced Problems in this issue should be typed (in duplicate, with double spacing) and should be mailed before February 29, 1980.

6276. *Proposed by R. K. Oliver, Pittsburgh, Pa.*

Let g and h be two screw motions of Euclidean three-space with positive angles less than $\pi/3$ and nonparallel axes. Show that the group generated by g and h is not discrete.

6277. *Proposed by Yasuo Watatani, Osaka Kyoiku University.*

If α and β are *-automorphisms of the algebra $B(H)$ of all bounded linear operators acting on a Hilbert space H such that

$$\alpha(x)+\alpha^{-1}(x)=\beta(x)+\beta^{-1}(x) \quad \text{for } x\in B(H),$$

then they commute.

6278. *Proposed by Stanley Wagon, Smith College, Northampton, Mass.*

Let X be the real vector space consisting of all bounded real-valued functions on the reals with bounded support. Is there a basis, B, for X which is closed under translation, i.e., if f is in B and t is real, then f_t is in B where $f_t(x)=f(x+t)$? (An affirmative answer would provide a new proof of Banach's theorem that a non-trivial, finitely additive, translation invariant measure defined for all bounded sets of reals exists.)

SOLUTIONS OF ADVANCED PROBLEMS

Totient Equation

6193 [1978, 121]. *Proposed by Robert E. Shafer, Berkeley, CA.*

Given that n is such that $2\phi(n)=n-1$ (ϕ is the Euler totient function), prove (1) $3\nmid n$; (2) If p

and q are distinct prime divisors of n, then $p \not\equiv 1 \pmod{2q}$; (3) n has at least 11 distinct prime divisors. (*Note.* In Beiler, *Recreations in the Theory of Numbers*, Dover, 1966, pp. 92–93, it is shown that n must have at least 7 distinct prime divisors.)

Graham Lord, Université Laval, Québec, remarks that the results are contained in a paper of Ronald Alter, this MONTHLY, 80 (1973) 192–193. Andrzej Makowski, Warsaw, states that they are contained in a paper of E. Lieuwens, *Nieuw Archief voor Wiskunde* (3), 18 (1970) 165–169. Lorraine L. Foster notes that Beiler does not prove that n must have at least seven distinct prime divisors, a result due to P. H. Lehner, *Bull. Amer. Math. Soc.*, 38 (1932) 745.

Solutions by Robert Breusch, Lorraine L. Foster, Emil Grosswald, L. Kuipers (Switzerland), N. Miku (Netherlands), Barry J. Powell (parts (1) and (2)), Blair Spearman, and the proposer. Grosswald says that he is quite certain that no integer satisfies the given relation but that he cannot prove it.

Expectation

6195 [1978, 122]. *Proposed by Andreas N. Philippou, University of Patras, Greece.*

For $j=1,2,\ldots$ and $n \geqslant j$, let X_{nj} and X_j be random variables defined on a probability space (Ω,A,P). Assume that $\sup_j \mathcal{E}|X_j|^r < \infty$ $(r>0)$, where $\mathcal{E}$ denotes expectation under P. Then $\max[\mathcal{E}|X_{nj}-X_j|^r,\ 1 \leqslant j \leqslant n] \to 0$, if and only if

$$\max\{P[|X_{nj}-X_j|>\varepsilon],\ 1 \leqslant j \leqslant n\} \to 0 \quad \text{and}$$
$$\max[\mathcal{E}|X_{nj}|^r - \mathcal{E}|X_j|^r,\ 1 \leqslant j \leqslant n] \to 0.$$

Solution by G. S. Rogers, New Mexico State University. From M. Loève, *Probability Theory*, D. Van Nostrand, 1963, p. 157, one notes that

$$\frac{\mathcal{E}|X_{nj}-X_j|^r - |a|^r}{\text{a.s. }\sup|X_{nj}-X_j|^r} \leqslant P(|X_{nj}-X_j|<a) \leqslant \frac{\mathcal{E}|X_{nj}-X_j|^r}{|a|^r}.$$

Then $\max_{1\leqslant j\leqslant n}\mathcal{E}|X_{nj}-X_j|^r<\varepsilon$ for $n>N_\varepsilon$ implies $\mathcal{E}|X_{nj}-X_r|^N<\varepsilon$ for $j\leqslant n>N_\varepsilon$ implies $P(|X_{nj}-X_j|\geqslant a)<\varepsilon'$ for $j\leqslant n>N_\varepsilon$ and (as in Loève) $|\mathcal{E}|X_{nj}|^r-\mathcal{E}|X_j|^r|<\varepsilon$ for $j\leqslant n>N_\varepsilon$. Hence, $\max_{1\leqslant j\leqslant n}P(|X_{nj}-X_j|\geqslant\varepsilon)\to 0$ and $\max_{1\leqslant j\leqslant n}(\mathcal{E}|X_{nj}|^r-\mathcal{E}|X_j|^r)\to 0$. Conversely, the last limit and $\sup_j\mathcal{E}|X_j|^r=F<\infty$ imply $E|X_{nj}|^r<\mathcal{E}|X_j|^r+\varepsilon<F+\varepsilon$ for $j\leqslant n\leqslant N_\varepsilon$ so by the c_r inequality (Loève, p. 155), $\mathcal{E}|X_{nj}-X_j|^r<c_r(2F+\varepsilon)$ and a.s. $\sup|X_{nj}-X_j|^r<\infty$. Then $P(|X_{nj}-X_j|^r\geqslant a)<\varepsilon$ for $j\leqslant n \leqslant N_\varepsilon$ implies $|\mathcal{E}|X_{nj}-X_j|^r-|a|^r|<\varepsilon$ for $j\leqslant n\leqslant N_\varepsilon$ so $\max_{1\leqslant j\leqslant n}\mathcal{E}|X_{nj}-X_j|^r\to 0$.

Harmonic in the Unit Disk

6198 [1978, 203]. *Proposed by Sanford S. Miller, State University College, Brockport, N.Y.*

Let $u(z)=u(x,y)$ be harmonic in the unit disk D with $u(0)=1$, and let $g(t)$ be a real-valued function satisfying $g(1)>0$ and $g(0)\leqslant\frac{1}{2}$. Show that if u satisfies $g(u)+xu_x+yu_y>0$, for $z\in D$, then $u(z)>0$, for $z\in D$. In particular, if $g(t)=\frac{1}{2}$, or $g(t)=t+\frac{1}{2}$, we obtain respectively

$$xu_x+yu_y>-\tfrac{1}{2} \Rightarrow u>0, \qquad u+xu_x+yu_y>-\tfrac{1}{2} \Rightarrow u>0.$$

Solution by Wolfgang Walter, Universität Karlsruhe, Germany. Assume that u is positive for $|z|<R<1$ and $u(z_0)=0$, $|z_0|=R$. It follows from Harnack's inequality that

$$u(z) \geqslant \frac{R-|z|}{R+|z|} \quad \text{for } |z|\leqslant R.$$

Putting $z=tz_0$, $f(t)=u(tz_0)$, $g(t)=(1-t)/(1+t)$, this inequality reads

$$f(t)\geqslant g(t) \quad \text{for } 0\leqslant t\leqslant 1, \qquad f(1)=g(1)=0,$$

hence $f'(1) \leqslant g'(1)$ or

$$xu_x + yu_y \leqslant -\tfrac{1}{2} \quad \text{at } z_0.$$

In other terms: If a function u harmonic in D satisfies

(i) $u(0)=1$,

(ii) $u(z)=0 \Rightarrow xu_x + yu_y > -\tfrac{1}{2}$,

then $u>0$ in D. In particular, the assertion of Problem 6198 is true, even with the assumption $g(1)>0$ removed.

REMARK. Let $u \in C^1(D)$ and

$$g(u) + xu_x + yu_y > 0 \quad \text{in } D, \tag{*}$$

where $g(s) \leqslant 0$ for $s \leqslant 0$. Then $u>0$ in D. (This theorem is true in any starshaped open set $D \subset \mathbb{R}^2$.)

Proof. It follows from (*) that $u(0)>0$. Assume that, for some $z_0 \in D$, $f(t)=u(tz_0)>0$ for $0 \leqslant t < 1$ and $f(1)=u(z_0)=0$. Then

$$f'(1) = xu_x + yu_y \leqslant 0,$$

which contradicts (*).

This remark is essentially due to Dr. R. Lemmert, Karlsruhe.

Also solved by Gustaf Gripenberg (Finland), Zane C. Motteler, and the proposer.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR., AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf, Carleton, and Macalester Colleges
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Introduction to Probability and Statistics. A multimedia program consisting of 10 filmstrips, audiotape cassettes, and study guide. Samuel L. Marateck. Harper & Row, New York, 1977. \$250.

These materials are designed to provide an introduction to topics in probability and statistics, assuming only a background of one year of high school algebra. The ten topics covered in the filmstrips and audiotapes are: Simple Probability; Permutations and Combinations; The Binomial Distribution; Distributions: The Normal Curve and the Normal Approximation to the Binomial Distribution; Sampling and the Central Limit Theorem; Confidence Intervals; The t Distribution; Hypothesis Testing; Chi-Square; and The F Distribution and the Analysis of Variance. The study guide contains the "concepts, formulas, and definitions used in each section of each filmstrip" (but no explanations), together with a set of (artificial) problems, with solutions, in five areas: General–Mathematics, Business–Economics, Psychology–Education, Politics–Social Science, and Nursing–Medicine. The author states that this program can be used

in two ways: "It can be used in the classroom to introduce and teach the major topics in probability and statistics. Or it can be used as a self-paced learning tool that individual students can use by themselves."

In my opinion, this program would be of little value if used alone. The materials contain no tables for solving the problems in the study guide, the statistical topics are barely introduced, there are essentially no realistic problems and little motivating material, and important topics are not included. Even when used as a supplement to a standard text its value is questionable. Besides the usual problems inherent in this sort of approach, and the lack of depth and coverage mentioned above, there are some flaws in the material itself.

Filmstrip 4 is particularly bad. Many of the normal curves are carelessly drawn, failing, for example, to indicate the effect of a change in standard deviation. The areas corresponding to various standard normal values are very inaccurately indicated. For instance, in one frame the area between standard normal values -1 and 2 appears symmetric. In one example, covering several frames, the tabulated value corresponding to a standard normal value of 1.05 is found and used instead of the correct value of 1.50.

The other filmstrips are much better; but occasional errors creep in, and the approach used in some instances may be different from that favored by the instructor. Some examples of each of these problems follow.

When the probability formulas in filmstrip 1 are obtained, no mention is made of the assumption that the outcomes are equally likely. The sample variance is presented in filmstrip 4, but the "computational formula" is not given. In several filmstrips what was intended to be a nu, representing degrees of freedom, comes out looking more like a gamma. Hypothesis testing is introduced in filmstrip 8 from the point of view of finding the actual values which would lead to rejection, rather than obtaining standard normal values as test statistics. In filmstrip 9 and in the study guide the formula for the chi-square statistic has the summation sign in the numerator instead of in front of the fraction. Analysis of variance is introduced in filmstrip 10 as a comparison of within and between sample variances, with no "computational formulas" developed. Then there are also the usual number of typographical errors.

On the positive side, the technical quality of the filmstrips and audiotapes is very good. The voices, alternately male and female, are clear and easy to understand. The filmstrips are in color and are easy to read. And, as far as they go, many of the topics are presented in a clear and careful manner. These materials could have some value for some students by providing an alternative form of delivery; but to be of any real value they must be corrected to eliminate the deficiencies indicated above and should be checked by the instructor in advance to see whether the approach is consistent with that used in the class.

RICHARD S. KLEBER, St. Olaf College

MISCELLANEA

29.

THE POWER OF POSITIVE THINKING

Since the estimated number of stars in the universe is 1 to the 20th power—that is, the number 1 followed by 20 zeros—since many billions of these stars resemble our sun and since our sun has a planetary system that gave rise to the race of man, many other suns could be expected to do likewise, scientists reason.

—Malcolm W. Browne, "Life May Exist Only on Earth, Study Says," *The New York Times*, 24 April 1979, p. C2. (Suggested by Joel E. Cohen.)

TELEGRAPHIC REVIEWS

Telegraphic reviews are designed to give prompt notice of new books with sufficient information to assist our readers in deciding whether to order an examination copy or to suggest library purchase. Possible uses are indicated as follows:

T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, P**, *The Role of Applications in the Undergraduate Mathematics Curriculum.* Ad Hoc Committee on Applied Mathematics Training. National Research Council, National Academy of Sciences, 1979, ix + 25 pp, (P). Report of the National Research Council's blue-ribbon Committee on Applied Mathematics Training, chaired by Peter Hilton. Its diagnosis includes a "pervasive tendency towards scholasticism in undergraduate mathematics education," dissolution of the traditional unity within mathematics, mutual suspicion and hostility between pure and applied mathematics, and undervaluing of mathematics by outsiders due to "withdrawn attitude of many mathematicians." "The principal problem is one of attitude." To correct these ills, they prescribe for mathematicians 14 changes in behavior and attitude that reflect the increasing importance of diverse applications in the mathematical sciences. An important policy statement that will help shape undergraduate mathematics in the 1980's. LAS

GENERAL, S(13-18), P, *Mathematics and Humor.* Aggie Vinik, Linda Silvey, Barnabas Hughes. NCTM, 1978, vii + 58 pp, $4 (P). [ISBN: 0-87353-137-X] Stories, jokes, riddles, quickies, Tom Swifties, verse, graffiti for grade to graduate school. Topic index provides easy access to humor for many lessons. Mostly (not all) good humor, new and old. Finishing in booklet's spirit--"To qualify as telegraphic (50 words or less) this review must terminate," said Tom haltingly. PJ

GENERAL, P, L, *Mathematical Reviews Index, Volumes 55 and 56 (1978).* Ed: J.L. Selfridge. AMS, 1979, $120 set. *Part 1, Author/Key Index,* iii + 768 pp, (P); *Part 2, Subject Index,* 623 pp, (P). This annual index is the first to be included with each subscription; also available separately under the title *Index of Mathematical Publication,* V. 10. LAS

BASIC, T(13: 1), *Intermediate Algebra for College Students, Second Edition.* Louis Leithold. Macmillan, 1979, x + 527 pp, $14.95. [ISBN: 0-02-369640-0] Standard topics in intermediate algebra. Suitable for first-timers at this level and for those in need of a review. Readable. Numerous worked-out examples. Graded exercises. Several collections of "story" problems. Additions to *First Edition* (TR, March 1975) include inequalities in two variables, systems of inequalities and an introduction to linear programming. JK

BASIC, T(13: 1), *The Language of Mathematics.* Nelda W. Roueche, Barbara Washburn Mink. P-H, 1979, xii + 500 pp, $14.95. [ISBN: 0-13-522920-0] Deals with the basic language and structure of the number system and introduces the student to some probability and statistics, the metric system, consumer math, and logic. LLK

PRECALCULUS, S, *Algebra and Trigonometry Refresher for Calculus Students.* Loren C. Larson. Freeman, 1979, xx + 192 pp, $4.95 (P). 36 brief review and reference sections, arranged into four groups: preliminaries, algebra for derivatives, algebra for applications, and logarithms, trigonometry. Each group begins with a diagonstic pre-test; each section includes 3-5 worked examples plus practice exercises. The volume concludes with post-tests for each group. A useful review volume for students in any beginning college course in mathematics or science. LAS

PRECALCULUS, T(13: 1), *Algebra and Trigonometry, A Pre-Calculus Approach.* Max A. Sobel, Norbert Lerner. P-H, 1979, xvi + 539 pp, $14.95. [ISBN: 0-13-021709-3] Very much a precalculus text. Many examples and problems. Well written with many helpful insights provided. Novel features are small exercise sets within the text to provide the student with a means to test his comprehension immediately without breaking up the continuity of the sections. Optional sections on complex numbers, matrices, and conics, among others, provide flexibility. TLS

EDUCATION, S(16-17), P, L, *The Status of Pre-College Science, Mathematics, and Social Science Education: 1955-1975 Volume II. Mathematics Education.* Marilyn N. Suydam, Alan Osborne. Information Reference Center for Science, Mathematics, and Environmental Education (Ohio St U, 1200 Chambers Rd., 3rd Floor, Columbus, OH 43212), 1977, 289 pp, (P). Describes the evidence bearing on the rationality of decision-making for educational policy that affects mathematics education. Summarizes practices in schools, teacher education and needs assessment, 1955-1975. Conclusion: progress has been made without systematic collection or usage of information about existing practice and without recognition of the effects of the diversity of values of the groups affected. Extensive bibliography. PJ

HISTORY, P*, *The Beginnings of Greek Mathematics.* Árpád Szabó. Texts and Stud. in Hist. of Logic and Philo., V. 17. Reidel, 1978, 358 pp, $47.50. [ISBN: 90-277-0819-3] Not a survey work, but a scholarly and original study of three specific developments: quadratic irrationality, proportions, and deductive mathematics. Very technical but readable, as would be expected from a philologist who is also mathematically literate. It is mildly unfortunate that the text is inconsistent in its use of Greek or Roman fonts: δυναμις alternates with *dynamis*. JAS

HISTORY, P, *Matematičeskie terminy. Spravočnik.* [Mathematical terms. A handbook.] N.V. Aleksandrova. Moscow, Vysšaya Skola, 1978, 190 pp, 35 kop. Not a dictionary but a collection of historical, etymological and notational essays on some 450 terms, mostly those in use by 1900 or thereabouts. A useful source for anyone looking for historical information on terms common in undergraduate mathematics. An appendix lists about 650 names of mathematicians in Russian with the original forms; very useful for anyone who is reading Russian mathematics. RPB

HISTORY, P, L. *Mark Kac: Probability, Number Theory, and Statistical Physics: Selected Papers.* Ed: K. Baclawski, M.D. Donsker. MIT Pr, 1979, xxxviii + 529 pp, $35. [ISBN: 0-262-11067-9] 52 papers, about one-third of Kac's work, selected by Kac as his most significant contributions to probability, number theory and statistical physics, preceded by an autobiographical note, an editor's introduction, a complete bibliography, and a mathematical commentary (with current references) setting Kac's work in a historical context. LAS

FOUNDATIONS, P. *Fondements des Mathématiques: Introduction à Une Philosophie Constructiviste.* Yvon Gauthier. Pr U Montreal, 1976, 460 pp, $18. [ISBN: 0-8405-0348-2] Wide ranging (and rambling) constructivist criticism of the foundations of mathematics, with special attention to axiomatic set theory, modern intuitionism, and theory of categories and topoi. Author follows Kreisel in seeing the philosopher's role as perennial critic of the concepts and principles of mathematics, with a goal of locating mathematics in the broad spectrum of human theoretical activities. Attempts to pursue this task taking full cognizance of modern developments in mathematical logic. GHM

FOUNDATIONS, P. *Physical Theory as Logico-Operational Structure.* Ed: C.A. Hooker. Reidel, 1979, xvii + 334 pp, $48.50. [ISBN: 90-277-0711-1] Nine research papers that aim to lay a new philosophical foundation for natural science by demonstrating that "physical theories are most significantly characterized by their abstract structural components...in opposition to the view that one should look to the applications of a theory in order to understand it." LAS

FOUNDATIONS, S(16-18), L. *Basic Set Theory.* Azriel Levy. Springer-Verlag, 1979, xiv + 391 pp, $24.90. [ISBN: 0-387-08417-7; 3-540-08417-7] Intermediate text between naive set theory and advanced research. Detailed exposition of basic material on ordinal and cardinal numbers. Advanced topics surveyed include descriptive set theory, Boolean algebras, and large cardinals, stopping short of applications involving model theory. Written in flowing expository style, with proofs omitted in some sections. Occasional exercises. GHM

FOUNDATIONS, S(18), P. *Ideals Over Uncountable Sets: Application of Almost Disjoint Functions and Generic Ultrapowers.* Thomas Jech, Karel Prikry. Memoirs No. 214. AMS, 1979, iii + 71 pp, $6.40 (P). [ISBN: 0-8218-2214-4] Very well written monograph investigating the effect of various properties of ideals on the arithmetic of cardinal numbers. Studies the deep relationship between saturation of ideals and cardinal exponentiation, as well as the consequences of the existence of precipitous ideals for Cantor's generalized continuum hypothesis and the singular cardinals problem. GHM

COMBINATORICS, S(16-17), P. *Graphentheoretische Methoden des Operations Research.* K. Hässig. B.G. Teubner, 1979, 160 pp, DM 26,80 (P). [ISBN: 3-519-02344-X] An introduction to the parts of graph theory (distance and flow problems) most useful in operations research. JD-B

COMBINATORICS, T*(16-17: 1, 2), S, L*. *Graphs & Digraphs.* Mehdi Behzad, Gary Chartrand, Linda Lesniak-Foster. Prindle, 1979, x + 406 pp, $23. [ISBN: 0-87150-261-5] The first two-thirds of this book is a reworking and updating of the first two authors' 1971 *Introduction to the Theory of Graphs* (TR, January 1973). The remainder is an introduction to digraphs. Lots of good problems and an excellent bibliography. A solid introductory course in graph theory for the advanced undergraduate or beginning graduate student. CEC

NUMBER THEORY, T(14: 1), S. *Number Theory for Beginners.* André Weil. Springer-Verlag, 1979, vii + 70 pp, $6 (P). [ISBN: 0-387-90381-X; 3-540-90381-X] Taken from 1949 lecture notes. A barebones introduction to number theory and algebra which is written in a very formal style. Includes a good collection of exercises. Notice the number of pages and the price. CEC

LINEAR ALGEBRA, T*(14-16: 1, 2), L. *Linear Algebra.* Stephen H. Friedberg, Arnold J. Insel, Lawrence E. Spence. P-H, 1979, xiv + 514 pp, $16.95. [ISBN: 0-13-537019-1] A sophisticated introductory text emphasizing the development of the theory through canonical forms and advanced material on inner product spaces. The approach is mostly algebraic; it begins with vector spaces and treats equations later. A number of significant applications is given, including more than introductory remarks on Markov matrices and convergence, and the Lorentz transformation. A good index and a significant collection of exercises together with clear but not simplistic writing offers strong students or an ambitious instructor a significantly different text. JAS

ALGEBRA, T*(14-17: 1, 2), S, L. *Algebra, Second Edition.* Saunders MacLane, Garrett Birkhoff. Macmillan, 1979, xv + 586 pp, $18.95. [ISBN: 0-02-374310-7] This is a completely rewritten descendent of the 1967 *First Edition* (TR, November 1967; ER, January 1971). The references have been brought up to date, and the text has been much simplified (or clarified) although it maintains its aggressively modern categorical background. However, chapters on the integers, groups, and rings now precede and support the unifying material on universal constructions. The affine geometry is gone, replaced by Galois theory, and there is more linear and multilinear algebra than in most competing books. This book will still make the naive nervous, but probably offers the serious student much better access to modern algebra than the first edition. JAS

ALGEBRA, T*(15-16: 1, 2), S, L. *A Concrete Introduction to Higher Algebra.* Lindsay Childs. Springer-Verlag, 1979, xiv + 338 pp, $14.80. [ISBN: 0-387-90333-X; 3-540-90333-X] A departure from the traditional approach to higher algebra in that number theory and the theory of equations make up the first two-thirds of the text, and the rest of the book focuses on fields. Several applications appear throughout. The writing style tends towards the formal. An adequate supply of exercises. An approach which deserves consideration. CEC

ALGEBRA, S(18), P. *Reflectors and Localization: Application to Sheaf Theory.* F. van Oystaeyen, A. Verschoren. Pure and Appl. Math., V. 41. Dekker, 1979, vii + 166 pp, $23.50 (P). [ISBN: 0-8247-6844-2] The authors aim "to show that the construction of a structure sheaf over the spectrum [of a non-commutative ring] may be regarded as the creation of a localization functor..., and to demonstrate how properties of this functor relate to ring theoretical...features of the ground ring." Bibliography, index. JS

FINITE MATHEMATICS, T(13-14: 1), *Finite Mathematics for Business, Economics and Social Science*. James Radlow. Duxbury Pr, 1979, xiii + 631 pp, $15.95. [ISBN: 0-87872-182-7] Informal and unintimidating. Classroom-tested material with flow charts and suggested step-by-step procedures. Loads of drill exercises and word problems with emphasis on real-world applications. Flexibility allows linear programming via linear algebra, Markov chains via probability, or mathematics of finance via exponentials and logarithms. Written for the students. JK

CALCULUS, T*(13: 1), *Calculus, A Practical Introduction*. Richard S. Millman, George D. Parker. McGraw, 1979, xv + 527 pp, $15.95. [ISBN: 0-07-042305-9] An intuitive but careful presentation with practical illustrations for students majoring in management, social or life sciences. Numerous examples introduce the derivative and motivate intuitive definitions of limit and continuity. An initial 37-page algebra review is supplemented with review of additional algebraic techniques as needed throughout the book. JNC

CALCULUS, T(13: 2), *Mathematics for the Biological Sciences*. Jagdish C. Arya, Robin W. Larcher. P-H, 1979, xviii + 709 pp, $19.95. [ISBN: 0-13-562439-8] A good two semester calculus course with applications from the life sciences. Some topics not usually included are probability and difference equations. Nice! LLK

CALCULUS, T(13-14: 3), L, *Calculus and Analytic Geometry, Fourth Edition*. John A. Tierney. Allyn, 1979, xiii + 786 pp, $24.95. [ISBN: 0-205-06454-X] Changes from the *Third Edition (First Edition*, TR, December 1968; *Second Edition*, TR, June/July 1972; *Third Edition*, TR, June/July 1975) are relatively minor, including additional problems, chapter review problems, vectors in Chapter 2, Lagrange multipliers, and some re-organization and re-writing. JS

NUMERICAL ANALYSIS, S(18), P, *Numerical and Quantitative Analysis*. G. Fichera. Trans: Sandro Graffi. Fearon-Pitman, 1978, x + 208 pp, $29.50. [ISBN: 0-273-00284-8] Two surveys: one on obtaining upper and lower bounds resulting from ordinary and partial differential equations and integral operators and some on a priori estimates of solutions to partial differential equations. The general settings are Hilbert and Banach spaces. The methods and the theory are well-motivated by interesting examples. The results stated include many due to the author and his students. RWN

FUNCTIONAL ANALYSIS, P, L, *Constructive Functional Analysis*. D.S. Bridges. Research Notes in Math., No. 28. Fearon-Pitman, 1979, 203 pp, $15 (P). [ISBN: 0-273-08418-6] A self-contained presentation of "constructive" proofs of major results in functional analysis and measure theory, such as the Hahn-Banach and Stone-Weierstrass theorems, the Lebesgue convergence theorems, and the functional calculus for self-adjoint operators on Hilbert space. An excellent introuduction to constructive methods. TRS

ANALYSIS, P, *Continuation Methods*. Ed: Hansjörg Wacker. Acad Pr, 1978, ix + 336 pp, $19.50. [ISBN: 0-12-729250-0] Proceedings of the symposium held at the Johannes Kepler University in Linz, Austria from October 3-4, 1977. JAS

ANALYSIS, P, *Fourier Analysis and Approximation Theory*. Ed: G. Alexits, P. Turán. North-Holland, 1978. *V. 1*, 458 pp; *V. 2*, 465 pp, $136.50 set. [ISBN: 963-8021-22-5] Most of the 88 lectures presented at the Colloquium on Fourier Analysis and Approximation Theory which was organized by the János Bolyai Society and held in Budapest from August 16-21, 1976. JAS

ANALYSIS, P, *Contributions to Analysis*. l'Enseignement Math, 1979, 106 pp, Frs. 29 (P). Nine papers (including ones by Ahlfors, Bauer, Bombieri, and Malgrange) from an April 1978 symposium honoring Albert Pfluger. LAS

DIFFERENTIAL GEOMETRY, T**(1, 2), S, L*, *Elementary Topics in Differential Geometry*. John A. Thorpe. Springer-Verlag, 1979, xiii + 253 pp, $16.80. [ISBN: 0-387-90357-7; 3-540-90357-7] This book exploits linear algebra via a vector field (not affine geometry) approach to co-dimension one manifolds. The organization into small coherent chapters extends a modern multivariable calculus course into a real geometric tool that offers a significant new approach to the growing need for geometric knowledge in mathematics and physics. The "dirty trick" aspects of the co-dimension one special case seem to be minimized. This yields a relatively simple approach which at the same time is a real improvement over classical curves and surfaces books. The overwhelming machinery of tensors and forms is kept at bay. Exercises, indices, and bibliography seem adequate, but the exposition does leave gaps in motivation and history for the instructor to fill in. JAS

DIFFERENTIAL GEOMETRY, P, *Differential Forms in General Relativity, Second Edition*. W. Israel. Dublin Inst for Adv Stud, 1979, vii + 80 pp, £3.15 (P). High-level exposition of some modern techniques in gravitational theory which makes use of differential forms. This edition has an added chapter on tensor-valued forms and Lie derivatives and an up-dated bibliography. JAS

TOPOLOGY, T(17-18: 1), S, P, *Lecture Notes in Mathematics-688: Shape Theory, An Introduction*. Jerzy Dydak, Jack Segal. Springer-Verlag, 1978, vi + 150 pp, $9.80 (P). [ISBN: 0-387-08955-1; 3-540-08955-1] Shape category and functor. Invariants and algebraic properties associated with shape theory. Pointed 1-movability, Whitehead and Hurewicz theorems, pointed ANSR's, cell-like maps. List of open problems. Bibliography. List of symbols. Index. RJA

TOPOLOGY, T(17: 1), S, P, *Degree Theory*. N.G. Lloyd. Tracts in Math., No. 73. Cambridge U Pr, 1978, x + 172 pp, $24.95. [ISBN: 0-521-21614-1] This book is designed as an introduction to degree theory for analysts and applied mathematicians. There is a special emphasis on applications. Includes a substantial list of references but no exercises. Provides one of the first expositions of generalized degree theories in English. CEC

TOPOLOGY, P, *Lecture Notes in Mathematica-664: Algebraic and Geometric Topology*. Ed: Kenneth C. Millett. Springer-Verlag, 1978, xi + 240 pp, $11 (P). [ISBN: 0-387-08920-9; 3-540-08920-9] Contains

the proceedings of a symposium held in honor of R.L. Wilder in July 1977. Includes three historical papers of more than usual interest on Wilder's contribution to various areas of mathematics. A list of Wilder's publications is provided. TLS

TOPOLOGY, T**(15-17: 1, 2), S, L*. *A Combinatorial Introduction to Topology.* Michael Henle. Freeman, 1979, xiv + 310 pp, $18.50. [ISBN: 0-7167-0083-2] A beautifully written, very flexible text for undergraduates or beginning graduate students. The book really illustrates (double entendre intended) what combinatorial topology is all about. The book provides a combinatorial viewpoint for geometric topology over a very wide range of topics, e.g., vector fields, map coloring, homology and homotopy theory, surfaces, and even a hint of physical applications. A good index, interesting problems, and a light yet clear style indicate the author's concern for the reader and student. JAS

PROBABILITY, T(16-17: 1). *Stochastische Matrizen.* F.-J. Fritz, B. Huppert, W. Willems. Springer-Verlag, 1979, ix + 191 pp, $17.60 (P). [ISBN: 0-387-09126-2; 3-540-09126-2] A text on stochastic matrices, with applications to random walks, card shuffling, queues, processes with absorbing states, and passage-times. Assumes a knowledge of elementary linear algebra but none of probability theory. Problems. JD-B

PROBABILITY, T*(15-18: 1, 2), S*, P, L. *Elementary Probability Theory with Stochastic Processes, Third Edition.* Kai Lai Chung. Springer-Verlag, 1979, xvi + 325 pp, $14.80. [ISBN: 0-387-90362-3; 3-540-90362-3] This new edition has some minor improvements and photographs of eight important probabilists. (*First Edition,* TR, January 1975; ER, August/September 1976; *Second Edition,* TR, May 1976.) FLW

STATISTICS, S(17-18), P. *Survival Probabilities, The Goal of Risk Theory.* Hilary L. Seal. Wiley, 1978, x + 103 pp, $24.50. [ISBN: 0-471-99683-1] A study, with programs in Fortran, of the survival of insurance (nonlife) companies with theoretical discussion of the techniques used in the computer models along with tables generated by these programs. JAS

STATISTICS, T(13-14: 1, 2). *Applied Statistics, Third Printing.* John Neter, William Wasserman, G.A. Whitmore. Allyn, 1978, xix + 743 pp, $17.95. [ISBN: 0-205-05982-1] Broad coverage of fundamental statistical concepts and methods with many applications to real-world problems. For students and professionals in management, economics and other social sciences. Only an occasional optional section requires calculus. Includes Bayesian decision making, linear statistical models, and time series analysis and index numbers. GHM

STATISTICS, T(14: 1). *Statistics for Technology, A Course in Applied Statistics, Second Edition.* Christopher Chatfield. Chapman and Hall, 1978, 370 pp, $9.50 (P). [ISBN: 0-470-26406-3] Modest revision of the 1970 first edition (1975 Reprint TR, November 1975). Primarily, the references have been updated, and a new appendix containing some miscellaneous topics omitted from the first edition has been added. RSK

STATISTICS, T*(16-17: 1, 2), P, L. *Analyzing Multivariate Data.* Paul E. Green. Dryden Pr, 1978, xvi + 519 pp, $24.95. [ISBN: 0-03-020786-X] Pragmatic approach to the statistical analysis of multivariate data, assuming a background of basic statistics and matrix algebra (reviewed in an appendix). Body of the text is devoted roughly half to single criterion, multiple predictor association, and half to multiple criterion, multiple predictor association and the analysis of interdependence. Techniques are first fully illustrated and discussed with a small numerical example, and then an appropriate BMD (Biomedical Data) computer program is used on a TV test commercial data set. RSK

STATISTICS, T(16-17: 2), S, P. *Statistische Analysen.* Volker Nollau. Birkhäuser, 1979, 378 pp, sFr. 36. [ISBN: 3-7643-1019-7] A modern and fairly sophisticated text and reference book on the design and analysis of experiments, written for engineers and scientists. No exercises. Algol programs for the tests covered. JD-B

COMPUTER PROGRAMMING, T(13-18: 1, 2), S, L. *ALGOL 68, A First and Second Course.* Andrew D. McGettrick. Cambridge U Pr, 1978, xii + 348 pp, $27.50; $9.95 (P). [ISBN: 0-521-21412-2; 0-521-29143-7] Intended for those who are learning Algol 68 as a first programming language. Chapters 1-5 form an Algol 60-type subset and are suitable for the first course. The remainder of the text comprises the advanced features of Algol 68 and can be used in the second course. The work is clearly written with many examples, exercises and problems in each chapter. Answers to exercises. An appendix on the Algol 68 environment and one containing a syntax chart. Index. RJA

COMPUTER PROGRAMMING, T(13-18: 1). *Applied Fortran IV Programming, Second Edition.* John R. Sturgul, Michael J. Merchant. Wadsworth, 1976, xvii + 391 pp, $11.95 (P). [ISBN: 0-534-00440-7] Text divided into two parts, one on the more elementary aspects of the language (constants, variables, arithmetic expressions, control statements, I/O, arrays, loops) and the other on advanced features (subprograms, logical expressions, conditional statements, alphameric data, double-precision, complex variables). Clear, easy to read format. Many examples, illustrations, exercises. Appendix on standard Fortran functions. Answers to exercises. Index. RJA

COMPUTER SCIENCE, S(13-18). *LINK68: An M6800 Linking Loader.* Robert D. Grappel, Jack E. Hemenway. BYTE Pub, 1978, v + 61 pp, $8 (P). [ISBN: 0-931718-09-0] Contains the source code and machine readable bar code listings of object code as well as descriptions of the various routines, I/O conventions, execution details, and error messages. Appendices. Index. RJA

COMPUTER SCIENCE, T(13-14: 1). *An Introduction to Computer Science, An Algorithmic Approach.* Jean-Paul Tremblay, Richard B. Bunt. McGraw, 1979, xx + 636 pp, $18.95. [ISBN: 0-07-065163-9] Intended for first courses in computer science. Computational problem solving, algorithm formulation; numeric and non-numeric applications; arrays, strings, linked lists, trees; control structures and programming style. PL/I-like notation. RWN

COMPUTER SCIENCE, T(16-17: 1), P. *Fuzzy Switching and Automata: Theory and Applications.* Abraham Kandel, Samuel C. Lee. Crane, Russak, 1979, x + 303 pp, $27.50. [ISBN: 0-7131-2670-1] Self-contained introduction to fuzzy sets and functions, an increasingly important tool in computer science and engineering. Treats decomposition of fuzzy switching functions and fuzzy automata and languages, with illustrative applications to neural networks, approximation, pattern recognition, and sociological role theory. The book itself is a testimony to fuzzy phenomena, with numerous ambiguities and awkward explanations. Extensive bibliography. GHM

COMPUTER SCIENCE, T(14-16), S, L. *Computer Principles of Modeling and Simulation.* T.G. Lewis, B.J. Smith. HM, 1979, vi + 393 pp, $19.50 [ISBN: 0-395-27143-6]; *Instructor's Manual,* vi + 76 pp, $.65 (P). [ISBN: 0-395-27144-4] Designed as a text for the ACM course A4 on discrete system simulation: covers both deterministic and stochastic simulation using space slice, time slice and discrete event techniques. Discusses simulation languages (GPSS, SIMSCRIPT, SIMPL/I) and related programming principles. *Instructor's Manual* summarizes highlights of the text, suggests additional problems and test questions, and presents solutions to selected problems from the text. LAS

COMPUTER SCIENCE, T*(13-18: 1, 2), S. *Fundamentals of the Computing Sciences.* Kurt Maly, Allen R. Hanson. P-H, 1978, xxi + 488 pp, $16.95. [ISBN: 0-13-335240-4] Presents appropriate, stimulating introductions to the major topics in computer science while establishing and using a specific problem solving methodology. Includes chapters on algorithms, their analysis and implementation, problem solving, computational complexity, errors, numerical applications, system processes, information structures, textual analysis, recursion, artificial intelligence. Chapter references. Exercises. Indices. There is a supplemental volume containing introductions to Fortran and SNOBOL, some advanced topic material, and some solutions to exercises. RJA

COMPUTER SCIENCE, T(13: 1), *Computer Science Fundamentals, An Algorithmic Approach Via Structured Programming.* E.A. Unger, Nasir Ahmed. Merrill, 1979, xii + 387 pp, $14.95. [ISBN: 0-675-08301-X] Intended for a general, introductory course. An over-view of computer systems, uses and effects. Problem solving using flowcharts. Arrays, strings, looping, subroutines, common algorithms. No particular programming language is presented, although programming is required in many of the exercises. RWN

APPLICATIONS (ARTIFICIAL INTELLIGENCE), S(15-18), P, L. *Artificial Intelligence: An MIT Perspective.* Ed: Patrick Henry Winston, Richard Henry Brown. MIT Pr, 1979. *V. 1,* xiii + 492 pp [ISBN: 0-262-23096-8]; *V. 2,* xiii + 486 pp, $25 each [ISBN: 0-262-23097-6] These volumes, inaugurating a series in artificial intelligence from MIT Press, present research of those at MIT who work in the area. Volume 1 is divided into three sections: problem solving; natural language understanding and intelligence computer coaches; and representation and learning. Volume 2 is divided into three sections: understanding vision; manipulation and productivity technology; and computer design and symbol manipulation. Short notes by the editors introduce the sections. All chapters have reference lists. Index. RJA

APPLICATIONS (PHYSICS), P. *Nonlinear Equations in Physics and Mathematics.* Ed: A.O. Barut. Reidel, 1978, ix + 473 pp, $39. [ISBN: 90-277-0936-X] Proceedings of the NATO Advanced Study Institute held in Istanbul, Turkey, August 1-13, 1977. This volume contains much current theory in nonlinear differential equations. JAS

APPLICATIONS (PHYSICS), P. *Quantum Electrodynamics.* Suraj N. Gupta. Gordon, 1977, xi + 226 pp, $38. [ISBN: 0-677-04240-X] "A concise and coherent presentation of the basic theory, calculational techniques, and important applications of quantum electrodynamics. Mathematical manipulations are explained in sufficient detail, and many derivations are provided in unusually simplified form." Prerequisite: one-year graduate course in quantum mechanics. TRS

APPLICATIONS (PHYSICS), P. *Die Grundstrukturen einer physikalischen Theorie.* G. Ludwig. Springer-Verlag, 1978, viii + 261 pp, $22 (P). [ISBN: 0-387-08821-0; 3-540-08821-0] Attempt at a new analysis of the structure of physical theories and proposal of a "new (axiomatic) form" for their formulation. Designed to facilitate a more precise treatment of the problems of interpreation of physical theory, introduction of physical concepts, and comparisons of different theories. Intended also as a basis for new formulations of philosophical issues relating to reality and possibility, particularly those raised by quantum theory. GHM

APPLICATIONS (PHYSICS), P. *Foundations of Mechanics, Second Edition.* Ralph Abraham, Jerrold E. Marsden. Benjamin/ Cummings, 1978, xxii + 806 pp, $39.50. [ISBN: 0-8053-0102-X] Revised and updated edition (*First Edition,* TR, December 1967). New sections on the rigid body, topology, and quantization. The bibliography has been fully updated. All in all, a fine revision of an already excellent book. TLS

APPLICATIONS (PHYSICS), T(18: 1), P. *Foundations of Theoretical Mechanics I: The Inverse Problem in Newtonian Mechanics.* Ruggero Maria Santilli. Springer-Verlag, 1978, xix + 266 pp, $29.80. [ISBN: 0-387-08874-1; 3-540-08874-1] A monograph on methodological foundations of Newtonian systems, with main emphasis on the existence and construction of Lagrangian or Hamiltonian representations of given equations of motion. A graduate text in mathematical physics. TRS

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; Ralph P. Boas, Northwestern; Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; John Dyer-Bennet, Carleton; Paul Jorgensen, Carleton; Lorraine L. Keller, St. Olaf; Richard S. Kleber, St. Olaf; Joseph Konhauser, Macalester; George H. Mills, St. Olaf; R.W. Nau, Carleton; Thomas R. Savage, St. Olaf; John Schue, Macalester; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Timothy L. Strotman, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, C.D. 20036

SUGGESTION BOX

Members of the MAA are encouraged to send in suggestions, questions, etc., about the operation of the Association. Communications will be referred to the appropriate officer of the Association for answering; from time to time, those of general interst may also be answered in one or both of the official journals. Communications should be addressed to: Suggestion Box, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036.

PERSONAL ITEMS

Assistant Professors Diane Resek and Vincent Bruno, San Francisco State University, have been promoted to Associate Professors.

Professor Robert Jacobsen has resigned as Chairman of the Mathematics Department at Luther College, Decorah, Iowa, but will continue to teach in the department. Professor Donald Pilgrim has been appointed Chairman.

Dr. Alan J. Goldman, long-time Chief of Operations Research at the National Bureau of Standards, has accepted an appointment as Professor of Mathematical Sciences at the Johns Hopkins University.

Professor Gail S. Young, University of Rochester, has been appointed Professor of Mathematics and Chairman of the Department of Mathematics and Statistics. He was President of the Mathematical Association in 1969-70.

Associate Professors Arthur T. White and Erik A. Schreiner, Western Michigan University, have been promoted to the rank of Professor.

Professor Gerald L. Alexanderson, Santa Clara University, has been elected to a five year term on the Board of Trustees, the governing body of the University.

Professor Kurt Bing, Renssalaer Polytechnic Institute, retired from the Department of Mathematical Sciences and was appointed Professor Emeritus.

INTERNATIONAL CONFERENCE ON NONLINEAR DYNAMICS

This conference, sponsored by the New York Academy of Sciences, will be held December 17-21, 1979, at the Barbizon Plaza Hotel, New York City. The central subject is the study of the *chaotic* and regular motion exhibited by nonlinear dynamical systems. A goal of this Conference is to expose theoreticians and mathematically oriented scientists to the experimental dynamics problems encountered in the natural sciences and to expose researchers in the natural sciences to recent more mathematical results. Another important objective is to expose graduate students, and young scientists in general, to this fascinating field. There will be about eight lectures per day.

For further information write Robert H. G. Helleman, School of Physics, Georgiatech, Atlanta, Georgia 30332.

ICME IV

The 4th International Congress on Mathematical Education will be held in Berkeley, California, on August 10-16, 1980. The scientific program will examine a broad spectrum of problems in mathematical education at all levels and for every variety of learner. Special emphasis will be given to questions of universal primary education, of research, technology, applications, the profession of teaching, and the relationship of language and mathematics. Preliminary plans include four plenary session invited addresses, at least fifteen other main invited speakers and about sixty panels and debates.

For further information write to ICME IV, Mathematics Department, University of California, Berkeley, California 94720.

NEW JOURNAL ON LEARNING PROBLEMS IN MATHEMATICS

A new journal entitled "Focus on Learning Problems in Mathematics" will be published four times a year. The objective of the journal is to bring the research and useful ideas from the areas of psychology, neurology, learning disability and mathematics education to the attention of classroom and special education teachers, diagnosticians and therapists, and researchers. The editor is Mahesh Sharma. Subscription information may be obtained from Lillian Travaglini, Marketing Director, CT/LM, P.O. Box 3149, Framingham, Massachusetts 01701.

FRONTIERS OF APPLIED GEOMETRY RESEARCH WORKSHOP

This workshop will be held at New Mexico State University, Las Cruces, New Mexico, January 7-11, 1980. The program will concentrate on applications of geometry. The main areas for discussion, with invited speakers are: Catastrophe Theory (J.J. Callahan, Smith College); Geometric Control Theory (H.G. Hermes, University of Colorado); Geometric Modelling of Physical Objects (H.B. Voelcker/ A. Requicha, University of Rochester); Curved Finite Elements (E.L. Wachspress, Knolls Atomic Power Laboratory). For further information and application forms contact, Frontiers of Applied Geometry, Department of Mathematical Sciences, New Mexico State University, Las Cruces, New Mexico 88003, Telephone: 505-646-3901.

COMPETITION ANNOUNCED

The Center for Women Scholars in San Francisco is offering a prize of $500 for the best article of not more than 5,000 words on solutions to the problems of the woman scholar. The winning article will be published in the 1980 edition of *The Woman Scholar's Handbook: Strategies for Success*, a project of the CFWS. Address submissions to Dr. Monika Kehoe, Editor, CFWS, AMERICAS, 300 Broadway, Suite 23, San Francisco, California 94133.

INTRODUCTORY BASIC LANGUAGE COURSE

Colorado State University's Engineering Renewal and Growth program is now offering a videotaped course entitled "An Introduction to Computer Programming Using the BASIC Language." The course consists of seven videotaped 15-minute lectures in color and supplemental Study Guide. It focuses on developing proficiency in the BASIC language. Material covered is equivalent to a university short course in programming, and encompasses programming techniques and a repertoire of BASIC statements. Topics covered include: Arithmetic and logical operations; Batch and interactive forms for reading and writing information; Loops; Predefined and user-defined programs; 1- and 2-dimensional arrays; and Attacking and debugging programs. The lecturer is Director of Computer Utilization in the CSU College of Agricultural Sciences. For more information contact: W.L. Somervell, ERG Director, Christman Field, Bldg. 1000, Colorado State University, Fort Collins, Colorado 80523.

ACM COMPUTER SCIENCE EMPLOYMENT REGISTER

The Eighth Annual Computer Science Employment Register will be conducted at the Kansas City Computer Science Conference, February 11-14, 1980. This Register, the only one of its kind, aids in matching computer scientists and data processing specialists with employer opportunities. The purpose of the Register is to provide a mechanism for establishing contact between applicant and employer in a professional manner.

Both applicants and employers must file their registration on official forms. Three different forms will be used: (1) applicant, (2) academic employer, and (3) business, industry, and government employer. These forms may be obtained from and completed forms should be returned to: Orrin E. Taulbee, ACM Computer Science Employment Register, Department of Computer Science, University of Pittsburgh, Pittsburgh, Pennsylvania 15260. Closing date for acceptance of forms is January 18, 1980.

SHORT COURSE IN PROBABILISTIC AND STATISTICAL METHODS IN MECHANICAL AND STRUCTURAL DESIGN

The course presented by the University of Arizona College of Engineering, will be held January 7-11, 1980, at the Ramada Inn, 404 North Freeway, Tucson, Arizona. The objective of this short course and workshop is to provide practical information on engineering applications of probabilistic and statistical methods, and design under random vibration environments. Modern methods of structural and mechanical reliability analysis will be presented. Special emphasis will be given to fatigue and fracture reliability. For more information contact: Dr. Paul H. Wirsching, Associate Professor of Aerospace and Mechanical Engineering, The University of Arizona, College of Engineering, Tucson, Arizona 85721.

DIDACTIC PROGRAMMING

This new journal of calculator-demonstrated math instruction has enlarged its scope. Beginning with the current (second) issue it will cover all levels of instruction and every field. It contains articles by teachers for teachers to implement calculator techniques in the classroom. Theory and error analysis, programs, and reviews and previews of books and calculators will be featured, with rapid publication a primary goal. Sample issue available to our readers free of charge. Send to: Educational Calculator Devices, Inc., Box 974, Laguna Beach, California 92562.

MATHEMATICAL ASSOCIATION OF AMERICA

1979 CONTRIBUTING MEMBERS AND SPECIAL GIFTS

The Association is deeply indebted to the generosity of the 104 members listed below who have elected to be Contributing Members, Sponsors, or Patrons for 1979 by making contributions beyond the normal dues.

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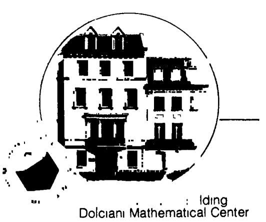
By June 15, 1979 approximately 1550 members of the Association had made gifts, pledges, and advance dues payments totaling over $362,000 to the MAA Building Fund and to special memorial funds to be used to assist in the purchase of the new MAA Headquarters. These gifts will be individually acknowledged elsewhere.



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Dolciani Mathematical Center

MARCH MEETING OF THE WISCONSIN SECTION

The Spring Meeting of the Wisconsin Section of the MAA was held at the University of Wisconsin at Eau Claire, March 30-31, 1979, with 82 in attendance. Chairwoman Lillian Gough (UW-Eau Claire) presided. The program was drawn up by Chairman-Elect, Norbert J. Kuenzi (UW-Oshkosh) and local arrangements were managed by Robert Langer (UW-Eau Claire).

The invited addresses were:

Computer Graphics, Pierre Malraison, Control Data Corporation, Minneapolis
Recursive Definitions in Mathematics, Donald Kreider, Dartmouth College

The following additional talks were given:

The Programmable Calculator as a Teaching Aid, Eli Maor, UW-Eau Claire
Investigating Divergent Series, Edward H. H. Gade III, UW-Oshkosh
Proving Computer Programs, Paul J. Campbell, Beloit College
Generalized Inner Products and Some Applications, Francis G. Florey, UW-Superior
Problem Solving in the Mathematics of Vocational Technical Schools, Neil Olsen, Northeast Wisconsin Technical Institute, Green Bay
Some Predecessors of Least Squares, David Lund, UW-Eau Claire
George Peacock's Principle of the Permancence of Equivalent Forms, Helena M. Pycior, UW-Milwaukee
How Good a Case Did Fermat Have?, Merrill Barnebey, UW-LaCrosse
Respondent Jeopardy and its Influence on Two Randomized Response Models and the Comprehension Factor, Dennis M. O'Brien, UW-LaCrosse
Conway's "Life" on a PET, Martin Engert, UW-Whitewater
Estimation of Survival Time from a Family Tree, V. Susarla, UW-Milwaukee
Line Equations of Curves; Duality in Analytic Geometry, Eli Maor, UW-Eau Claire
Can Merit Money Be Fairly Distributed? or, *The Arrow Paradox,* Harvey Fox, UW-Waukesha Center
Statistical Computing for Microcomputers, Jerome Klotz, UW-Madison
On a Class of Analytic Functions, Ghulam Shah, UW-Waukesha Center

The film program included a videotape *Einstein,* and the films *Inferential Statistics - Sampling and Estimation* and *Inferential Statistics - Hypothesis Testing,* all obtained from the Oshkosh Public Library.

Norbert J. Kuenzi, UW—Oshkosh, Chairman-Elect, succeeded to the office of Section Chairman. Anthony E. Barkauskas, UW-LaCrosse, was elected Chairman-Elect, and Paul J. Campbell, Beloit College, was elected Secretary-Treasurer. Gary B. Klatt, UW-Whitewater, continues as Section Governor.

It was decided to hold a Fall Workshop (date, topic, location to be announced) and to hold the Spring Meeting in late March at UW-Milwaukee.

PAUL J. CAMPBELL, *Secretary-Treasurer*

APRIL MEETING OF THE NEBRASKA SECTION

The annual meeting of the Nebraska Section was held on April 20-21, 1979, at Nebraska Wesleyan University in Lincoln in conjunction with the Eighty-Ninth Meeting of the Nebraska Academy of Sciences. Some seventy-five members and guests were in attendance including six high school students whose scores on the 1979 Annual High School Mathematics Examination placed them on the Contest Honor Roll. The high school students, their teachers, and their parents were guests of the Nebraska Section of the Mathematical Association of America at the Academy Banquet on Friday, April 20.

Professor David Roselle, Secretary of the Mathematical Association of America, reported on the new headquarters, on new programs, and on other activities. Officers for 1979-1980 were elected as follows: Chairman, Mildred L. Gross, Doane College; Chairman Elect, Alexander Mehaffey, Jr., University of South Dakota; Secretary-Treasurer, Henry Miot Cox, University of Nebraska; Chairman of High School Contest Committee, Stanley D. Luke, Nebraska Wesleyan University.

Professor Thomas S. Shores, Chairman of the Section, presided at both sessions of the meeting and introduced the following program:

Characterizing Objective Values in Multiple Objective Maximization Problems, Jerald Dauer, University of Nebraska-Lincoln (UN-L)
Several Stability Concepts for Dynamical Systems, Bruce F. Sloan, University of Nebraska at Omaha (UNO)
The Solution of BVPS for Ordinary Differential Equations by Newton's Method, Dwight V. Sukup, University of South Dakota
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Report on the 1979 Annual High School Mathematics Contest in Nebraska and in South Dakota, Stanley D. Luke, Nebraska Wesleyan University
Combinatorial Problems: Solved and Unsolved, Davis Roselle, Secretary of the Mathematical Association of America, Virginia Polytechnic Institute.

HENRY MIOT COX, *Secretary*

APRIL MEETING OF THE MARYLAND-D.C.-VIRGINIA SECTION

The Maryland-District of Columbia-Virginia Section of the MAA held its spring meeting at George Mason University on Saturday, April 21, 1979. Approximately eighty-nine people were in attendance including eighty-six members of the Section. Section Chairman Orville Thomas of the U.S. Naval Academy presided over the program and a brief business meeting. Joseph E. Kent, Vice Chairman for programs, served as Program Chairman.

The program featured a Panel Discussion on "Prospects in Mathematical Education in the 1980's." The panelists were Dorothy L. Bernstein, President of the MAA, Goucher College; Ron Davis, Northern Virginia Community College; Robert L. Wilson, Jr., Washington and Lee University; and Joseph F. Kent, University of Richmond, who served as moderator. The program also included a well-planned Student-Centered Program which ran concurrently with the program.

The following contributed papers were presented:
The Program of the Joint Board on Science and Engineering Education for the Greater Washington Area, by Joseph Leep, Washington Academy of Sciences
Generating Binary Operations, by William P. Wardlaw, U.S. Naval Academy
Re-Teaching Mathematics, by Vera R. Grandlund, University of Virginia
Extrema for Functions of Several Variables, by Charles H. Heiberg, U.S. Naval Academy
The Placement Testing Program of the MAA, by Richard H. Prosl, College of William and Mary
Mathematics for Economists: A Two Course Sequence, by Richard P. Rozek, The FTC and the University of Pittsburg
An Application of Probability to the Politics of Richmond, Virginia, by Milton P. Eisner and Forrest B. Wall, J. Sargeant Reynolds Community College
The Bias of a Popular But Not So Random Card Shuffle, by David P. Robbins, Washington and Lee U
Counting Labeled Trees, by Louis W. Shapiro, Howard University
Physical Properties of the Fundamental Curves, by Allen Barwick, Woodrow Wilson High School (D.C.)
An Application of Markov Processes to the Time Development of Systems, by Joseph T. Siewick, University of Maryland
Some Unsolved Variants of the Traveling Salesman Problem, by Bruce Golden and Arjang Assad, University of Maryland

The Student Centered Program included the following:
Film: *Cycloidal Curves or Tales from the Wanklenberg Woods,* MAA
Is Reincarnation Mathematically Inevitable?, by Arthur Charlesworth, University of Richmond
How Students Can Do "Research", by Allen Barwick, Woodrow Wilson High School (D.C.)
Math and Careers, by Richard Bolstein, George Mason University

REUBEN C. DRAKE, *Secretary*

APRIL MEETING OF THE OHIO SECTION

The Ohio Section of MAA held its sixty-third annual Spring meeting at the Middletown Campus-Miami University, Middletown, Ohio, April 20-21, 1979. One hundred and thirty people registered for the meeting, held jointly with The Ohio Mathematics Association of Two-Year Colleges. Section Chairperson M.D. Wetzel presided; C. A. Long was the Program Chairman.

Invited addresses included: *The Development of Algebraic Topology - A Study in Evolution,* by Peter Hilton, Case Western Reserve University and Battelle Memorial Institute, Seattle, Washington; and *History of Mathematics as a Pedagogical Tool,* by V. Frederick Rickey, Bowling Green State U.

The following contributed papers were also presented:
Hjelmslev Planes, A. Cronheim, Ohio State University
Geometry Through Isometrics, K. Cummins, Kent State University
Green's Theorem is a Two-Color Theorem, R. M. Dieffenbach, Miami University-Middletown
How E. S. Pearson Helped Me Teach Statistics, T. A. Hern, Bowling Green State University
Solutions of Ordinary Differential Equations by Finite Element Methods, J. Jones, Jr., R. A. Adams, and R. W. Newsom, Jr., Air Force Inst. Tech
Inversive, Laguerre, and Minkowski Planes, J. Kahn, Ohio State University
Three Dimensional Kaleidoscopes, J. Kennedy, Miami University
Mathematics and Literature, D. O. Koehler, Miami University
The Utility Problem, D. E. Kullman, Miami University
The Minimum Values of Pi, R. G. Laatsch, Miami University
Extracting Roots by Successive Subtraction and the Difference Triangle, A. P. Malone, University of Cincinnati
Representations of Gaussian Processes, C. Park, Miami University
Characterizing Generalized Quadrangle via Planes, S. Payne, Miami University
Were There Phythagoreans in Great Britain in 2500 B.C.?, L.C. Peck, Miami University
Hyperbolic Planes, N. Raber, University of Akron
Problems from the 1979 MAA Contest, L. J. Schneider, John Carroll University
Error Aspects of a Computational Method for Barrier Crossing Probabilities of Wiener Paths, F. Schuurmann, Miami University

Counter Cultural and Individual Development in Geometry, W. Weber, Bowling Green State University
PRIME 80 Recommendations - Will They Save Your Math Department from Annihilation in 1984?, K. L. Whipkey, Westminster College, Pennsylvania
A Geometrical Development of Some Familiar Results, R. L. Wilson, Ohio Wesleyan University
Affine and Projective Planes, J. Yaqub, Ohio State University

The meeting agenda also included meetings of The Executive Committee and of *ad hoc* committees — Committee on Co-operation Among Colleges and Universities, Committee on Curriculum, and Committee on Teacher Training and Certification. Business meetings of MAA Ohio Section and of The Ohio Math Association of Two-Year Colleges were also held.

Meeting highlights included discussion sessions. A Panel Discussion on *The Place of Geometry in the Curriculum* was led by C.A. Long, Bowling Green State University (moderator); K. Cummins, Kent State University; P. Hilton, Case Western Reserve University; V.F. Rickey, Bowling Green State University; and J. Smith, Muskingum College. A Panel Discussion on *Two-Year College Mathematics* was led by J. Karnes, Ohio University-Lancaster (moderator); J. Carney, Lorain County Community College; B. Samimy, Ohio College of Applied Science, University of Cincinnati; and E. Young, Sinclair Community College. 'Swap' sessions included: *Math Anxiety*, led by J. Palagallo, University of Akron; and *Devices to Improve Classroom Presentations* to membership of The Ohio Math Association of Two-Year Colleges.

MAA Ohio Section officers elected for the academic year 1979-80 include: D.O. Koehler, Miami University, Chairman; D. Deever, Otterbein College, Chairman-Elect; M.D. Wetzel, Denison University, Past Chairman; G. Mavrigian, Youngstown State University, Secretary-Treasurer; H.W. Vayo, University of Toledo, Program Chairman; D.J. Horwath, John Carroll University and A. Poorman, Ashland College, Program Committee Members. S.W. Hahn, Wittenberg University, Serves as Sectional Governor; L.J. Schneider, John Carroll University, as Supervisor of the MAA High School Mathematics Competition; C.P. Yang, Miami University-Middletown, as representative to the Two-Year College Mathematics Journal; and R.A. Little, Baldwin-Wallace College, as Section Newsletter Editor.

GUS MAVRIGIAN, *Secretary-Treasurer*

SPRING MEETING OF NEW JERSEY SECTION

The New Jersey Section of the Mathematical Association of America held its annual Spring Meeting on Saturday, April 28, 1979 at Monmouth College, West Long Branch, New Jersey. This was a joint meeting in association with AMTNJ and MATYCNJ.

After a brief welcome by the officers of the three groups present, Alan Tucker of SUNY at Stony Brook gave the keynote address. His topic was *A Proposed Unified Mathematical Sciences Program*. The members could then choose between five concurrent sessions. The papers presented at these sessions were:

Subassociativity, by Michael Aissen of Rutgers Newark
A Mathematical Analysis of Romantic Love, by Ronald Ruemmler of Middlesex County College
Limits from a Topological Viewpoint for the Advanced Placement Student, by James Shafer, Glassboro State College
Teaching the Bright Student, by Mary Dell Morrison of Columbia High School and Darryl Walke, Somerset Community College
Basic Skills Testing and Teaching the Basics, by Robert Smith of Lawrence High School North and Linda Tappin of Middlesex County College.

The luncheon speaker was Neil A.J. Sloane of Bell Labs. His paper was entitled, *The Orchard Problem*. After the luncheon there was a choice of two panel discussions. The first panel, which discussed *Problems with Developmental Mathematics Courses*, consisted of Shirley Beller, Burlington County College; James Maglian, Union College; Susan Marchand, Kean College; Thomas Marlowe, St.Peter's College; and Karen Rappaport, Essex County College. The second panel, made up of Robert Canavan, Monmouth College; Jack DeTalvo, Asbury Park Schools; Henry Peterson, Wayne Hills High School; and Richard Perchner, Monmouth College discussed *The Impact of Computers, Electronic Calculators, and Computer Science on Mathematics Education*. Discussions followed both panels.

JAMES MAGLIANO, *Secretary*

APRIL MEETING OF THE ILLINOIS SECTION

The fifty-eighth annual meeting of the Illinois Section was held on the campus of Northern Illinois University, DeKalb, on Friday and Saturday, April 27-28 with sixty-five members registered. The following addresses were presented:

Latin Squares: Complete and Incomplete, Charles Vanden Eynden, Illinois State University
The Rates of Convergence and Divergence of Some Infinite Series, Richard Johnsonbaugh, Chicago State U
Math Anxiety: What is it? What to Do?, Ed Schulz, Elgin Community College; Elaine Sullivan, Oakton Community College; Joe Sullivan, Elgin Community College
What is Topos Theory?. Lawrence N. Stout, Illinois Wesleyan University
Time Management: Motivation In and Out of the Classroom, Ralph Hannon, Kishwaukee College
Placement Tests in Practice, Philip E. Milos, University of Wisconsin, Madison Member, MAA Committee on the Placement Testing Program.

At the annual banquet on Friday evening, MAA President Dorothy Bernstein of Goucher College, Baltimore, spoke on *A Differential Equation of Literary Criticism*. At the annual business meeting, Professor Gary Tippett of Bradley University was elected chairman for 1979-80, Professor R.N. Pendergrass of Southern Illinois University - Edwardsville chairman-elect (1980-81), Professor Barbara Juister of Elgin Community College 2nd vice-chairperson, and Professor Howard Saar was re-elected secretary-treasurer.

HOWARD C. SAAR, *Secretary-Treasurer*

CALENDAR OF FUTURE MEETINGS

Sixty-third Annual Meeting, San Antonio, Texas, January 5–7, 1980.

Sixtieth Summer Metting, University of Michigan, Ann Arbor, August 18–20, 1980.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, West Virginia Wesleyan College, Buckhannon, April 25–26, 1980.

FLORIDA, Jacksonville University, Jacksonville, March 7–8, 1980.

ILLINOIS, John A. Logan College, Carterville, April 25–26, 1980.

INDIANA, Wabash College, Crawfordsville, October 27, 1979.

INTERMOUNTAIN, Utah State University, Logan, late April or early May 1980.

IOWA, Simpson College, Indianola, April 18–19, 1980.

KANSAS, March or April. Deadline for papers January 1.

KENTUCKY, Western Kentucky University, Bowling Green, April 11–12, 1980.

LOUISIANA–MISSISSIPPI, Louisiana Tech University, Ruston, February 15–16, 1980.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Prince Georges Community College, Largo, Maryland, November 10, 1979.

METROPOLITAN NEW YORK, Mercy College, Dobbs Ferry, May 4, 1980.

MICHIGAN, Hope College, Holland, May 2–3, 1980.

MISSOURI, late March/early April. Deadline for papers January 31.

NEBRASKA, April.

NEW JERSEY, Essex County College, Newark, November 3, 1979.

NORTH CENTRAL, University of North Dakota, Grand Forks, October 26–27, 1979.

NORTHEASTERN, University of Hartford, Connecticut, November 18, 1979.

NORTHERN CALIFORNIA, Naval Postgraduate School, Monterey, February 23, 1980.

OHIO, College of Wooster, Wooster, October 26–27, 1979.

OKLAHOMA-ARKANSAS, Westark Community College, Fort Smith, Arkansas, March 28–29, 1980.

PACIFIC NORTHWEST, Central Washington University, Ellensburg, Washington, June 20–21, 1980.

PHILADELPHIA, Drexel University, Philadelphia, November 17, 1979.

ROCKY MOUNTAIN, University of Colorado, Boulder, March 28–29, 1980.

SEAWAY, SUNY at Albany, October 26–27, 1979.

SOUTHEASTERN, Appalachian State University, Boone, North Carolina, April 11–12, 1980.

SOUTHERN CALIFORNIA, University of California., Riverside, November 16–17, 1979.

SOUTHWESTERN, usually in April. Deadline for papers 2 weeks before meeting.

TEXAS, East Texas State University, Commerce, April 4–5, 1980.

WISCONSIN, University of Wisconsin, Milwaukee, March 28–29, 1980.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, San Francisco, January 3–8, 1980.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES, Town & Country Hotel, San Diego, California, October 17–20, 1979.

AMERICAN MATHEMATICAL SOCIETY, San Antonio, Texas, January 3–6, 1980.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Amherst, Massachusetts, June 23–26, 1980.

ASSOCIATION FOR COMPUTING MACHINERY, Kansas City, Missouri, February 12–14, 1980.

ASSOCIATION FOR SYMBOLIC LOGIC, New York City, December 28–29, 1979.

ASSOCIATION FOR WOMEN IN MATHEMATICS

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHEMATIQUES

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Seattle, Washington, April 16–19, 1980.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Shoreham Hotel, Washington, D.C., May 5–7, 1980.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Radisson Muehlebach, Kansas City, Missouri, November 1–3, 1979.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Denver Marriott, Denver, Colorado, November 12–14, 1979.

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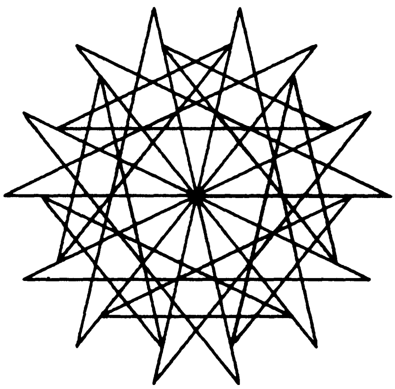

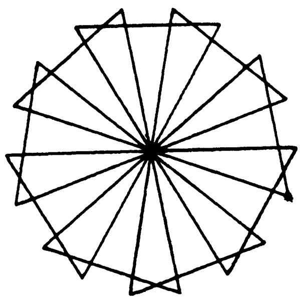

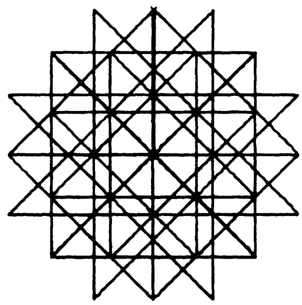

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THE AMERICAN MATHEMATICAL MONTHLY

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Research papers per se are unsuitable; see statement of policy (Vol. 85, p. 1).

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PICTURESQUE EXPONENTIAL SUMS, I

D. H. LEHMER AND EMMA LEHMER

Abstract. Let k be a fixed positive integer $\geqslant 2$ and let $\zeta = \exp(2\pi i/k)$. For any integer $n \geqslant 0$ we define $b(n)$ to be the sum of the digits of n when written to the base k. In this paper we consider the exponential sum

$$S_j(m) = \sum_{n=0}^{m} \zeta^{b(n)+jn}.$$

The properties of $S_j(m)$ are exhibited by plotting their graphs $G_j(k)$ for $k \leqslant 9$. These properties are proved for general k. The paper is the forerunner of a more complicated paper [2] in which $b(n)$ is replaced by the sum of products of all pairs of two consecutive digits of n.

1. Introduction and Motivation. The following identity is familiar

$$(1+x)(1+x^2)(1+x^4)(1+x^8)\cdots(1+x^{2^{\alpha-1}}) = \sum_{n=0}^{2^\alpha - 1} x^n$$

and is easily established by multiplying the left-hand side by $(1-x)$. It demonstrates that every integer n is the sum of distinct powers of two in just one way. If we look at the product

$$(1-x)(1-x^2)(1-x^4)(1-x^8)\cdots(1-x^{2^{\alpha-1}}) = \sum_{n=0}^{2^\alpha - 1} c_n x^n,$$

we may ask for the meaning of c_n. It is seen that we use the unique binary representation of n, where

$$n = d_0 + d_1 2 + d_2 2^2 + d_3 2^3 + \cdots,$$

but weigh this partition of n by $+1$ or -1 according as the number of parts is even or odd. There being only one partition of n, we see that

$$c_n = (-1)^{b(n)}$$

where

$$b(n) = d_0 + d_1 + d_2 + d_3 + \cdots.$$

How do we generalize this result? We can take a base $k \geqslant 2$ and $\zeta = \exp(2\pi i/k)$ and consider the product

$$\prod_{\nu=0}^{\alpha-1} (1 + \zeta x^{k^\nu} + \zeta^2 x^{2k^\nu} + \cdots + \zeta^{k-1} x^{(k-1)k^\nu}) = \sum_{n=0}^{k^\alpha - 1} c_n^{(k)} x^n.$$

Here we are considering partitions of n into powers of k with no part repeated more than $k-1$ times. For each n there is but one such partition

$$n = d_0 + d_1 k + d_2 k^2 + \cdots \qquad (0 \leqslant d_i < k).$$

The part or digit d_i of n is here given the weight ζ^{d_i} and the whole partition is given the weight

$$\zeta^{d_0}\zeta^{d_1}\zeta^{d_2}\cdots = \zeta^{b(n)}$$

where

$$b(n) = d_0 + d_1 + d_2 + \cdots$$

is the digit sum of n. Thus

$$c_n^{(k)} = \zeta^{b(n)}$$

The authors have been writing on number theory, both individually and in collaboration, for many years. Their first papers in this MONTHLY appeared, respectively, in 1924 and in 1937.—*Editors*

is the desired generalization. Hence we can write

$$\prod_{\nu=0}^{\alpha-1}(1-x^{k^{\nu+1}})/(1-\zeta x^{k^{\nu}})=\sum_{n=0}^{k^{\alpha}-1}\zeta^{b(n)}x^{n}. \tag{1}$$

As $\alpha\to\infty$ the series on the right converges when $|x|<1$, but diverges for $|x|\geqslant 1$, since in this case the terms do not tend to zero. We are interested in the partial sum

$$S_j(m)=\sum_{n=0}^{m}\zeta^{b(n)+nj}\qquad(0\leqslant j<k) \tag{2}$$

which is obtained when x is replaced by ζ^j, an arbitrary kth root of unity. We see that if we make this substitution in (1) we get

$$S_j(k^{\alpha}-1)=\begin{cases}0 & \text{if } j\neq k-1\\ 0 & \text{if } j=k-1 \text{ and } \alpha>1\\ k & \text{if } j=k-1 \text{ and } \alpha=1.\end{cases}$$

We proceed to take a closer look at $S_j(m)$.

2. Fundamental Theorem. We preface the theorem by the following lemma.

LEMMA 1. *If d_0 is any digit then*

$$S_j(km+d_0)=\begin{cases}\zeta^{b(m)}S_j(d_0) & (j<k-1)\\ kS_0(m)-(k-1-d_0)\zeta^{b(m)} & (j=k-1).\end{cases}$$

Proof. Since

$$b(\mu k+\nu)=b(\mu)+\nu\quad\text{for } \nu<k,$$

we can write, letting $n=\mu k+\nu$ in (2),

$$S_j(km+d_0)=\sum_{\mu=0}^{m-1}\zeta^{b(\mu)}\sum_{\nu=0}^{k-1}\zeta^{(j+1)\nu}+\sum_{\nu=0}^{d_0}\zeta^{b(m)+(j+1)\nu}.$$

If $j<k-1$, the inner sum vanishes and the lemma follows by (2). If $j=k-1$, the inner sum is k and we have

$$S_{k-1}(km+d_0)=kS_0(m-1)+(1+d_0)\zeta^{b(m)}=kS_0(m)-(k-1-d_0)\zeta^{b(m)},$$

which is the lemma.

We have the following useful corollaries:

COROLLARY 1. *If $m=d_1$ is a digit, then*

$$S_j(d_1k+d_0)=\begin{cases}\zeta^{d_1}\sum_{\nu=0}^{d_0}\zeta^{(1+j)\nu} & \textit{if } j<k-1\\ k\sum_{\nu=0}^{d_1-1}\zeta^{\nu}+(1+d_0)\zeta^{d_1} & \textit{if } j=k-1.\end{cases}$$

COROLLARY 2.

$$S_j(km)=\zeta^{b(m)}\quad \textit{if } j<k-1$$

COROLLARY 3.

$$S_j(km+k-1)=\begin{cases}0 & \textit{if } j<k-1\\ kS_0(m) & \textit{if } j=k-1.\end{cases}$$

ROTATION THEOREM.

$$S_j(nk^2+d_1k+d_0)=\zeta^{b(n)}S_j(d_1k+d_0).$$

Proof. If $j<k-1$, then Lemma 1 with $m=kn+d_1$ gives

$$S_j(nk^2+d_1k+d_0)=\zeta^{b(kn+d_1)}\sum_{\nu=0}^{d_0}\zeta^{(1+j)\nu}$$
$$=\zeta^{b(n)}S_j(d_1k+d_0)$$

by Corollary 1.

If $j=k-1$, then using Lemma 1 first for $j=k-1$ and then for $j=0$, we have

$$\begin{aligned}S_{k-1}(nk^2+d_1k+d_0)&=kS_0(kn+d_1)-(k-1-d_0)\zeta^{b(n)+d_1}\\&=k\zeta^{b(n)}\sum_{\nu=0}^{d_1}\zeta^{\nu}-k\zeta^{b(n)+d_1}+(1+d_0)\zeta^{b(n)+d_1}\\&=\zeta^{b(n)}\left(k\sum_{\nu=0}^{d_1-1}\zeta^{\nu}+(1+d_0)\zeta^{d_1}\right)\\&=\zeta^{b(n)}S_{k-1}(d_1k+d_0)\end{aligned}$$

by Corollary 1, which completes the proof of the theorem.

As a consequence of this theorem we see that the exponential sum $S_j(m)$, although not a periodic function of m, cannot take on more than k^3 values. In fact all the different values of $S_j(m)$ will be found among its first k^3 values. These are given explicitly by letting $n=d_2$ in the Rotation Theorem, using Corollary 1.

COROLLARY 4.

$$\begin{aligned}S_j(d_2k^2+d_1k+d_0)&=\zeta^{d_1+d_2}\sum_{\nu=0}^{d_0}\zeta^{(j+1)\nu} && \text{if } j<k-1\\&=k\zeta^{d_2}\sum_{\nu=0}^{d_1-1}\zeta^{\nu}+(d_0+1)\zeta^{d_1+d_2}, && j=k-1.\end{aligned}$$

A special case of this with $d_0=d_1=k-1$ and $d_2=d-1$ gives

COROLLARY 5.

$$S_{k-1}(dk^2-1)=0.$$

An immediate consequence of Corollaries 3 and 5 is the crude inequality

$$|S_j(m)|\leqslant\begin{cases}k/2 & \text{if } j<k-1\\ k^2/2 & \text{if } j=k-1\end{cases}\tag{3}$$

which will be improved later.

3. The Graph G_j of S_j. We understand by the graph $G_j=G_j(k)$ of $S_j(n)$ the graph whose vertices are the points

$$S_j(0),S_j(1),S_j(2),\dots$$

of the complex plane and whose edges are the straight line arcs joining $S_j(m)$ with $S_j(m+1)$ for $m=0,1,2,\dots$. We see from the Rotation Theorem that the graph of $S_j(n)$ is finite.

For $k=2$, $S_j(m)=0$ or ±1 for $j=0$ and 0, ±1 or ±2 for $j=1$, so that $G_0(2)$ and $G_1(2)$ are line segments $-1\leqslant x\leqslant1$ and $-2\leqslant x\leqslant2$, respectively. In fact by Corollaries 2 and 3,

$$S_0(2m)=(-1)^{b(m)},\qquad S_0(2m+1)=0,$$

while by Lemma 1,

$$S_1(4m)=S_1(4m+2)=(-1)^{b(m)},\ S_1(4m+1)=2(-1)^{b(m)},\ S_1(4m+3)=0.$$

We dismiss this real valued function as trivial and from now on consider $k\geqslant 3$.

We next prove two simple properties of the graphs $G_j(k)$.

PROPERTY 1. *The graph $G_{k-1}(k)$ is the k-fold enlargement of the graph $G_0(k)$.*

Proof. By Corollary 3

$$S_{k-1}(km+k-1)=kS_0(m),$$

so that for every vertex $S_0(m)$ on $G_0(k)$ there is a corresponding vertex $S_{k-1}(km+k-1)$ on $G_{k-1}(k)$ which is a k-fold enlargement. By (2), the difference

$$S_{k-1}(N)-S_{k-1}(N-1)=\zeta^{b(N)-N}$$

and is independent of the last digit of N. Thus the $k+1$ consecutive points

$$S_{k-1}(km+k-1), S_{k-1}(k(m+1)),\dots,S_{k-1}(k(m+1)+k-1)$$

are collinear on $G_{k-1}(k)$. This exhibits a straight edge of length k on the graph joining the points $S_{k-1}(km+k-1)$ and $S_{k-1}(k(m+1)+k-1)$.

PROPERTY 2. *The graphs $G_j(k)$ and $G_{k-2-j}(k)$ are complex conjugates of each other.*

Proof. For every integer $m=d_2k^2+d_1k+d_0$ define m' by

$$m'=\begin{cases}(k-d_2)k^2+(k-d_1)k+d_0 & \text{if } d_0\neq k-1\\ (k-d_2)k^2+(k-2-d_1)k+k-1 & \text{if } d_1\neq k-1\\ (k-2-d_2)k^2+(k-1)k+k-1 & \text{if } d_2\neq k-1\end{cases}$$

where $k-0$ is to be taken as 0. By Corollaries 2, 3, 4, and 5

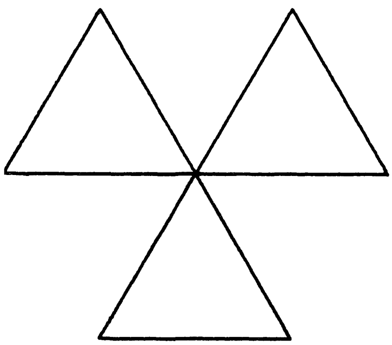
$$S_{k-2-j}(m')=\overline{S_j}(m)$$

and

$$S_{k-2-j}(m'+1)=\overline{S}_j(m+1).$$

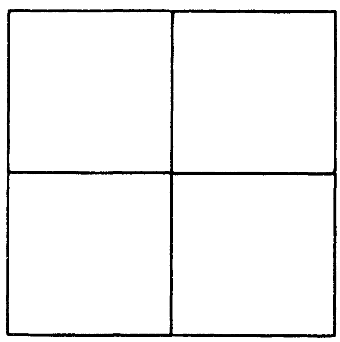
Thus for $m<k^3$ the vertices of G_j are complex conjugates of those of G_{k-2-j} and the edges of G_j are reflections in the real axis of the edges of G_{k-2-j}. Since the graph $G_j(k)$ is finished in $<k^3$ steps the proof is also complete.

By the first two properties of $G_j(k)$ it is unnecessary to plot $G_j(k)$ for more than the first $[k/2]$ values of j. We give the graphs of $G_j(k)$ for these values of j for $k<10$. These graphs were



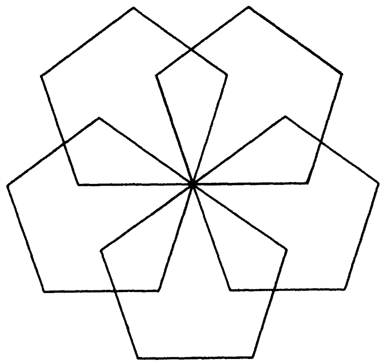
$G_0(3)$

FIG. 1



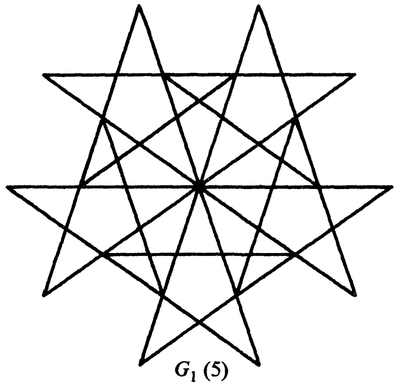
$G_0(4)$

FIG. 2



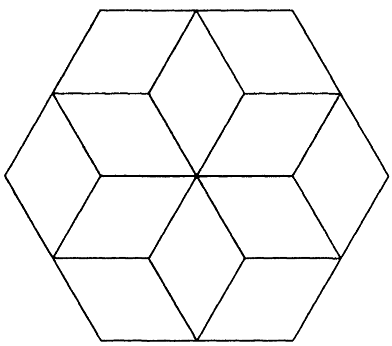
G_0 (5)

FIG. 3



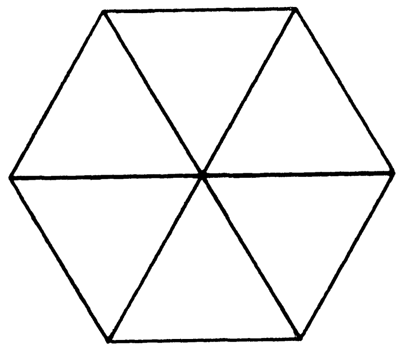
G_1 (5)

FIG. 4



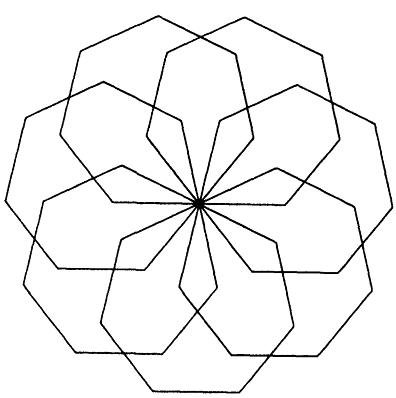
G_0 (6)

FIG. 5



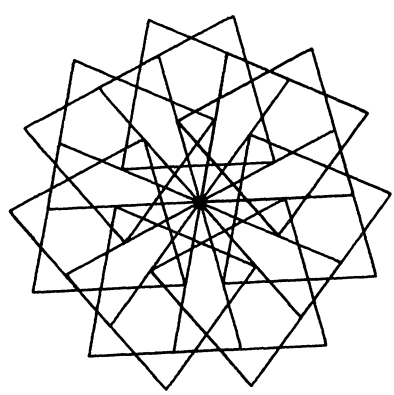
G_1 (6)

FIG. 6



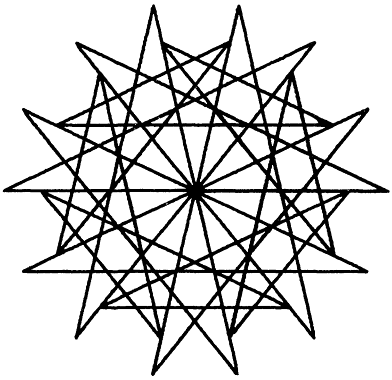
G_0 (7)

FIG. 7



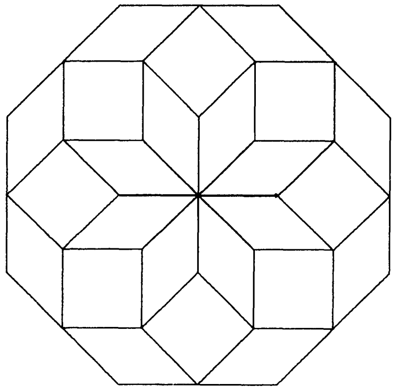
G_1 (7)

FIG. 8



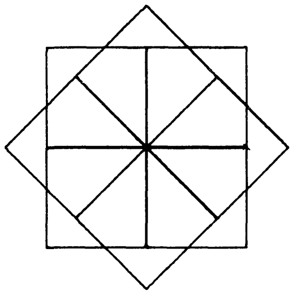
$G_2(7)$

FIG. 9



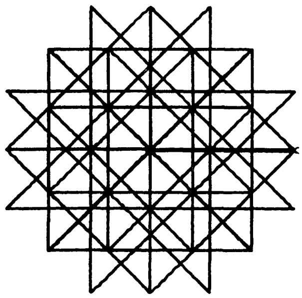
$G_0(8)$

FIG. 10



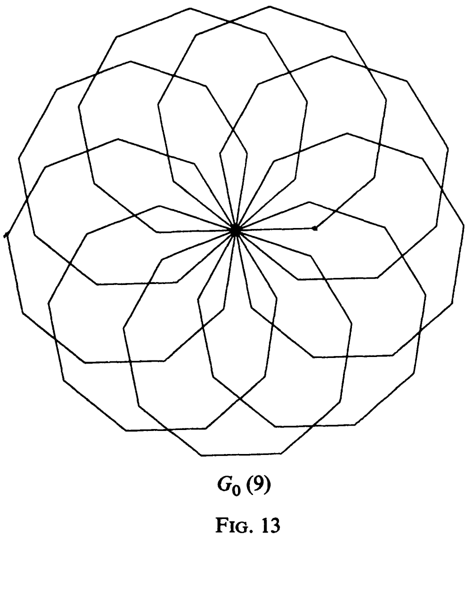
$G_1(8)$

FIG. 11



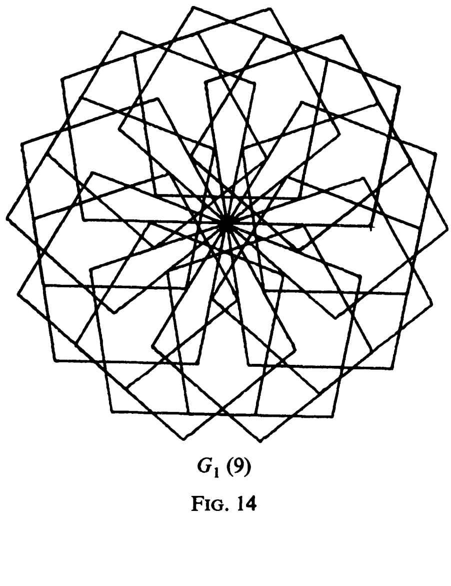
$G_2(8)$

FIG. 12



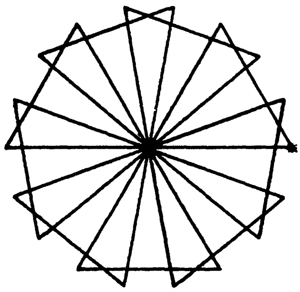
$G_0(9)$

FIG. 13



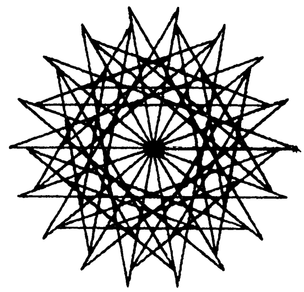
$G_1(9)$

FIG. 14



$G_2(9)$

FIG. 15



$G_3(9)$

FIG. 16

plotted on the Calcomp equipment at the Computer Center of the University of California in Berkeley.

Inspection of these plots leads to the following further properties of $G_j(k)$.

PROPERTY 3. *For k even, the graph $G_{(k-2)/2}$ degenerates into a set of k lines joining the origin to the kth roots of unity.*

Proof. If we put $j=(k-2)/2$ into Lemma 1, we get

$$S_{(k-2)/2}(km+d_0)=\zeta^{b(m)}\sum_{\nu=0}^{d_0}(-1)^{\nu}=\begin{cases}\zeta^{b(m)} & d_0 \text{ even}\\ 0 & d_0 \text{ odd.}\end{cases}$$

As d_0 varies with m fixed, we see that the sum assumes alternately the values 0 and a certain root of unity. As m varies, $b(m)$ assumes all k values modulo k.

To develop further properties it is convenient to introduce polygons $P_j(k)$ for $j=1$ to $[(k-2)/2]$, which are simple generalizations of the regular convex polygon $P_1(k)$. In general $P_j(k)$ is obtained by joining

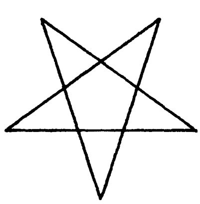
$$1=\zeta^0 \text{ to } 1+\zeta^j \text{ to } 1+\zeta^j+\zeta^{2j} \text{ to} \cdots \text{ to } 0 \text{ to } 1$$

by straight lines. In other words $P_j(k)$ is the graph of the exponential sum

$$\sum_{\nu=0}^{k}\zeta^{\nu j}.$$

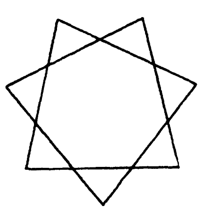
We give plots of these k-gons $P_j(k)$ for $k<10$. They appear in the next property of $G_j(k)$.

PROPERTY 4. *For $j<k-1$, $G_j(k)$ is the union of the k k-gons $P_{j+1}(k)$ equally spaced around the origin.*



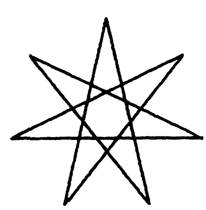
$P_2(5)$

FIG. 17



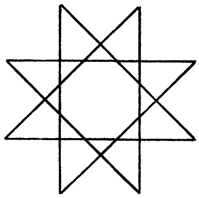
$P_2(7)$

FIG. 18



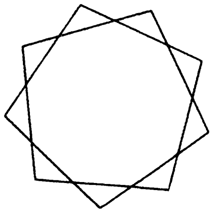
$P_3(7)$

FIG. 19



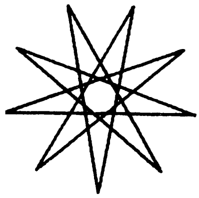
$P_3\,(8)$

FIG. 20



$P_2\,(9)$

FIG. 21



$P_4\,(9)$

FIG. 22

Proof. By Lemma 1

$$S_j(km+d_0)=\zeta^{b(m)}\sum_{\nu=0}^{d_0}\zeta^{\nu(1+j)}.$$

As d_0 varies from 0 to $k-1$ the graph $G_j(k)$ describes the k-gon $P_{j+1}(k)$ returning to the origin. As m varies, $b(m)$ ranges over a complete set of values modulo k. Thus Property 4 is evident.

PROPERTY 5. *If k is even, $G_j(k)$ is symmetric with respect to the real and imaginary axes. If k is odd, $G_j(k)$ is symmetric only with respect to the imaginary axis.*

Proof. This follows from Property 4, since if k is even $\zeta^{k/2}=-1$ and the kth roots of unity are symmetric with respect to both the real and imaginary axes. In case k is odd the symmetry with respect to the imaginary axis follows from the symmetry of the centers of the pairs of k-gons and the symmetry of the k-gon at the bottom of $G_j(k)$, with respect to the imaginary axis.

PROPERTY 6. *Let $R_j(k)$ denote the radius of the smallest circle which covers $G_j(k)$ for $j<k-1$, then*

$$R_j(k)=\begin{cases}\csc(\pi(j+1)/k) & \text{if } k/(k,j+1) \text{ is even}\\ \csc(\pi(j+1)/k)\cos(\pi(k,j+1)/2k) & \text{otherwise.}\end{cases}$$

Proof. By Property 4 the desired radius $R_j(k)$ is equal to the maximum diameter d_{j+1} of the k-gon $P_{j+1}^{(k)}$. Letting $j+1=h$, $\delta=(k,h)$, $k=k_1\delta$, $h=h_1\delta$, $(k_1,h_1)=1$, we have

$$d_h=\max_r\left|\sum_{\nu=0}^{r-1}\zeta^{h\nu}\right|=\max_r|\zeta^{rh}-1|/|\zeta^h-1|.$$

To evaluate the denominator we note that

$$|\zeta^h-1|=|\zeta^{h/2}||\zeta^{h/2}-\zeta^{-h/2}|=2\sin(\pi h/k).$$

To evaluate the numerator we split two cases.

Case 1. k is even. In this case

$$\max_r|\zeta^{rh}-1|=2$$

which is achieved for $r=(k_1/2)\bar{h}_1 \pmod{k_1}$ where

$$h_1\bar{h}_1=1 \pmod{k_1}.$$

Case 2. k_1 is odd. In this case we can choose r so that

$$rh_1=(k_1-1)/2 \pmod{k_1}.$$

Then

$$|\zeta^{rh}-1|=2|\sin(\pi(k_1-1)/(2k_1)|=2\cos(\pi/(2k_1)). \tag{4}$$

Hence

$$R_j(k)=d_h=\begin{cases}\csc(\pi h_1/k_1) & \text{if } k_1 \text{ is even}\\ \csc(\pi h_1/k_1)\cos(\pi/2k_1) & \text{if } h_1 \text{ is odd}\end{cases}$$

which proves Property 6.

It follows from this property that

$$|S_j(n)|\leqslant k/(2j+2)\quad \text{for } j\leqslant(k-2)/2$$

In fact by (4)

$$|S_j(n)|\leqslant R_j(k)<\csc(\pi h/k)\leqslant k/2h=k/(2j+2),$$

which is an improvement on (3) when $j>0$.

In a second paper [2] we consider the sums

$$S_j(m)=\sum_{n=0}^{m}\zeta^{e(n)+jn}$$

where

$$e(n)=\sum_{i=0}^{\infty}d_i d_{i+1}$$

plays the role of $b(n)$. These sums were considered by Brillhart and Morton [1] for $k=2$. For $k>2$ they lead to infinite graphs [2] which are much more complex than those of this paper.

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2. D. H. and Emma Lehmer, Picturesque exponential sums, II, J. Reine Angew. Math. (to appear).

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DISCOVERING THEOREMS WITH A COMPUTER: THE CASE OF $y'=\sin(xy)$

WENDELL MILLS, BORIS WEISFEILER, AND ALLAN M. KRALL

1. Introduction. The problem $y'=\sin(xy)$, $y(0)=A$, arose during an attempt to find suitable numerical examples to present to a class in differential equations and proved to be quite

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A. M. Krall did his graduate work at the University of Virginia under E. J. McShane and Marvin Rosenblum, receiving his Ph.D. in 1963. Since then, he has served at the Pennsylvania State University, where he has been Professor since 1971. The main thrust of his work has been concerned with general boundary value problems. More recently he has become interested in special functions. —*Editors*

fascinating. The equation was analyzed numerically using a computer. At first the behavior of the solutions was quite baffling. They oscillated for a while (the longer the greater A) then approached zero with x tending to infinity. The conjectures describing the behavior of solutions were formulated only after the solutions for various values of A were calculated in detail.

Using geometric ideas we were then able to give a qualitative explanation of the numerical results. The additional feature which appeared during the analysis is the existence of separatrices which in the first quadrant tend to the hyperbolas $xy=2n\pi$ from below as $x\to\infty$. All other solutions tend to the hyperbolas $xy=(2n+1)\pi$ from above.

The methods of our analysis can easily be generalized to equations of the form $y'=f(g(x,y))$, where f is a function which has an infinite number of zeros without accumulation point and satisfies certain growth conditions. The function $g(x,y)$ is such that the curves $g(x,y)=c$ are concave and approach the x-axis asymptotically. There are, of course, additional conditions, connecting f and g. We chose to deal with our original $y'=\sin(xy)$ in order to leave our method as transparent as possible.

2. Properties of $y'=\sin(xy)$, $y(0)=A$. We note that the differential equation is such that the set of solution curves is symmetric with respect to the x-axis, the y-axis, and the origin. Consequently it is sufficient to consider the first quadrant. Further, Picard's theorem holds, so a unique solution passes through each point in the plane. Since $y\equiv 0$ is a solution (with $y(0)=0$), no solution satisfying $y(0)=A>0$ ever crosses the x-axis.

We refer the reader to Figure 1, which was determined numerically. We shall now attempt to verify analytically that what the figure suggests is, in fact, always true.

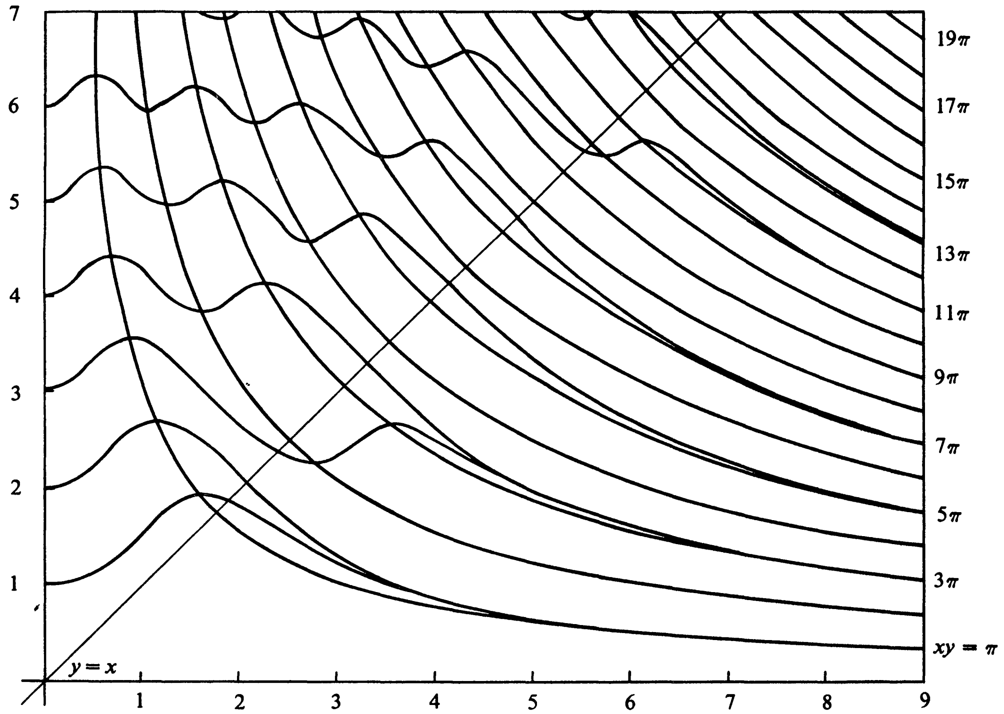

FIG. 1

LEMMA 1. *Let $y(x)$ be a solution of $y'=\sin(xy)$. Then*

(a) *If $y(x)$ intersects $xy=n\pi$, it does so with slope* 0.

(b) *If $y(x)$ intersects $xy=(n+\frac{1}{2})\pi$, it does so with slope $(-1)^n$.*

(c) $\sup|y'(x)|=1$.

(d) *If $y(x)$ intersects $xy=2n\pi$, then it also intersects $xy=(2n+1)\pi$.*

Proof. Only (d) is not completely clear. To prove (d), denote by (a,b) the intersection point of $y(x)$ and $xy=2n\pi$. Let L_1,L_2 be straight lines through (a,b) with slopes 1 and 0, respectively. Both L_1 and L_2 intersect $xy=(2n+1)\pi$ and because $0\leqslant y'(x)\leqslant 1$ for (x,y) between the curves $xy=2n\pi$ and $xy=(2n+1)\pi$, we have that $y(x)$ is below L_1 and above L_2. Thus it must intersect $xy=(2n+1)\pi$.

THEOREM 2. *Let $y(x)$ be a solution of $y'=\sin(xy)$. Then $y(x)$ intersects the hyperbola $xy=\alpha$*

(a) *at most once if $2n\pi\leqslant\alpha\leqslant(2n+1)\pi$,*

(b) *at most twice if $(2n-1)\pi<\alpha<2n\pi$.*

Proof. (a) Suppose (x_1,y_1), (x_2,y_2) are two intersection points. Then $f(x)=y(x)-(\alpha/x)$ satisfies $f(x_1)=f(x_2)=0$. Furthermore, $2n\pi\leqslant\alpha\leqslant(2n+1)\pi$ gives $y'(x_1)=\sin x_1y_1=\sin\alpha\geqslant 0$ and $y'(x_2)=\sin x_2y_2=\sin\alpha\geqslant 0$. Thus $f'(x_1)>0$, $f'(x_2)>0$, and by the intermediate value theorem there exists $x_3, x_1<x_3<x_2$, such that $f(x_3)=0$. Inductively, there exists a bounded sequence $\{x_i\}_1^\infty$ such that $f(x_i)=0$. Hence, there exists an accumulation point, c, of $\{x_i\}_1^\infty$, $x_1\leqslant c\leqslant x_2$. Since $f(x)$ is analytic about c, $f(x)=0$ in a neighborhood of c, a contradiction.

(b) Let $(2n-1)\pi<\alpha<2n\pi$ and let $x=c$ be the (unique) positive solution of $\sin\alpha=-\alpha/x^2$. Let $f(x)=y(x)-(\alpha/x)$. Then any intersection point, x_L, $0\leqslant x_L<c$, satisfies $f'(x_L)>0$, and any intersection point, $x_R, c<x_R<\infty$, satisfies $f'(x_R)<0$. An argument identical to that in (a) shows there is at most one intersection point in each interval $[0,c]$ and $[c,\infty]$.

COROLLARY 3. *If a hyperbola, $xy=\alpha$, is tangent to a solution, $y(x)$, then the point of tangency is unique and is the only intersection point of $y(x)$ and $xy=\alpha$.*

Proof. The tangency point occurs at the point, c, in the proof of Theorem 2 (b).

THEOREM 4. *For all initial values $y(0)=A$ the solution $y(x)$ to $y'=\sin(xy)$ intersects the line $y=x$.*

Proof. Let L be the broken line obtained as follows (Fig. 2):

1. From $(0,A)$ it has slope 1 until it intersects $xy=\pi$.
2. Between $xy=(2n-1)\pi$ and $xy=2n\pi$ it has slope 0.
3. Between $xy=2n\pi$ and $xy=(2n+1)\pi$ it has slope 1.
4. L is continuous.

Let (a_n,b_n) be the point of intersection of L with $xy=n\pi$. Then for n odd,

$$a_{n+1}=\frac{\pi(n+1)}{b_n}=a_n\left(\frac{n+1}{n}\right).$$

All that is necessary is to show L intersects $y=x$, since the solution satisfying $y(0)=A$ lies below L. This follows, since for the horizontal components of L

$$\sum_{n\,\mathrm{odd}}(a_{n+1}-a_n)=\sum_{n\,\mathrm{odd}}a_n(1/n)>a_1\sum_{n\,\mathrm{odd}}(1/n)=\infty.$$

So while the diagonal components parallel $y=x$, the horizontal components push it relatively ever farther to the right and ultimately beyond.

Until the solution intersects $y=x$, it crosses the regions between the hyperbolas $xy=n\pi$. The slopes of y within these regions are alternately positive and negative giving

THEOREM 5. *Until a solution $y(x)$ crosses $y=x$, it is alternately increasing and decreasing. The solution oscillates.*

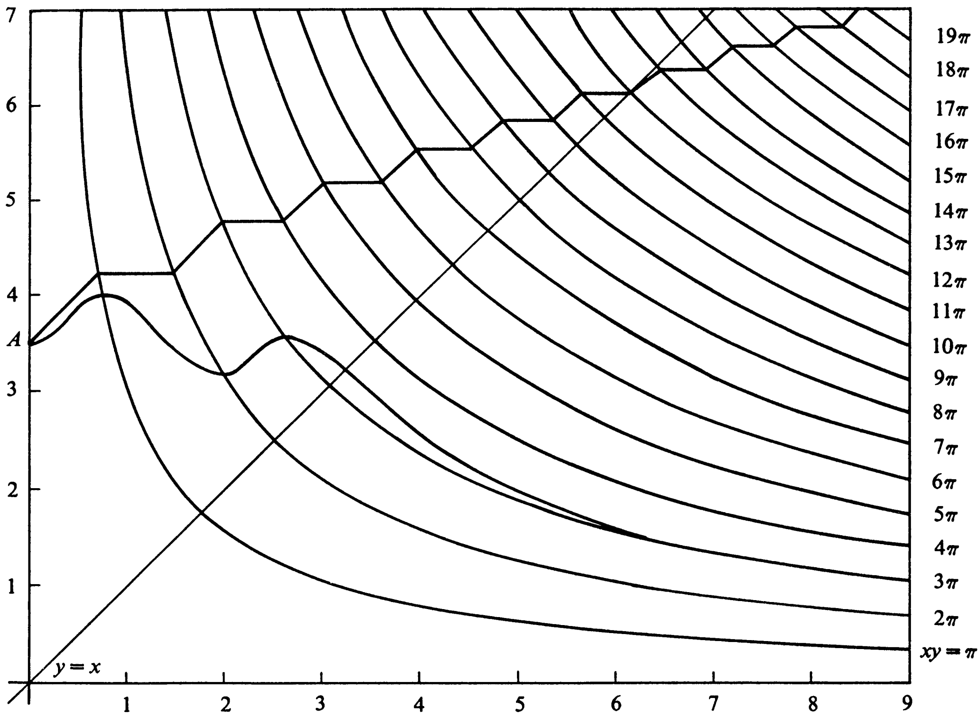

FIG. 2

THEOREM 6. *Let a solution $y(x)$ intersect $y=x$ at $x=x_0$. Let $x_0y(x_0)<(2n+\frac{3}{2})\pi$. Then the solution $y(x)$ does not intersect $xy=(2n+\frac{3}{2})\pi$.*

Proof. First note that once the solution lies below $y=x$, it remains below by Lemma 1(c). Assume the conclusion is false. Let (a,b) be the first point of intersection with $xy=(2n+\frac{3}{2})\pi$ below $y=x$. Such a first point exists by Theorem 2. The slope $y'(x)$ at a is -1, while the slope of the hyperbola is greater than -1. This implies that to the left of a the solution is above the hyperbola. This further implies that either the solution and the hyperbola intersect to the left of a and below $y=x$ or $x_0y(x_0)>(2n+\frac{3}{2})\pi$. Both are impossible.

Likewise, if the solution intersects $xy=(2n+\frac{3}{2})\pi$ above $y=x$, then in order to pass through $(x_0,y(x_0))$ it would have to intersect $xy=(2n+\frac{3}{2})\pi$ again above $y=x$. Again slope considerations make this impossible.

COROLLARY 7. *For each solution $y(x)$, there exists a maximum n such that the solution intersects $xy=(2n+1)\pi$. The solution lies between $xy=(2n+1)\pi$ and $xy=(2n+2)\pi$ for all sufficiently large x.*

Proof. In the open region bounded by the x-axis, the y-axis, and $xy=\pi$, $0<y'<1$. So $y(x)$ is increasing and is bounded above by $y=A+x$, below by $y=A$. Since both intersect $xy=\pi$, $y(x)$ does so as well.

The existence of such an n is now guaranteed by Theorem 6.

Once $y(x)$ intersects $xy=(2n+1)\pi$, it cannot do so again according to Theorem 2. By Lemma 1(d) it cannot intersect $xy=(2n+2)\pi$. Therefore it remains between $xy=(2n+1)\pi$ and $xy=(2n+2)\pi$.

3. The Asymptotic Nature of Solutions. In order to adequately describe the asymptotic nature of the solutions as $x\to\infty$, let us consider the following regions (see Fig. 3): Let

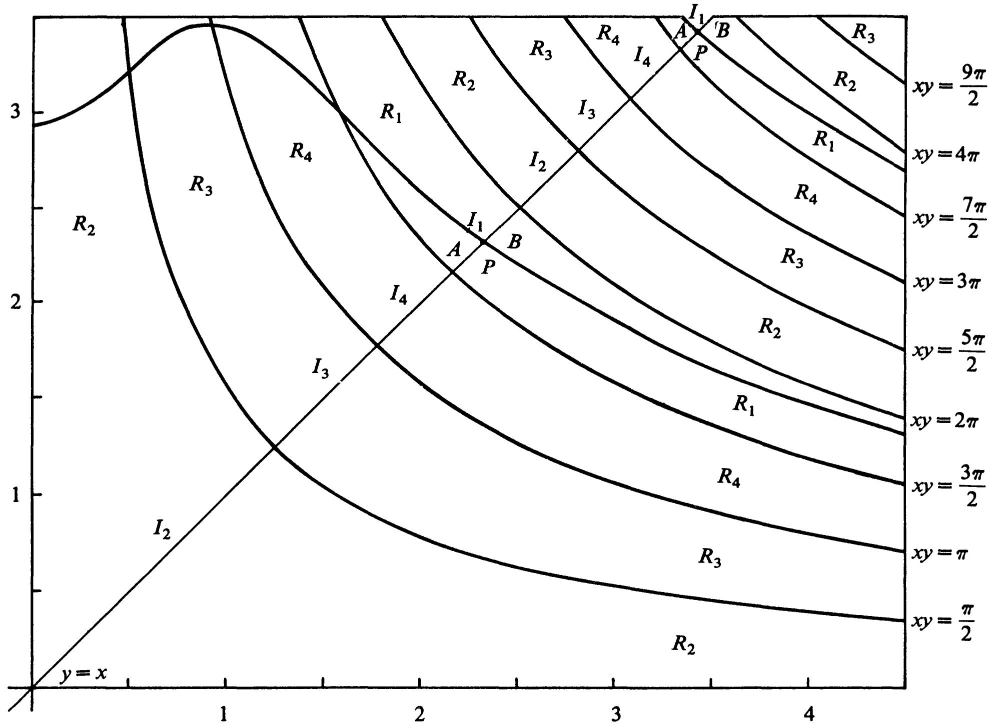

FIG. 3

$$R_1=\left\{(x,y):\left(2n-\tfrac{1}{2}\right)\pi\leqslant xy\leqslant 2n\pi\right\},$$
$$R_2=\left\{(x,y):2n\pi\leqslant xy\leqslant\left(2n+\tfrac{1}{2}\right)\pi\right\},$$
$$R_3=\left\{(x,y):\left(2n+\tfrac{1}{2}\right)\pi\leqslant xy\leqslant(2n+1)\pi\right\},$$
$$R_4=\left\{(x,y):(2n+1)\pi\leqslant xy\leqslant\left(2n+\tfrac{3}{2}\right)\pi\right\},$$
$$R'_1=\left\{(x,y):\left(2n+\tfrac{3}{2}\right)\pi\leqslant xy\leqslant(2n+2)\pi\right\}.$$

Let

$$I_1=R_1\cap\{\text{line}\,y=x\},$$
$$I_2=R_2\cap\{\text{line}\,y=x\},$$
$$I_3=R_3\cap\{\text{line}\,y=x\},$$
$$I_4=R_4\cap\{\text{line}\,y=x\},$$
$$I_1'=R_1'\cap\{\text{line}\,y=x\}.$$

We shall examine in succession the solutions $y(x)$ which pass through I_1, I_2, I_3, I_4, I_1'.

I_1: The solution passing through the point $\left(\sqrt{\left(2n-\frac{1}{2}\right)\pi}\,,\sqrt{\left(2n-\frac{1}{2}\right)\pi}\,\right)$, the left end of I_1, must drop below $xy=(2n-\frac{1}{2})\pi$, since its slope at that point is -1. By Corollary 3 any other passage through $xy=(2n-\frac{1}{2})\pi$ is impossible. Likewise, solutions intersecting I_1 near the left end of I_1 must intersect $xy=(2n-\frac{1}{2})\pi$ below $y=x$, leaving R_1 to remain in the region below, by Theorem 2. We shall show that these solutions become asymptotic to $xy=(2n-1)\pi$ when we consider the regions I_1' and R_4.

Similarly the solution passing through the point $(\sqrt{2n\pi}\,,\sqrt{2n\pi}\,)$, the right end of I_1, must

pass into R_2 and never return to R_1. Likewise, solutions intersecting I_1 near the right end of I_1 must intersect $xy=2n\pi$ below $y=x$, leaving R_1, and remaining above.

Thus there exist three sets contained in I_1,

$A=\{(x,x)$: the solution passing through (x,x) intersects $xy=(2n-\frac{1}{2})\pi$ below $y=x$ and remains below $xy=(2n-\frac{1}{2})\pi\}$,

$B=\{(x,x)$: the solution passing through (x,x) does not intersect $xy=(2n-\frac{1}{2})\pi$ or $xy=2n\pi$ below $y=x\}$,

$C=\{(x,x)$: the solution passing through (x,x) intersects $xy=2n\pi$ below $y=x$ and remains above $xy=2n\pi\}$.

A moment's reflection establishes that A and C are nonempty intervals

$$\left(\sqrt{(2n-\tfrac{1}{2})\pi}\,,\sqrt{(2n-\tfrac{1}{2})\pi}\,\right)\in A,\ \ (\sqrt{2n\pi}\,,\sqrt{2n\pi}\,)\in C$$

with B in between. The boundary points between A and B, B and C are neither in A nor in C, since if the boundary p between A and B were in A, then points above p would be in A. If the boundary p between B and C were in C, then points below p would be in C. B is nonempty since there cannot be a last point of A or a first point of C.

THEOREM 8. (a) *Solutions passing through A in I_1 intersect $xy=(2n-\frac{1}{2})\pi$ and remain below, becoming asymptotic to $xy=(2n-1)\pi$.*

(b) *The set B consists of exactly one point p_n. The solution passing through p_n remains in I_1. This solution $y(x)$ becomes asymptotic to $xy=2n\pi$, and $xy(x)\to 2n\pi$.*

(c) *Solutions passing through C in I_1 intersect $xy=2n\pi, xy=(2n+\frac{1}{2})\pi$ and $xy=(2n+1)\pi$, passing through R_2, through R_3, into R_4, where they become asymptotic to $xy=(2n+1)\pi$.*

Proof. (a) We shall establish the asymptotic nature of the solutions when R_4 is examined in detail.

(b) Let $y(x)$ be a solution passing through B.

If $xy(x)$ does not approach $2n\pi$, then by Theorem 2 it must ultimately be bounded away from $2n\pi$. In that case there is an $\varepsilon>0$ such that $\sin(xy)<-\varepsilon<0$. This implies that $y'<-\varepsilon$, and forces $y(x)$ to intersect $xy=(2n-\frac{1}{2})\pi$, which is contrary to assumption.

It is apparent that B is closed. Let $y(x)$ represent the solution passing through the left end of B, and let $Y(x)$ be any other solution passing through B. Then $Y(x)>y(x)$, and $xY(x)$ and $xy(x)$ both approach $2n\pi$. Thus

$$Y'-y'=\sin(xY)-\sin(xy)=\int_{xy}^{xY}\cos t\,dt.$$

Since xY and xy are eventually close to $2n\pi$, there is a $\delta>0$ such that $\cos t>1-\delta>0$, for $xy<t<xY$, x sufficiently large. Hence,

$$\begin{aligned}Y'-y'&>\int_{xy}^{xY}(1-\delta)\,dt,\\ &=x(1-\delta)(Y-y).\end{aligned}$$

This implies for some $C>0$,

$$Y-y>C\exp\left[(1-\delta)x^2/2\right]$$

and $Y\to\infty$ as $x\to\infty$. This is impossible, so B contains only one point.

(c) We shall establish the asymptotic nature of the solutions when R_4 is examined in detail.

I_2: Solutions passing through I_2 have positive slope at that point. According to Corollary 7, they must intersect $xy=(2n+1)\pi$ and enter R_4. By Theorems 2 and 6 they must remain in R_4.

THEOREM 9. *Solutions passing through I_2 intersect $xy=(2n+\frac{1}{2})\pi$ and $xy=(2n+1)\pi$, passing through R_3 into R_4, where they remain in R_4 and become asymptotic to $xy=(2n+1)\pi$.*

Proof. The asymptotic nature of the solutions will be established when R_4 is examined in detail.

I_3: Solutions passing through I_3 also have positive slope and must intersect $xy=(2n+1)\pi$ and enter R_4 by Corollary 7. They must remain in R_4 by Theorems 2 and 6.

THEOREM 10. *Solutions passing through I_3 intersect $xy=(2n+1)\pi$ passing into and remaining in R_4, where they become asymptotic to $xy=(2n+1)\pi$.*

I_4: According to Theorem 2 solutions passing through I_4 must remain above $xy=(2n+1)\pi$. According to Theorem 6 they must remain below $xy=(2n+\frac{3}{2})\pi$. Thus they remain in R_4.

THEOREM 11. *Solutions passing through I_4 remain in R_4 and become asymptotic to $xy=(2n+1)\pi$.*

I_1': Solutions passing through I_1' divide themselves into three classes, just as those passing through I_1. Those which are of interest to us intersect $xy=(2n+\frac{3}{2})\pi$ below $y=x$ and remain in R_4.

THEOREM 12. *Solutions passing through I_1' within the interval A of that region intersect $xy=(2n+\frac{3}{2})\pi$, remain in R_4 and become asymptotic to $xy=(2n+1)\pi$.*

The behavior in R_4: We have established that only the solution passing through p_n, which becomes asymptotic to $xy=2n\pi$, fails to enter R_4 (or its counterpart, associated with the integer $n-1$).

THEOREM 13. *All solutions passing through a point in R_4 below $y=x$ remain in R_4 and become asymptotic to $xy=(2n+1)\pi$. For such solutions $xy(x)\to(2n+1)\pi$ as $x\to\infty$.*

Proof. If, for a solution $y(x)$, $xy(x)$ did not approach $(2n+1)\pi$, then by Theorem 2 there exists an $\varepsilon>0$ such that $\sin xy<-\varepsilon<0$. This immediately forces an intersection of such a solution with $xy=(2n+1)\pi$, which is impossible.

MAIN THEOREM. *Let $y(x)$ be a solution of $y'=\sin(xy)$, $y(0)=A$, $A>0$, in the first quadrant.*

(a) *$y(x)$ intersects the line $x=y$ at some point (a,a). It oscillates until it intersects this line.*

(b) *If $(a,a)=p_n$ of Theorem 8, then $y(x)$ approaches the hyperbola $xy=2n\pi$ asymptotically from below and $x\cdot y(x)\to 2n\pi$ as $x\to\infty$.*

(c) *If (a,a) lies between p_{n-1} and p_n, then $y(x)$ approaches the hyperbola $xy=(2n-1)\pi$ from above, and $x\cdot y(x)\to(2n-1)\pi$ as $x\to\infty$. Moreover $y(x)$ intersects the hyperbola $xy=(2n-\frac{1}{2}\pi$ exactly once if $(a,a)=\left(\sqrt{(2n-\frac{1}{2})\pi}, \sqrt{(2n-\frac{1}{2})\pi}\right)$ and exactly twice if (a,a) is between $\left(\sqrt{(2n-\frac{1}{2})\pi}, \sqrt{(2n-\frac{1}{2})\pi}\right)$ and p_n.*

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THE GÖDEL INCOMPLETENESS THEOREM FROM A LENGTH-OF-PROOF PERSPECTIVE

JOHN DAWSON

1. Introduction. The Gödel Incompleteness Theorem is one of the most profound and sensational results of twentieth-century mathematics. Its appearance in 1931 ([**9**], translated in [**5**, pp. 5–38] and [**11**, pp. 596–616]), shattered the hopes of those committed to Hilbert's formalist program and added impetus to the development of mathematical logic. Yet it is doubtful even today that many non-logicians appreciate the significance of Gödel's theorem or understand its basic content.

In part, this lack of understanding may reflect a lingering belief that foundational matters are not really relevant to the concerns of the "working" mathematician. Aside from such prejudice, however, most expositions of the Incompleteness Theorem have focused on the paradoxical nature of its proof, with the result that the significance of the theorem has tended to become lost amid technical intricacies. Some may even regard the theorem as an abstruse curiosity, like the Paradox of the Liar, to which it is closely related.

It is the aim of the present article to examine Gödel's theorem from a wholly different point of view that illuminates its meaning while largely ignoring its proof. Along the way we shall survey a number of important results in logic and consider recent progress toward the solution of an outstanding open problem.

2. Logical background. Although the impact of Gödel's theorem extends beyond the confines of number theory, we shall restrict attention to that original context. At the outset, we must distinguish between the informal Peano axioms familiar to non-logicians and their logical formalization in first-order number theory (to which Gödel's theorem applies). Following Landau [**14**], we take the former in the form:

1. Zero is a natural number.
2. Associated to each natural number is a unique natural number called its successor.
3. Zero itself is not the successor of any natural number.
4. Natural numbers having equal successors are themselves equal.
5. (The induction axiom) Any set of natural numbers containing zero, and containing the successor of every number it contains, contains all natural numbers.

On the basis of these axioms, together with definitions of the arithmetical operations, we can proceed by rules of informal logic (as Landau does) to derive all the basic theorems of elementary arithmetic. Further, as is well known, we can prove that any two structures satisfying all the axioms must be isomorphic; to do so, however, we must make essential use of the induction axiom, which alone among the axioms refers to *sets* of numbers rather than to numbers themselves.

In formalizing the axioms it is desirable to avoid set-theoretic notions. This can be done by rephrasing the induction axiom in terms closer to Peano's original formulation: "Any *property* true of zero and true of the successor of a number whenever true of the number itself is true of all natural numbers." We interpret "property" to mean "property expressible by a formula" of a suitable formal language $\mathfrak{L}$, and correspondingly we replace Peano's single induction axiom by an infinite induction *schema*, introducing an induction axiom for each particular formula of our language.

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The "suitable formal language" can be constructed in a variety of ways, but in any case we understand that it shall be a first-order language. That is:

1. The symbols of $\mathfrak{L}$ shall consist of a countably infinite number of distinct variable symbols, zero or more constant and function symbols, the equality symbol and zero or more other relation symbols, and finitely many logical connective and quantifier symbols (say $\neg$, $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$, $\forall$, and $\exists$). It is understood that each relation or function symbol applies to only finitely many arguments. Parentheses may also be included to enhance readability.
2. The syntax of $\mathfrak{L}$ shall be as follows: **Terms** ("nouns") are obtained inductively by applying zero or more function symbols to variables, constants, or previously defined terms. **Atomic formulas** ("simple sentences") are obtained by taking terms as arguments of relation symbols. More complex **formulas** are built up inductively from atomic formulas by applying the connectives and quantifiers.
3. The semantics of $\mathfrak{L}$ shall be as follows: A **structure** $\mathfrak{A}$ for $\mathfrak{L}$ shall consist of a non-empty set A (regarded as the domain over which the variable symbols are allowed to vary) together with a distinguished element of A corresponding to each constant symbol of $\mathfrak{L}$, an n-ary relation on A corresponding to each n-ary relation symbol of $\mathfrak{L}$ (with the identity relation corresponding to the equality symbol), and an n-ary function from A to A corresponding to each n-ary function symbol of $\mathfrak{L}$.

Note that in first-order languages, variables are always interpreted as ranging over elements of structures, never over sets of such elements. Note also that although from a set-theoretic point of view functions can be treated as a special kind of relations, in formal logic function and relation symbols are distinguished because they play syntactically different roles. We shall see in §6 that this distinction can be critical in some contexts.

Formal languages are vehicles for the precise expression of mathematical theories; their great virtue is that they permit a rigorous distinction to be made between the fundamental mathematical notions of truth and provability. Thus the semantical notion of truth is defined relative to structures: given an assignment of specific elements of a structure to correspond to the variable symbols of the language, the relations, functions, and constants of the structure provide a natural framework for interpreting the meaning (and hence the truth or falsity) of the atomic formulas. The truth or falsity of more complex formulas can then be defined inductively (a crucial idea due to Tarski [**24**], translated in [**28**, pp. 152–278]) by interpreting the meaning of the connectives and quantifiers according to the rules of classical logic. A formula is **true in the structure** $\mathfrak{A}$ if it is true there regardless of the particular assignment of elements to its variable symbols; it is **logically valid** if it is true in every structure for the language.

To define the concept of proof, we first select a subset (perhaps all) of the logically valid formulas as **logical axioms**. We then construct a particular first-order theory by selecting another set of formulas (perhaps empty) as **proper** (or non-logical) **axioms**. Finally we specify a set of **rules of inference** (finitary truth-preserving functions from tuples of formulas to formulas). A **proof** in the theory then consists of a finite sequence of formulas, each of which is either an axiom or the result of applying a rule of inference to previous formulas in the list. The **length** ("number of lines") of the proof is the number of formulas in the sequence.

The choice of a particular language and of the set of proper axioms is dictated to some extent by the particular mathematical theory to be formalized. For number theory we assume that the corresponding formal language $\mathfrak{L}$ has the binary relation symbol $<$, the binary function symbols $+$ and $\cdot$, the unary function symbol S (for successor), and the constant symbol $\bar{0}$. Following Shoenfield [**22**, pp. 22 and 204], we take as proper axioms:

1. $(\forall x)\neg(Sx=\bar{0})$
2. $\forall x \forall y(Sx=Sy \rightarrow x=y)$
3. $\forall x(x+\bar{0}=x)$
4. $\forall x \forall y(x+Sy=S(x+y))$
5. $\forall x(x\cdot\bar{0}=\bar{0})$

6. $\forall x \forall y(x \cdot Sy = (x \cdot y) + x)$
7. $(\forall x) \neg (x < \bar{0})$
8. $\forall x \forall y(x < Sy \leftrightarrow x < y \vee x = y)$
9. (Induction schema) For every formula $\Phi(x)$ in which the variable symbol x is free (unquantified), the induction axiom $(\Phi(\bar{0}) \wedge \forall x(\Phi(x) \rightarrow \Phi(Sx))) \rightarrow \forall x \Phi(x)$.

Inessentially, we may for convenience expand $\mathfrak{L}$ to include constant symbols $\bar{n}$ for each natural number n, adding as axioms the definitions

$$\bar{n} = \underbrace{S \ldots S}_{n \text{ times}}\bar{0}.$$

We shall refer to the resulting formalization of Peano arithmetic as PA.

3. Limitations of formalization: Completeness and incompleteness. We posit that the aim of mathematical research is to ascertain what is true about certain mathematically interesting structures, while the role of proofs is secondary, a means to an end. In particular, the formal definition of truth in purely semantic terms suggests the possibility of dispensing with proofs altogether. This can indeed be done for the logic of connectives (propositional logic), where truth tables provide an effective means for deciding the validity of an arbitrary propositional statement. However, when quantifiers and relations are introduced, it is no longer clear how to determine whether a statement holds in all possible interpretations: a fundamental result in first-order logic (**Church's Undecidability Theorem**) states that there is no effective procedure (algorithm) for determining whether an arbitrary formula of PA is logically valid. Indeed, the same is true for any first-order language containing , in addition to the equality symbol, at least one n-ary relation symbol for some $n \geqslant 2$. (See [**15**, p. 293].)

Church's theorem has far-reaching implications. It justifies the use of proofs in establishing arithmetical truth, yet at the same time it poses a problem for the selection of an adequate set of logical axioms—for in any satisfactory proof system, it must be possible to *recognize* the axioms (a condition known as **recursive axiomatizability**), and this will not be possible if the logical axioms are taken to consist of *all* logically valid formulas. So the question arises: Is it possible to choose a recursive set of logical axioms and a finite set of rules of inference from which all logically valid formulas can be derived as theorems?

Tentative solutions to this problem were proposed by Hilbert and his followers during the early decades of this century. The capstone to their efforts was provided by Gödel in his doctoral dissertation ([**8**], translated in [**11**, pp. 582–591]): he showed that a scheme can be so chosen that in *any* first-order theory, the theorems coincide exactly with those formulas true in all **models** of (that is, structures satisfying) the proper axioms. This result, the **Gödel Completeness Theorem**, is a corollary to Gödel's result that every *consistent* first-order theory (one in which no contradiction is provable) *has* a model. For if a formula is not provable in a given theory, its negation can be consistently added as a new axiom; a model for the augmented theory is then a model of the original theory in which the formula in question is false. (Strictly speaking, Gödel obtained his results only for languages with countably many symbols; extensions to uncountable languages were made later by Mal'cev and others.)

The Completeness Theorem seemingly vindicated formalist hopes. But, like the notion of logical validity, it contained a weakness: reference to *all* models of a set of axioms. Unfortunately, it is seldom apparent how all models of a theory behave. Indeed, the discovery of non-Euclidean geometries became a turning point in the history of mathematics because it awakened recognition of the existence of non-standard models of axiomatic systems. Modern consistency and independence proofs in set theory (work in which Gödel's name again figures prominently) are based on the same idea.

Still, the problem of unintended interpretations may appear irrelevant to number theory in view of the informal proof that Peano's axioms characterize the natural numbers up to

isomorphism. But the Completeness Theorem itself implies the existence of non-isomorphic models of *formal* number theory. For, since proofs are finite in length, a theory is consistent if every finite subset of its axioms is consistent. Hence by completeness we obtain the **Compactness Theorem** for first-order logic: A first-order theory has a model if each of its finite subtheories does. In particular, adjoining to PA a new constant symbol c together with the axioms $c \neq \bar{n}$ for each natural number n gives a consistent theory whose models satisfy PA but are not isomorphic to the natural numbers. (This argument is due to Henkin; the existence of non-standard models of PA was first established by Skolem some ten years prior to Gödel's work.)

The mere existence of non-standard models did not of itself dash the hope of discovering exactly what is true of the natural numbers, since isomorphism is a sufficient but not a necessary condition that two structures satisfy the same sentences of PA. It remained conceivable that all models of PA possessed the same first-order properties. But in 1931, only a year after the Completeness Theorem, Gödel astounded the mathematical world by giving an example of a sentence of PA true in the natural numbers but unprovable in PA. Similar examples can be constructed in all consistent, recursively axiomatizable extensions of PA. This (First) **Incompleteness Theorem** thus established that the formulas true in all models of formal number theory form a proper subset of those true of the natural numbers; the special properties that distinguish the natural numbers from non-standard "impostors" cannot be discovered by means of formal proofs in the given deductive system.

One may still ask: Is there an alternative to the use of formal proofs in establishing truth in number theory? Paradoxically, the answer is yes, since Gödel's proof of the Incompleteness Theorem exemplifies the possibility of demonstrating the truth of a formula via an *informal* proof of its *formal* undecidability (see §5 below). However, the potential for discovering number-theoretically significant truths by this method has only very recently begun to be realized (an exciting advance due to J. B. Paris [**18**]). More to the point, Gödel's techniques also yield that there is no effective procedure for deciding the truth or falsity (in the natural numbers) of an arbitrary formula of PA. This analogue of Church's theorem stands in sharp contrast to a positive result of Presburger [**19**], who had shown only two years earlier that there *is* an effective decision procedure for arithmetic without multiplication. The interested reader may consult [**2**, pp. 220–228], [**6**, pp. 188–192], or [**15**, pp. 237–240] for further details.

4. Universal statements and the induction axiom. In attempting to understand how a statement can be true yet unprovable, it is natural to study the relative complexity of first-order formulas. The most obvious measure of complexity is in terms of quantifier structure. A standard theorem of first-order logic states that every formula is logically equivalent to one in **prenex form**, that is, to a formula of the form $Q_0x_0 \cdots Q_nx_n\Phi$, where the Q's denote a sequence (possibly empty) of quantifiers and Φ is a quantifier-free formula. The most significant classification of prenex forms is according to the number of alternations of quantifier. In particular, formulas with prenex equivalents $\forall x_0 \cdots \forall x_n\Phi$ are called **universal** (or Π_1) and those with prenex equivalents $\exists x_0 \cdots \exists x_n\Phi$ are called **existential** (or Σ_1). Formulas with no free variables are said to be **closed**, and we shall refer to closed formulas of the form $\Phi(\bar{n})$ as **numerical instances** of the formulas $\Phi(x)$, $\exists x\Phi$, and $\forall x\Phi$. (Closed formulas are also called **sentences**.)

It is reasonable to hope that PA is at least strong enough to prove every closed, quantifier-free formula that is true in the natural numbers. In fact, every true closed *existential* formula is a theorem of PA. (For a proof of this assertion see [**22**, pp. 209–211]. The result is especially pertinent to Gödel's *Second* Incompleteness Theorem, not considered here, concerning the impossibility of internally formalized consistency proofs.) Trouble first arises when we consider universal formulas. Indeed, if we extend PA by adjoining new function symbols for certain (primitive) recursive functions, adding as axioms the definitions of these functions in PA, then in the extended theory Gödel's canonical example of a true but unprovable sentence is a Π_1 formula. In PA itself there are true unprovable sentences of the form $\forall x_0 \cdots \forall x_n(P(x_0, \ldots, x_n) \neq$

$Q(x_0,\ldots,x_n)$), where P and Q denote polynomials in several variables with positive integer coefficients. That is, there exist Diophantine equations having no positive integer solutions, whose unsolvability cannot be proved in PA. (The existence of such equations follows from Matiyasevič's celebrated negative solution (1970) of Hilbert's Tenth Problem—see **[4]** for a complete account.)

The main tool for proving universal statements in PA is the induction schema, which we may regard as the source both of the strength and the weaknesses of the theory. Without induction, PA would be emasculated. Moreover, no finite set of induction axioms can replace the full induction schema, as Ryll-Nardzewski showed in **[21]** using non-standard models. In a sense, even the full schema contains too *few* formulas, since there are only countably many formulas of PA, but uncountably many properties (subsets) of the natural numbers to which (informal) induction should apply. This fundamental conflict underlies the apparent contradiction between the existence of non-standard models and the informal proof of the categoricity of Peano's axioms.

We prefer to regard the induction axiom as a sort of *compression principle*, since it provides a means of proving in finitely many steps the truth of infinitely many numerical instances of a number-theoretic statement. As such, however, the induction axioms are of restricted scope, since to apply them one must somehow establish the premise $\forall x(\Phi(x)\to\Phi(Sx))$. Of course, *if* $\forall x\Phi$ is true, then so is $\forall x(\Phi(x)\to\Phi(Sx))$, but the problem confronting the theorem prover is to justify independently the general passage from n to $n+1$, so as to *conclude* $\forall x\Phi$ by induction.

Now there is a rather obvious way around this difficulty: namely, extend the concept of proof to include infinite well-ordered sequences of formulas and introduce the infinitary rule "From $\Phi(\bar{0}),\Phi(\bar{1}),\ldots,$ deduce $\forall x\Phi$." This so-called **ω-rule** is obviously sound, but just as obviously worthless from a practical standpoint, since it is not clear how we can ever know each of $\Phi(\bar{0}),\Phi(\bar{1}),\ldots$ apart from already knowing $\forall x\Phi$. The ω-rule has nonetheless been recommended as a useful principle for introducing induction to high school students **[27]**, and it has a most important philosophical significance. For it turns out that the resulting **ω-logic** is the proper vehicle for investigating the structure of the standard natural numbers, in the sense that the theorems provable in PA using ω-logic coincide exactly with the formulas true in $\mathbb{N}$. This special case of the **ω-Completeness Theorem** (see §5 below) may be paraphrased as follows.

METATHEOREM: *Any formula true in the natural numbers but unprovable in* (*ordinary*) PA *must be true by virtue of an infinite number of "special cases."*

It is reasonable to suspect that there may be universal statements of PA that hold in $\mathbb{N}$ merely through the dovetailing of an infinite number of inherently different instances. What is remarkable is that the ω-Completeness Theorem implies that this is the *only* reason for the proof-theoretic inadequacy of PA.

It is our contention that the ω-Completeness Theorem provides the best approach to understanding Gödel's Incompleteness Theorem. The next section is devoted to a more detailed consideration of this idea.

5. ω-completeness and omission of types. The ω-Completeness Theorem grew out of concepts employed by Gödel in his 1931 paper, yet it did not appear until twenty-five years later. The history of the evolution of the ideas involved is an interesting subject deserving critical study, for it is punctuated by repeated instances of discovery, dormancy, rediscovery, and application. We pause here to offer a capsule summary of the course of events.

In 1927, Tarski introduced the concepts of ω-consistency and ω-completeness (cf. **[28**, footnote 2, p. 279]): A theory in the language of number theory is **ω-consistent** if whenever each of $\Phi(\bar{0})$, $\Phi(\bar{1}),\ldots$ is provable, we cannot also prove $\exists x(\neg\Phi)$; if the provability of $\Phi(\bar{0})$, $\Phi(\bar{1}),\ldots$ actually implies the provability of $\forall x\Phi$ (without the ω-rule), then the theory is **ω-complete**. Gödel invoked ω-consistency as a hypothesis in his original statement of the Incompleteness

Theorem, and Tarski noted some further consequences of ω-consistency and ω-completeness in 1933 ([**25**], translated in [**28**, pp. 279–295]). But in 1936 Rosser ([**20**], reprinted in [**5**, pp. 231–235]) showed that by appeal to a comparison of proof *lengths* (of Φ and $\neg\Phi$), the assumption of ω-consistency in Gödel's theorem could be replaced by ordinary consistency.

Whether Rosser's result was responsible for stifling interest in the ω-concepts is unclear; in any event the first forms of the ω-Completeness Theorem were enunciated by Orey [**16**] in 1956 and Henkin [**12**] in 1957. A period of "ripening" then ensued during which the theorem quietly reappeared in various guises. Eventually the several variants coalesced into the form:

Every consistent, ω-complete theory in the language of number theory has a model whose domain consists precisely of the elements interpreting the constant symbols $\bar{n}$ (a so-called **ω-model**). (*)

There followed another period of benign indifference. Then, with the sudden growth of model theory (currently the most active and exciting research area in logic), the theorem underwent a remarkable metamorphosis, emerging as the **Omitting Types Theorem**. In the latter form, the theorem has become an important tool for model construction (see [**3**, pp. 78–84]). We shall not need this more general version, but its curious name deserves some words of explanation. A $(1-)$ **type** Γ is a set of formulas in a single free variable x which is finitely satisfiable (that is, $\exists x(\sigma_1(x)\wedge\cdots\wedge\sigma_n(x))$ is consistent for any finite subset $\{\sigma_1(x),\ldots,\sigma_n(x)\}$ of Γ). A type is **realized** in a structure $\mathcal{Q}$ if some assignment to x causes all its formulas to be satisfied in $\mathcal{Q}$; otherwise $\mathcal{Q}$ is said to **omit** the type. A type Γ is **principal** with respect to a theory T if some formula $\Phi(x)$ consistent with T **generates** Γ, that is, if $T\cup\{\exists x\Phi(x)\}$ is consistent and the formula $\forall x(\Phi(x)\rightarrow\theta(x))$ is provable in T for every θ in Γ. The Omitting Types Theorem asserts the existence of models omitting *non*-principal types. In particular, an ω-model is a model omitting the type $\{x\neq\bar{n}:n<\omega\}$. (At this point the reader should recall our construction of a non-standard model of PA.)

To conclude this section, we will use (*) to justify our earlier assertion that the theorems provable in PA using ω-logic coincide with the formulas true in $\mathbb{N}$. (For a proof of (*) itself, see [**26**, p. 27].) Additionally, we will show that any universal sentence neither provable nor disprovable in (ordinary) PA must be true in $\mathbb{N}$. Toward those ends, we introduce the following notation. Given a theory T and a sentence θ, we write $T\vdash\theta$ to mean "T proves θ in ordinary logic" and $T\vdash_\omega\theta$ to mean "T proves θ in ω-logic." We also write T^ω for $\{\theta:T\vdash_\omega\theta\}$. Finally, we let Th $\mathbb{N}$ (the **theory of** $\mathbb{N}$) denote the set of all closed formulas true in the standard natural numbers. Suppose then that $\sigma\in\text{Th }\mathbb{N}$ but $\sigma\notin\text{PA}^\omega$. (Note that we certainly have $\text{PA}^\omega\subseteq\text{Th }\mathbb{N}$.) Then the theory $T=\text{PA}^\omega\cup\{\neg\sigma\}$ is consistent. Moreover, T is ω-complete. (For if $T\vdash\Phi(\bar{0}),T\vdash\Phi(\bar{1}),\ldots$ yet T fails to prove $\forall x\Phi(x)$, then by the Deduction Theorem of propositional logic, $\text{PA}^\omega\vdash\neg\sigma\rightarrow\Phi(\bar{0})$, $\text{PA}^\omega\vdash\neg\sigma\rightarrow\Phi(\bar{1}),\ldots,$ but PA^ω does not prove $\neg\sigma\rightarrow\forall x\Phi(x)$. However, $\text{PA}^\omega\vdash\forall x(\neg\sigma\rightarrow\Phi(x))$, and since $\neg\sigma$ is closed, $\forall x(\neg\sigma\rightarrow\Phi(x))$ is logically equivalent to $\neg\sigma\rightarrow\forall x\Phi(x)$.) Hence by (*), T has an ω-model. But since T extends PA, any ω-model of T is isomorphic to $\mathbb{N}$. This is a contradiction, since $\neg\sigma\in T$ but $\sigma\in\text{Th }\mathbb{N}$.

Finally, suppose σ is any universal statement undecidable in PA. Then $\sigma\in\text{Th }\mathbb{N}$. For otherwise $\mathbb{N}$ would satisfy $\exists x_0\cdots\exists x_n(\neg\Phi)$, where $\forall x_0\cdots\forall x_n\Phi$ is a prenex equivalent of σ. But then $\exists x_0\cdots\exists x_n(\neg\Phi)$ would be a true closed existential formula, so PA would prove $\exists x_0\cdots\exists x_n(\neg\Phi)$ and thereby refute σ.

6. Proof lengths—A conjecture of Kreisel and a theorem of Parikh. Throughout the model-theoretic "gestation period" described in the preceding section, a parallel development was taking place in proof theory. In one year (1936) three important papers appeared, all based on considerations of the length or complexity of proofs in formal systems. One, Rosser's improvement of the Incompleteness Theorem, has already been mentioned. The others were Gentzen's consistency proof for arithmetic ([**7**], translated in [**23**, pp. 132–213]), the first of its kind,

employing a transfinite induction on the complexity of derivations; and a brief and little-noticed article by Gödel himself ([10], translated in [5, pp. 82–83]). The latter, titled simply "On the Length of Proofs," pointed out that, by passing to systems of "higher type" (allowing sets of integers, sets of sets of integers, etc.), not only can new theorems be proved, but "it becomes possible to shorten extraordinarily infinitely many of the proofs already available."

Translated only in 1965, Gödel's length-of-proof paper was largely ignored until after the advent of the computer revolution, when concern for efficiency of computation led to Manuel Blum's creation of computational complexity theory, and Gödel's proof-shortening result was resurrected to become the progenitor of a whole class of **speed-up theorems** (see [1, pp. 253 and 261–263]). Somewhat earlier (1955) Kreisel and Wang [13] also studied proof shortening. They considered various measures of "proof of length $\leqslant n$," including proof in at most n lines, and obtained results related to Gödel's Second Incompleteness Theorem.

The possibility of some kind of synthesis of the model-theoretic and proof-theoretic perspectives must have suggested itself at about this time. To see how this might be possible, let us return to the ideas of §4. There we saw how a universal statement could fail to be provable despite the provability of each of its numerical instances. In particular, it can happen that the lengths of proofs of the instances $\Phi(\bar{n})$ depend on n. Thus, given $\Phi(x)$, let $\mathrm{PL}(\Phi)=\{m$: For some n, the shortest proof of $\Phi(\bar{n})$ requires $\geqslant m$ lines$\}$. Using Gödel's techniques, one can construct (in a recursive extension of PA) a true universal statement $\forall x\Phi$ for which $\mathrm{PL}(\Phi)$ contains *all* natural numbers; it follows that $\forall x\Phi$ will not be provable in PA, since the schema $\forall x\Phi\rightarrow\Phi(\tau)$, for *any* term τ, is a part of the underlying logic of any first-order theory. On the other hand, what of the converse problem? If there is a uniform bound m to the lengths of shortest proofs of the instances $\Phi(\bar{n})$ (a situation we abbreviate by $(\forall n)(\mathrm{PA}\vdash_m\Phi(\bar{n}))$), is it necessary that $\forall x\Phi$ be provable in PA? A conjecture attributed to Kreisel asserts that this is indeed the case.

Kreisel's conjecture is certainly a bold one, since even if each $\Phi(\bar{n})$ *is* provable in at most m lines there is no reason to think that the individual proofs might not still incorporate an infinite number of different arguments. The conjecture has gained greater credence, however, from a remarkable recent result of Parikh [17]. Parikh's theorem applies not to PA, but to a seemingly inessential variant which, following Parikh, we denote by PA*. PA* differs from PA only in that the binary function symbols $+$ and $\cdot$ are replaced by ternary relation symbols A and M, with corresponding axioms added stating that A and M represent functions. It follows that each formula Φ of PA has a translation Φ^* in PA* having the same meaning, and that $\mathrm{PA}\vdash\Phi$ if and only if $\mathrm{PA}^*\vdash\Phi^*$, so that PA and PA* prove the "same" theorems. Proofs in PA* tend to be more "cumbersome" (in particular, longer) than those in PA, but the terms of PA* are much simpler, consisting just of the successor function S applied some finite number of times to either $\bar{0}$ or a variable symbol. As a result, Parikh was able to establish that if $(\exists m)\ (\forall n)\ (\mathrm{PA}^*\vdash_m\Phi(\bar{n}))$, then $\mathrm{PA}^*\vdash\forall x\Phi$, that is, the analogue of Kreisel's conjecture holds for PA*.

Parikh's result comes maddeningly close to settling Kreisel's conjecture, but the complexity of terms in PA blocks any obvious extension of Parikh's proof. It is certainly true that if the translates $\Phi^*(\bar{n})$ of each $\Phi(\bar{n})$ all have proofs in PA* of lengths $\leqslant m$, then $\forall x\Phi^*$ (which is just $(\forall x\Phi)^*$) will be provable in PA*, and hence $\forall x\Phi$ will be provable in PA. But it is not clear that boundedness of proof lengths in PA implies any corresponding boundedness for proof lengths of translates in PA*. With Parikh's theorem we have in a sense come full circle, for Parikh's proof depends, of all things, on Presburger's early decidability result (cf. §3 above). We have only to make a final speculation. The mathematical literature already contains examples of proofs obtained by computer verification of myriads of cases (most notably the recent solution of the Four-Color Problem), while the history of number theory is rife with similar instances of unsolved conjectures (Fermat's Last "Theorem," Goldbach's Conjecture, etc.) proved by a variety of methods to hold in a great many particular cases. Could some of these famous conjectures be natural examples of the undecidability phenomena we have been discussing?

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THE TWENTIETH INTERNATIONAL MATHEMATICAL OLYMPIAD

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The Twentieth International Mathematical Olympiad took place in Bucharest, Romania, on July 6 and 7, 1978. Thanks to the Office of Naval Research, our team had its three-week training session at the U.S. Naval Academy, Annapolis. The Army Research Office provided funds for our travel, and on June 30 we started our journey. Unfortunately, a long delay at Dulles Airport resulted in our arriving at Frankfurt too late to make our connecting flight to Bucharest, where we arrived one day late.

As has been usual, your delegate and secretary were separated from the team members and transported to the small town of Busteni. The selection of the six problems for the Olympiad began on July 3 and was completed on July 5. In spite of United States objections against having more than one problem from any one nation on the contest, the jury did select two problems from one country—the United States!

The Olympiad itself took place in Bucharest, at the Institutul Agronomic. There were impressive opening ceremonies on July 6 at which notables from Romania spoke, welcoming the contestants. Then the contest began. The problems will be found at the end of this article.

Grading of the problems actually began the same evening. Murray Klamkin and I completed our work by July 9, and even graded the two geometry problems for the Romanian students. On July 11, we were off to Bucharest, where we also stayed at the Institutul Agronomic. Our final meeting, with farewell ceremonies and a "solemn" dinner, took place on July 12 at the Institute. Our return trip on July 14 was without incident.

Our first surprise on arriving in Bucharest was to learn that the Soviet Union, Hungary, and the German Democratic Republic would not participate. Hungary did send a representative, who explained that the invitation to them had arrived too late. There was no word from either of the other two countries. We all had our own ideas as to why this situation had occurred. I myself managed to think of six reasons, most political in nature. At any rate, seventeen nations did take part. A list of the nations with their scores is appended. It should be noted that the United States was second, behind Romania, the host nation.

Austria	174	Germany	184	Romania	237
Bulgaria	182	Great Britain	201	Sweden	117
Cuba (4)	68	Mongolia	61	Turkey	66
Czechoslovakia	195	Netherlands	157	United States	225
France	179	Poland	156	Vietnam	200
Finland	118			Yugoslavia	171

Our second surprise was the participation, for the first time, of Turkey, and our third was the excellent showing of Vietnam, which had a complete team of eight students for the first time.

Our team consisted of the following students:

Andrew Bernoff	Upper Dublin H.S.	Fort Washington, Pa.
Daniel Bloch	Bellport H.S.	Brookhaven, N.Y.
Randall Dougherty	W. T. Woodson H.S.	Fairfax, Va.
Mark Kleiman	Stuyvesant H.S.	New York City, N.Y.
Victor Milenkovic	New Trier East H.S.	Winnetka, Ill.
David Montana	Lawrenceville School	Lawrenceville, N.J.
Ehud Reiter	T. S. Wootton H.S.	Rockville, Md.
Charles Walter	Centennial H.S.	Champaign, Ill.

The students performed nobly. First, Mark Kleiman had the only perfect score among all the contestants. Next, our team earned one "First Place" award, three "Second Place" awards, and three "Third Place" awards. The table below shows how our team performed, problem by problem. Researchers may draw some conclusions from comparison of the scores with the type of problem.

Our final surprise occurred at our farewell meeting on July 12, when we were informed that the 1979 Olympiad would be held in Great Britain.

Our own plans at present call for the Annual High School Mathematics Exam, which starts the Olympiad process, to take place on March 6, 1979, the U.S.A. Olympiad to take place on May 1, the training session to be at West Point in June, and the Twenty-first International Olympiad in early July. Meanwhile discussions are under way about the possibility of holding the International Olympiad in the United States in 1981.

Student	1	2	3	4	5	6	Total
Bernoff	3	4	4	5	6	0	22
Bloch	6	4	1	4	6	0	21
Dougherty	6	4	8	5	6	0	29
Kleiman	6	7	8	5	6	8	40
Milenkovic	5	7	6	5	6	1	30
Montana	6	4	4	5	6	1	26
Reiter	6	6	8	5	6	0	31
Walter	6	0	8	5	6	1	26
							225

Twentieth International Mathematical Olympiad

Budapest, July 6–7, 1978

First Day

1. m and n are natural numbers with $n > m \geqslant 1$. In their decimal representation, the last three digits of 1978^m are equal, respectively, to the last three digits of 1978^n. Find m and n such that $m+n$ has its least value. (Cuba)

2. P is a given point inside a given sphere and A, B, C, are any three points on the sphere such that PA, PB, and PC are mutually perpendicular. Let Q be the vertex diagonally opposite to P in the parallelepiped determined by PA, PB, and PC. Find the locus of Q. (United States)

3. The set of all positive integers is the union of two disjoint subsets $\{f(1), f(2), \ldots, f(n), \ldots\}$, $\{g(1), g(2), \ldots, g(n), \ldots\}$ where

$$f(1) < f(2) < \cdots < f(n) < \cdots,$$
$$g(1) < g(2) < \cdots < g(n) < \cdots,$$

and $g(n) = f(f(n)) + 1$ for all $n > 1$. Determine $f(240)$. (Great Britain)

Second Day

4. In triangle $ABC, AB = AC$. A circle is tangent internally to the circumcircle of triangle ABC and also to sides AB, AC at P, Q, respectively. Prove that the midpoint of segment PQ is the center of the incircle of triangle ABC. (United States)

5. Let $\{a_k\}$ $(k = 1, 2, 3, \ldots, n, \ldots)$ be a sequence of distinct positive integers. Prove that, for all natural numbers n,

$$\sum_{k=1}^{n} \frac{a_k}{k^2} \geqslant \sum_{k=1}^{n} \frac{1}{k}. \qquad \text{(France)}$$

6. An international society has its members from six different countries. The list of members contains 1978 names, numbered 1, 2, ..., 1978. Prove that there is at least one member whose number is the sum of two members from his own country, or twice as large as the number of one member from his own country. (Netherlands)

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THE WILLIAM LOWELL PUTNAM MATHEMATICAL COMPETITION

L. F. KLOSINSKI, G. L. ALEXANDERSON, A. P. HILLMAN

The following results of the thirty-ninth William Lowell Putnam Mathematical Competition, held on December 2, 1978, have been determined in accordance with the governing regulations. This annual contest is supported by the William Lowell Putnam Prize Fund for the Promotion of Scholarship, left by Mrs. Putnam in memory of her husband, and is held under the auspices of the Mathematical Association of America.

The first prize, five thousand dollars, was awarded to the Department of Mathematics of **Case Western Reserve University**, Cleveland, Ohio. The members of its winning team were Edward J. Branagan, Jr., Paul M. Herdeg, and Russell D. Lyons; each was awarded a prize of two hundred fifty dollars.

The second prize, two thousand five hundred dollars, was awarded to the Department of Mathematics of **Washington University**, St. Louis, Missouri. The members of its team were George T. Gilbert, Philip I. Harrington, and Tim J. Steger; each was awarded a prize of two hundred dollars.

The third prize, one thousand five hundred dollars, was awarded to the Department of Mathematics of the **University of Waterloo**, Waterloo, Ontario. The members of its team were Michael H. Albert, Rajiv Gupta, and Geoffrey Mess; each was awarded a prize of one hundred fifty dollars.

The fourth prize, one thousand dollars, was awarded to the Department of Mathematics of **Harvard University**, Cambridge, Massachusetts. The members of its team were Daniel S. Freed, Stephen W. Modzelewski, and Alan S. Stern; each was awarded a prize of one hundred dollars.

The fifth prize, five hundred dollars, was awarded to the Department of Mathematics of the **California Institute of Technology**, Pasadena, California. The members of its team were Christopher S. Bretherton, Steve C. Jackson, and Peter W. Shor; each was awarded a prize of fifty dollars.

The five highest ranking individual contestants, in alphabetical order, were **Randall L. Dougherty**, University of California, Berkeley; **Mark P. Kleiman**, Princeton University; **Russell D. Lyons**, Case Western Reserve University; **Peter W. Shor**, California Institute of Technology; and **Steven T. Tschantz**, University of California, Berkeley. Each of these students was designated a Putnam Fellow by the Mathematical Association of America and awarded a prize of five hundred dollars by the Putnam Prize Fund.

The next five highest ranking individuals, in alphabetical order, were *Daniel V. D'Eramo*, Massachusetts Institute of Technology; *Philip I. Harrington*, Washington University; *Paul M. Herdeg*, Case Western Reserve University; *Stephen W. Modzelewski*, Harvard University; and *Miller S. Puckette*, Massachusetts Institute of Technology. Each of these five students was awarded a prize of two hundred fifty dollars.

The following teams, named in alphabetical order, received honorable mention: *University of California, Santa Barbara*, with team members Daniel N. Abellera, Timothy Redmond, and John R. Rose; *McGill University*, with team members Neal N. Madras, Patrick R. Sheppard, and Patrick A. N. Smith; *University of Pennsylvania*, with team members Robin Forman, Hal M. Switkay, and Deane Yang; *Princeton University*, with team members Raymond A. Coley, Bruce D. Ladendorf, and Matthew P. Wiener; and *Stanford University*, with team members Nicholas E. Baxter, Timothy M. Quey, and Lloyd E. Sakazaki.

Honorable mention was achieved by the following thirty-two individuals, named in alphabetical order: *Michael H. Albert*, University of Waterloo; *Kenneth S. Alexander*, University of Washington; *Michael J. Barall*, Princeton University; *Adam J. Beneschan*, University of California, Irvine; *Joshua D. Bernoff*, Pennsylvania State University; *Edward J. Branagan, Jr.*, Case Western Reserve University; *Christopher S. Bretherton*, California Institute of Technology; *David D. Chambliss*, Princeton University; *Raymond A. Coley*, Princeton University; *Boris A. Datskovsky*, Columbia University; *Matthew T. Delevoryas*, Rice University; *Robin Forman*, University of Pennsylvania; *Mark Haiman*, Massachusetts Institute of Technology; *David A. Huse*, University of Massachusetts, Amherst; *Daniel G. Knierim*, University of California, Davis; *Richard J. La Sota*, University of Illinois, Urbana-Champaign; *Neal N. Madras*, McGill University; *Geoffrey Mess*, University of Waterloo; *Richard T. Mifflin*, Rice University; *Renato E. Mirollo*, Columbia University; *David J. Montana*, Harvard University; *Larry J. Romans*, University of California, Davis; *John R. Rose*, University of California, Santa Barbara; *Lorenzo A. Sadun*, Massachusetts Institute of Technology; *Harry A. Smith*, University of Michigan, Ann

Arbor; *Patrick A. N. Smith*, McGill University; *Tim J. Steger*, Washington University; *John R. Stembridge*, California Institute of Technology; *Alan S. Stern*, Harvard University; *Charles H. Walter*, Princeton University; *David S. Witte*, University of Wisconsin, Madison; and *Carl K. Woll*, University of Rochester.

The other individuals who achieved ranks among the top 100, in alphabetical order of their schools, were: University of Arizona, *Michael A. Filaseta*; Augustana College, *Richard E. Moore*; Brigham Young University, *Alan K. Harrison*; California Institute of Technology, *Sangtae Kim* and *Brett A. Spivey*; Carnegie-Mellon University, *Margaret A. Lepley*; Case Western Reserve University, *James W. Davis*; University of Dallas, *Jeffrey A. Elmer*; Drexel University, *Stephen P. Yankovich*; Emory University, *Larry N. Brantley*; Harvard University, *Daniel L. Bloch, Bruce M. Fleischer, Michael P. Mattis, Victor J. Milenkovic, James G. Propp, William A. Titus, Daniel H. Ullman,* and *Peter W. Wallace*; Haverford College, *Ben Finkelstein*; University of Hawaii, *Ping T. Tang*; University of Kansas, *Arthur S. Parker*; Massachusetts Institute of Technology, *Tobias B. Orloff* and *Paul G. Weiss*; McGill University, *Thomas S. Salisbury*; Michigan State University, *Lee D. Mosher* and *Stephen J. Scherr*; University of Missouri, *Gary K. Cobb*; University of Nebraska, *Jim R. Johnson*; North Carolina State University, *Bruce H. Sigmon*; Northwestern University, *Charles E. Bantz*; Oberlin College, *Alan D. Frank*; Princeton University, *Paul M. Anderson, Lawrence M. Ausubel, Michael R. Avidon, Thomas A. Blackadar,* and *David R. Grant*; Purdue University, *Kendall S. Stanley*; Rice University, *Ricardo L. Diaz*; Samford University, *Steven M. Hudson*; Stanford University, *Nicholas E. Baxter, Richard Beigel,* and *Kenneth D. Olum*; University of Texas, *Roger L. Strong*; University of Toronto, *Nicholas A. Martin* and *Norman J. Wildberger*; Vanderbilt University, *Hartwig P. Arenstorf*; Washington University, *George T. Gilbert, Adam D. Helfer, Nathan E. Schroeder,* and *David A. Williams*; Washington State University, *Kenneth S. McElvain;* University of Waterloo, *Rajiv Gupta, Bradd T. Hart, Michael Hollosi, Robert J. Tibshirani,* and *David J. Wright*; Yale University, *Leslie D. Saper*.

There were 2,019 individual contestants from 339 colleges and universities in Canada and the United States in the competition of December 2, 1978. Teams were entered by 246 institutions.

The winner of the annual William Lowell Putnam Fellowship for the competition of the previous year (December 3, 1977) was Stephen W. Modzelewski of Harvard University.

The Questions Committee of the thirty-ninth competition consisted of R. T. Bumby (Chairman), E. J. Barbeau, and L. A. Zalcman; they proposed the problems listed below and were most prominent among those suggesting solutions.

PROBLEMS, PART A

Problem A-1

Let A be any set of 20 distinct integers chosen from the arithmetic progression 1, 4, 7,..., 100. Prove that there must be two distinct integers in A whose sum is 104.

Problem A-2

Let $a,b,p_1,p_2,\ldots,p_n$ be real numbers with $a\neq b$. Define $f(x)=(p_1-x)(p_2-x)(p_3-x)\cdots(p_n-x)$. Show that

$$\det\begin{bmatrix} p_1 & a & a & a & \cdots & a & a \\ b & p_2 & a & a & \cdots & a & a \\ b & b & p_3 & a & \cdots & a & a \\ b & b & b & p_4 & \cdots & a & a \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ b & b & b & b & \cdots & p_{n-1} & a \\ b & b & b & b & \cdots & b & p_n \end{bmatrix} = \frac{bf(a)-af(b)}{b-a}.$$

Problem A-3

Let $p(x)=2+4x+3x^2+5x^3+3x^4+4x^5+2x^6$. For k with $0<k<5$, define

$$I_k=\int_0^\infty \frac{x^k}{p(x)}\,dx.$$

For which k is I_k smallest?

Problem A-4

A "bypass" operation on a set S is a mapping from $S\times S$ to S with the property

$$B(B(w,x),B(y,z))=B(w,z) \qquad \text{for all} \quad w,x,y,z \text{ in } S.$$

(a) Prove that $B(a,b)=c$ implies $B(c,c)=c$ when B is a bypass.
(b) Prove that $B(a,b)=c$ implies $B(a,x)=B(c,x)$ for all x in S when B is a bypass.
(c) Construct a table for a bypass operation B on a finite set S with the following three properties:
 (i) $B(x,x)=x$ for all x in S.
 (ii) There exist d and e in S with $B(d,e)=d\neq e$.
 (iii) There exist f and g in S with $B(f,g)\neq f$.

Problem A-5

Let $0<x_i<\pi$ for $i=1,2,\ldots,n$ and set

$$x=\frac{x_1+x_2+\cdots+x_n}{n}.$$

Prove that

$$\prod_{i=1}^{n}\frac{\sin x_i}{x_i}\leqslant\left(\frac{\sin x}{x}\right)^n.$$

Problem A-6

Let n distinct points in the plane be given. Prove that fewer than $2n^{3/2}$ pairs of them are unit distance apart.

PROBLEMS, PART B

Problem B-1

Find the area of a convex octagon that is inscribed in a circle and has four consecutive sides of length 3 units and the remaining four sides of length 2 units. Give the answer in the form $r+s\sqrt{t}$ with r, s, and t positive integers.

Problem B-2

Express

$$\sum_{n=1}^{\infty}\sum_{m=1}^{\infty}\frac{1}{m^2n+mn^2+2mn}$$

as a rational number.

Problem B-3

The sequence $\{Q_n(x)\}$ of polynomials is defined by

$$Q_1(x)=1+x,\ Q_2(x)=1+2x,$$

and, for $m\geqslant 1$, by

$$Q_{2m+1}(x)=Q_{2m}(x)+(m+1)xQ_{2m-1}(x),$$
$$Q_{2m+2}(x)=Q_{2m+1}(x)+(m+1)xQ_{2m}(x).$$

Let x_n be the largest real solution of $Q_n(x)=0$. Prove that $\{x_n\}$ is an increasing sequence and that $\lim_{n\to\infty}x_n=0$.

Problem B-4

Prove that for every real number N, the equation

$$x_1^2+x_2^2+x_3^2+x_4^2=x_1x_2x_3+x_1x_2x_4+x_1x_3x_4+x_2x_3x_4$$

has a solution for which x_1, x_2, x_3, x_4 are all integers larger than N.

Problem B-5

Find the largest A for which there exists a polynomial

$$P(x)=Ax^4+Bx^3+Cx^2+Dx+E,$$

with real coefficients, which satisfies

$$0\leqslant P(x)\leqslant 1 \quad \text{for} \quad -1\leqslant x\leqslant 1.$$

Problem B-6

Let p and n be positive integers. Suppose that the numbers $c_{h,k}(h=1,2,\ldots,n; k=1,2,\ldots,ph)$ satisfy $0\leqslant c_{h,k}\leqslant 1$. Prove that

$$\left(\sum \frac{c_{h,k}}{h}\right)^2\leqslant 2p\sum c_{h,k},$$

where each summation is over all admissible ordered pairs (h,k).

SOLUTIONS

In the 12-tuples $(n_{10},n_9,\ldots,n_0,n_{-1})$ following each problem number below, n_i for $10\geqslant i\geqslant 0$ is the number of students among the top 203 contestants achieving i points for the problem and n_{-1} is the number of those not submitting solutions.

A-1. (155, 16, 15, 0, 0, 0, 0, 0, 13, 1, 3, 0)

Each of the twenty integers of A must be in one of the eighteen disjoint sets

$$\{1\},\ \{52\},\ \{4,100\},\ \{7,97\},\ \{10,94\},\ldots,\{49,55\}.$$

Hence some (at least two) of the pairs $\{4,100\},\ldots,\{49,55\}$ must have two integers from A. But the sum for each of these pairs is 104.

A-2. (46, 13, 11, 2, 0, 0, 0, 6, 6, 20, 31, 68)

Let M_t be the matrix obtained by subtracting t from each entry of the given matrix and let $G(t)$ be the determinant of M_t. By subtracting the entries of any row from the corresponding entries of each other row, one sees that $G(t)$ is linear in t. Then one notes that $G(a)=f(a)$ and $G(b)=f(b)$ using the fact that they are determinants of triangular matrices. Then linear interpolation shows that the desired determinant $G(0)$ is

$$[bG(a)-aG(b)]/(b-a)=[bf(a)-af(b)]/(b-a).$$

A-3. (10, 17, 8, 7, 0, 0, 0, 16, 8, 7, 24, 106)

Since the integral converges for $-1<k<5$, one can consider I_k to be defined on this open interval. Letting $x=1/t$, one finds that

$$I_k=\int_\infty^0 \frac{t^{-k}}{t^{-6}p(t)}\left(\frac{-dt}{t^2}\right)=\int_0^\infty \frac{t^{4-k}dt}{p(t)}=I_{4-k}.$$

Then

$$I_k=(I_k+I_{4-k})/2=\int_0^\infty \frac{[(x^k+x^{4-k})/2]dx}{p(x)}\geqslant\int_0^\infty \frac{x^2dx}{p(x)}=I_2,$$

since $(x^k+x^{4-k})/2\geqslant\sqrt{x^k\cdot x^{4-k}}=x^2$ by the Arithmetic Mean–Geometric Mean Inequality. Thus I_k is smallest for $k=2$.

A-4. (32, 0, 5, 1, 1, 1, 0, 124, 0, 28, 2, 9)

(a) The defining property with $[w,x,y,z]=[a,b,a,b]$ and the hypothesis $B(a,b)=c$ give us

$$B(c,c)=B(B(a,b),B(a,b))=B(a,b)=c.$$

(b) The defining property with $[w,x,y,z]=[a,b,x,x]$ and $B(a,b)=c$ give

$$B(c,B(x,x))=B(B(a,b),B(x,x))=B(a,x).$$

Then using the result in (a) and $[w,x,y,z]=[c,c,x,x]$, one has

$$B(c,B(x,x))=B(B(c,c),B(x,x))=B(c,x).$$

Together, these show that $B(a,b)=c$ implies $B(a,x)=B(c,x)$ for all x in S.

(c) An easy way to obtain a bypass with property (i) is to let S be a cartesian product $I\times J$ and to define the operation B by

$$B((i,j),(h,k))=(i,k).$$

Properties (ii) and (iii) will hold if I and J, respectively, have more than one element. Except for notation, every bypass is obtained this way.

Tables with $S=\{a,b,c,d\}$ and with $S=\{u,v,w,x,y,z\}$ follow:

	a or c	b or d
a or b	a	b
c or d	c	d

	u or x	v or y	w or z
u or v or w	u	v	w
x or y or z	x	y	z

A-5. (30, 1, 2, 0, 0, 0, 0, 0, 4, 4, 48, 114)

Let $g(x)=\ln[(\sin x)/x]=\ln(\sin x)-\ln x$. Then

$$g''(x)=-\csc^2 x+\frac{1}{x^2}=\frac{1}{x^2}-\frac{1}{\sin^2 x}<0 \quad \text{for } 0<x<\pi$$

since $x>\sin x$ for $x>0$. Thus the graph of $g(x)$ is concave down and hence

$$\frac{1}{n}\sum_{i=1}^{n} g(x_i)\leqslant g\left[\frac{\sum_{i=1}^{n} x_i}{n}\right]=g(x),$$

or $\Sigma\, g(x_i)\leqslant ng(x)$. Since e^x is an increasing function, this implies

$$\prod_{i=1}^{n}\frac{\sin x_i}{x_i}=e^{\Sigma g(x_i)}\leqslant e^{ng(x)}=\left(\frac{\sin x}{x}\right)^n.$$

A-6. (0, 1, 1, 0, 0, 1, 0, 2, 1, 5, 47, 145)

For a set $\{p_1,\ldots,p_n\}$ of points in the plane, let e_i be the number of p_j one unit from p_i. Then $E=(e_1+\cdots+e_n)/2$ is the number of pairs with unit distance. Let C_i be the circle with center at p_i and radius 1. Each pair of circles has at most 2 intersections, so the C_i intersect in at most $2\binom{n}{2}=n(n-1)$ points. It suffices to treat the case in which each $e_i\geqslant 1$.

The point p_i occurs $\binom{e_i}{2}$ times as an intersection of C_j. Hence

$$\text{(A)}\qquad n(n-1)\geqslant\sum\binom{e_i}{2}=\sum e_i(e_i-1)/2\geqslant(1/2)\sum(e_i-1)^2.$$

In (A) and what follows, all sums are over $i=1,2,\ldots,n$. Using the Cauchy-Schwarz Inequality

and (A) one has

$$\left[\sum(e_i-1)\right]^2 \leqslant \left[\sum 1\right]\left[\sum(e_i-1)^2\right] \leqslant n\cdot 2n(n-1) < 2n^3.$$

Hence $\Sigma(e_i-1)\leqslant \sqrt{2}\, n^{3/2}$ and so

$$E=\left(\sum e_i\right)/2 \leqslant (n+\sqrt{2}\, n^{3/2})/2 < 2n^{3/2}.$$

B-1. (107, 5, 8, 0, 0, 0, 0, 0, 13, 8, 46, 16)

The area is the same as for an octagon inscribed in a circle and with sides alternately 3 units and 2 units in length. For such an octagon, all angles measure $3\pi/4$ and one can augment the octagon into a square with sides of length $3+2\sqrt{2}$ by properly placing a $\sqrt{2}$, $\sqrt{2}$, 2 isosceles right triangle on each of the sides of length 2. Hence the desired area is

$$(3+2\sqrt{2}\,)^2-(4\sqrt{2}\cdot\sqrt{2}\,/2)=13+12\sqrt{2}\,.$$

A second solution follows. Let r be the radius of the circle and let α and β be half of the central angles for the chords of lengths 3 and 2, respectively. Then $8\alpha+8\beta=2\pi$ and so $\beta=(\pi/4)-\alpha$. Also

$$\frac{3}{2r}=\sin\alpha,\qquad \frac{1}{r}=\sin\beta=\sin\left(\frac{\pi}{4}-\alpha\right)=\frac{\cos\alpha-\sin\alpha}{\sqrt{2}},$$

$$\frac{2}{3}=\frac{2r}{3}\cdot\frac{1}{r}=\frac{\cos\alpha-\sin\alpha}{\sqrt{2}\,\sin\alpha}=\frac{\cot\alpha-1}{\sqrt{2}}.$$

Now $\cot\alpha=(3+2\sqrt{2}\,)/3=[(3+2\sqrt{2}\,)/2]/(3/2)$ and hence the distance from the center of the circle to a chord of length 3 is $h_3=(3+2\sqrt{2}\,)/2$. Similarly the distance to a chord of length 2 is $h_2=(2+3\sqrt{2}\,)/2$. Finally, the desired area is

$$4(3h_3+2h_2)/2=(9+6\sqrt{2}\,)+(4+6\sqrt{2}\,)=13+12\sqrt{2}\,.$$

B-2. (43, 12, 7, 5, 0, 1, 5, 8, 5, 1, 13, 103)

Let S be the desired sum. Then

$$S=\sum_{n=1}^{\infty}\frac{1}{n}\sum_{m=1}^{\infty}\frac{1}{n+2}\left(\frac{1}{m}-\frac{1}{m+n+2}\right)$$

$$=\sum_{n=1}^{\infty}\frac{1}{n(n+2)}\left[\left(1-\frac{1}{n+3}\right)+\left(\frac{1}{2}-\frac{1}{n+4}\right)+\left(\frac{1}{3}-\frac{1}{n+5}\right)+\cdots\right]$$

$$=\frac{1}{2}\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)\left[\left(1-\frac{1}{n+3}\right)+\left(\frac{1}{2}-\frac{1}{n+4}\right)+\cdots\right].$$

Hence

$$2S=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)\lim_{k\to\infty}\left[1+\frac{1}{2}+\cdots+\frac{1}{n+2}-\frac{1}{k}-\frac{1}{k+1}-\cdots-\frac{1}{k+n+1}\right]$$

$$=\sum_{n=1}^{\infty}\left(\frac{1}{n}-\frac{1}{n+2}\right)\left(1+\frac{1}{2}+\cdots+\frac{1}{n+2}\right)$$

$$=\lim_{h\to\infty}\left[\left(1-\frac{1}{3}\right)\left(1+\frac{1}{2}+\frac{1}{3}\right)+\left(\frac{1}{2}-\frac{1}{4}\right)\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)\right.$$

$$\left.+\cdots+\left(\frac{1}{h}-\frac{1}{h+2}\right)\left(1+\frac{1}{2}+\cdots+\frac{1}{h}\right)\right]$$

$$=\lim_{h\to\infty}\left[1\cdot\left(1+\frac{1}{2}+\frac{1}{3}\right)+\frac{1}{2}\left(1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}\right)+\frac{1}{3}\left(\frac{1}{4}+\frac{1}{5}\right)+\frac{1}{4}\left(\frac{1}{5}+\frac{1}{6}\right)\right.$$

$$+\cdots+\frac{1}{h}\left(\frac{1}{h+1}+\frac{1}{h+2}\right)-\frac{1}{h+1}\left(1+\frac{1}{2}+\cdots+\frac{1}{h-1}\right)-\frac{1}{h+2}\left(1+\frac{1}{2}+\cdots+\frac{1}{h}\right)\Big]$$
$$=\frac{6+3+2}{6}+\frac{12+6+4+3}{2\cdot 12}+\left(\frac{1}{3\cdot 4}+\frac{1}{4\cdot 5}+\cdots\right)+\left(\frac{1}{3\cdot 5}+\frac{1}{4\cdot 6}+\cdots\right)$$
$$=\frac{11}{6}+\frac{25}{24}+\frac{1}{3}+\frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}\right)=\frac{7}{2}.$$

Thus $S=7/4$.

B-3. (9, 2, 2, 2, 5, 44, 14, 4, 1, 1, 20, 99)

Clearly, $x_1=-1$, $x_2=-\frac{1}{2}$. An easy induction shows that each Q_n is positive for $x\geqslant 0$. Hence $x_n<0$, if Q_n has zeros.

Assume inductively that $x_1<x_2<\cdots<x_{2m-1}<x_{2m}$. Then $Q_{2m-1}(x)>0$ for $x>x_{2m-1}$. In particular, $Q_{2m-1}(x_{2m})>0$. Hence

$$Q_{2m+1}(x_{2m})=Q_{2m}(x_{2m})+(m+1)x_{2m}Q_{2m-1}(x_{2m})$$
$$=(m+1)x_{2m}Q_{2m-1}(x_{2m})<0.$$

This implies that $Q_{2m+1}(x)=0$ for some $x>x_{2m}$, i.e., $x_{2m+1}>x_{2m}$. Similarly, one shows that $x_{2m+2}>x_{2m+1}$.

Let $a=-1/(m+1)$. Using the given recursive definition of the $Q_n(x)$, one finds that

$$Q_{2m+2}(a)=Q_{2m+1}(a)-Q_{2m}(a)=-Q_{2m-1}(a).$$

Hence at least one of $Q_{2m+2}(a)$ and $Q_{2m-1}(a)$ is nonpositive. Thus either $x_{2m+2}\geqslant a$ or $x_{2m-1}\geqslant a$. But each of these implies that both $x_{2m+2}\geqslant -1/(m+1)$ and $x_{2m+3}\geqslant -1/(m+1)$. It follows that $-2/n\leqslant x_n<0$ for all n and then that $\lim_{n\to\infty}x_n=0$.

B-4. (12, 3, 0, 0, 0, 0, 1, 0, 0, 0, 26, 161)

Clearly (1, 1, 1, 1) is a solution. Thinking of x_1, x_2, x_3 as fixed, the equation is quadratic in x_4 and one sees that the x_4 of a solution can be replaced by $x_4'=x_1x_2+x_1x_3+x_2x_3-x_4$ to obtain a new solution when $x_4'\neq x_4$. Also, the x_i may be permuted arbitrarily since the equation is symmetric in the x_i. Thus we may assume that $x_4\leqslant m=\min(x_1,x_2,x_3)$. Also assume that each $x_i\geqslant 1$. Then $x_4'\geqslant 3m^2-m>m$. This implies that one can start with the solution (1, 1, 1, 1) and through repeated use of the procedures stated above obtain a solution with each x_i an integer greater than N.

B-5. (13, 4, 1, 1, 0, 1, 9, 2, 2, 15, 53, 102)

The solution is very easy if one knows that the Chebyshev polynomial $C(x)=8x^4-8x^2+1=\cos(4\operatorname{Arccos}x)$ has the largest leading coefficient of all fourth degree polynomials $f(x)$ satisfying $-1\leqslant f(x)\leqslant 1$ for $-1\leqslant x\leqslant 1$; then one lets $P(x)=[C(x)+1]/2$ and has 4 as the largest A.

Without this information, one can use various substitutions to change the problem into equivalent ones of maximizing A in simpler functions satisfying conditions over intervals. With $Q(x)=[P(x)+P(-x)]/2$, the condition becomes

$$0\leqslant Q(x)=Ax^4+Cx^2+E\leqslant 1 \quad\text{over}\quad -1\leqslant x\leqslant 1.$$

Letting $x^2=y$, this becomes

$$0\leqslant R(y)=Ay^2+Cy+E\leqslant 1 \quad\text{over}\quad 0\leqslant y\leqslant 1.$$

Letting $y=(z+1)/2$ and $S(z)=R[(z+1)/2]$, one has

$$0 \leqslant S(z) = (A/4)z^2 + Fz + G \leqslant 1 \quad \text{over} \quad -1 \leqslant z \leqslant 1.$$

With $T(z) = [S(z) + S(-z)]/2$, one obtains

$$0 \leqslant (A/4)z^2 + G \leqslant 1 \quad \text{over} \quad -1 \leqslant z \leqslant 1.$$

Finally, letting $z^2 = w$, it becomes

$$0 \leqslant (A/4)W + G \leqslant 1 \quad \text{over} \quad 0 \leqslant w \leqslant 1.$$

Now it is clear that the maximum A is 4 and that this maximum is achieved with $G = 0$, i.e., with

$$T(z) = z^2, \qquad R(y) = (2y-1)^2, \qquad Q(x) = 4x^4 - 4x^2 + 1.$$

B-6. (31, 4, 0, 0, 0, 1, 0, 0, 5, 1, 30, 131)

Let $a_h = (\Sigma_{k=1}^{ph} c_{h,k})/h$. Clearly, $0 \leqslant a_h \leqslant p$. We now prove that $(\Sigma_{h=1}^{n} a_h)^2 \leqslant 2p\Sigma_{h=1}^{n}(ha_h)$, which is equivalent to the assertion of the problem, by induction on n.

For $n = 1$, one has $a_1^2 \leqslant pa_1 \leqslant 2pa_1$ as required. Suppose the inequality established for $n = m$. Then

$$\begin{aligned}
\left(\sum_{h=1}^{m+1} a_h\right)^2 &= \left(\sum_{h=1}^{m} a_h\right)^2 + 2a_{m+1}\sum_{h=1}^{m} a_h + a_{m+1}^2 \\
&\leqslant 2p\sum_{h=1}^{m}(ha_h) + 2a_{m+1}pm + 2pa_{m+1} \\
&\leqslant 2p\left[(m+1)a_{m+1} + \sum_{h=1}^{m}(ha_h)\right] = 2p\sum_{h=1}^{m+1}(ha_h),
\end{aligned}$$

as desired.

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EUCLIDEAN AND NON-EUCLIDEAN GEOMETRIES WITHOUT CONTINUITY

MARVIN JAY GREENBERG

Introduction. David Hilbert, in his classic *Foundations of Geometry* **[11]**, formulated axioms for plane Euclidean and hyperbolic (Bolyai-Lobachevskian) geometries. The common part of these axiom systems consists of the incidence, order, and congruence axioms. A model of these three groups of axioms is called an *H-plane* (terminology from **[10]**). If, in addition, the familiar Euclidean parallel postulate is assumed, the H-plane will be called *Euclidean*. If, on the other hand, the following non-Euclidean parallel postulate is assumed, the plane will be called *hyperbolic* (terminology from Felix Klein).

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Hyperbolic Axiom. *For any line l and point P not on l, there are two asymptotic parallels to l through P, i.e., two parallels n and n' such that if $\mathfrak{R}$ is the open angular region determined by n and n' that contains l, then the lines through P which meet l are exactly the lines that pass through $\mathfrak{R}$ (see Fig.* 1).

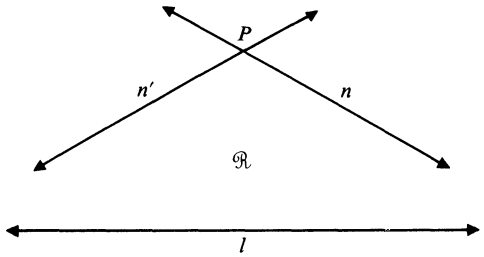

Fig. 1

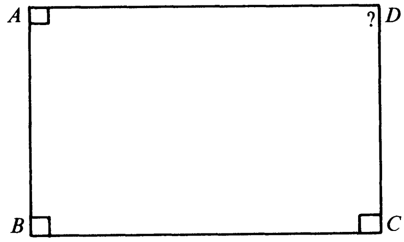

Fig. 2

Hilbert did not consider any other H-planes besides these two types, for he was primarily concerned with *continuous* H-planes, i.e., ones in which Dedekind's cut axiom holds for the ordered set of points on a line; and it can be proved that a continuous H-plane must be Euclidean or hyperbolic [**8**]. In fact, the only continuous H-planes (up to isomorphism) are the familiar Cartesian plane coordinatized by the field $\mathbf{R}$ of real numbers and the real hyperbolic plane, represented by either the Klein or the Poincaré models over $\mathbf{R}$ [**6**], [**8**]. At the same time, in developing the theorems of these two geometries, Hilbert was very careful never to use Dedekind's axiom. He was anxious to preserve the "purity" of the methods of proof, and in the conclusion to his Grundlagen asserted [**11**, p. 106]:

> The *impossibility* of certain solutions and problems thus plays a prominent role in modern mathematics and the drive to answer questions of this type was oftentimes the cause for the discovery of new and fruitful areas of investigation. Recall only Abel's proof of the impossibility of solving the fifth degree equation by radicals, the realization of the impossibility of proving the axiom of parallels, and Hermite and Lindemann's theorem on the impossibility of constructing the numbers e and π algebraically. . . . In fact, the present geometric investigation seeks to uncover which axioms, hypotheses or aids are necessary for the proof of a fact in elementary geometry, and in every decision the question as to which method of proof is to be preferred, from the adopted point of view, remains open.

One consequence of Dedekind's cut axiom is the axiom of Archimedes, which asserts that given any segment AB, there exists a positive integer n such that the segment formed from n juxtaposed copies of a chosen unit segment is longer than AB. If Dedekind's axiom is abandoned, this opens the possibility of non-Archimedean geometries, which were first studied in 1891 by G. Veronese [**15**], [**16**]. The simplest example is the Cartesian plane coordinatized by the field $\mathbf{R}\{t\}$ of formal power series (allowing finitely many negative powers of t); here t is a positive infinitesimal, smaller than any positive real number, and t^{-1} is infinitely large.

In 1900, Max Dehn [7] constructed further non-Archimedean H-planes, using them to establish independence of certain theorems from the H-plane axioms. Consider a quadrilateral $\square ABCD$ with at least three right angles, called a *Lambert quadrilateral.* (See Fig. 2.)

The theorem of Saccheri-Legendre implies that the fourth angle cannot be obtuse (it is a right angle in a Euclidean H-plane and is acute in a hyperbolic H-plane). Analysis of the proof of this theorem shows that Archimedes' axiom is used [**8**]. Dehn constructed a non-Archimedean H-plane (called *the non-Legendrean plane*) in which the fourth angle is obtuse. Thus the Saccheri-Legendre Theorem is independent of the H-plane axioms. (Note that the fourth angle is also obtuse in plane elliptic geometry or in spherical geometry, but these geometries do not satisfy the H-plane axioms.)

Suppose we then assume the hypothesis of the right angle (respectively, acute angle); must the

H-plane be Euclidean (respectively, hyperbolic)? Dehn answered both these questions in the negative, with his *semi-Euclidean* and *semi-hyperbolic* planes, both of which are non-Archimedean. Suppose we also assume Archimedes' axiom. Then the answer is positive for the right angle but is still negative for the acute angle; this answer was obtained over 60 years later by W. Pejas [**14**] and F. Bachmann [**4**]. In fact, Pejas gave a complete classification of all H-planes! We will sketch this classification in the next sections and apply it to deduce further independence results.

1. Projective embedding. Following Arthur Cayley's prophetic vision in 1858 that "projective geometry is all geometry," the classification of H-planes will be based on a canonical embedding in a projective plane $\mathcal{P}$ coordinatized by an ordered Pythagorean field K. Let $\mathcal{H}$ be the given H-plane. The points of $\mathcal{P}$ will be the pencils of lines from $\mathcal{H}$, which are of three types:

(i) for each point $A \in \mathcal{H}$, the pencil $G(A)$ of all lines through A; these pencils are called *proper points* of $\mathcal{P}$, and the embedding sends A to $G(A)$;

(ii) for each line $a \in \mathcal{H}$, the pencil $G(a)$ of all lines perpendicular to a; $G(a)$ is called the *pole* of a, and all such points of $\mathcal{P}$ will be called *ultra-ideal points*;

(iii) for each pair l, m of parallel lines which do not have a common perpendicular, the pencil $G(lm)$ of all lines n such that the product of the reflections in l, m and n is a reflection; these pencils will be called *ideal points* of $\mathcal{P}$.

The lines of $\mathcal{P}$ are certain sets of pencils that we will describe, and incidence is set membership. For each line $a \in \mathcal{H}$, the set $g(a)$ of all pencils containing a is a line of $\mathcal{P}$ called *proper*, and the embedding sends a to $g(a)$. For each point $A \in \mathcal{H}$, the set $g(A)$ of all poles of lines of $\mathcal{H}$ through A is a line of $\mathcal{P}$ called the *polar* of A. Finally, there may be other lines of $\mathcal{P}$ that can only be described indirectly in terms of Hjelmslev's half-rotations about a chosen point $O \in \mathcal{H}$(see [**10**], p. 123).

The embedding sends $\mathcal{H}$ into the affine plane $\mathcal{A}$ obtained from $\mathcal{P}$ by removing the polar $g(O)$. The image of the point set of $\mathcal{H}$ is a convex open subset of $\mathcal{A}$. If we identify these sets, the lines in $\mathcal{H}$ become identified with all the affine lines that pass through points of $\mathcal{H}$. Congruence in $\mathcal{H}$ is determined by the group of motions of $\mathcal{H}$, which is generated by the reflections; and the reflection in a line l is just the harmonic homology having $g(l)$ as axis and the pole of l as center.

It can be proved that there are enough affine translations of $\mathcal{A}$ so that E. Artin's method of canonically constructing a field K works: that is, K is the field of all trace-preserving endomorphisms of the group of affine translations ([**13**, p. 211] and [**1**, p. 58]). Then $\mathcal{A}$ is just the affine plane coordinatized by K. The proof of the existence of such a coordinatization is quite subtle, using deep insights of G. Hessenberg and J. Hjelmslev.

The ordering relation for points on a line in $\mathcal{H}$ is induced by a uniquely determined ordering on the field K [**13**, p. 223]. The field K must be Pythagorean (the sum of two squares is a square —[**10**, p. 224]). $\mathbf{R}\{t\}$ is an example of a Pythagorean ordered field.

We choose two perpendicular lines through our origin O to serve as our x and y axes, and we make the convention that once the x-axis is coordinatized by choosing a unit point $(1,0)$, the corresponding unit point $(0,1)$ on the y-axis will be its reflection across the symmetry line of these axes. With this convention, it can be shown that there is a *metric constant* $k \in K$ such that the pole of a line with homogeneous coordinates $[a,b,c]$ has homogeneous coordinates (a,b,kc) [**10**, p. 221]. If we chose a different unit point of the x-axis, the metric constant would be changed to $\lambda^2 k$, for some $\lambda \neq 0$. So the metric constant is only determined up to multiplication by a square factor. Nevertheless, it makes sense to say that the metric constant is either positive, negative, or zero. The metric constant and the set of points $\mathcal{H}$ completely determine $\mathcal{H}$.

THEOREM 1 (Pejas [**14**]). *The metric constant is positive, negative, or zero according as the fourth angle of a Lambert quadrilateral is obtuse, acute, or right.*

COROLLARY. *If Archimedes' axiom holds in $\mathcal{H}$, the metric constant is zero or negative (Saccheri-Legendre).*

2. Archimedean H-Planes. Consider an embedding of one H-plane $\mathcal{H}$ into another H-plane $\mathcal{H}'$: it is a one-to-one mapping of the points and lines of $\mathcal{H}$ to points and lines of $\mathcal{H}'$ such that the incidence, betweenness, and congruence relations hold for configurations in $\mathcal{H}$ if and only if they hold for their images in $\mathcal{H}'$. Call the embedding *strict* if whenever a line in $\mathcal{H}'$ passes through the image of a point in $\mathcal{H}$ then that line is the image of a line in $\mathcal{H}$. The embedding is called an *isomorphism* if it maps $\mathcal{H}$ onto $\mathcal{H}'$. $\mathcal{H}$ (respectively, $\mathcal{H}'$) is called *maximal* (respectively, *minimal*) if every strict embedding $\mathcal{H}\to\mathcal{H}'$ is an isomorphism. A strict embedding $\mathcal{H}\to\mathcal{H}'$ induces an isomorphism $\mathcal{P}\xrightarrow{\sim}\mathcal{P}'$ of the projective-metric closures.

THEOREM 2 (Bachmann). *The following conditions are equivalent for an H-plane $\mathcal{H}$:*
1. *Archimedes' axiom holds in $\mathcal{H}$.*
2. *K is an Archimedean ordered field.*
3. *$\mathcal{H}$ is minimal.*

Moreover, a minimal H-plane is maximal (but not conversely).

The non-trivial part of this theorem is the implication $3\Rightarrow 1$, whose proof requires Pejas' classification [4].

Pejas gave the following example of an Archimedean H-plane in which the acute angle hypothesis holds but which is not hyperbolic; it is an example of a *semi-elliptic plane*, defined by the property that any two parallel lines have a *unique* common perpendicular (in a hyperbolic plane, two asymptotically parallel lines have no common perpendicular, whereas two divergently parallel lines have a unique one): Let K_0 be an Archimedean ordered field with two distinct orderings $<$ and $<'$ (for example, $K_0=\mathbf{Q}(\sqrt{2}\,)$). Let L, L' be the *real closures* **[12]** of K_0 with respect to these orderings within a given algebraic closure. Set $K=L\cap L'$. Then K is Pythagorean, Archimedean, and contains an element k such that $k<0$ and $0<'k$. We take k as metric constant and the points of $\mathcal{H}$ to be all (x,y) in the affine plane over K for which $k(x^2+y^2)+1>0$. $\mathcal{H}$ is the interior of the "absolute conic" $x^2+y^2=-k^{-1}$, which is empty because $\sqrt{-k}\notin K$; since $\mathcal{H}$ is maximal, that conic is the locus of all ideal points, so asymptotic parallels do not exist and the plane is semi-elliptic.

THEOREM 3. *An Archimedean H-plane must be either Euclidean, hyperbolic, or non-modular semi-elliptic.*

An H-plane is called *modular* if the fourth vertex of any Lambert quadrilateral in $\mathcal{P}$ having sides in $\mathcal{H}$ belongs to $\mathcal{H}$, i.e., if a, b, c, d are four lines in $\mathcal{H}$ such that $a\perp b\perp c\perp d$, then a meets d. Among Archimedean H-planes, only the Euclidean is modular. For an algebraic characterization of modular H-planes, see the Corollary to Theorem 5.

3. The Circle Axiom. One consequence of continuity is the following.

CIRCLE AXIOM. *If a line passes through a point inside a circle, then it intersects the circle (in two points).*

Having abandoned continuity, we can no longer expect this axiom to hold. It turns out that there is a nice algebraic criterion for this axiom to hold **[9]**.

THEOREM 4. *The circle axiom holds in $\mathcal{H}$ if and only if the field K is Euclidean (i.e., every positive element of K has a square root in K). In that case the metric constant can be normed to be* 0, 1, *or* -1.

COROLLARY 1. *An H-plane in which the circle axiom holds cannot be semi-elliptic.*

For in case $k=0$, pairs of parallels exist having infinitely many common perpendiculars; in case $k=1$ which we know can only occur if K is non-Archimedean (Corollary to Theorem 1), we will see later that ideal points always exist. In case $k=-1$, the points on the absolute conic $x^2+y^2=1$ are ideal. Combining this with Theorem 3 gives the following important corollary.

COROLLARY 2. *An Archimedean H-plane in which the circle axiom holds must be either Euclidean or hyperbolic.*

In particular, the conjunction of Archimedes' axiom, the circle axiom, and the negation of Euclid's parallel postulate implies the existence of asymptotic parallels—a remarkable result for which no elementary proof is known. We will discuss in section 5 a construction for the asymptotic parallels by straightedge and compass that was given by Janos Bolyai.

The circle axiom need not hold in a Euclidean H-plane, for Euclidean models exist over any Pythagorean field. However, as F. Schur first observed in 1904, the circle axiom must hold in a hyperbolic plane. Of course Archimedes' axiom need not hold in a hyperbolic plane, as the Klein model can be constructed over any Euclidean field.

Here is an example of a non-Archimedean Euclidean field: Let K_0 be any Euclidean field (such as **R** or the subfield of real numbers constructible by straightedge and compass from rational numbers). Then the power series field $K_0\{t\}$ can be shown to be Pythagorean. It is not Euclidean since, e.g., $\sqrt{t}$ is not in it. But the union of power series fields

$$\bigcup_{n=1}^{\infty} K_0\{t^{1/2^n}\}$$

can be shown to be Euclidean [**10**, p. 204].

4. Pejas Classification. It begins with the observation that the *abscissa set*

$$M=\{x\in K|(x,0)\in\mathcal{H}\}$$

completely determines the point set of $\mathcal{H}$ by the condition

$$(x,y)\in\mathcal{H}\Leftrightarrow\sqrt{x^2+y^2}\in M.$$

The problem is thus reduced to determining all possible abscissa sets for a given metric constant k in the ordered Pythagorean field K.

Let N denote the set of positive integers, and let

$$R=\{x\in K|\exists n\in\mathrm{N}\text{ with }|x|<n\}$$

$$I=\left\{x\in K|\ |x|<\frac{1}{n}\forall n\in\mathrm{N}\right\}.$$

Then R is a valuation ring of K (i.e., for each $x\neq 0$ in K, either $x\in R$ or $x^{-1}\in R$), and I is its unique maximal ideal. The elements of I are called *infinitesimal* and the elements of $K-R$ are called *infinite*. If the ordering of K is Archimedean, then $I=\{0\}$ and $R=K$.

The abscissa set M is convex and symmetric about O. From this and the fact that M cannot contain the pole $(1/kx,0)$ of one of its points $(x,0)$, it follows that M must be contained in the *fundamental interval*

$$M_1=\{x\in K|\ |kx^2|<1\}$$

(in case $k=0$, $M_1=K$). Furthermore, if $x\in M$, $x\neq 0$, the midpoint of $(x,0)$ and the origin must have its abscissa m in M, and a calculation gives

$$m=\frac{x}{1+\sqrt{1+kx^2}}.$$

So we see that $x\in M$ implies that $1+kx^2$ is a square in K (automatic if k is a square), another restriction on M. Skipping the remainder of the lengthy analysis, we state the final result [**10**, p. 235].

THEOREM 5 (Pejas). *A subset M of the fundamental interval M_1 is an abscissa set for an H-plane with metric constant k if and only if either*

(1) *M is a non-zero R-module for whose elements x the elements $1+kx^2$ are squares in K;*

(2) *or only in case $k<0$ and there exist units in the ring R of the form ka^2: M is obtained from M_1 by deleting an infinitesimal neighborhood U_J of the boundary*

$$U_J=\{a\in K\mid 1+ka^2\in J\}$$

where J is a prime R-ideal having the property that

$$0<1+kx^2\notin J\Rightarrow\sqrt{1+kx^2}\in K.$$

In this case M is not an R-module.

COROLLARY (Bachmann). *$\mathcal{H}$ is modular if and only if M is an R-module.*

Pejas' classification takes a much simpler form in case the circle axiom holds. By Theorem 4, we can norm k to be 0, 1, or -1. The fundamental interval M_1 is just the open interval $-1<x<1$ in case $k\neq 0$, and the point set of $\mathcal{H}$ is therefore contained in the interior of the unit circle. *A subset M of M_1 is an abscissa set for some $\mathcal{H}$ with metric constant k if and only if either*

(1) *M is a non-zero R-module*; *or*

(2) *only in case $k=-1$*:

$$M=\{x\in M_1 \mid 1-|x|\notin J\}$$

for some prime R-ideal J (M is not an R-module).

The Klein model for the hyperbolic plane is obtained by taking $k=-1$ and M non-modular with $J=\{0\}$. The semi-hyperbolic plane is obtained by taking $k=-1$ and M non-modular with $J=I$. The semi-Euclidean plane is obtained by taking $k=0$ and $M=I$. The non-Legendrean plane is obtained by taking $k=1$ and $M=I$; Figure 3 illustrates the distribution of proper, ideal and ultra-ideal points on the x-axis in this plane.

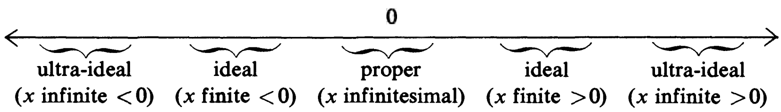

FIG. 3

A similar figure illustrates the distribution for the semi-hyperbolic plane, where the ideal points

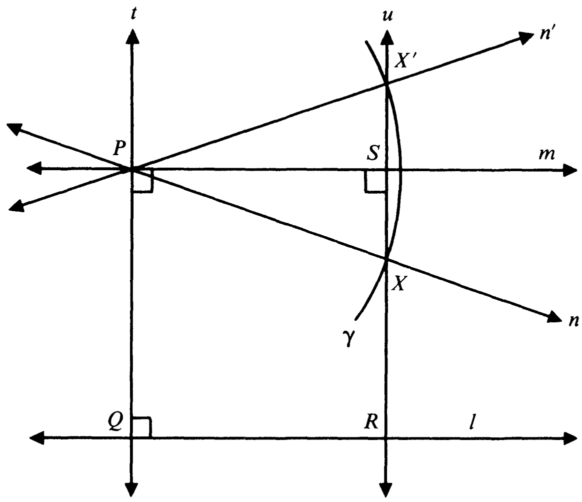

FIG. 4

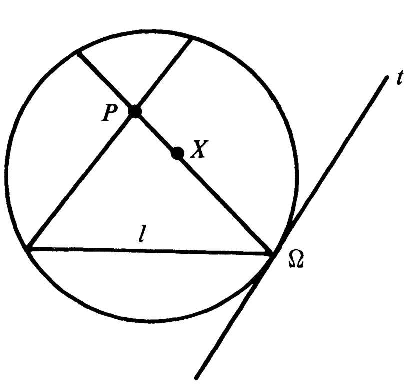

FIG. 5

inside M_1 are those for which $|x| \leqslant 1$ and $1-|x|$ is infinitesimal. Since the minimum $|x|$ for such x does not exist, asymptotic parallels do not exist in this plane.

5. J. Bolyai's Parallel Construction. If the hypothesis of the acute angle holds, a side of a Lambert quadrilateral adjacent to the acute angle is longer than its opposite side **[8]**. Given point P not on line l, let Q be the foot of the perpendicular t from P to l. Let m be the perpendicular to t through P, let R be any other point on l, and let S be the foot of the perpendicular u from R to m. Then $\square SPQR$ is a Lambert quadrilateral with acute angle at R. If a circle γ is then drawn with center P and radius congruent to QR, the circle axiom guarantees that it will cut line u in two points X, X', since u passes through point S inside γ (see Fig. 4). J. Bolyai then claimed that lines PX and PX' are the asymptotic parallels to l through P. This is correct if we assume Archimedes' axiom (previous proofs used the full continuity axiom). However, if the field K is non-Archimedean, the semi-hyperbolic plane shows that Bolyai's construction does not yield asymptotic parallels. It can be proved **[9]** that Bolyai's construction gives the two parallels to l passing through the ideal ends of l that lie on the absolute conic (see Fig. 5). If we draw the tangent t at one of these ends Ω, then Ω is the pole of t, and since PX and l pass through the pole of t, t could be considered the "common perpendicular at infinity" of the parallels PX and l. This is what Girolamo Saccheri discovered in the eighteenth century when he attempted to prove Euclid's parallel postulate by a reductio ad absurdum argument, and when arriving at this discovery exclaimed that it was "absolutely repugnant to the nature of a straight line." If he could have accepted this surprise, historians might have credited him with having discovered non-Euclidean geometry.

6. Absolute Geometry. J. Bolyai gave this name to the common part of real Euclidean and hyperbolic geometries. I would argue that this should be called *neutral geometry* (following Prenowitz and Jordan), because we remain neutral about the parallel postulate and because it is not "absolute enough" to include elliptic geometry. F. Bachmann, in his monumental treatise **[2]**, gave a definition of plane absolute geometry that is broad enough to include elliptic geometry and any other geometry for which the projective embedding of section 1 can be carried out. In his definition, he not only abandons the axiom of continuity, but he drops the axioms of order as well. He builds his system on the fact that congruence is determined by the group $\mathcal{G}$ of motions, which is generated by its subset $\mathbb{S}$ of (line) reflections, each of which is involutory (equal to its inverse). We have already used reflections in the very definition of ideal points, and they play a crucial role in constructing the field K. Here is his foundation for plane absolute geometry.

Given a group $\mathcal{G}$ generated by an invariant subset $\mathbb{S}$ of involutions the elements of $\mathbb{S}$ are called *lines* and denoted by lower case letters. Those products $P = ab$ of two lines which are again involutions are called *points* and denoted by capital letters. P is *incident* with c (denoted $P\mathrm{I}c$) if Pc is involutory. a is *perpendicular* to b (denoted $a \perp b$) if ab is involutory, i.e., ab is a point.

AXIOM 1. Two distinct points are incident with a unique line.

AXIOM 2. If P is incident with a, b, and c, then abc is a line.

AXIOM 3. If d is perpendicular to a, b, and c, then abc is a line.

AXIOM 4. There exist three lines, a, b, c, such that $a \perp b$ (let $ab = P$), and $P \not\mathrm{I} c, a \not\perp c, b \not\perp c$.

Special Cases:

Generalized Elliptic Planes. There exist three pairwise perpendicular lines.

Generalized Euclidean Planes. There exist four lines, a, b, c, d, such that a and b are both perpendicular to both c and d.

Generalized Hyperbolic Planes. There exist a pair of lines which have neither a point nor a perpendicular in common; for any line l and any point P, there pass at most two lines through P which have neither a point nor a perpendicular in common with l.

Planes with Free Mobility. If $P\mathrm{I}a$ and $P'\mathrm{I}a'$, then there exists $\sigma \in \mathcal{G}$ such that $P' = \sigma P \sigma^{-1}$ and $a' = \sigma a \sigma^{-1}$.

An H-plane $\mathcal{H}$ gives rise to a plane with free mobility when $\mathcal{G}$ is taken to be its group of motions and $\mathcal{S}$ its subset of line reflections. Conversely, a plane with free mobility having a compatible order structure (in the sense of Hilbert) arises from a uniquely determined H-plane in this way. Thus Hilbert's approach is incorporated into Klein's Erlanger Program, whereby the group $\mathcal{G}$ becomes the primordial object of interest.

Much work has been done by Bachmann and his students in studying the models of these axioms (which models he calls "metric planes"). (Wouldn't "absolute planes" be a better name, since these planes have no metric in the usual sense of distances?) For a survey of this work see **[3]**; for an introduction in English, see **[5]**. The classification problem remains unsolved.

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MATHEMATICAL NOTES

Edited by Deborah Tepper Haimo and Franklin Tepper Haimo

Material for this department should be sent to Professor Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis, MO 63121.

Advice to prospective authors: The editors have recently been receiving about ten times as many Mathematical Notes as can be used. It will simplify our work if authors will submit only especially interesting manuscripts. Mathematical Notes should be short papers of one to four printed pages which give new insights, new and improved proofs of old theorems, brief bits of mathematical folklore that have not found a home in the literature, or (occasionally!) new results that are not too technical. The topics should be of wide current interest. Papers that have already been rejected by a research journal are only very rarely suitable as Mathematical Notes.

R.P.B.

ON A PROBLEM OF STEINHAUS ON LATTICE POINTS

I. J. SCHOENBERG

Let Q denote the field of rational numbers. Define in R^n the *rational lattice*

$$Q^n = \{(x_i); \quad \text{all} \quad x_i \in Q\},$$

the set of *rational planes*

$$\Pi_n = \left\{ \pi : \sum_1^n c_i x_i + d = 0; \quad \text{all} \quad c_i \quad \text{and} \quad d \in Q, \ \sum c_i^2 > 0 \right\},$$

and the set Ω_n of points of R^n that are left if we remove all rational planes:

$$\Omega_n = R^n \setminus \bigcup_{\pi \in \Pi_n} \pi.$$

Clearly,

$$\Omega_n = \{(x_i); \quad \text{the numbers } 1, x_1, \ldots, x_n \text{ are linearly independent over } Q\}.$$

Points of Ω_n are easily obtained in terms of logarithms of primes: *If $p_k (k = 0, 1, \ldots, n)$ are distinct prime numbers and*

$$x_i = \frac{\log p_i}{\log p_0} (i = 1, \ldots, n),$$

then $(x_i) \in \Omega_n$. This is an immediate consequence of the unique factorization of integers into prime factors.

Our main result is a kind of duality between Q^n and Ω_n:

THEOREM 1. *The following two conditions on a point $X = (x_i)$ are equivalent*:
(1) *X is at different distances from all points of Q^n, and*
(2) $X \in \Omega_n$.

Proof. We begin with two geometric remarks.

I. If $A = (a_i)$, $B = (b_i)$ are two *distinct* points of R^n, then the equation of the plane π which is the perpendicular bisectrix of the segment $[A, B]$ has coefficients that are rational functions of the a_i and the b_i.

II. If $B = (b_i)$ is the mirror image of $A = (a_i)$ with respect to the plane π, then the coordinates b_i are rational functions of the a_i and of the coefficients of the plane π.

1. (1) *implies* (2), for if $X \notin \Omega_n$, then $X = (x_i)$ would belong to some plane $\pi \in \Pi_n$. If we pick $A = (a_i) \in Q^n$, then, by II, also the mirror image B of A, belongs to Q^n. Moreover, if $A \notin \pi$, then $A \neq B$. By symmetry in π it is clear that X is equidistant from A and B, in contradiction to (1).

2. (2) *implies* (1), for if X were equidistant from the distinct points A and B of Q^n, then $X = (x_i)$ would belong to the perpendicular bisectrix π, of A and B, while $\pi \in \Pi_n$ by I. This contradicts the assumption (2) and completes the proof.

REMARKS. 1. An obvious corollary: *Let $X \in \Omega_n$. If $A = (a_i) \in Q^n$, then the sphere with center X and radius $|A - X|$ contains the point A, but contains no other point of Q^n.*

2. Again let $X \in \Omega_n$. *If we join the point X to all distinct points of Q^n by straight lines, then no two of these lines can coincide.* This follows easily from our assumption that $X \in \Omega_n$.

3. The entire note remains valid if we replace throughout the rational field Q, by any fixed real field F, hence Q^n by F^n, etc.

4. This note was suggested by a problem of Steinhaus solved by Sierpiński ([**2**, pp. 8–9], [**1**, pp. 117–118]) by observing that the point $X = (\sqrt{2}, 1/3)$ has different distances from all points

of the lattice $\mathbb{Z}^2=\{(m,n);\ m,n \text{ in } \mathbb{Z}\}$. Notice that X does not belong to Ω_2, because X belongs to the rational line π: $-3x_2+1=0$, and therefore X is equidistant from any two points A and B, of Q^2, that are symmetric with respect to the line $x_2=1/3$.

By Theorem 1 it is clear that any point $X\in\Omega_n$ is at different distances from all points of the lattice $\mathbb{Z}^n=\{(x_i);\ \text{all } x_i\in\mathbb{Z}\}$ because $\mathbb{Z}^n\subset Q^n$.

5. A last remark is as follows. Let t be a real transcendental number. Clearly

$$T=(t,t^2,\ldots,t^n)\in\Omega_n.$$

By Theorem 1 the distances $\theta=|R-T|$, where

$$R=(r_1,r_2,\ldots,r_n)\in Q^n,$$

are all distinct. The following holds:

(3) *The distances $\theta=|R-T|$ are all transcendental numbers.*

Since

(4)
$$\theta^2=|R-T|^2=(t-r_1)^2+\cdots+(t^n-r_n)^2=t^{2n}+\sum_0^{2n-1}c_jt^j$$

with rational c_j, it follows that $\theta^2=|R-T|^2$ is transcendental, because an equation

$$(\theta^2)^k+d_1(\theta^2)^{k-1}+\cdots+d_k=0,\quad \text{with}\quad d_i\in Q,$$

would yield, by (4), a similar equation of degree $2nk$ for t, which is impossible. Finally, the transcendentality of θ^2 implies (3).

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THE BORSUK-ULAM THEOREM AND ORTHOGONALITY IN NORMED LINEAR SPACES

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This note is concerned with two well-known theorems: the Borsuk-Ulam antipodal mapping theorem and a theorem concerning the deviation between subspaces of normed linear spaces which can be stated in terms of "orthogonality." We will say that a point of a normed linear space is orthogonal to a linear subspace if its distance from the subspace is equal to its norm. The Borsuk-Ulam theorem is topological and there is no simple elementary proof. The second theorem is stated and proved in [3] and also in [1] where the proof is an essentially simple application of the Borsuk-Ulam theorem. The second theorem has applications in approximation theory (e.g., [4]), in the perturbation theory of linear operators ([1], [2]), and in Banach space theory. The two theorems will be stated and it will then be shown that the Borsuk-Ulam theorem can be deduced from the second theorem. This will show that the two theorems are equivalent in the sense that each can be obtained by a simple argument from the other.

THE BORSUK-ULAM THEOREM. *If $\phi: S^n\to R^n$ is a continuous map of the euclidean sphere S^n into R^n with the property that $\phi(-x)=-\phi(x)$ for all $x\in S^n$, then $\phi(x_0)=0$ for some $x_0\in S^n$.*

THEOREM ON THE DEVIATION OF SUBSPACES. *Let E be a real normed linear space and let M and N be finite dimensional subspaces of E with* $\dim M < \dim N$. *Then there exists a non-zero $f \in N$ such that f is orthogonal to M.*

Let $\phi: S^n \to R^n$ be as in the statement of the Borsuk-Ulam theorem. Let $f_j: S^n \to R (j=1,\ldots, n+1)$ be the restrictions of the coordinate projections on R^{n+1} and let $g_k: R^n \to R (k=1,\ldots,n)$ be the coordinate projections. Let $C(S^n)$ be the space of real continuous functions on S^n with the uniform norm and let $N = \mathrm{sp}\{f_1,\ldots,f_{n+1}\}$ and $M = \mathrm{sp}\{g_1\phi,\ldots,g_n\phi\}$ so that $\dim N = n+1$ and $\dim M \leqslant n$. Let $f \in N$ be a function whose existence is asserted by the second theorem. Then f is the restriction to the euclidean sphere S^n of a linear functional on R^{n+1} and so attains its norm $\|f\|$ at a *unique* pair $\{x_0, -x_0\}$ of antipodal points on S^n. Suppose that $\phi(x_0) \neq 0$. Then $(g_k\phi)(x_0) \neq 0$ for some k. Now $(f - \varepsilon(g_k\phi))(-x_0) = -(f - \varepsilon(g_k\phi))(x_0)$. It follows that if ε is small and of the right sign then $\|f - \varepsilon(g_k\phi)\| < \|f\|$. This contradicts the choice of f. Therefore $\phi(x_0) = 0$.

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EXTREMA FOR FUNCTIONS OF SEVERAL VARIABLES

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Introduction. The nature of a critical point a of a function f of two variables may be determined if the polynomial F consisting of the second-order terms of the Taylor expansion $T(f,a)$ of $f(x)-f(a)$ about a is definite or indefinite. (See [1].) Since all non-zero terms $\alpha(x_1-a_1)^{q_1}(x_2-a_2)^{q_2}$ of $T(f,a)$ satisfy $q_1/2+q_2/2 \geqslant 1$, whereas all terms of F satisfy $q_1/2+q_2/2 = 1$, the polynomial F is one possible outcome if one were to select a pair of positive numbers p_1, p_2 such that $q_1/p_1+q_2/p_2 \geqslant 1$ for all non-zero terms of $T(f,a)$ and then form the polynomial ρ equal to the sum of those terms $T(f,a)$ for which $q_1/p_1+q_2/p_2=1$. Call any such polynomial a critical polynomial of f. The main result of this paper implies that if any critical polynomial is positive (respectively, negative) definite then f has a strict relative minimum (respectively, maximum) at a. Moreover if f has a relative minimum (respectively, maximum), then each critical polynomial of f is positive (respectively, negative) semi-definite.

In order to apply these results it is useful to interpret the idea of critical polynomial geometrically. Let Q be the set consisting of all pairs (q_1,q_2) of non-negative integers such that $\alpha(x_1-a_1)^{q_1}(x_2-a_2)^{q_2}$ is a non-zero term of $T(f,a)$. If L is a line in the q_1-q_2 coordinate plane such that (i) L intersects Q, (ii) L intersects each positive axis, and (iii) no point of Q lies strictly between L and the origin, say that L is a critical line. Then the following observation aids in identifying critical polynomials. A polynomial ρ is critical if and only if for some critical line L, ρ equals the sum of those terms $\alpha(x_1-a_1)^{q_1}(x_2-a_2)^{q_2}$ of $T(f,a)$ such that $(q_1,q_2) \in L$.

1. Examples. To illustrate the results stated above we assume their validity and use them to determine the nature of the critical point $(0,0)$ of the function f defined by $f(x,y) = ax^{10}+bx^4y^9$

$+cy^{14}$ where a, b, and c are non-zero constants. Plotting the points of the set $Q=\{(10,0),(4,9),(0,14)\}$ in the q_1-q_2 coordinate plane, one sees that the critical lines given by the three equations $q_1/10+q_2=1$, $q_1+q_2/14=1$ and $q_1/10+q_2/14=1$ give rise to the three critical polynomials ax^{10}, cy^{14} and $ax^{10}+cy^{14}$ and that other critical lines, such as the one given by the equation $q_1/10+2q_2/3=1$, do not give rise to any other critical polynomials. Since the critical polynomial $ax^{10}+cy^{14}$ is positive definite if $a>0$, $c>0$, negative definite if $a<0$, $c<0$, and indefinite if $a\cdot c<0$, the main result of this paper implies that f has a strict relative minimum at $(0,0)$ if $a>0$, $c>0$, a strict relative maximum at $(0,0)$ if $a<0$, $c<0$, and neither a relative minimum nor a relative maximum if $a\cdot c<0$.

As a further illustration, consider the function f given by $f(x,y)=ax^{10}+bx^4y^8+cy^{14}$ where a, b, and c are non-zero constants, not all of the same sign. The origin is a critical point, the set Q equals $\{(10,0),(4,8),(0,14)\}$ and the critical polynomials of f are ax^{10}, cy^{14}, bx^4y^8, $ax^{10}+bx^4y^8$ and $cy^{14}+bx^4y^8$. Since a, b, and c are not all of the same sign, one of the first three critical polynomials is negative semi-definite and another is positive semi-definite. Therefore the main result of this paper will imply that the origin is neither a relative maximum nor a relative minimum for f.

As a final illustration consider the function f defined by $f(x,y)=2x^4-3x^2y+y^2$, an example due to Goursat [3, p. 31] and Peano [2], [3, p. 61]. The set Q equals $\{(4,0),(2,1),(0,2)\}$. One critical line is $q_1/4+q_2/2=1$. Hence f itself is a critical polynomial. Since $f(1,0)>0$ and $f(.75,1)<0$, f is indefinite and the main result of this paper implies that the origin is neither a relative maximum nor a relative minimum for f.

2. Terminology and notation. Let σ and τ denote points of R^n. Alternatively, denote a point σ of R^n by using the vector notation $\langle\sigma_i\rangle_{i=1}^n$ where σ_i represents the ith component of σ. Let $\vec{0}$ denote the origin of R^n. For any n-tuple l of non-negative integers, let $l!$ denote the real number $l_1!\cdot l_2!\cdots l_n!$. Finally, let δ_{ij} denote the Kronecker delta.

A real-valued polynomial ρ defined on R^n is said to be positive definite if $\rho(\tau)$ is positive for all $\tau\neq\vec{0}$ and is said to be positive semi-definite if $\rho(\tau)$ is non-negative for all τ. In the former case we write $\rho>0$ and in the latter case we write $\rho\geqslant 0$. Similarly, define $\rho<0$ and $\rho\leqslant 0$. If a polynomial assumes both positive and negative values, it is said to be indefinite.

For each n-tuple p of positive numbers, let L_p denote the hyperplane in R^n determined by the n points $\langle p_i\delta_{ij}\rangle_{j=1}^n$, $i=1,2,\dots,n$. We shall say that L_p supports a subset Q of R^n away from $\vec{0}$ if $L_p\cap Q$ is non-empty and if Q is contained in the closed half-space determined by L_p that does not contain $\vec{0}$.

Let m be a positive integer, f be of class $C^{(m)}$ on an open subset of R^n, and a be a point of that open subset which is also a critical point of f. Let Q_f denote the set of n-tuples l of non-negative integers such that $l_1+l_2+\cdots+l_n\leqslant m$ and such that the lth term in Taylor's formula for $f(\sigma)-f(a)$ about a is not identically zero, where the lth term is

$$(l!)^{-1}D_lf(a)(\tau-a)^l \quad \text{if } l_1+l_2+\cdots+l_n<m$$

and is

$$(l!)^{-1}D_lf(a+s(\tau-a))(\tau-a)^l,\ 0<s<1, \quad \text{if } l_1+\cdots+l_n=m.$$

This last expression is zero if f is a polynomial and m is chosen to be larger than the degree of f. If for some n-tuple p of positive numbers the hyperplane L_p supports Q_f away from $\vec{0}$, then say that L_p is a critical hyperplane of f at a and that the polynomial ρ defined by

$$\rho(\tau)=\sum_{l\in Q_f\cap H_p}(l!)^{-1}D_lf(a)\tau^l,\ \tau\in R^n,$$

is a critical polynomial of f at a.

3. Theorem. *Let f be of class $C^{(m)}$ on an open subset of R^n and let a be a point of that open subset which is also a critical point of f. Then:*

(i) $\rho \geqslant 0$ *for every critical polynomial* ρ *of* f *at* a *is necessary for a relative minimum at* a.
(ii) $\rho > 0$ *for some critical polynomial* ρ *of* f *at* a *is sufficient for a strict relative minimum at* a.
(iii) $\rho \leqslant 0$ *for every critical polynomial* ρ *of* f *at* a *is necessary for a relative maximum at* a.
(iv) $\rho < 0$ *for some critical polynomial* ρ *of* f *at* a *is sufficient for a strict relative maximum at* a.

4. Remarks. For certain functions the theorem may reveal the nature of a critical point only if one rotates coordinates. For example, if one tests the critical point $(0,0)$ of the function $f(x,y)=(x-y)^2-x^2y^2+y^4$, one finds that there is exactly one critical polynomial, namely, $\rho(x,y)=(x-y)^2$. The theorem appears to yield no information about the nature of the critical point $(0,0)$. However, if one considers the function $f\circ\lambda$, where λ is the rotation given by $\lambda(x,y)=2^{-\frac{1}{2}}(x+y,y-x)$, one finds that to the critical point $(0,0)$ there correspond three critical polynomials, one of which is given by $\rho(x,y)=-xy^3$. Since ρ is indefinite, the theorem implies that the function $f\circ\lambda$ and hence the function f have neither a relative maximum nor a relative minimum at $(0,0)$.

To see that rotating coordinates will not always reveal the nature of a critical point, consider the functions $f(x,y)=(x-y^2)^2-y^8$ and $f(x,y)=(x-y^2)^2+y^8$. The former has neither a relative minimum nor a relative maximum at the critical point $(0,0)$, while the latter has a strict relative minimum at the critical point $(0,0)$. Neither of these facts will be revealed by the theorem even if one rotates coordinates.

5. Lemma. *Let* f *be of class* $C^{(m)}$ *on a closed ball* $B(\vec{0},r_0)$ *centered at* $\vec{0}$ *of radius* r_0, ρ *be a critical polynomial of* f *at* $\vec{0}$ *and* $f(\vec{0})=0$. *Then*

$$\max_{\sigma\in\partial B} r^{-1}|(f-\rho)(\langle r^{1/p_i}\sigma_i\rangle_{i=1}^n)|\to 0,$$

as $r\to 0+$.

In proving the lemma and the theorem we shall use the following facts. Their proofs are straightforward and therefore omitted.
(1) L_p supports Q_f away from $\vec{0}$ only if $\Sigma l_i/p_i \geqslant 1$ for every $l\in Q_f$.
(2) $l\in L_p$ if and only if $\Sigma l_i/p_i=1$.
(3) If ρ is the critical polynomial determined by a critical hyperplane L_p, then $\rho(\langle r^{1/p_i}\sigma_i\rangle_{i=1}^n)=r\rho(\sigma)$ for all ρ,σ and for all $r>0$.

Proof of lemma. Using Taylor's formula, we may write

$$(4)\quad (f-\rho)(\tau)=\Sigma(l!)^{-1}D_lf(\vec{0})\tau^l+\Sigma(l!)^{-1}D_lf(s\tau)\tau^l+\Sigma(l!)^{-1}(D_lf(s\tau)-D_lf(\vec{0}))\tau^l,$$

where $0<s<1$, and the index sets for the sums are $\{l\in Q_f\setminus L_p:\Sigma l_i<m\}$, $\{l\in Q_f\setminus L_p:\Sigma l_i=m\}$, and $\{l\in Q_f\cap L_p:\Sigma l_i=m\}$, respectively.

The first two sums in (4) are bounded in absolute value by a sum of finitely many terms of the form $C\Pi_{i=1}^n|\tau_i|^{l_i}$, where $\Sigma l_i/p_i>1$, since by (1) and (2) this last inequality holds for all l in $Q_f\setminus L_p$. Since $l\in Q_f\cap L_p$ implies that D_lf is continuous on B and by (2) that $\Sigma l_i/p_i=1$, the last sum is bounded in absolute value by a sum of finitely many terms of the form $C_\tau\Pi_{i=1}^n|\tau_i|^{l_i}$, where $\Sigma l_i/p_i=1$ and $C_\tau\to 0$, as $\tau\to\vec{0}$. So for some $\varepsilon>0$ and all r in $(0,1)$,

$$\max_{\sigma\in\partial B}|(f-\rho)(\langle r^{1/p_i}\sigma_i\rangle_{i=1}^n)|\leqslant c_*(r^\varepsilon+k_r)r,$$

where k_r is defined to be the maximum of C_τ as τ varies over the set $\{\tau:|\tau|\leqslant r^{1/\max p_i}r_0\}$. This proves the lemma since k_r tends to 0 as $r\to 0+$.

In proving the theorem one may assume that $a=\vec{0}$ and that $f(a)=0$ since the function $f(\tau+a)-f(a)$ is zero at $\tau=0$, since a polynomial ρ is a critical polynomial of f at a if and only if ρ is a critical polynomial of $f(\tau+a)-f(a)$ at $\vec{0}$ and since the critical points a and $\vec{0}$ of f and $f(\tau+a)-f(a)$, respectively, must be identical in nature.

6. Proof of theorem. (i) Suppose that there exists a critical polynomial ρ which is not positive semi-definite.

There exists a point σ such that $\rho(\sigma)<0$. By (3) and the lemma, $(|f-\rho|/|\rho|)(\langle r^{1/p_i}\sigma_i\rangle) \leqslant C_r/|\rho(\sigma)|$, where $C_r \to 0$, as $r \to 0+$. Thus, $f(\langle r^{1/p_i}\sigma_i\rangle)$, which equals $\rho(1+(f-\rho)/\rho)(\langle r^{1/p_i}\sigma_i\rangle)$, is negative for all r sufficiently small, proving (i).

(ii) Suppose that there exists a critical polynomial ρ which is positive definite.

Let $\varepsilon>0$ be given. By (3) and the lemma, there exists r_ε in $(0,1)$ such that $(|f-\rho|/\rho)(\langle r^{1/p_i}\sigma_i\rangle) <\varepsilon$ for all r in $(0,r_\varepsilon)$ and all σ in $\partial B(\vec{0},r_\varepsilon)$. To prove (ii) it suffices to show that

(5) if $|\tau|<\delta \equiv \min_{\sigma\in\partial B}|\langle r_\varepsilon^{1/p_i}\sigma_i\rangle|$ then there exists $r\in(0,r_\varepsilon)$, $\sigma\in\partial B(\vec{0},r_\varepsilon)$ such that $\tau=\langle r^{1/p_i}\sigma_i\rangle$, because we would then have $(|f-\rho|/\rho)(\tau)<\varepsilon$ for all τ such that $|\tau|<\delta$, which would imply that f has a strict relative minimum at $\vec{0}$ since $f=\rho(1+(f-\rho)/\rho)$.

To prove (5), let $|\tau|<\delta$. Since $|<r^{1/p_i}\tau_i>|$ is a continuous function of r, tending to $+\infty$ as r tends to $+\infty$, there exists $r_*>0$ such that $|\langle r_*^{1/p_i}\tau_i\rangle|=1$. But then

(6) $\tau=\langle r^{1/p_i}\sigma_i\rangle_{i=1}^n$ where $\sigma=\langle r_*^{1/p_i}\tau_i\rangle_{i=1}^n\in\partial B(\vec{0},1)$ and where $r=1/r_*$. Since $|\tau|<\delta$, $|\langle r^{1/p_i}\sigma_i\rangle| <|\langle r_\varepsilon^{1/p_i}\sigma_i\rangle|$. But $|\langle r^{1/p_i}\sigma_i\rangle|$ is a strictly increasing function of r, so $r<r_\varepsilon$, which with (6) proves (5).

Parts (iii) and (iv) of the theorem follow from (i) and (ii), respectively.

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CLASSROOM NOTES

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Material for this department should be sent to Professor Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis, MO 63121.

Advice to prospective authors: The editors have recently been receiving about **ten times** as many Classroom Notes as can be used. It will simplify our work if authors will submit only especially interesting manuscripts. R.P.B.

ALGEBRAIC EQUATIONS WHOSE ROOTS FORM REGULAR n-GONS IN THE COMPLEX PLANE

BOLESH J. SKUTNIK

If the coefficients of the n-ic equation

$$a_0z^n+a_1z^{n-1}+\cdots+a_n=0$$

obey the relations

$$n^k a_k a_0^{k-1} = \binom{n}{k} a_1^k \qquad (1<k<n) \tag{1}$$

then the roots of the algebraic equation will form a regular n-sided polygon in the Argand plane. With no loss in generality we can set $a_0 = 1$, and thereby simplify the required relationships to

$$n^k a_k = \binom{n}{k} a_1^k \qquad (1<k<n). \tag{2}$$

To establish this, let us begin with the case where the polygon is in standard position. The form of the equation is

$$z^n + a_n = 0$$

$$a_1 = a_2 = \cdots = a_{n-1} = 0$$

$$\text{or } z^n = r^n \text{ with the roots } re^{2\pi i\nu/n} \qquad (\nu = 0, 1, \ldots, n-1). \tag{3}$$

and where the projection of the first root on the real axis is $r\cos 2\pi/n$. This is analogous to the cyclotomic equation

$$z^n = 1$$

whose roots form the vertices of a regular n-gon in the Argand plane.

An arbitrary position for the resulting polygon is achieved by a linear transformation

$$z' = az + b'. \tag{4}$$

Thus

$$z'^n = (az + b')^n = 1 \tag{5}$$

is the required form of the n-ic equation whose roots form a regular n-gon. Dividing (5) by a^n and altering b' to b yields the equation

$$(Z+b)^n - C = 0 \tag{5'}$$

which may be compared with the n-ic equation

$$Z^n + a_1 Z^{n-1} + \cdots + a_n = 0.$$

Comparison of the coefficients of the two equations reveals that

$$a_k / a_1^k = \binom{n}{k} b^k / (nb)^k = \binom{n}{k} / n^k \tag{6}$$

which can be rewritten in the form of equation (2), i.e.,

$$n^k a_k = \binom{n}{k} a_1^k \qquad (1<k<n).$$

Thus we find that equation (2), and thus (1), is a necessary and sufficient condition for an nth degree equation with complex coefficients to have roots which form a regular n-sided polygon in the complex plane.

The quartic and quintic cases can serve as illustrative and interesting examples of the $n-2$ relationships that must be satisfied in each case:

$$\begin{array}{llll} n=4, & k=2, & 16a_2 = 6a_1^2, & (8/3)a_2 = a_1^2; \\ & k=3, & 64a_3 = 4a_1^3, & 16a_3 = a_1^3; \end{array} \tag{7}$$

$$\begin{array}{llll} n=5, & k=2, & 25a_2 = 10a_1^2, & (5/2)a_2 = a_1^2; \\ & k=3, & 125a_3 = 10a_1^3, & (25/2)a_3 = a_1^3; \\ & k=4, & 625a_4 = 5a_1^4, & 125a_4 = a_1^4. \end{array} \tag{8}$$

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PRODUCTS OF COMMUTATORS ARE NOT ALWAYS COMMUTATORS: AN EXAMPLE

PHYLLIS JOAN CASSIDY

Explicit, conceptually simple examples of groups whose commutator subgroups contain non-commutators do not abound [1]. Indeed, Ore showed that in the symmetric groups S_∞ and S_n, every product of commutators is a commutator [2]. The following matrix group provides an easy example of a group whose commutator group contains, for every positive integer n, elements that cannot be written as products of n commutators.

Let f be a polynomial in x, g be a polynomial in y, and h be a polynomial in x and y—all with rational coefficients. Let $m(f,g,h)$ be the matrix

$$\begin{bmatrix} 1 & f(x) & h(x,y) \\ 0 & 1 & g(y) \\ 0 & 0 & 1 \end{bmatrix},$$

and let G be the set of all such matrices. Since the matrix product $m(f_1,g_1,h_1)\cdot m(f_2,g_2,h_2)$ is $m(f_1+f_2,g_1+g_2,h_1+h_2+f_1g_2)$ and the matrix inverse of $m(f,g,h)$ is $m(-f,-g,-h+fg)$, it is evident that G is a group. The commutator $[m(f_1,g_1,h_1),m(f_2,g_2,h_2)]$ is $m(0,0,f_1g_2-f_2g_1)$. Thus, the commutator subgroup is a subgroup of the commutative group consisting of all matrices $m(0,0,h)$, where h is a polynomial in x and y with rational coefficients. The matrix multiplication is described by the simple formula $m(0,0,h_1)\cdot m(0,0,h_2)=m(0,0,h_1+h_2)$.

We observe that $m(0,0,\Sigma a_{ij}x^iy^j)=\Pi[m(a_{ij}x^i,0,0),m(0,y^j,0)]$. Therefore, for every polynomial h in x and y with rational coefficients, $m(0,0,h)$ is in the commutator subgroup of G. Thus, the commutator subgroup is identifiable with the additive group of polynomials in x and y with rational coefficients.

Let n be any positive integer and let $h(x,y)=\Sigma_{i=0}^{2n+1}x^iy^i$. We shall show that $m(0,0,h)$ cannot be written as a product of n commutators. Suppose

$$h(x,y)=\sum_{j=1}^{n}(f_j(x)g_j(y)-h_j(x)k_j(y)). \qquad (*)$$

View both sides of the equation as polynomials in x with coefficients that are polynomials in y. Write $f_j(x)=\Sigma_i a_{ij}x^i$ and $h_j(x)=\Sigma_i b_{ij}x^i$, and in (*) equate coefficients of $1,x,\dots,x^{2n+1}$. We then have

$$\sum_{j=1}^{n}(a_{ij}g_j(y)-b_{ij}k_j(y))=y^i \qquad i=1,\dots,2n+1.$$

Therefore, $1,y,\dots,y^{2n+1}$, which are linearly independent, are in the space generated by the $2n$ elements $g_j(y),k_j(y),j=1,\dots,n$. This is a contradiction.

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COMPARISON OF ESTIMATES FOR e AND e^{-1}

R. B. DARST

Let $a_n=[1+(1/n)]^n$, $b_n=[1+(1/n)]^{n+1}$, $c_n=[1-(1/n)]^n$, $f_n=1+1+1/(2!)+\cdots+1/(n!)$, and $g_n=1-1+1/(2!)+\cdots+(-1)^n/(n!)$.

It is well known that a_n increases to e, b_n decreases to e, $c_n = 1/b_{n-1}$ increases to e^{-1}, and $a_n < f_n$. A colleague asked the author for an easy proof that $g_n > c_n$. While looking for an answer, the following interesting proposition appeared.

PROPOSITION. *If* $0 \leqslant \alpha < 1/2$, *then eventually* $\phi_\alpha(n) = [1 + (1/n)]^{n+\alpha}$ *increases to* e; *if* $1/2 \leqslant \alpha$, *then* $\phi_\alpha(n)$ *decreases to* e.

Before establishing the proposition we notice that $\phi_0(n) = a_n$ and $\phi_1(n) = b_n$. Also observe that $\phi_{(1/2)}(n) = [a_n b_n]^{1/2}$, so the fact that $[a_n b_n]^{1/2}$ decreases to e follows from the proposition.

Proof. Let ϕ_α be defined for each $x > 0$ by

$$\phi_\alpha(x) = [1 + (1/x)]^{x+\alpha} = \exp\{(x+\alpha)\ln[1 + (1/x)]\}.$$

Then $\phi_\alpha^{(1)}(x) = \phi_\alpha(x)\psi_\alpha(x)$, where

$$\psi_\alpha(x) = \{[\ln(x+1) - \ln(x)] - (x+\alpha)/[(x+1)x]\}.$$

Let $f(x) = \ln(x+1) - \ln(x)$. Taylor's theorem with remainder tells us that for some $y = y_x \in (x, x+1)$,

$$1/x - 1/(2x^2) < f(x) = 1/2 - 1/(2x^2) + 1/(3y^3) < 1/x - 1/(2x^2) + 1/(3x^3).$$

Hence $\psi_\alpha(x) > 0$ if $(2x-1)/(2x^2) > (x+\alpha)/[(x+1)x]$: if $(1-2\alpha)x > 1$; and $\psi_\alpha(x) < 0$ if $(6x^2 - 3x + 2)/(6x^3) < (x+\alpha)/[(x+1)x]$: if $(6\alpha - 3)x^2 + x > 2$. For $\alpha \geqslant 1/2$, $(3/4)^\alpha \leqslant (3/4)^{1/2} < 8/9$, so $2^{1+\alpha} > [1 + (1/2)]^{2+\alpha}$.

COROLLARY. *If* $n > 1$, *then* $g_n > c_n$.

Since $g_{2n} > e^{-1} > c_{2n}$, the corollary is verified by showing that $g_{2n+1} > c_{2n+1}$, $n \geqslant 1$. To this end, let $d_n = [\phi_{1/2}(n)]$. Then $1/d_n$ increases to e^{-1} and

$$(1/d_n) - (1/b_n) = (b_n - d_n)/b_n d_n = \{[1 + 1/n]^{1/2} - 1\}/b_n > \{1/(4n)\}/b_n \geqslant (16n)^{-1},$$

since $(1+x)^{1/2} > 1 + x/4$, $1 \geqslant x > 0$. Thus $e^{-1} - (16n)^{-1} > (1/b_n) = c_{n+1}$. But $g_n > e^{-1} - (n!)^{-1}$. Hence $g_{2n+1} > c_{2n+1}$ if $(2n+1)! > 32n$: $n > 1$; $g_3 > c_3$.

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MATHEMATICAL EDUCATION

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GRADE POINT AVERAGES AND THE CENTRAL LIMIT THEOREM

ROBERT M. KOZELKA

1. Introduction and Summary. Among the various "real-world" applications of the Central Limit Theorem, no example gets students' attention better than grade point averages. One can use various models of grading variability, and study of the models brings out some of the implications of various grading systems. In particular, one is led to question certain uses of the grade point average, especially for comparisons among students. With hand calculators and

will all average out over four years." Since the sum S takes only integral values (regardless of the value of p), the second decimal place must be off by at least 3 units ($S = \pm 1$, average $e = 1/30$) if it is not exactly correct ($S = 0$). The model above suggests (at least) two substitutes for wishful thinking. One is to increase grading accuracy within the given grade format. The difficulty of doing this increases with the number of cutting points; this is shown clearly by the differences in accuracy probabilities between the percentage and the five-point grading scales. The second possibility is to change the format. While "honors-pass-fail" has its own difficulties, the fewer the cutting points the more probable it becomes that there is no error. In particular, extreme GPAs may conceivably have increased accuracy due to endpoint phenomena.

Not all of the difference between grade and true worth is due to grading error. The probability that a student does worse (or better) than what he or she really knows can easily be incorporated into the model. One can simply add another (independent?) random variable, give it an appropriate probability distribution, and then consider the convolution with the grading error. A simpler way is to consider p as made up of two components, one representing student error and the other grading error. If one assumes that student errors are *not* symmetrically distributed, the mathematics becomes slightly more formidable and the probabilities of accuracy decrease considerably.

"Grade inflation" has caused many colleges to revise standards for honors and/or Dean's List nominations. It has also caused some uneasiness with the rank-in-class concept when students are grouped with nearly identical GPAs. If this model prompts teachers to be more precise in their grading, well and good. If in addition it suggests that both faculty and administration re-think awarding of important distinctions on the basis of GPAs computed to two (or more) decimal places, so much the better. At the very least, the results may give some weight to the arguments of those who would substitute "ranked in xth quantile" (for small x) for a specific student rank. Subjectivity in grading neither can nor should be eliminated, but a model like that presented here may help to make obvious some undesirable and potentially avoidable ramifications.

A final note: Many student evaluations of course and/or instructor are much more subjective than grades. In the model this means larger values of p and hence increased variance.

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A SEMINAR IN MATHEMATICAL MODEL-BUILDING

David A. Smith

Genesis. Those of us trained in the "pure" tradition of mathematics have become (sometimes painfully) aware of the need for a more "applied" orientation in our undergraduate program. We have had to recognize the following facts:

a. Most of our students will become (at most) users, not developers, of mathematics.
b. For those capable of becoming good mathematicians, there are strong incentives to find majors in other departments, such as computer science, operations research, and management sciences.
c. For those who remain in mathematics through a graduate program, there may not be enough positions in the academic world; the few existing academic positions require teachers equipped to cope with circumstances (a) and (b).

For these and other reasons, the college mathematics teacher should have some skill in mathematical model-building (not to mention digital computation, statistics, and at least some of the applications of the subjects in our "standard" curriculum). But how does the over-burdened professor acquire these skills?

Here CUPM has shown the way [**6**]:

> *The first recommendation of the Panel* [on Applied Mathematics] *is that each department should offer a course or two in applied mathematics which treat some realistic situations completely, beginning with a careful analysis of the nonmathematical origin of the problem; giving extremely careful consideration to formulation of a mathematical model, solution of the mathematical problem, and relevant computations; and presenting thoughtful interpretations of the theoretical results to the original problem.* In other words, there should be a few courses which give the students the experience of grappling with an entire problem from beginning to end. (Section III)
>
> The panel explicitly calls attention to the possibility of conducting these courses as seminars in which students and faculty members work cooperatively. (Section IV)

When I read this in 1972, it occurred to me that most students must know *something* to which mathematics can be applied. However, this thought was quickly followed by, "But I don't know how to teach a seminar." So I demurred until January 1973, when I heard Professor Robert Thrall talk about a course on model-building offered at Rice University. With advice from Professor Thrall, I organized and have been offering a similar course at Duke. The course has changed somewhat as I have learned new things from my students.

Structure. A number of modeling courses have been described in this MONTHLY ([**2**], [**11**], [**12**], [**17**], [**21**], [**26**]; also see [**27**]), but none closely resembles this one. Its central feature is the term project. Each student must be involved with one major modeling problem for most of the semester. The work may be (but rarely is) done jointly by two or more students and may be done in conjunction with another course or a job.

Of course most students are not prepared to plunge into a project at the start of the semester, so what happens during the first few weeks is crucial. I have used Maki and Thompson [**15**] as a text since it appeared in 1973; we spend about two weeks reading and discussing the first two chapters. Thereafter, the text serves as a reference, as a source of ideas, and as background for some of my lectures. (Lectures are not forbidden in a seminar, but they can't go on all the time.) The third week is devoted to outside readings, selected from a "Chinese menu" (one from column A, etc.) which has gradually grown to more than 150 items, roughly classified by subject. Students must read at least one item on the modeling process (to get a point of view different from that of the text), one item from an area in which they have an interest, and one from an area in which they think they have no interest; they must write a one-page summary of each. (Communication skills are very important in this course, and early feedback on students' efforts at writing pays dividends later.) During this third week the students are also preparing lists of possible topics (with no limit on how wild the ideas may be); these topics, as well as the outside readings, are discussed in class.

In the fourth week, classes are suspended and each student meets privately with me, as long and as often as necessary, to negotiate a deal on what the project topic will be. It is at this time that I have to force them to be realistic about what can be accomplished in the rest of the semester. Some feeling for that comes with experience, but a good guideline for beginners is to make sure the project can be segmented in some way, so that actual progress can be written up, even if the desired goals have not all been reached. During the fifth week we have a group discussion of the selected topics (necessarily more realistic and duller than the "blue sky" discussion two weeks earlier), a one-page summary of the topic and goals is prepared by each student, and I lecture on the subject of long-range planning. It is essential that the student think through and prepare a detailed plan for the project. It is hard enough to stick to a schedule that exists; but if the student does not know what tasks are to be completed this week, and next, and so on, there is no way to know whether he or she is on schedule.

Thereafter, the critical stages are:

—Submission of the detailed written plan, outlining the entire project development, and listing all resources known to the student, including an initial bibliography (week 7)

—Submission of an outline of the paper and complete bibliography (week 9, following vacation)
—Submission of a rough draft of the paper (week 11)
—Oral presentation of the project to the group (last two weeks of classes)
—Submission of the finished project report (during the exam period and in lieu of an exam)

Class time after the projects have been selected is divided about evenly between lecture and discussion. The students have something to discuss, namely, their own and their classmates' projects. They are expected to make regular progress reports, to ask questions about others' work, and to provide criticism and suggestions. They learn a lot from each other, and everyone has a stake in all the projects. The topics for my lectures have some relevance to the selected projects. With only one or two hours on any single mathematical topic, I can't possibly provide all the background needed by the student using that topic; but I can provide enough background for the others in the group to follow what the student is doing.

The assumed common background of the students is three semesters of calculus and one of linear algebra. My lectures have dealt with differential equations, linear programming, graph theory, least squares, statistics, probability, combinatorics, differential geometry, game theory, queueing theory, random number generation, and a variety of applications I've learned about along the way, many of which are discussed in **[15]**. I have also found it very useful to invite colleagues from other departments to lecture on their uses of mathematics (e.g., decision analysis in political science) or specific techniques in which my background was lacking (e.g., computer simulation packages). Often the students are eager to suggest people in other departments who are working in areas related to their projects.

Pitfalls. Perhaps the most obvious and intimidating pitfall for the teacher whose prior experience is limited to lecturing on a set body of material is the seminar format itself. How does one "discuss" material that is not clearly defined and that in some cases may be better known to some of the students than to the instructor? The seminar has to be approached as a learning experience for the faculty member as well as for the students. In particular, that means the answers don't all have to be known in advance. In fact, one should temporarily shed the role of "teacher" and replace it with that of "guide" or "enabler." At the very least, we have more experience than our students in learning new subjects, finding information in the library, organizing thoughts logically, and presenting our conclusions. That, plus the help available from a textbook, is enough to get started. Attitude is very important, however. In particular, one must consider the areas of application as important in their own right, and one must be willing to *listen* as students talk and to take their ideas seriously. Don't be afraid to say, "I don't know" (and add "but I'll find out," if appropriate), and don't try to keep steering the conversation toward topics you're already sure about. On the other hand, enter each session prepared to talk about *something*, because the students will be expecting some guidance. A specific assignment, such as the written summaries of outside readings mentioned above, is a good way to get started on the next discussion.

For the student attempting to choose a project, the first big pitfall is indecisiveness. Students are seldom called upon to make decisions early in a course that may directly affect their success or failure for the whole semester. But a semester is already too short for a real project, and the student cannot be permitted any more time than indicated above for getting started. It is important that the topic be the student's own idea, not one imposed by the instructor. This is the purpose of the "blue sky" list prior to the time of commitment. It gives the instructor room to maneuver during negotiations, to insist on commitment without actually selecting the topic.

The opposite danger is plunging into a project that is too ambitious. I have already noted the antidote above. Let the student have grand designs, but insist on subgoals that can be identified as progress along the way. What's left over at the end of the semester may turn into a lifelong career.

Don't expect, or let your students expect, that their projects must resemble published papers they read as samples. After the first offering of the course, past papers can be made available. These will make the point that student work is very uneven.

It is easy to seduce students into choosing topics that turn out to be too difficult, even though they look easy at first glance. For example, a standard topic for a modeling course is queueing theory, usually limited to the case of single server, first-come–first-served, Poisson arrival and service rate [**15**, pp. 391–400]. An important "problem" for students is waiting in lines, and they are very eager to "solve" such problems. However, queueing theory can get very difficult very quickly. My students have written about queues in dining halls (two-line and free-flow cafeterias), keypunch rooms, and the language laboratory, all with results that were somewhat less than hoped for. At the very least, one should be prepared to make available the tools for computer simulation of queues if students are going to work in this area.

Another serious pitfall is the temptation to be a "nice guy" when students come in with tales of woe about whatever is keeping them from staying on schedule. The first few times I taught the course, I had a high incidence of "incomplete" grades. Most of the students eventually completed their projects, but it is always an added burden (not to mention risk) for students to be completing work from previous semesters. My solution is to make all the preliminary work along the way part of the semester grade, with deductions (modest ones, to be sure) for late work. Grade-conscious students cannot always see the importance of the schedule, but they can produce prodigious amounts of work to keep their grades up.

The offer to accommodate joint projects can also be a pitfall. Most of our students are more accustomed to competing than to cooperating. For some, the idea of having a helper is attractive, but they usually find that they can't really work together; this may result in wheel-spinning, followed by a painful process of starting over, necessarily behind schedule. I still think a goal for this type of course should be to teach cooperation, which is the normal mode for successful endeavors in the real world, but the best I've been able to achieve is cooperation with me or with an outside agency or person with whom the student has some other incentive to work (e.g., money). In fact, I have not had a single joint project with two or more students come to fruition.

On the other hand, many of the projects, including most of the best ones, have been done in conjunction with another course, a research lab, or a consulting project on which the student is employed. However, there is a pitfall here, too. If the student depends on others to perform certain tasks on time (e.g., data gathering), the schedule may slip through no fault of the student.

Data gathering in general may be a pitfall. A good model requires good data, first for model selection and construction, then for fitting parameters, finally for validation. But time usually does not allow for extensive data gathering, so the project may be limited to data already in hand or available from published sources. (One attraction of campus-based projects is that data *can* be gathered in a reasonable amount of time, perhaps with the help of other students. Use of fictitious but plausible data is not out of the question for an academic exercise, but of course this makes validation impossible.)

A minor pitfall is built into our traditional 50-minute class period, which is frequently not long enough for a good discussion. The Duke class schedule permits the option of two 75-minute periods per week, and this works very nicely for one discussion and one lecture.

Samples. What do they write about? I have mentioned the projects dealing with campus waiting lines. The following additional examples illustrate the broad range of interests the students bring to the seminar. They also indicate how varied are the possibilities for modeling projects and the opportunities for the instructor to learn something new.

a. Urban problems (all involving Durham, N.C.)

—Municipal expenditures, 1938-1974 (How are they related to inflation, population growth, tax base, rising expectations, etc.?)

—Racial segregation in housing patterns, 1920-1970 (viewed as a network flow problem;

Durham has many mini-ghettos that shift from time to time)
—A rate study of the Durham water works (part of a study commissioned by the City Council to determine equitable pricing for water and sewer services—one of the consequences of the study, if it had been fully implemented by the Council, would have been to double Duke University's water bill, but the political clout that comes with the largest payroll in town moderated the increase somewhat)
—A street is expected to become a feeder for a major new shopping mall; should it be widened? (a question of more than academic interest, since the street in question is the university's link to its "faculty ghetto," and it has very heavy traffic at rush hours)

b. Biology and ecology
—The path followed by migrating hawks along an irregular ridge in Pennsylvania called Hawk Mountain (Do they use updrafts to minimize energy expenditure?)
—The population dynamics of the black bean aphid (an insect with an interesting life cycle that includes migration and both sexual and parthenogenic reproduction)
—A population model of Ponderosa pines and bark beetles along the rim of the Los Angeles basin (Air pollution from automobiles weakens the resistance of the pines to their natural enemies, the beetles, which are threatening to wipe them out.)
—A linear programming model for forest management alternatives in the Pisgah National Forest in western North Carolina (a study commissioned by the U.S. Forest Service)

c. Economics
—A forecasting model for the stock market (how to double your money in three months by buying and selling Pizza Hut stock—however, I haven't heard any reports of one of my student's becoming a millionaire)
—Aggregate saving behavior (a micro- and macro-economic model that explains cross-sectional data as well as long-run trends and short-term fluctuations)
—A linear programming model for optimizing business cash flow

d. Psychology
—An analysis of Hoffman's model for perception of optical illusions (This study stimulated my interest in deeper study of the subject, leading to serious research that was eventually published [25]. In fact, I have four or five other research projects arising from the seminar waiting until I can find time to work on them.)

e. Games and gaming
—Can one place the eight chess pieces of one color on the board in such a way that every square is under attack? (This is still an unsolved problem, as far as I know, but it generated a lot of interesting combinatorial mathematics.)
—A computer model (game-playing program) for three-dimensional tic-tac-toe
—A strategy for winning at jai alai
—A strategy for Master Mind (a commercial code-breaking game)

f. Cosmology
—The probability of communication with extraterrestrial intelligent beings (very low, within the lifetime of our civilization, in spite of a high probability of existence of intelligent life elsewhere)
—What is the shape of the universe? (Would you believe a cosmic bagel?)

g. Medicine
—A model for the spread of bubonic plague in England, 1348-1349 (an interesting application of modern techniques for modeling epidemics because of the way plague was spread, by fleas traveling on rats, and because a lot of data is actually available)

—A computer simulation of electron spin resonance (ESR) studies of red blood cells (a support function for a research team in the chemistry department and the medical school—they were studying red blood cell membranes, abnormality of which was thought to be a possible cause for muscular dystrophy)

h. History
—Can the outcome of Civil War battles be "predicted" retrospectively, using a deterministic model that accounts for numbers of troops, morale, weaponry, supply lines, leadership, etc.?

i. Computer Science
—Correctness proofs for structured programs

j. Energy
—The solar energy input to a standing body of water (integration of seven submodels to accommodate factors such as the path of the sun across the sky, latitude, season, varying sky conditions, and reflected radiation)
—A prediction of plutonium proliferation resulting from nuclear electric power generation
—The efficiency of solar home heating (an adaptation to local weather data of a model developed by NASA at Langley, Virginia)

k. Music
—An analysis of Bluegrass banjo (the physics of the banjo string and the patterns of rhythms and notes that are typical of Bluegrass)

Resources. The CUPM recommendations **[6]** include outlines for courses in optimization, graph theory and combinatorics, and fluid mechanics, each with a list of references. Some of those references are useful for a seminar, but by no means all of them. If it were not for the help provided by Professor Thrall, based on substantial experience, I probably would not have attempted this venture in 1973, with the resources that were then available. The situation is different today, since there are now several textbooks, with various objectives, suitable for a variety of audiences **[1]**, **[4]**, **[5]**, **[10]**, **[14]**, **[15]**, **[16]**, **[20]**, **[22]**. Other books include **[7]**, **[23]**, **[24]**, **[28]**. Of course, many of the newer textbooks for traditional subjects have "applications" sections that deal with relatively simple models.

Articles in this MONTHLY on the use of mathematics in the real world **[3]**, **[9]**, **[18]**, **[30]** are helpful in establishing an attitude and a framework for a modeling course. The published experiences of others, already noted, will help one decide what is possible, what level of students to try to reach, where the course belongs in the curriculum, etc.

Colleagues in other departments may be teaching more specialized modeling courses and have extensive bibliographies to share. For example, I was provided a bibliography of some 20 books and over 100 articles relating to the biological and earth sciences by a friend teaching a course in this area **[19]**. My own reading list is available to interested teachers. In formulating it, I have relied on all the other sources mentioned, and I have added articles from *Science*, *Nature*, *Scientific American*, this MONTHLY, *Mathematics Magazine*, *SIAM Review* (Classroom Notes), and *Applied Mathematical Modelling* (a relatively new journal, published in England).

The materials available from or indexed by the Undergraduate Mathematics Applications Project (UMAP) **[29]** are also very helpful. There are conference proceedings **[8]**, **[13]**, from which one may glean useful and current concepts in modeling practice.

In any of the disciplines of natural or social science, one will find quantitatively oriented journals devoted in part to mathematical models. It sometimes takes some digging to get through the technical jargon of the discipline, and sometimes there is no substance left when that has been done, but there is also a lot of interesting wheat among the chaff. Thus it is no longer true that published materials to support a modeling course are lacking, if indeed that was ever the case.

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PROBLEMS AND SOLUTIONS

EDITED BY A. P. HILLMAN AND VLADIMIR DROBOT

The editors receive more problems than they can use, but they are always on the lookout for interesting and novel problems. The task of selection will be much easier if proposers will keep the following guidelines in mind:

Send all proposed problems, in duplicate if possible, to Prof. Vladimir Drobot, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95051. Please include solutions and any information that will help the editors, including reasons why the problem is interesting. Problems in well-known textbooks and results in generally accessible sources are not acceptable.

Solutions should be sent to the addresses given at the head of each problem set.

An asterisk (*) indicates that the proposer did not supply a solution. If you submit a problem without a solution, you should tell the editors whether you know (or somebody else knows) how to solve the problem. If you are uncertain, the problem might be more suitable for the Research Problems section; see the instructions given there.

Proposers are asked to aim for the same audience as for the rest of the MONTHLY; *a rule of thumb is to think of people who have had at least a year of graduate work in Mathematics, very likely some years ago. Proposals that can be both understood and solved by high school students, or by those in the first two years of college, are usually more appropriate for the Mathematics Magazine or the Two-Year College Mathematics Journal. No precise line can be drawn between elementary and advanced problems, but in general an elementary problem will involve notions no more difficult than those encountered in the first year of graduate work and should be solvable by methods common at the same level. An advanced problem may deal with more advanced concepts or require more sophisticated techniques for its solution.*

A problem that takes half a page or more to state, or that involves several ad hoc definitions, is usually unsuitable. The statement of a problem should avoid specialized jargon and symbols whenever the English language is more readily understood: for example, "f is a continuous function" is preferable to "$f \in C$."

Before submitting a problem, be sure that it says exactly what you intend it to say. Solvers are not telepathic and will usually solve what was proposed, not what you had in mind.

PROBLEMS DEDICATED TO EMORY P. STARKE

Solutions of these problems should be mailed to Prof. A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, N.M. 87131 (U.S.A.) by March 31, 1980. To facilitate consideration, solutions should be typed with double spacing. If acknowledgment is desired, include a self-addressed card or label.

S 21. *Proposed by Paul Erdös, Technion, Haifa, Israel.*

Let $A(n,k)=(n+1)(n+2)\cdots(n+k)$, $B(n,k) = \operatorname{lcm}\ [n+1,n+2,\ldots,n+k]$, and $\alpha(n,k)=A(n,k)/B(n,k)$.

(a) How many distinct values can $\alpha(n,k)$ take for fixed k?

(b)* Do m, n, k exist with $m>n+k-1$ and $B(m,k)=B(n,k)$?

ELEMENTARY PROBLEMS

Solutions of Elementary Problems should be mailed in duplicate to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA 94303 (U.S.A.) by March 31, 1980. Please type with double spacing and place the solver's name and mailing address on each sheet. If acknowledgment is desired, include a self-addressed card or label.

E 2797. *Proposed by Barry J. Powell, Kirkland, WA.*

Show that there are infinitely many primes p such that for any pair of coprime odd positive integers x and y with no two of p, x, y, congruent modulo p, the exponent of p in $x^{p-1}-y^{p-1}$ is odd. [The exponent of p in m is the largest integer e such that $p^e|m$.]

E 2798. *Proposed by Doug Hensley, Texas A & M University, College Station, TX.*

Prove that there are infinitely many pairs (p,q) of primes such that $(q-1)/p$ is an integer k and 2 is a kth power modulo q.

E 2799. *Proposed by Marlow Sholander, Case Western Reserve University.*

For n a positive integer, let $n!!$ denote the *superfactorial* $\prod_{i=1}^{n} i!$ and let $0!!=1$. Set $A_n=(2n-1)!!/[(n-1)!!]^4$. Prove that A_n is an integral multiple of $(2n-1)!$. (A_n is the reciprocal of the determinant of the n by n Hilbert matrix $H_n=(h_{ij})$ with $h_{ij}=(i+j-1)^{-1}$. See Pólya-Szegö, *Aufgaben und Lehrsätze aus der Analysis*, Dover, 1945, vol. 2, Chap. 7, Prob. 3.)

E 2800. *Proposed by B. de la Rosa, University of the Orange Free State, Bloemfontein, South Africa.*

Show that an odd positive integer c is composite if and only if there exists a positive integer $a \leqslant (c-3)/3$ such that $(2a-1)^2+8c$ is a square.

E 2801. *Proposed by Louis Nirenberg, N.Y.U.; D. Kinderlehrer, University of Minnesota; and J. Spruck, University of Minnesota and Brooklyn College.*

Let $P_1(z)$ and $P_2(z)$ be monic polynomials with complex coefficients of degree $m+k$, $0 \leqslant k < m$, such that $z_1,\ldots,z_m$ in the upper half-plane are zeros of P_1 while $\bar{z}_1,\ldots,\bar{z}_m$ are zeros of P_2. Show that P_1-P_2 has degree greater than $m-k-2$.

E 2802. *Proposed by M. Slater, University of Bristol, England.*

Given a triangle ABC (in the Euclidean plane), construct similar isosceles triangles ABC' and ACB' outwards on the respective bases AB and AC, and BCA'' inwards on the base BC (or ABC'' and ACB'' inwards and BCA' outwards). Show that $AB'A''C'$ (respectively, $AB''A'C''$) is a parallelogram.

SOLUTIONS OF ELEMENTARY PROBLEMS

An Impossible Triangle

E 2687 [1977, 820]. *Proposed by Ronald J. Evans, UCSD, La Jolla.*

Does there exist a triangle with rational sides whose base equals its altitude?

Solution by J. G. Mauldon, Amherst College. Suppose, for the purpose of contradiction, that such a triangle exists. The obtuse and acute cases are pictured in Fig. 1. In either case, both relations $(x\pm y)^2=4a^2+(a\pm k)^2$ hold, so that $xy=ak$, x,y,a and k are rational and nonzero, $x^2+y^2=5a^2+k^2$, and consequently $k^2x^2+k^2y^2=5x^2y^2+k^4$. Writing $u=v=k$, we find that the equation

$$u^2y^2+v^2x^2=u^2v^2+5x^2y^2 \tag{1}$$

has a solution in nonzero rationals, and consequently a solution in positive integers. This will now be shown to be impossible.

Let (U,V,X,Y) be a solution of (1) in positive integers which minimizes the product XY. Then clearly $(U,X)=(V,Y)=1$ and, consequently, $(X,Y)=1$. Furthermore XY cannot be odd,

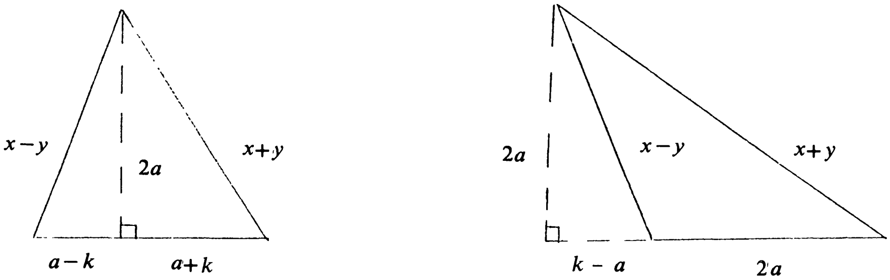

FIG. 1

for this would imply the impossibility $(U^2-1)(V^2-1)\equiv 4 \pmod 8$. Hence we may and shall assume that X is even and UY is odd, so that $U^2+Y^2\neq X^2$, U^2-X^2 is odd, and $(U^2-X^2, U^2-5X^2)=1$.

Since $(U^2-X^2)V^2=(U^2-5X^2)Y^2$ and $(U^2-X^2,\ U^2-5X^2)=(V^2,Y^2)=1$ and $U^2-X^2\neq -Y^2$, it follows that

$$U^2-X^2=Y^2 \quad \text{and} \quad U^2-5X^2=V^2. \tag{2}$$

Methods expounded in standard books on number theory, or in the 1914 reference cited below, show that the most general solution of (2) in positive integers with $(U,X)=1$ is

$$U=p^2+q^2,\quad X=2pq,\qquad U=r^2+5s^2,\quad X=2rs\qquad (p,q,r,s>0).$$

Since $pq=rs$ we may write $p=uy$, $q=vx$, $r=uv$, $s=xy$ and so, equating the two resulting expressions for U, we obtain (1) with $0<xy=s<X\leqslant XY$. This contradicts the postulated minimality of XY, and so establishes the asserted impossibility.

Also solved by Robert Breusch, Nathan Fine, L. E. Mattics, David Penney, Karl Stellmacher, Aleksandras Zujus, and the proposer. Two incomplete solutions were received.

Editor's Comment. The problem was reduced to that of showing the impossibility of $x^4+Nx^2y^2+y^4=z^2$ for positive integers x,y,z, by Penney ($N=-18$) and by Zujus and the proposer ($N=3$). The impossibility for these and other values of N is proved in H. C. Pocklington's paper "Some Diophantine Impossibilities" (Proc. Camb. Phil. Soc., 17 (1914), 108–121).

The proposer asks what other integral ratios of base to altitude (or altitude to base) besides 1 are impossible in triangles with rational sides.

Periodic Patterns of Signs

E 2719 [1978, 495]. *Proposed by John S. Lew, IBM Watson Research Center, Yorktown Heights, New York.*

For a fixed positive integer m let S_m be the sum of the series

$$\pm 1 \pm \tfrac{1}{3} \pm \tfrac{1}{5} \pm \tfrac{1}{7} \pm \tfrac{1}{9} \pm \cdots$$

where the first m terms have sign $+$, the next m terms have sign $-$, then the succeeding m terms have sign $+$, etc.

Gregory and Leibniz, independently, found $S_1=\pi/4$. On learning of this result, Newton (24 October 1676) wrote to Oldenburg [D. T. Whiteside (ed.): *The Mathematical Papers of Isaac Newton*, vol. 4, Cambridge Univ. Press, London, 1971, p. 372], a letter intended also for Leibniz, asking for the more difficult sum S_2. Leibniz apparently did not solve the problem.

Evaluate S_2 and S_3.

Solution by Bruce C. Berndt, University of Illinois. We shall prove a general theorem from which the desired evaluations will follow as corollaries.

Let $\{a_k\}$, $-\infty<k<\infty$, be an odd, periodic sequence of complex numbers with period n, i.e., $a_k=-a_{-k}$ and $a_{k+n}=a_k$ for every integer k. Since $\{a_k\}$ has period n, a_k may be expanded in a finite Fourier series,

$$a_k=\sum_{r=0}^{n-1} b_r e^{2\pi irk/n}, \qquad -\infty<k<\infty. \tag{1}$$

In fact, it is easily verified that (1) is valid if and only if

$$b_k=\frac{1}{n}\sum_{r=0}^{n-1} a_r e^{-2\pi irk/n}, \qquad -\infty<k<\infty.$$

Since $\{a_k\}$ is odd and $\{b_k\}$ is also odd, we see that $a_0=0=b_0$.

THEOREM. *With $\{a_k\}$ and $\{b_k\}$ defined as above, we have*

$$\sum_{k=1}^{\infty}\frac{a_k}{k}=-\frac{\pi i}{n}\sum_{r=0}^{n-1} b_r r.$$

Proof. Since $b_0=0$, it is easy to show by partial summation that the infinite series on the left side above converges.

Let C_N denote the rectangle with its center at the origin, its horizontal sides of length $2N+1$, and its vertical sides of length $\sqrt{N}$, where N is a positive integer. Let

$$g(z)=\sum_{r=0}^{n-1} b_r e^{2\pi irz/n}, \qquad f(z)=e^{-\pi iz}g(z)/\sin(\pi z),$$

and, integrating around C_N,

$$I_N=\frac{1}{2\pi i}\int_{C_N}\frac{\pi f(z)}{z}\,dz.$$

By the residue theorem,

$$\begin{aligned} I_N&=\frac{2\pi i}{n}\sum_{r=0}^{n-1} b_r r+\sum_{\substack{k=-N\\k\neq 0}}^{N} g(k)/k\\ &=\frac{2\pi i}{n}\sum_{r=0}^{n-1} b_r r+2\sum_{k=1}^{N} a_k/k. \end{aligned} \tag{2}$$

From the definition of f and the fact that $b_0=0$, we deduce that there exists a positive constant A, independent of N, such that for all $z=x+iy$ on the horizontal sides of C_N, $|f(z)|\leq A\exp(-2\pi|y|/n)$. Also, $f(z)$ has period $2n$. Thus, there is a positive constant B, independent of N, such that for all z on the vertical sides of C_N, $|f(z)|\leq B$. Using these two bounds for $|f|$, we readily find that

$$I_N=O(\sqrt{N}\exp(-\pi\sqrt{N}/n))+O(1/\sqrt{N})=o(1), \tag{3}$$

as N tends to ∞. Letting N tend to ∞ in (2) and using (3), we complete the proof.

We now use the Theorem to determine S_2 and S_3. To evaluate S_2, let $a_k=1$, -1, and 0, according as $k\equiv 1$, 3 (mod 8), $k\equiv 5$, 7 (mod 8), or $k\equiv 0$ (mod 2), respectively. Here $n=8$. Elementary calculations show that $b_k=-i\sqrt{2}/4$, $i\sqrt{2}/4$, and 0, according as $k\equiv 1$, 3 (mod 8), $k\equiv 5$, 7 (mod 8), and $k\equiv 0$ (mod 2), respectively. Our theorem then yields $S_2=\pi\sqrt{2}/4$. To

evaluate S_3, let $a_k=1$, -1, and 0, according as $k\equiv 1$, 3, 5 (mod 12), $k\equiv 7$, 9, 11 (mod 12), or $k\equiv 0$ (mod 2), respectively. Here $n=12$. Elementary calculations give $b_k=-i/3$, $i/3$, $-i/6$, $i/6$, 0, according as $k\equiv 1$, 5 (mod 12), $k\equiv 7$, 11 (mod 12), $k\equiv 3$ (mod 12), $k\equiv 9$ (mod 12), $k\equiv 0$ (mod 2). The theorem then gives $S_3=5\pi/12$.

Editorial Notes: Many solvers found explicit formulas for S_m, for instance

$$S_m=\frac{\pi}{4m}\sum_{k=0}^{m-1}\operatorname{cosec}\left(\frac{\pi}{2m}+\frac{k\pi}{m}\right)$$

or equivalent expressions.

Readers located the problem in various sources: H. Koppers (Switzerland) found it as Aufgabe 780, *Elem. Math.*, 32 (1977) 192, proposed by I. Paasche (Germany) and solved by M. Vowe (Switzerland). M. Gerstell found the formulas for S_2 and S_3 in I. S. Gradshteyn and I. M. Ryzhik, *Table of Integrals, Series, and Products Formulae*, 0.239 (5) (6) of the 4th Russian edition and page 9 of the 1965 English translation; P. Henrici (Zurich) located a formula for S_2 and some of its applications in Courant-Hilbert, *Methods of Math. Physics*, vol. 1, p. 350; Bob Prielipp found the problem for S_2 in this MONTHLY (E 704, 1946, 36; 467–468); R. Johnsonbaugh, Wayne Boucher and B. A. Troesch (independently) evaluated the sum for the sign pattern $++-/+--/\cdots$. R. B. Nelson found the formula for S_2 in L. B. W. Jolley, the *Summation of Series*, Dover, 1961, p. 16; one of the editors located an equivalent problem on page 30 of N. Koblitz, *p-adic Numbers, p-adic Analysis and Zeta-Functions*, Springer-Verlag, 1977.

D. Shelupsky, B. L. R. Shawyer, and R. Breusch (independently) obtained an asymptotic formula for S_m, namely

$$S_m=\frac{1}{2}\log m+\frac{1}{2}\left(\gamma+\log\left(\frac{8}{\pi}\right)\right)+\mathcal{O}\left(\frac{1}{m}\right)\qquad(m\to\infty).$$

W. Boucher, B. A. Troesch, and I. Kotlarski investigated other patterns of signs and derived related formulas.

Also solved by 87 other readers.

An Insensitive Central Limit Theorem

E 2723 [1978, 496]. *Proposed by Allen Moy, University of Illinois, Chicago.*

For a fixed $t>0$, find $\lim_{n\to\infty}e^{-nt}\sum_{k=0}^{n-1}(nt)^k/k!$.

Solution I by Robert Breusch, Amherst, Mass. The following facts are needed.
(a) For $n\geqslant 1$, the relation $n!>e^{-n}n^n$ holds.
(b) For $t>0$, $t\neq 1$, $p_t=te^{1-t}$, the relation $0<p_t<1$ holds.
Set

$$E_n(t)=e^{-nt}\sum_{k=0}^{n-1}(nt)^k/k!.$$

Then if $0<t<1$,

$$\begin{aligned}1-E_n(t)&=e^{-nt}\sum_{k=n}^{\infty}(nt)^k/k!\\&=e^{-nt}\left[(nt)^n/n!\right]\left[1+nt/(n+1)+(nt)^2/(n+1)(n+2)+\cdots\right]\\&<e^{-nt}\left[(nt)^n/n!\right]\left[1+t+t^2+\cdots\right]\\&<e^{-nt}\left[(nt)^ne^n/n^n\right]\left[(1-t)^{-1}\right],\quad\text{by (a)},\\&=e^{-nt}[te]^n\left[(1-t)^{-1}\right]\\&=(p_t)^n(1-t)^{-1}.\end{aligned}$$

Therefore by (b),

$$\lim_{n\to\infty}E_n(t)=1\quad\text{if }0<t<1.$$

If $t>1$, then $(nt)^k/k! < (nt)^n/n!$ $(k<n)$, so that $E_n(t) < ne^{-nt}(nt)^n/n! < ne^{-nt}[(nt)^n e^n/n^n]$, by (a), thus $E_n(t) < n(p_t)^n$. Hence by (b), $\lim_{n\to\infty} E_n(t) = 0$.

For $t=1$, see solutions II, III.

Solution II by Michael Barr, McGill University. Take $t=1$. Set $E_n = \Sigma_{k=0}^{n-1} e^{-n} n^k/k!$, $R_n = 1 - E_n = \Sigma_{k=n}^{\infty} e^{-n} n^k/k!$. Note $E_n = \frac{1}{2}(E_n + R_n) + \frac{1}{2}(E_n - R_n)$. It will be proved that $\lim E_n = 1/2$, by establishing that $\lim(E_n - R_n) = 0$. To see this, write $E_n - R_n = A_n[B_n - C_n - D_n]$, where

$$A_n = n^n e^{-n}/n!, \qquad B_n = \sum_{k=0}^{n-1} n!\,n^{k-1}/k!,$$

$$C_n = \sum_{n}^{2n-1} n!\,n^{k-n}/k!, \qquad D_n = \sum_{2n}^{\infty} n!\,n^{k-n}/k!.$$

First

$$\begin{aligned} D_n &= \sum_{j=n}^{\infty} n!\,n^j/(n+j)! \\ &= \sum_{j=n}^{\infty} n^j/\Pi_{i=1}^{j}(n+i) < \sum_{j=n}^{\infty} n^{j+1-n}/\Pi_{i=n}^{j}(n+i) \\ &< \sum_{j=n}^{\infty} (1/2)^{j+1-n} = 1. \end{aligned}$$

Thus $0 < D_n < 1$.

Now set $B(n,0)=1$, $B(n,k) = 1 + (k/n)B(n,k-1)$. It can be checked that $B_n = B(n,n-1)$. Also, if $C(n,0)=1$, $C(n,k) = 1 + [n/(2n-k)]C(n,k-1)$, then $C_n = C(n,n-1)$.

LEMMA 1. *$B(n,k) \leqslant n/(n-k)$ if $k<n$.*

Proof. True if $k=0$. Also, $B(n,k+1) = 1 + [(k+1)/n]B(n,k)$, and by induction hypothesis, $B(n,k+1) \leqslant 1 + [(k+1)/n][n/(n-k)] = (n+1)/(n-k) < n/[n-(k+1)]$.

LEMMA 2. *If $k<n$, then $0 \leqslant C(n,k) - B(n,k) \leqslant 1$.*

True if $k=0$. But

$$\begin{aligned} C(n,k+1) - B(n,k+1) &= [n/(2n-k-1)]C(n,k) - [(k+1)/n]B(n,k) \\ &= [n/(2n-k-1)][C(n,k) - B(n,k)] \\ &\quad + \left[(n-k-1)^2/\{n(2n-k-1)\}\right]B(n,k). \end{aligned}$$

Application of lemma 1 to the second term, and of the induction hypothesis to the first term, proves lemma 2.

In particular (with $k=n-1$ in lemma 2) $0 \leqslant C_n - B_n \leqslant 1$. Thus $B_n - C_n - D_n$ is bounded. By Stirling's formula, $\lim A_n = 0$, so $E_n - R_n$ does approach 0.

Solution III by Ole Jørsboe, Matematisk Institut, Danmarks Tekniske Højskole. The limit is 1 if $0<t<1$, 1/2 if $t=1$, and 0 if $t>1$. This can be seen as follows. Let $X_1, X_2, \ldots$ be a sequence of i.i.d. (identically but independently distributed) random variables, with Poisson distribution having mean and variance t. Then $S_n = \Sigma_1^n X_j$ has Poisson distribution with mean and variance nt. By the central limit theorem, the relation

$$\lim_{n\to\infty} P\left\{(S_n - nt)/\sqrt{nt} \leqslant x\right\} = \Phi(x) \equiv (2\pi)^{-1/2}\int_{-\infty}^{x} \exp(-\tfrac{1}{2}t^2)\,dt$$

holds for every real number x. Since S_n is Poisson distributed, this may be paraphrased:

$$\lim_{n\to\infty} e^{-nt}\Sigma(nt)^k/k! = \Phi(x), \qquad (*)$$

the sum running from $k=0$ to $k=[nt+x\sqrt{nt}\,]$. Here $[\mu]$ is the largest integer not exceeding μ.

First consider the case $t=1$. Put $x=0$ in (*): $\lim_{n\to\infty} e^{-n}\sum_{k=0}^{n} n^k/k! = \Phi(0) = 1/2$. But by Stirling's formula, $\lim_{n\to\infty} e^{-n}n^n/n! = 0$, so the assertion is proved in this case.

Now take t fixed, $0<t<1$. For given $\epsilon>0$, choose x so that $\Phi(x)>1-\epsilon$. For n sufficiently large, it will be true that $n-1>[nt+x\sqrt{nt}\,]$, so that

$$1 \geqslant \lim_{n\to\infty} e^{-nt}\sum_{k=0}^{n-1}(nt)^k/k!$$
$$\geqslant \lim_{n\to\infty} e^{-nt}\sum (nt)^k/k! = \Phi(x) > 1-\epsilon,$$

and thus the limit is 1. (The last sum again runs from $k=0$ to $k=[nt+x\sqrt{nt}\,]$.)

If $t>1$, it is shown, in a similar manner, that the limit is 0. (Choose x with $\Phi(x)<\epsilon$).

Norton refers to his paper "Estimates for Partial Sums of the Exponential Series," *J. Math. Anal. Appl.*, 63 (1978) 265–296, for a proof that $e^{-n}\sum_{k=0}^{n-1} n^k/k! = \frac{1}{2} + O(n^{-\frac{1}{2}})$ and that this estimate is the best possible. Shelupsky remarks that the result is an insensitive version of the Central Limit Theorem resulting from use of the "wrong" upper limits in the sum. Block and Brunson remark that the weak law of large numbers suffices for the case $t\neq 1$. Several readers cited the result as a standard problem in probability theory; others noted that the case $t=1$ is a standard result concerning the incomplete gamma function.

Also solved by Ed Adams, Henry Block, D. Borwein (Canada), Paul Bracken (Canada), Barry Brunson, Mike Chamberlain, Ted Cox, Ed Dixon, Michael Dixon, John Gillespie, Gustaf Gripenberg (Finland), Richard Groeneveld, P. Henrici (Switzerland), Myron Hlynka, Felix Jenni (Switzerland), Ignacy Kotlarski, Joel Levy, Peter Lindstrom, George Markowsky, Kirby McMaster, Karl Norton, Nicholas Phillips, Daniel Rosenblum, Otto Ruehr, St. Olaf Problems Group, Robert Shafer, David Shelupsky, Michael Skalsky, Ronald Thisted, University of South Alabama Problem Group, Jack Wilson, and the proposer.

Bounded Prime Factors for Terms in an Arithmetic Sequence

E 2725 [1978, 593]. *Proposed by Solomon W. Golomb, University of Southern California.*

Given positive integers a and b show that there exists a positive integer c such that infinitely many numbers of the form $an+b$ (n a positive integer) have all their prime factors $\leqslant c$.

Solution by 22 solvers. All numbers in the sequence $\{b(a+1)^k, k=1,2,\ldots\}$ are distinct and of the form $an+b$. Thus $c=\max(b,a+1)$ satisfies the condition of the problem.

Also solved by 51 others, including the proposer.

Editor's Comments. (1) H. L. Abbott and D. Tibbels identified this as problem #68 in Sierpiński's "250 Problems in Elementary Number Theory."

(2) Many solvers discussed minimizing c. Without loss of generality, we can assume $1\leqslant b\leqslant a$. Let $c(a,b)$ denote the minimal value of c for the sequence $\{an+b\}$. For $a>2$, the sequence $\{b(a-1)^{2k}, k=1,2,\ldots\}$ shows $c(a,b)\leqslant a$, while if $a=b=$prime, then $c(a,b)=a$. (For $a\leqslant 2$, we have $c(1,1)=2$, $c(2,1)=3$, $c(2,2)=2$.) If $\gcd(a,b)=d$, with $a=dx$, $b=dy$, and $p_0=$ maximum prime divisor of d, then $c(a,b)=\max(p_0, c(x,y))$. If $\gcd(a,b)=1$, the sequence $\{b^{k\phi(a)+1}, k=1,2,\ldots\}$ ($\phi=$ Euler function) shows $c(a,b)\leqslant$ maximum prime divisor of b.

(3) If a is a prime (say p) which does not divide b, the following method usually gives a much better value for c. Let g be a primitive root mod$p$, so that no two of $g,g^2,g^3,\ldots,g^{p-1}$ are congruent mod$p$. Thus $g^l\equiv b$ (mod$p$) for some $1\leqslant l\leqslant p-1$. The sequence $\{g^{l+(p-1)k}, k=1,2,\ldots\}$ shows $c(a,b)\leqslant$ maximum prime divisor of g. The smallest such g is generally small compared to p. D. H. Lehmer communicates that every prime $<10^{13}$ has a primitve root g less than 137.

D. A. Burgess has proved that for each $\varepsilon>0$ there is a $K=K(\varepsilon)$ such that for all primes p there is a primitive root g with $0<g<K(\varepsilon)p^{1/4+\varepsilon}$. See Proc. London Math. Soc. (3) 12 (1962) 179–192.

Equivalence of Triangles

E 2727* [1978, 594]. *Proposed by David P. Robbins, Hamilton College.*

Two triangles $A_1A_2A_3$ and $B_1B_2B_3$ in $\mathbf{R}^3$ are equivalent if there exist three different parallel

lines p_1, p_2, p_3 and rigid motions σ, τ such that $\sigma(A_i)$ and $\tau(B_i)$ lie on $p_i (i = 1,2,3)$.

Find necessary and sufficient conditions for equivalence of the two triangles.

Solution by Michael Goldberg, Washington, D.C. The problem is interpreted to mean: Find conditions on the triangles so that they are both sections of the same triangular prism. Then one triangle can be turned in three-space and the other translated so that they become the bases of a truncated triangular prism.

Let $a \leqslant b \leqslant c$ be the lengths of the sides of the smaller triangle and $d \leqslant e \leqslant f$ the lengths of the sides of the larger triangle. Let the smaller triangle be the normal section of a triangular prism, while the larger triangle is an inclined section. Then, as seen in Figure 1, it is necessary and sufficient that $\sqrt{f^2 - c^2} = \sqrt{d^2 - a^2} + \sqrt{e^2 - b^2}$.

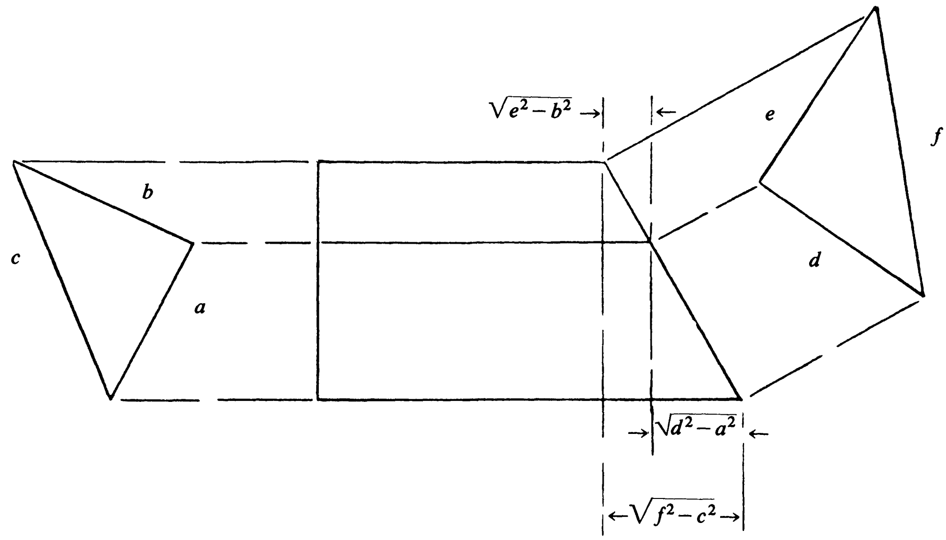

FIG. 1

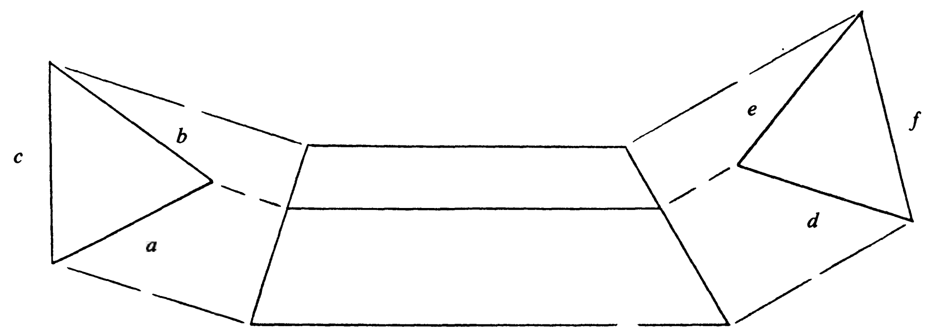

FIG. 2

Both sections can be inclined, as shown in Figure 2, to obtain a similar condition.

Also solved by Jeffrey M. Cohen, Jordi Dou (Spain), Walter Effross, L. Kuipers (Switzerland), and Albert Nijenhuis.

Editor's Note. Jordi Dou observed that this "equivalence" is not an equivalence relation. In fact, two different equilateral triangles are "equivalent" to an isosceles triangle.

Mutually Tangent Cylinders

E 2728 [1978, 594]. *Proposed by J. G. Mauldon, Amherst College*

Let a, b, c, d be radii of four mutually externally tangent circular right cylinders whose axes are parallel to the four principal diagonals of a cube. Characterize all quadruples a, b, c, d which arise in this way.

Solution by Robert Breusch, Amherst College. If the edges of a cube are parallel to the coordinate axes, then the direction ratios of the four diagonals are $\pm 1 : \pm 1 : 1$. If A, B, C, D are the axes of the cylinders with radii a, b, c, d, respectively, assume that

$$\left.\begin{matrix} A \\ B \\ C \\ D \end{matrix}\right\}\text{has direction ratio}\left\{\begin{matrix} 1: & 1:1 \\ 1: & -1:1 \\ -1: & 1:1 \\ -1: & -1:1 \end{matrix}\right\}\text{and passes through}\left\{\begin{matrix} (a_1,a_2,0) \\ (b_1,b_2,0) \\ (c_1,c_2,0) \\ (d_1,d_2,0) \end{matrix}\right. .$$

Two cylinders touch externally iff the shortest distance between their axes is equal to the sum of their radii. It is found that the distances between pairs of axes are

$$\begin{array}{ll} \delta(A,B)=|b_1-a_1|/\sqrt{2} & \delta(A,C)=|a_2-c_2|/\sqrt{2} \\ \delta(A,D)=|d_2-a_2-d_1+a_1|/\sqrt{2} & \delta(B,C)=|c_1-b_1+c_2-b_2|/\sqrt{2} \\ \delta(B,D)=|b_2-d_2|/\sqrt{2} & \delta(C,D)=|d_1-c_1|/\sqrt{2} \end{array}$$

Thus the four cylinders will be mutually externally tangent iff

$$\text{(I)}\quad \left\{\begin{array}{ll} b_1-a_1=\varepsilon_{ab}\sqrt{2}\cdot(a+b) & a_2-c_2=\varepsilon_{ac}\sqrt{2}\cdot(a+c) \\ d_2-a_2-d_1+a_1=\varepsilon_{ad}\sqrt{2}\cdot(a+d) & c_1-b_1+c_2-b_2=\varepsilon_{bc}\sqrt{2}\cdot(b+c) \\ b_2-d_2=\varepsilon_{bd}\sqrt{2}\cdot(b+d) & d_1-c_1=\varepsilon_{cd}\sqrt{2}\cdot(c+d) \end{array}\right.$$

(All the ε are ± 1.)

Editor's Note: At this point the problem has been completely reduced to the analytic geometry problem of determining when four points can be suitably located in the plane; and also to the linear algebra problem of determining when the six equations (I) in eight unknowns can be solved.

Because the sum of the left-hand members of these six equations is zero, it follows that necessarily

$$\text{(II)}\quad S\equiv\varepsilon_{ab}(a+b)+\varepsilon_{ac}(a+c)+\varepsilon_{ad}(a+d)+\varepsilon_{bc}(b+c)+\varepsilon_{bd}(b+d)+\varepsilon_{cd}(c+d)=0.$$

Since a, b, c, d are all positive, (II) is certainly not satisfied if all six ε are positive, nor if just one is negative. If two ε are negative, (II) will again not be satisfied if the subscripts of the two negative ε are disjoint (e.g., if $\varepsilon_{ab}=\varepsilon_{cd}=-1$) because in this case, $S=a+b+c+d>0$.

If $\varepsilon_{ab}=\varepsilon_{ac}=-1$, then $S=-a+b+c+3d$, which *can* be zero.

If three of the ε are negative, we may assume that $\varepsilon_{ab}=\varepsilon_{ac}=-1$. If in addition

$$\begin{array}{ll} \varepsilon_{ad}=-1, & \text{then } S=-3a+b+c+d; \\ \varepsilon_{bc}=-1, & \text{then } S=-a-b-c+3d; \\ \varepsilon_{bd}=-1, & \text{then } S=-a-b+c+d. \end{array}$$

(The case $\varepsilon_{cd}=-1$ is essentially the same as $\varepsilon_{bd}=-1$.)

It follows that, aside from permutations, S can be zero only if the largest of a, b, c, d (call it d) satisfies one of the three conditions

$$\text{(III)} \quad \begin{cases} d=a+b+3c \\ d=a+b-c \\ d=3a-b-c \end{cases}.$$

Conversely, in each of these three cases (making appropriate choices for the ε's), equations (I) provide solutions for the eight unknowns $\{a_i,b_i,c_i,d_i|i=1,2\}$.

Thus we get the final result: The four cylinders can be mutually externally tangent iff the radii a, b, c, d satisfy one of the three conditions in (III).

Editor's Note: To establish the converse rigorously it is necessary to note that the system (I) is of rank 5. When one of the equations in (III) holds, the system is consistent, and has three degrees of freedom, corresponding to the possibility of translating the configuration of cylinders rigidly in space relative to the arbitrarily chosen coordinate system.

Also solved by O. P. Lossers (Netherlands), L. Kuipers (Switzerland), and the proposer.

ADVANCED PROBLEMS

Solutions of Advanced Problems should be sent to Professor Roger C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109. To facilitate their consideration, solutions of Advanced Problems in this issue should be typed (in duplicate, with double spacing) and should be mailed before March 31, 1980.

6279. *Proposed by Lee A. Rubel, University of Illinois, Urbana-Champaign.*

Let $f(z)$ be an entire function such that the maximum modulus over every closed line segment L is achieved at one of the endpoints a and b of L; that is,

$$\max\{|f(z)|:z\in L\}=\max\{|f(a)|,\ |f(b)|\}.$$

Prove that $f(z)$ has either the form $A(z-B)^n$ or the form $A\exp Bz$, where A and B are constants and n is a nonnegative integer.

6280. *Proposed by David Siegel, Stanford University.*

Let u be a harmonic function in a regular n-gon with sides $s_1,\dots,s_n$ and radii $r_1,\dots,r_n$ joining the center to the vertices. Show that

$$\sum_{i=1}^{n}\int_{s_i}u\,ds=2\sin\frac{\pi}{n}\sum_{i=1}^{n}\int_{r_i}u\,ds,$$

where the integrals are taken with respect to arc length.

6281*. *Proposed by Clark Kimberling, University of Evansville.*

If $A=(1,a_1,a_2,\dots)$ is a sequence of 1's and 2's, let $B=(1,b_1,b_2,\dots)$ where b_n is the length of the nth maximal string of identical symbols in A. If $B=A$, then A must be $(1,2,2,1,1,2,1,2,2,1,\dots)$. By a *run* is meant a finite subsequence of consecutive terms of A. Its *complement* is obtained by interchanging all 1's and 2's.

Prove or disprove:

(1) The complement of every run is also a run;

(2) Every run occurs infinitely many times.

(This remarkable sequence appears in Problem 5304 [1965, 674, and 1966, 681]. The definition above of A comes from N. J. A. Sloane's book, *A Handbook of Integer Sequences*, p. 10, in which it is used as an example of a "self-generating" sequence. Both results would follow if one could show that, for every initial run $1,a_1,a_2,\dots,a_n$, both this run and its complement occur later in A. The initial run of 42 symbols and its complement both occur later within the first 550 terms of A.)

SOLUTIONS OF ADVANCED PROBLEMS

Limits Identified

6196* [1978, 122]. *Proposed by Daniel Shanks, National Bureau of Standards.*

(A) Let $-5<x_0<0$ and let

$$x_n=\begin{cases}\sqrt{x_{n-1}+5} & \text{if} \quad n\not\equiv 0 \pmod 4\\ -\sqrt{x_{n-1}+5} & \text{if} \quad n\equiv 0 \pmod 4).\end{cases}$$

Identify the numbers toward which x_{4m}, x_{4m+1}, x_{4m+2}, and x_{4m+3} converges as $m\to\infty$.

(B) Let p be a prime for which $(5|p)=+1$, so that $\sqrt{5}$ exists, modulo p. Then

$$(15\pm 6\sqrt{5}\,|p)=+1, \quad \text{or} \quad -1,$$

according as $p\equiv\pm 1 \pmod{15}$, or $p\equiv\pm 4 \pmod{15}$, respectively.

(C) What is the relationship between problems A and B?

Solution by R. W. K. Odoni, University of Exeter, U.K. (A) We write $\phi(x)=\sqrt{(x+5)}$ for $x>-5$, so that $x_{4n}=-\phi(x_{4n-1})$ and $x_{n+1}=\phi(x_n)$ otherwise. Since $-5<x_0<0$, it is clear that $x_{4n}<0$ for all n. Let us write $F(x)$ for $-\phi(\phi(\phi(\phi(x))))$ when $x>-5$. A simple calculation shows that f maps $[-5.0]$ into $[-2.85,-2.75]=I$, and that $F(I)\subseteq I$. When $x,y<0$, it is clear that $|\phi(x)-\phi(y)|<(1/4)|x-y|$, from which we deduce that F is a contraction mapping on the compact set I, and thus has a unique fixed point there (call this ξ_0), to which x_{4n} must tend. Since ϕ is continuous, $x_{4n+1}\to\phi(\xi_0)=\xi_1$, $x_{4n+2}\to\phi(\xi_1)=\xi_2$ and $x_{4n+3}\to\phi(\xi_2)=\xi_3$. From the relations $\xi_3^2-5=\xi_2$, $\xi_2^2-5=\xi_1$, $\xi_1^2-5=\xi_0$ and $\xi_0^2-5=\xi_3$, we see that the ξ_i are algebraic integers. Hoping against hope we make the lucky guess that they are a complete set of algebraic conjugates over the rational field $\mathbb{Q}$. Using a hand calculator we find $\xi_0=-2.7833861$, $\xi_1=1.4888297$, $\xi_2=2.5473181$ and $\xi_3=2.7472381$. Their elementary symmetric functions are very close to integers, and we are led to conjecture that the ξ_i are the roots of

$$x^4-4x^3-4x^2+31x-29=0. \tag{1}$$

Direct calculation shows that $x\mapsto x^2-5$ does indeed permute the roots of (1). The substitution $x=1+y$ reduces (1) to $y^4-10y^2+15y-5=0$, which is irreducible over $\mathbb{Q}$ (by Eisenstein's criterion, with $p=5$). It now seems a fair bet that the Galois group of (1) is the cyclic 4-group generated by $\xi_1\mapsto\xi_0\mapsto\xi_3\mapsto\xi_2\mapsto\xi_1$. If this were so then $\xi_0+\xi_2$ and $\xi_1+\xi_3$ would be quadratic surds. The above numerical values for the ξ_i suggest that $\xi_0+\xi_2$ is very nearly $2-\sqrt{5}$ and $\xi_1+\xi_3$ very nearly $2+\sqrt{5}$. This would yield quadratic equations for the ξ_i, from which we should have

$$2\xi_0=2-\sqrt{5}-\sqrt{15+6\sqrt{5}}\,, \qquad 2\xi_1=2+\sqrt{5}-\sqrt{15-6\sqrt{5}}\,,$$
$$2\xi_2=2-\sqrt{5}+\sqrt{15+6\sqrt{5}}\,, \qquad 2\xi_3=2+\sqrt{5}+\sqrt{15-6\sqrt{5}}\,.$$

Calculation of these surds does indeed give close agreement with the earlier numerical values. Finally, it is easily seen that $\frac{1}{2}(2-\sqrt{5}-\sqrt{(15+6\sqrt{5})})$ lies in I, and is a fixed point of F, thus proving all our conjectures.

(B) The congruences $x^2\equiv 5$, $y^2\equiv 15+6x$ are equivalent to $y^4-30y^2+45\equiv 0 \pmod p$. Since we are asked to relate this to $p \pmod{15}$, it is natural to consider the cyclotomic field $\mathbb{Q}(\mu)$, where $\mu=\exp(2\pi i/15)$. We shall show that the quartic field $\mathbb{R}\cap\mathbb{Q}(\mu)=\mathbb{Q}(\cos 2\pi/15)$ is identical with $\mathbb{Q}(\sqrt{(15+6\sqrt{5})})=\mathbb{Q}(\xi_0)$; the behavior of the congruence $y^4-30y^2+45\equiv 0 \pmod p$ is then governed by the prime ideal factorization of p in $\mathbb{Q}(\cos 2\pi/15)$, by a classical result of Kummer. Let $\beta=(\sqrt{5}-1)/2$, so that $\beta/2=\cos 2\pi/5=\cos 3\phi$, where $\phi=2\pi/15$. Writing $\cos 3\phi$ as a cubic in $\cos\phi$, we find that $4\cos^2\phi-2(1+\beta)\cos\phi+\beta-1=0$ so that $\mathbb{Q}(\cos\phi)=\mathbb{Q}(\sqrt{5})(\sqrt{(6-3\beta)})$.

But $(6-3\beta)(15+6\sqrt{5})=9(3+\beta)^2$ is a perfect square in $\mathbb{Q}(\sqrt{5})$, so $\mathbb{Q}(\cos 2\pi/15)=\mathbb{Q}(\sqrt{(15+6\sqrt{5})})=\mathbb{Q}(\xi_0)$, as asserted.

Now assume that p is unramified in $\mathbb{Q}(\cos 2\pi/15)$ and does not divide the conductor of the order $\mathbb{Z}[\sqrt{15+6\sqrt{5}}]$. Then, for $p>3$, the congruence $y^4-30y^2+45\equiv 0 \pmod p$ is soluble if and only if p splits completely in $\mathbb{Q}(\cos 2\pi/15)$, which holds if and only if the Artin symbol $\left[\dfrac{\mathbb{Q}(\mu)/\mathbb{Q}}{p}\right]$ lies in the subgroup of $\mathrm{Gal}\,\mathbb{Q}(\mu)/\mathbb{Q}$ generated by complex conjugation, and this is equivalent to $p\equiv\pm 1 \pmod{15}$.

(C) The solution $y \pmod p$ in (B) can be refined to a root of $y^4-30y^2+45=0$ in the field $\tilde{Q}_p$ of p-adic numbers, provided $p\equiv\pm 1 \pmod{15}$ and $p>p_0$. Writing $2\tilde{\xi}_0=2-y-(y^2-15)/6$, $\tilde{\xi}_3=\tilde{\xi}_0^2-5$, $\tilde{\xi}_2=\tilde{\xi}_3^2-5$, and $\tilde{\xi}_1=\tilde{\xi}_2^2-5$, we have $\tilde{\xi}_0=\tilde{\xi}_1^2-5$, and the algorithm of (A) (interpreted p-adically) has $x_{4n+1}\to\tilde{\xi}_i$ if x_0 is p-adically close enough to $\tilde{\xi}_0$.

Also solved by B. Ferrero (parts A and B), Emil Grosswald (A and B), and L. van Hamme (Belgium) (part A). Grosswald considers all 16 variants of the problem obtained from part A by modifying the pattern of pluses and minuses in the recursion formula, and observes that the corresponding limits are the roots of the equation $x=(((x^2-5)^2-5)^2-5)^2-5$; van Hamme solves the problem obtained from part A by replacing 4 by 3.

Isoptic Curves

6223* [1978, 600]. *Proposed by Harry D. Ruderman, Hunter College Campus School.*

Let C be a convex curve. Let Q be a curve such that the two tangents to C from each point P of Q form an angle θ fixed in size. Assume that all points are in the same plane.

(1) If $\theta=90°$ and Q is a circle, must C be a circle or an ellipse?

(2) If C is an ellipse and $\theta\neq 90°$ what is the nature of Q?

Solution by J. H. E. Cohn, Royal Holloway College, London. (1) The answer is no. Choose as origin the center of the circle. It is easily seen that C must be centrally symmetric in O. Draw a line from O in direction θ and let it meet the perpendicular tangent at a distance $f(\theta)>0$ from O. Then the required conditions on $f(\theta)$ are (i) $f(\theta)+f''(\theta)>0$, for convexity, and (ii) $f^2(\theta)+f^2(\theta+\frac{1}{2}\pi)=R^2$, where R is the radius of the circle. Now let $f^2(\theta)=\lambda+F(\theta)$ where $F(\theta)+F(\theta+\frac{1}{2}\pi)$ is constant. This will satisfy condition (ii) and also condition (i) provided λ is sufficiently large. Examples are $F(\theta)\equiv 0$ for the circle; $F(\theta)=\cos 2\theta$ for the ellipse and $F(\theta)=\cos 6\theta$ with $\lambda>17$ for a curve which is neither an ellipse nor a circle, there being just six points on the curve at maximum distance from the center.

(2) Let C be $x^2a^{-2}+y^2b^{-2}=1$. Then the two tangents through a point (ξ,η) have equation

$$(x^2a^{-2}+y^2b^{-2}-1)(\xi^2a^{-2}+\eta^2b^{-2}-1)=(x\xi a^{-2}+y\eta b^{-2}-1)^2$$

or

$$x^2a^{-2}(\eta^2b^{-2}-1)-2xya^{-2}b^{-2}\xi\eta+y^2b^{-2}(\xi^2z^{-2}-1)+\cdots=0.$$

The condition for these tangents to meet at an angle of θ or $\pi-\theta$ is

$$\{a^{-2}(\eta^2b^{-2}-1)+b^{-2}(\xi^2a^{-2}-1)\}^2\sin^2\theta=4\cos^2\theta\{a^{-4}b^{-4}\xi^2\eta^2-a^{-2}b^{-2}(\eta^2b^{-2}-1)(\xi^2a^{-2}-1)$$

and so the equation of Q reduces to

$$(x^2+y^2-a^2-b^2)^2\sin^2\theta=4\cos^2\theta(a^2y^2+b^2x^2-a^2b^2).$$

If say $0<\theta<\pi/2$ then this quartic curve splits into two closed curves,

$$(x^2+y^2-a^2-b^2)=\pm 2\cot\theta(a^2y^2+b^2x^2-a^2b^2)^{\frac{1}{2}}.$$

The one with the upper sign lies outside the director circle and, viewed from any point on this, the ellipse has an angular width of θ; the one with lower sign lies betweeen the ellipse and the director circle and yields an angular width of $\pi-\theta$ for the ellipse.

Also solved by Michael Goldberg, O. P. Lossers (Netherlands), and N. Miku (Netherlands). Goldberg notes that more detailed discussion of isoptic curves is given in the following references.

Michael Goldberg, Rotors in polygons and polyhedra, Math. Comp., 14 (1960) 236–237.

______, Double-contact cam mechanisms, U. S. Patent no. 2,741,132, April 16, 1956.

J. W. Green, Sets subtending a constant angle on a circle, Duke Math. J., 17 (1950) 263–267.

W. Wunderlich, Kurven mit isoptischen Kreis, Aequationes Math., 6 (1971) 71–81.

A Functional Equation

6226 [1978, 600]. *Proposed by Marlow Sholander, Case Western Reserve University.*

Domain D consists of the real numbers R from which a finite set is deleted. On domain D, the functions f, F, and G are continuous and satisfy the identity

$$f(r)-f(s)=(r-s)F(r)G(s).$$

Describe $f(x)$ on domain R.

Composite solution. Nothing can be said about f outside D. We show that, on D, f is either constant or a linear fractional transformation of the form $f(x)=(ax+b)/(cx+d)$, for some a, b, c, d such that $ad-bc\neq 0$. Assume that f is not constant and choose r and s in D such that $f(r)\neq f(s)$. Then from

$$f(r)-f(s)=[f(x)-f(s)]-[f(x)-f(r)]=[(x-s)G(s)-(x-r)G(r)]F(x)$$

we conclude that, for all x in D, $F(x)=1/(cx+d)$ for some c and d. Now the equation $f(x)-f(r)=(x-r)F(x)G(r)$ shows that f is of the required form.

Solutions were given by K. F. Andersen, John Bryant & Robert Gilmer, L. E. Clarke (England), Jeffrey M. Cohen, Roger Cooke, Evin Joyce Cramer, G. Crofts, Gustaf Gripenberg (Finland), O. Hajek, Eli L. Isaacson, Ignacy Icchak Kotlarski, Peter W. Lindstrom, J. G. Mauldon, N. Miku (Netherlands), Albert Nijenhuis, Ronaldo Folegatti Poubel (Brazil), I. J. Schoenberg, F. B. Strauss, University of Southern Alabama Problem Group, Gregory P. Wene, J. B. Wilker, David Witte, and the proposer. Several solvers noted that the proof does not require continuity. Kotlarski established the conclusion for the complex case, and Mauldon noted that the result holds for any subset D of any field.

MISCELLANEA

30. On noticing a man with delirium tremens: "... 'dt,' God help this old tattooed man, meant also a time differential, a vanishingly small instant in which change had to be confronted at last for what it was, where it could no longer disguise itself as something innocuous like an average rate, where velocity dwelled in the projectile though the projectile be frozen in midflight, where death dwelled in the cell though the cell be looked in on at its most quick."

—From the book *The Crying of Lot 49,* p. 129, by Thomas Pynchon. Copyright © 1966, 1965 by Thomas Pynchon. Reprinted by permission of J. B. Lippincott Company. (Suggested by Howard E. Reinhardt.)

TELEGRAPHIC REVIEWS

Edited by J. Arthur Seebach, Jr. and Lynn A. Steen
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Asteriks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, T(13: 1), *Mathematics, An Informal Approach*. Albert B. Bennet, Jr., Leonard T. Nelson. Allyn, 1979, xviii + 556 pp, $15.95. [ISBN: 0-205-06519-8]; *Mathematics, An Activity Approach*. 1979, ix + 227 pp, $9.95 (P). [ISBN: 0-205-06518-X] For liberal arts students or prospective elementary teachers; numerous photographs, comic strips, quotations, puzzles, etc., enliven this inductive approach to number systems, geometry, probability and statistics. Uses the metric system and includes sections on the use of calculators. The supplementary workbook contains a sequence of inductive activities and experiments for each section of the text. JNC

GENERAL, T(14-16: 1, 2), S, *Fundamental Concepts of Mathematics, Second Edition*. R.L. Goodstein. Pure and Appl. Math., V. 107. Pergamon Pr, 1979, ix + 323 pp, $16.50 (P). [ISBN: 0-08-021666-8] A survey for teachers of mathematics and "cultivated amateurs" of number theory, algebra, calculus (called, more figuratively, "numbers for counting," "numbers for sharing" and "numbers unending"), set theory and graph theory. Changes from the first (1962) edition include updates on recently solved problems, and--more important--the introduction of numerous sets of exercises with detailed solutions. Although interestingly written, the book has dulled with age and overexposure: its topics are now all routine parts of the regular teacher-preparatory curriculum. LAS

BASIC, T(13: 1, 2), *Modern Elementary Mathematics, Third Edition*. Malcolm Graham. HarBrace J, 1979, x + 470 pp, $14.50. [ISBN: 0-15-561041-4] Truth tables, sets, relations, number systems, measurement, a little statistics, and analytic geometry. (*First Edition*, TR, January 1971; ER, January 1972; *Second Edition*, TR, October 1975.) FLW

PRECALCULUS, T(13: 1), L, *Precalculus, Second Edition*. S.L. Salas, C.G. Salas. Wiley, 1979, xii + 356 pp, $14.95. [ISBN: 0-471-03124-0] Major changes from the first edition (TR, June/July 1975) include the addition of material on the exponential functions, inverse trigonometric functions, Cramer's rule, and additional exercises at the end of each chapter. JS

PRECALCULUS, T?(13: 1), *Mathematics for Electronics*. Donald P. Leach. Reston Pub, 1979, xv + 431 pp, $15.95. [ISBN: 0-8359-4277-5] A "how-to-do-it" approach to algebra and trigonometry which focuses on rules and procedures instead of understanding...a step in the wrong direction. The language is often imprecise and inaccurate; the exercises are mostly mindless. LCL

PRECALCULUS, T(13: 1), *College Algebra, Second Edition*. Mustafa A. Munem, William Tschirhart, James P. Yizze. Worth, 1979, xi + 416 pp, $14.95. [ISBN: 0-87901-098-3]; *Study Guide*, ix + 320 pp, $4.95 (P). [ISBN: 0-87901-099-1] A less formal approach than the first edition (TR, December 1974). Optional material now in Chapters 7 and 8 rather than being included throughout the text. Good problems to emphasize the uses of algebra. LLK

EDUCATION, P, *Mathematics Teacher Education: Critical Issues and Trends*. Ed: Douglas B. Aichele. NEA, 1978, 64 pp, $4.50 (P); $8.35. An overview by the editor of several issues facing mathematics educators with a summary of NACOME recommendations for teacher education plus five different and diverse chapters on topics such as kindergarten mathematics, teacher education, math anxiety, learning theory, recommendations for preparation of general mathematics teachers, in-service for K-12, and urban teacher preparation. Bibliography. PJ

HISTORY, L, *Collected Papers of G.H. Hardy, Including Joint Papers with J.E. Littlewood and Others, Volume VII*. Clarendon Pr, 1979, xviii + 897 pp, $78. [ISBN: 0-19-853347-0] Final volume in the *Collected Papers*, including papers on Fourier transforms, integral equations, miscellany (e.g., the Hardy-Weinberg law), problems (with solutions), reviews, and obituary notices (by, not about, Hardy). LAS

FOUNDATIONS, T, *Méthodes et Concepts de la Logique Formelle*. Yvon Gauthier. Pr U Montreal, 1978, 238 pp, (P). [ISBN: 0-8405-0395-4] Introduction to formal logic intended primarily for students of philosophy. Expanded lecture notes published as a pioneering venture to enable French speaking students to take up logic without reliance on English texts. Quantificational semantics given via consistency trees (or tableau, à la Beth, Smullyan). Touches on decidability and incompleteness. Takes a brief look at intuitionist, modal, multivalued and higher order logics, and concludes with some remarks on the traditional schools of philosophy of logic. GHM

FOUNDATIONS, P, L*, *Selected Papers in Logic and Foundations, Didactics, Economics*. Karl Menger. Vienna Circle Collection, V. 10. Riedel, 1979, xii + 341 pp, $49.50; $24.95 (P). [ISBN: 90-277-0320-5; 90-277-0321-3] 26 papers, written over a span of 50 years, including several innovative discussions of

the notion of a variable, several on the foundations and teaching of geometry, and several on the theory of applications of mathematics to science. All papers are in or translated into English. An imaginative and still relevant collection of seminal ideas, the remarkable legacy of one member of the Vienna Circle. LAS

FOUNDATIONS, T(16-17: 2), L. *Introduction to Mathematical Logic: Set Theory, Computable Functions, Model Theory*. Jerome Malitz. Springer-Verlag, 1979, xii + 198 pp, $14. [ISBN: 0-387-90346-1; 3-540-90346-1] Survey of basic topics on the semantical side of mathematical logic, omitting the classical syntactic questions of proofs, completeness, etc. Suitable for a general mathematical audience, with a good variety of exercises. Concludes each chapter with a brief indication of current research trends. Marred by excessive specialized notation. GHM

FOUNDATIONS, P. *Absoluteness of Intuitionistic Logic*. D.M.R. Leivant. Math. Centre Tracts, No. 73. Math Centrum, 1979, ix + 137 pp, Dfl. 17 (P). [ISBN: 90-6196-122-X] Author's doctoral dissertation in proof theory for intuitionistic logic. Deals with technical questions of the following sort: an (intuitionistic) logic L is "absolute" for an (arithmetic) theory T iff for every schema S of L, if every arithmetical instance of S in the language of T is a theorem of T, then S is a theorem of L. GHM

COMBINATORICS, T(16-18), S, P*, L*. *Selected Topics in Graph Theory*. Ed: Lowell W. Beineke, Robin J. Wilson. Acad Pr, 1978, xii + 451 pp, $44.50. [ISBN: 0-12-086250-6] Fourteen expository articles serve as both an introduction to "pure" graph theory (e.g., four color theorem, Hamiltonian graphs, tournaments, Ramsey graph theory, graph enumeration) and a reference to latest results. Uniform terminology, style and notation; suitable for advanced undergraduate seminars. LCL

NUMBER THEORY, T(18: 1), S, P. *Elliptic Curves,Diophantine Analysis*. Serge Lang. Grund. math. Wissenschaften, B. 231. Springer-Verlag, 1978, xi + 261 pp, $37.40. [ISBN: 0-387-08489-4; 3-540-08489-4] This book splits into two parts. The first deals with the ordinary arithmetic of the elliptic curve: transcendental parametrization, p-adic parametrization, points of finite order, and reductions of certain diophantine problems to diophantine inequalities involving logarithms. The second part deals with proofs of inequalities which are strong enough to obtain the finiteness of integral points. A good list of references. CEC

ALGEBRA, P. *Global Subdirect Products*. Peter H. Krauss, David M. Clark. Memoirs No. 210. AMS, 1979, 109 pp, $6.80 (P). [ISBN: 0-8218-2210-1] A sheaf theoretic treatment of subdirect products. The framework is that of universal algebra. JAS

ALGEBRA, P. *Studies in Algebra and Number Theory*. Ed: Gian-Carlo Rota. Acad Pr, 1979, xi + 369 pp, $38. [ISBN: 0-12-599153-3] A supplementary volume of the journal *Advances in Mathematics*. Included are eight research papers from a fairly wide range of topics in algebra and number theory. CEC

FINITE MATHEMATICS, T(13: 1, 2). *Fundamentals of Mathematics for Business, Social, and Life Sciences*. William J. Adams. P-H, 1979, xvii + 697 pp, $18.95. [ISBN: 0-13-341073-0] Linear mathematics (especially linear programming), probability, algebra, and calculus (including some multivariable, but not trigonometric). Thorough, comprehensive coverage with clear definitions, careful consideration of concepts (e.g., properties of mathematical constructs), plus applications. LCL

FINITE MATHEMATICS, T(13), S, L. *Applied Mathematics for Business, Economics, and the Social Sciences*. Frank S. Budnick. McGraw, 1979, xxii + 649 pp, $16.95. [ISBN: 0-07-008851-9] Comprehensive overview of algebraic principles and limited calculus techniques that are utilized extensively in undergraduate business and economics. Terse emphasis on techniques. Excellent examples with extensive cross-referencing. Good reference or text. WC

FINITE MATHEMATICS, T*(13: 1, 2). *Mathematics with Applications*. Laurence D. Hoffmann, Michael Orkin. McGraw, 1979, x + 438 pp, $15.95. [ISBN: 0-07-029301-5] Upfront applications of linear mathematics, probability and statistics, and calculus, to situations of practical importance especially for students in the social, management, or life sciences. An usually rich collection of meaningful examples and exercises. Requires only high school algebra prerequisite. LCL

DIFFERENTIAL EQUATIONS, P. *Topological Degree Methods in Nonlinear Boundary Value Problems*. J. Mawhin. CBMS Reg. Conf. in Math., No. 40. AMS, 1979, v + 122 pp, $10 (P). [ISBN: 0-8218-1690-X] Exposition of the use of topological degree in treating existence and bifurcation results. This volume originated in the conference held at the Claremont University Center, Claremont, California in June 1977. JAS

DIFFERENTIAL EQUATIONS, T(15-16: 1), L. *Boundary Value Problems, Second Edition*. David L. Powers. Acad Pr, 1979, xi + 351 pp, $15.95. [ISBN: 0-12-563760-8] Remains a very attractive text for first course in boundary value problems. Additions to and changes from first edition (TR, January 1973; ER, March 1974) include Chapter 0 on ordinary differential equations, more examples on heat equation, numerical method for wave equation, 200 new exercises, updated bibliography, greatly expanded answer section and minor rearrangement of some sections. JK

DIFFERENTIAL EQUATIONS, P. *Lecture Notes in Mathematics-679: Numerical Treatment of Differential Equations in Applications*. Ed: R. Ansorge, W. Törnig. Springer-Verlag, 1978, ix + 163 pp, $9.80 (P). [ISBN: 0-387-08940-3; 3-540-08940-3] 14 papers from a conference held at Oberwolfach in December, 1977. Applications include elasticity, fluid flow, Maxwell's time-dependent equations, sound rays, evolution, fields and singular perturbations. Some of the methods employed are Galerkin,Tan, SOR, and Newton-like as well as discretization. RWN

NUMERICAL ANALYSIS, P. *Numerische Behandlung von Eigenwertaufgaben, Band 2*. J. Albrecht, L. Collatz. Int. Ser. Num. Math., V. 43. Birkhäuser, 1979, 203 pp, sFr. 42 (P). [ISBN: 3-7643-1067-7] Papers from the meeting held at the Technische Universität Clausthal from May 18-20, 1978. JAS

NUMERICAL ANALYSIS, P, *Lecture Notes in Mathematics-701: Functional Analysis Methods in Numerical Analysis.* Ed: M. Zuhair Nashed. Springer-Verlag, 1979, vii + 333 pp, $16 (P). [ISBN: 0-387-09110-6; 3-540-09110-6] Proceedings of a special session on applications of functional analysis to numerical analysis and approximation theory, which was held at the 1977 A.M.S. meeting in St. Louis.TRS

NUMERICAL ANALYSIS, S(18), P, *Praktische Mathematik II: Methoden der Analysis.* H. Werner, R. Schaback. Springer-Verlag, 1979, viii + 388 pp, $20.90 (P). [ISBN: 0-387-09193-9; 3-540-09193-9] A major rewriting in response to the rapid developments in interpolation theory, approximation theory, splines and numerical differential equation problems. JAS

NUMERICAL ANALYSIS, P, *Numerische Methoden der Approximationstheorie, Band 4.* L. Collatz, G. Meinardus, H. Werner. Int. Ser. Num. Math., V. 42. Birkhäuser, 1978, 344 pp, sFr. 54 (P). [ISBN: 3-7643-1025-1] Proceedings of the fourth conference on this topic at Oberwolfach which took place November 13-19, 1977. JAS

NUMERICAL ANALYSIS, T(17-18: 1, 2), S, *Theoretical Numerical Analysis, An Introduction to Advanced Techniques.* Peter Linz. Wiley, 1979, xi + 228 pp, $17.95. [ISBN: 0-471-04561-6] Includes fundamentals from functional analysis and approximation theory. Theoretical concepts of numerical analysis for integration, linear operator equations of the first and second kinds, nonlinear inverse problems, the finite element method and improperly posed problems. RWN

FUNCTIONAL ANALYSIS, P, *Singular Perturbation Techniques Applied to Integro-Differential Equations.* H. Grabmüller. Fearon-Pitman, 1978, 148 pp, $12.50 (P). [ISBN: 0-273-08409-7] Develops the analytic apparatus for the application of matched asymptotic expansions to boundary layer problems and extends the Wiener-Hopf techniques to integral equations with operator-valued kernels. TRS

FUNCTIONAL ANALYSIS, P, *Approximation Theory and Functional Analysis.* Ed: João B. Prolla. Math. Stud., V. 35. North-Holland, 1979, viii + 449 pp, $44 (P). [ISBN: 0-444-85264-6] This book contains the Proceedings of the International Symposium on Approximation Theory held at the Universidade Estadual de Campinas (UNICAMP), Brazil, during August 1-5, 1977. TRS

FUNCTIONAL ANALYSIS, T(18: 2), P, *Lectures on von Neumann Algebras.* Serban Strătilă, László Zsidó. Editura Academiei (Romania) (US Distr: ISBS, P.O. Box 555, Forest Grove, OR 97116), 1979, 478 pp, $59. [ISBN: 0-85626-109-2] A self-contained exposition of the main results of the global theory of von Neumann algebras, including the Tomita-Takesaki theory of standard forms and the Connes cocycle theorem, as well as the essential elements of the spectral theory of operators on Hilbert space. Presumes only basic functional analysis. Exercises, comments, bibliographical notes, and 120 pages of references. TRS

FUNCTIONAL ANALYSIS, P, *Symmetric Structures in Banach Spaces.* W.B. Johnson, *et al.* Memoirs No. 217. AMS, 1979, v + 298 pp, $10 (P). [ISBN: 0-8218-2217-9] Detailed investigation of spaces with a symmetric basis of finite length and of rearrangement invariant function spaces. The emphasis is on questions arising naturally out of the theory of L_p-spaces. TRS

FUNCTIONAL ANALYSIS, P, *Closed Graph Theorems and Webbed Spaces.* M. De Wilde. Research Notes in Math., No. 19. Fearon-Pitman, 1978, 158 pp, $13.50 (P). [ISBN: 0-273-08403-8] A self-contained treatment of modern extensions of the closed graph theorem which focuses on the role of webbed spaces, which were introduced by the author. JAS

FUNCTIONAL ANALYSIS, P, *Lecture Notes in Mathematics-692: Barrelledness in Topological and Ordered Vector Spaces.* T. Husain, S.M. Khaleelulla. Springer-Verlag, 1978, ix + 258 pp, $14.30 (P). [ISBN: 0-387-09096-7; 3-540-09096-7] An up-to-date account of certain classes of topological vector spaces which include the class of barrelled spaces and for which certain classical Banach space theorems (e.g., open mapping, Banach-Steinhaus) remain valid. TRS

FUNCTIONAL ANALYSIS, P, *A Theory of Differentiation in Locally Convex Spaces.* S. Yamamuro. Memoirs No. 212. AMS, 1979, v + 82 pp, $6.40 (P). [ISBN: 0-8218-2212-8] The theory rests on considering two different sets of continuous linear maps, giving two different derivatives. After proving the inverse mapping theorem, there are chapters on differential equations, analytic maps and Fredholm maps. TLS

OPTIMIZATION, S(16-17), P, L, *La théorie des sous-gradients et ses applications à l'optimisation.* R.Tyrrell Rockafellar. Pr U Montreal, 1979, 168 pp, $11 (P). [ISBN: 2-7606-0436-5] French translation of lecture series delivered at University of Montreal, February-March 1978. Surveys the modern theory of subdifferentials and subgradients for nondifferentiable functions on Euclidean space, with applications to the theory of optimization, variational inequalities and monotone operators. GHM

ANALYSIS, P, *Lecture Notes in Mathematics-698: Bessel Polynomials.* Emil Grosswald. Springer-Verlag, 1978, xiv + 182 pp, $9.80 (P). [ISBN: 0-387-09104-1; 3-540-09104-1] An accessible, systematic, and largely self-contained account of the most important theorems on Bessel polynomials. Includes a brief history, some open problems, and a large bibliography. TRS

ANALYSIS, P, *Orthogonal Polynomials.* Paul G. Nevai. Memoirs No. 213. AMS, 1979, v + 185 pp, $8.40 (P). [ISBN: 0-8218-2213-6] A study of Christoffel functions and their generalizations with applications to orthogonal polynomials, Fourier series, eigenvalues of Toeplitz matrices and Lagrange interpolation. Includes a positive solution to Turán's problem. TRS

ANALYSIS, P, *Lecture Notes in Mathematics-699: Function Theory on Manifolds Which Possess a Pole.* R.E. Greene, H. Wu. Springer-Verlag, 1979, 213 pp, $12.50 (P). [ISBN: 0-387-09108-4; 3-540-09108-4] This paper provides the beginning of a function theory on complete simply-connected Riemannian manifolds of non-positive sectional curvature (Cartan-Hadamard manifolds). JAS

ANALYSIS, P. *Grundstrukturen der Analysis II.* Werner Gähler. Birkhäuser, 1978, viii + 623 pp, sFr. 92. [ISBN: 3-7643-0966-0] This volume builds from the fundamental concepts of set theory, filters and limits in the first volume (TR, February 1978) through algebraic structures with limits (e.g., pseudo-topological groups) and mapping spaces to a discussion of differentiation. The presentation includes categorical concepts of great generality. Extensive bibliography and index. JAS

GEOMETRY, T*(16-17), S, P, L. *Join Geometries, A Theory of Convex Sets and Linear Geometry.* Walter Prenowitz, James Jantosciak. Springer-Verlag, 1979, xxi + 534 pp, $19.80. [ISBN: 0-387-90340-2; 3-540-90340-2] A splendid recasting of classical geometry in terms of the join and extension operations introduced by Prenowitz in his Slaught Paper. Of special interest are the dimension-free development of linear spaces and much on convexity. Many excellent examples, illustrations, and exercises. SS

TOPOLOGY, T(17-18: 1), S. *An Introduction to Homotopy Theory via Groupoids and Universal Constructions.* Philip R. Heath. Pure and Appl. Math., No. 49. Queen's U, 1978, 118 pp, (P). Lecture notes for a one-semester course covering extension, lifting, and classification problems. Numerous exercises. Many examples and figures. No index. TLS

TOPOLOGY, P. *Geometric Topology.* Ed: James C. Cantrell. Acad Pr, 1979, xiii + 698 pp, $36. [ISBN: 0-12-58860-2] Proceedings of the Georgia Topology Conference held at the University of Georgia, from August 1-12, 1977. JAS

PROBABILITY, T(17-18), P. *Probabilităţi şi Procese Stocastice, Volume I.* George Ciucu, Constantin Tudor. Editura Academiei (Romania), 1978, 336 pp. An expository but advanced (measure theoretic) treatise on probability theory. The lack of an index would militate against its use as a reference even in Romania. JAS

STATISTICS, T(13: 1). *Basic Applied Statistics.* B.L. Raktoe, J.J. Hubert. Statistics, V. 27. Dekker, 1979, x + 426 pp, $14.50 (P). [ISBN: 0-8247-6537-0] Classical text with a strong emphasis on developing underlying concepts. Contains essentially no nonparametric material, but does include a chapter on sample surveys in addition to the usual topics (including regression and experimental design). Makes use of a large number of real world examples and exercises. RSK

STATISTICS, P*. *Developments in Statistics, Volume 2.* Ed: Paruchuri R. Krishnaiah. Acad Pr, 1979, xiii + 322 pp, $33.50. [ISBN: 0-12-426602-9] Second in a series containing papers which are too long for journals, but too short to be separate monographs (*Volume 1*, TR, March 1979). Contains six invited papers which give "authoritative reviews of the present state of the art, as well as some new material, on some aspects of the areas of Markovian fields, random vibrations, sequential analysis, simultaneous tests, time series, and turbulence." RSK

STATISTICS, P. *Lattice Path Combinatorics with Statistical Applications.* T.V. Narayana. Math. Expos., No. 23. U of Toronto Pr, 1979, xii + 106 pp, $15. [ISBN: 0-8020-5405-6] Primarily concerned with "problems which can be conveniently treated in terms of 'dominance'" (e.g., Smirnov test-statistics). Also includes material on the combinatorics of knock-out tournaments and a miscellany of further research problems. RSK

STATISTICS, T(13: 1, 2). *Elementary Business Statistics.* Donald R. Byrkit. D. Van Nostrand, 1979, vii + 524 pp, $15.95. [ISBN: 0-442-21408-1] Covers all standard introductory topics plus selection of optional supplementary material. Mathematical exposition is minimal, with emphasis on reasonable demonstration by example. Appendices give review of prerequisite mathematics and supplementary material on probability. Special features include confidence intervals for differences and for standard deviation, analysis of variance in greater depth than usual, introduction to decision theory, early exposure to hypothesis testing. GHM

COMPUTER PROGRAMMING, S, P. *SARGON: A Computer Chess Program.* Dan Spracklen, Kathe Spracklen. Hayden, 1978, 114 pp, $14.95 (P). [ISBN: 0-8104-5155-7] Fully documented chess program written in Z-80 assembly language using the TDL Macro Assembler. It occupies 8K of RAM. The program took first place in the Chess Tournament at the 1978 West Coast Computer Faire by winning 5 games out of 5. LCL

COMPUTER PROGRAMMING, T(13), S. *Fundamentals of FORTRAN for Management.* Roy Ageloff, Richard Mojena. Wadsworth, 1979, xvi + 443 pp, $11.95 (P). [ISBN: 0-534-00710-4] An extensive, nontechnical introduction featuring many examples and exercises, meaningful applications, both batch and time sharing approaches. LCL

COMPUTER PROGRAMMING, S(13-14). *Programming Techniques, Volume 2: Simulation.* Ed: Blaise W. Liffick. BYTE Pub, 1979, v + 125 pp, $6 (P). [ISBN: 0-931718-13-9] 12 articles extracted from *BYTE* magazine, intended for personal computer users. Classified into three categories: artificial intelligence, simulation of motion, and experimentation. RWN

COMPUTER PROGRAMMING, T(13: 1). *QWIKTRAN: Quick FORTRAN for Micros, Minis, and Main-frames.* C. Kevin McCabe. Dilithium Pr, 1979, xvi + 220 pp, $9.95 (P). [ISBN: 0-918398-24-X] A slowly-paced introduction to ANSI Fortran IV written in a popular style. Covers most of the language. RWN

COMPUTER PROGRAMMING, T(13-16: 1), S. *A Clear Introduction to FORTRAN IV, Including Standard FORTRAN, WATFOR, and WATFIV, Second Edition.* Richard M. Jaffe. Duxbury Pr, 1979, xv + 289 pp, $13.95. [ISBN: 0-87872-175-4] Designed as a text and a well-organized reference, this *Second Edition* emphasizes writing of complete programs. Stresses development of appropriate flowcharts, and gives guidelines that lead the student towards structured programming. LAS

COMPUTER PROGRAMMING, T(13), S*. *BASIC.* Steven C. Lawlor. Wadsworth, 1979, xi + 218 pp, $8.95 (P). [ISBN: 0-534-00694-9] A very readable introduction to Basic programming. Numerous examples and exercises. Supplements available for a number of machine-specific variations. Excellent for self study or as a text. TAV

COMPUTER SCIENCE, T*(13-18: 1, 2), S. *Foundations of Computer Science.* M.S. Carberry, *et al.* Computer Sci Pr, 1979, ix + 317 pp, $15.95. [ISBN: 0-914894-18-8] A refreshing approach to a first course in computer science. Begins with scientific and historical perspectives and proceeds to the usual programming topics--algorithms, flowcharts, program correctness, stepwise refinement. Then come the nonconventional topics--hardware, parsing, control structures, data structures, artificial intelligence, Turing machines, social issues, numerical and non-numerical applications, pocket calculator programming--all presented in an attractive, understandable way for the beginner. Many examples. Chapter references and exercises. Index. RJA

COMPUTER SCIENCE, S(15-18), P, L. *Proving Programs Correct.* Robert B. Anderson. Wiley, 1979, viii + 184 pp, $8.95 (P). [ISBN: 0-471-03395-2] Emphasizes informal correctness proofs that are valuable in program testing. Contains material on mathematical induction, inductive assertions, and structural induction as techniques for proving program correctness. Inductive assertions are applied to iterative programs and, in particular, to some Fortran and PL/1 programs. Structural induction applies to correctness of recursive programs. The final chapter pertains to current research. Good as a supplementary text. Descriptor-indexed bibliography. Index. RJA

COMPUTER SCIENCE, T(14-16: 1, 2), S, P, L*. *Principles of Interactive Computer Graphics, Second Edition.* William M. Newman, Robert F. Sproull. McGraw, 1979, xvi + 541 pp, $24.95. [ISBN: 0-07-046338-7] A thorough revision of the 1973 original, including much new material on raster (i.e., TV-like) graphics, three dimensional displays, and graphics package design. Program examples are now written in Pascal. A comprehensive and very readable text, with multi-chapter sections devoted to graphics packages, interactive graphics, raster graphics, three dimensional graphics, and graphics systems. LAS

COMPUTER SCIENCE, T*(14-18: 1, 2), S, L. *Compiler Construction: Theory and Practice.* William A. Barrett, John D. Couch. SRA, 1979, xvii + 661 pp, $17.95. [ISBN: 0-574-21160-8] Contains both mathematical theory of language translation and practical concerns required to produce high quality compilers. The text seems to achieve a balanced treatment of both of these aspects that makes it ideally suited as a teaching text. All aspects of compiler construction are treated, including optimization and error recovery. Three specific machines are discussed in the object code chapter: the Hewlett-Packard 3000, the CDC 6000 computer, and the IBM System/360. Chapters have bibliographical notes. Annotated bibliography. Index. RJA

SYSTEMS THEORY, S(18), P. *Monotone Operators and Applications in Control and Network Theory.* Vaclav Dolezal. Stud. in Automation and Control, V. 2. Elsevier Sci Pub, 1979, ix + 174 pp, $43.50. [ISBN: 0-444-41791-5] The first half of this book discusses the basic theory of monotone operators on a Hilbert space. The second half describes applications of this theory to control and network theory. Includes a brief list of references. No exercises. CEC

APPLICATIONS, S(18), P. *Interdisciplinary Mathematics, Volume XX: Cartanian Geometry, Nonlinear Waves, and Control Theory, Part A.* Robert Hermann. Math Sci Pr, 1979, xv + 501 pp, $50 (P). [ISBN: 0-915-69227-9] This book brings much of the geometric Lie theory to bear on problems from nonlinear waves and control theory. After a brief review of the theory, the author turns to various topics, providing the geometry as it is needed. A wide range of topics. No index, but the table of contents is detailed. Large bibliography. TLS

APPLICATIONS (ASTRONOMY), P. *Mathematical Astronomy with a Pocket Calculator.* Aubrey Jones. Wiley, 1978, 254 pp, $15.95. [ISBN: 0-470-26552-3] A collection of algorithms for solving astronomical problems (e.g., involving time and position) for the working astronomer. Computational procedures illustrated by examples, given for both programmable and nonprogrammable calculators in both algebraic and reverse Polish notation. LCL

APPLICATIONS (BIOLOGY), P. *Some Mathematical Questions in Biology, X.* Ed: Simon A. Levin. Lect. on Math. in Life Sci., V. 11. AMS, 1979, viii + 179 pp, $13.20 (P). [ISBN: 0-8218-1161-4] Proceedings of Twelfth Symposium on Mathematical Biology held at the Annual Meeting of the American Association for the Advancement of Science in Washington, D.C., on February 14, 1978. JAS

APPLICATIONS (MODELLING), T(15-16: 1, 2), S*, P, L*. *Mathematical Modelling Techniques.* R. Aris. Research Notes in Math., No. 24. Fearon-Pitman, 1978, 191 pp, $15 (P). [ISBN: 0-273-08413-5] This is not a problem solving text but rather a literate and readable discussion with a skeleton of classical examples of the "craft of mathematical modelling." The technical mathematics referred to includes fuzzy sets, differential equations, and linear algebra. The chapter headings help to describe this book, which is both technical and philosophical: What Is a Model?, The Different Types of Model, How to Formulate a Model, How Should a Model Be Manipulated Into Its Most Responsive Form?, How Should a Model Be Evaluated? JAS

APPLICATIONS (PHYSICS), T(17-18: 1, 2), S, P. *Special Relativity, The Foundation of Macroscopic Physics.* W.G. Dixon. Cambridge U Pr, 1978, x + 261 pp, $34.50. [ISBN: 0-521-21871-3] An exposition with the aim of showing that "an understanding of the basic laws of macroscopic systems can be gained more easily within relativistic physics than within Newtonian physics." The treatment and especially the notation is that of physics, rather than that of modern geometry. JAS

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; William Carlson, St. Olaf; Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; Paul Jorgensen, Carleton; Lorraine L. Keller, St. Olaf; Richard S. Kleber, St. Olaf; Joseph Konhauser, Macalester; Loren C. Larson, St. Olaf; George H. Mills, St. Olaf; R.W. Nau, Carleton; Thomas R. Savage, St. Olaf; Seymour Schuster, Carleton; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Timothy L. Strotman, St. Olaf; T.A. Vessey, St. Olaf; Frank L. Wolf, Carleton.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036

PERSONAL ITEMS

Adrian College, Adrian, Michigan: Associate Professor Margaret O. Marchand has been appointed Chairman of the Mathematics Department. Assistant Professor James O. Watson has been promoted to Associate Professor.

Rockford College, Rockford, Illinois: Dr. Gerald Lee Caton of the University of Montana has been appointed Associate Professor. Associate Professor George M. Whitson has accepted appointment at the University of Michigan at Dearborn.

Alfred University, Alfred, New York: Assistant Professor Dean Hoover has been promoted to Associate Professor. Associate Professor Roger H. Moritz has been appointed Chairman.

Stanford University: Associate Professors Gregory Brumfiel and Mary V. Sunseri have been promoted to Professors. Professor Hans Samelson has been appointed Chairman of the Mathematics Department.

Oklahoma State University: Associate Professor John M. Jobe has been promoted to Professor. Dr. Garrett S. Sylvester, New Mexico State University, has been appointed Assistant Professor.

Sweet Briar College, Sweet Briar, Virginia: Assistant Professor John F. Daughtry is on leave at the University of North Carolina, Chapel Hill, during the 1979-80 academic year. Associate Professor Judith Molinar Elkins has been promoted to Professor.

Syracuse University: Dr. J. Theodore Cox of the University of Southern California-Los Angeles has been appointed Assistant Professor. Professor Jack E. Graver has been appointed Chairman of the Mathematics Department.

Associate Professors Charles Votaw and Ervin Eltze, Fort Hays State University, Hays, Kansas, have been promoted to Professors.

Associate Professors William H. Graves and Kenneth I. Gross, University of North Carolina, Chapel Hill, have been promoted to Professors.

Associate Professor Franz X. Hiergeist, West Virginia University, has been promoted to Professor.

Dr. Temple H. Fay, New Mexico State University, has been appointed Associate Professor at the University of Southern Mississippi, Hattiesburg.

Dr. Eric C. Nummela, St. Cloud State University, St. Cloud, Minnesota, has been appointed Associate Professor at New England College, Henniker, New Hampshire.

Professor David L. Armacost, Amherst College, has been elected Chairman of the Department of Mathematics.

Professor Ignacio L. Iribarren, Simon Bolivar University, Caracas, Venezuela, has been appointed Academic Vice-Rector.

Dr. John W. Baker, Bloomsburg State College, Bloomsburg, Pennsylvania, has been appointed Assistant Professor at Duquesne University, Pittsburgh, Pennsylvania.

Dr. John J. Dranchak, University of Portland, Portland, Oregon, has been appointed Chairman of the Department of Mathematics.

Dr. Pierre Malraison has moved from Control Data Corporation, St. Paul, Minnesota, to Manufacturing Data Systems, Inc., Ann Arbor, Michigan.

Assistant Professor Simeon Reich, University of Southern California, has been promoted to Associate Professor.

Assistant Professor Frank Servas, Jr., St. John's University, Jamaica, New York, has been promoted to Associate Professor.

Assistant Professor Thomas Cecil, Vassar College, has been appointed Assistant Professor at Holy Cross College.

Professor Peter Hilton, Batelle Research Center, Seattle, Washington, has been elected a Corresponding Member of the Brazilian Academy of Sciences.

Mr. Harris Jones, Asheville, North Carolina, died on July 1, 1977. He was a member of the Association for forty five years.

Professor Emeritus George N. Garrison, City College of New York, died on July 22, 1979, at the age of seventy three. He was a member of the Association for twenty five years.

Dr. Harold S. Grant, West Dennis, Massachusetts, died on May 20, 1979. He was a member of the Association for forty two years.

Retired Professor Sidney W. McCusky, Case Western Reserve University, died on April 22, 1979, at the age of seventy two. He was a member of the Association for thirty three years.

Professor Gideon Peyser, New Jersey Institute of Technology, died in May, 1979. He was a member of the Association for twenty one years.

Professor Emeritus Sumner I. Vrooman, Pratt Institute, died in December, 1978.

SHORT COURSE ON MODERN STATISTICS; METHODS AND APPLICATIONS

The American Mathematical Society, in conjunction with its eighty-sixth annual meeting in San Antonio, Texas, will continue its short course series with a course entitled "Modern Statistics: Methods and Applications." The one and one-half day course will be held on Monday afternoon and evening, January 7, 1980, and on Tuesday morning, January 8, 1980.

The program is under the direction of Robert V. Hogg of the Department of Statistics, University of Iowa. It will consist of six lectures of seventy-five minutes each and two general discussion periods.

The speakers and the titles of their talks are:

Samples and Surveys, Wayne A. Fuller, Iowa State University
Analysis of variance, Peter W.M. John, University of Texas-Austin
Nonparametric statistical tests of hypotheses, Ronald H. Randles, University of Iowa
Rank estimates from nonparametric tests, Thomas P. Hettmansperger, Pennsylvania State University
Statistical inferences for ordered parameters, F.T. Wright, University of Missouri-Rolla
Time series: Model estimation, data analysis, and robust procedures, R. Douglas Martin, University of Washington.

Further information may be obtained by writing or calling the American Mathematical Society's Meeting Arrangements Department, P.O. Box 6887, Providence, RI 02940; Telephone (401) 272-9500, Ext. 239.

SECURE COMMUNICATIONS AND ASYMMETRIC CRYPTOSYSTEMS

There will be a special session on "Secure Communications and Asymmetric Cryptosystems" at the annual meeting of the AAAS in San Francisco on January 5, 1980. Gustavus J. Simmons, Sandia Laboratories, will preside. The speakers and their topics will be:

An Introduction to the New Cryptography, Leonard M. Adleman, M.I.T.
Computationally "Hard" Problems as a Source for Cryptosystems, Hugh C. Williams, Univ. of Manitoba
Message Authentication without Secrecy, Gustavus J. Simmons, Sandia Laboratories
New Developments in Public-Key Cryptography, Whitfield Diffie and Ralph Merkle, Bell Northern Research, Inc.

ADVANCED SEMINAR ON SINGULAR PERTURBATIONS AND ASYMPTOTICS

The Mathematics Research Center at the University of Wisconsin-Madison will hold a conference on recent developments in the theories of singular perturbations and singular-point asymptotics of differential equations and their applications to the sciences on May 28-30, 1980. The main program will consist of approximately fourteen lectures, and the invited speakers include J. D. Cole, P. P. N. deGroen, W. Eckhaus, O. A. Ladizhenskaya, J. L. Lions, B. Matkowsky, A. H. Nayfeh, F. W. J. Olver, R. O'Malley and Y. Sibuya. Professor D. R. Smith will give an introductory short course of about five lectures on multivariate approximation methods for singular perturbations and their applications on May 27. A detailed program will be available in February and further information may be obtained from Mrs. Gladys Moran, Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706.

AMS-MAA-SIAM CONGRESSIONAL SCIENCE FELLOWSHIP FOR 1980-81

Applications are invited from candidates in the mathematical sciences for a Congressional Science Fellowship to be supported jointly by the American Mathematical Society, the Mathematical Association of American and the Society for Industrial and Applied Mathematics for the twelve-month period beginning 1 September 1980. The AMS-MAA-SIAM Fellow will serve, along with several Fellows selected by the American Association for the Advancement of Science and around fifteen Fellows sponsored by other scientific societies, under an annual program coordinated by the AAAS. The stipend for the 1980-81 AMS-MAA-SIAM Fellowship is $19,000, which may be supplemented by a small amount toward relocation and travel expenses. It may also be supplemented by sabbatical salary or other employer contribution in the case of a person on sabbatical leave for the 1980-81 year.

The 1979-80 AMS-MAA-SIAM Fellowship has been awarded to Dr. Robert T. Smythe, Associate Professor of Mathematics at the University of Oregon. Dr. Smythe, who is 38 years old, holds bachelor's degrees from Oberlin College and from Oxford University, where he was a Rhodes Scholar in 1963-65. He received his doctorate in 1969 from Stanford University, specializing there in probability and statistics. His current research interests involve stochastic models arising in physical and biological contexts, and he also has an interest in questions of scientific, environmental, and political policy. The AMS-MAA-SIAM Congressional Science Fellowship was awarded for the first time in 1978-79, to Dr. Edmund Gregory Lee of Fordham University.

Congressional Science Fellows spend their fellowship year working on the staff of an individual congressman or a congressional committee or in the congressional Office of Technology Assessment. Based on information on available congressional staff positions gathered by the AAAS during the summer, each Fellow's assignment is worked out by the Fellow and the congressional office concerned following an intensive two-week orientation and interview procedure organized by the AAAS during which the Fellows encounter many facets of Congress, the Executive Brand, and people and organizations on the Washington scene. The AAAS provides advice and assistance during the process and remains in frequent and regular contact with all the Fellows throughout the fellowship year. More detailed

information about the overall program is available from AAAS Congressional Science Fellow Program, 1776 Massachusetts Avenue, N.W., Washington, C.D. 20036; telephone (202) 467-4475.

The AMS-MAA-SIAM Congressional Science Fellowship is to be awarded competitively to a mathematically trained person at the postdoctoral to mid-career level without regard to sex, race, or ethnic group. Selection will be made by a panel of the AMS-MAA-SIAM Joint Projects Committee for Mathematics with the cooperation and advice of the overall AAAS Program. *Applications should be sent to the Conference Board of the Mathematical Sciences, 1500 Massachusetts Avenue, N.W., Suite 457-458, Washington, D.C. 20005. The deadline for receipt of applications is 15 February 1980.* It is anticipated that the award will be made by around 1 April 1980.

In addition to demonstrating exceptional competence in some areas of the mathematical sciences, an applicant for the AMS-MAA-SIAM Congressional Science Fellowship should have a rather broad scientific and technical background and a strong interest in the uses of the mathematical and other sciences in the solution of societal problems. He or she would also be articulate, literate, flexible and able to work effectively with a wide variety of people. An application should state why the applicant wants to be a Congressional Science Fellow, should summarize his or her qualifications, and should be accompanied by a resume. Also, CBMS should receive by 15 February 1980 three letters from persons knowledgeable about the applicant's competence and suitability for the award.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

MAY MEETING OF THE INTERMOUNTAIN SECTION

The meeting was held at Idaho State University, Pocatello, on May 4-5, 1979. Chairman was Steven Parker, Idaho State University with committee members T. L. Williams, Idaho State University; S. S. Terry, Ricks College; S. J. Leon, Weber State College; and M. P. Windham, Utah State University. Sessions for presented papers were held Friday afternoon and Saturday morning. Titles and authors are listed below.

The invited guest of honor for the banquet was Dr. James D. Murray, Professor of Mathematics at Oxford University and currently a Visiting Professor at the Massachusetts Institute of Technology, University of Utah and California Institute of Technology. His invited address was: *How the Leopard Got Its Spots and Some Other Biological and Ecological Stories: A Mathematician's View.*

The invited address on Saturday morning was given by Dr. Keith Reed, University of Utah, titled: *The Mathematics of Chance—A Mathematics Course in a Liberal Education Program.*

JUNE MEETING OF THE PACIFIC NORTHWEST SECTION

The Annual Meeting of the Pacific Northwest Section was held June 15-16, 1979 at the University of British Columbia, Vancouver, B.C. in conjunction with meetings of the AMS and SIAM. Highlight of the meetings was Constance Reid's dinner talk, "The Answer to the Question Everyone Asks."

In addition to Mrs. Reid's talk, the following invited addresses were given

An Introduction to Cognitive Mapping, T. Pletcher, Vancouver Community College, Langara, B.C.

Furstenberg's Proof of Szemeredi's Theorem (or, Ergodic Theory Strikes Again), Douglas A. Lind, University of Washington, Seattle, Washington

Mathematics in the Open University, B. Coates, British Open University and J. Koumi, British Broadcasting Company

Cooperative Education in Mathematics, D. Dale Olesky, University of Victoria

Geometrical Theorems in Slides—An Innovative Approach for Teaching Geometry, Ved P. Madan, Red Deer College, Red Deer, Alberta

Compartmentalization of Mathematical Cognition, Hazel Jo Reed, The Evergreen State College, Olympia, Washington

A Probabilistic Approach to Studying Groups, Kenneth A. Ross, University of Oregon, Eugene, Oregon

The AMS program included two invited addresses.

One-Dimensional Transformations, Oscar E. Lanford III, University of California, San Diego, California

Some Geometrical Aspects of General Relativity, Theodore T. Frankel, University of California, San Diego, California

In addition there were Special Sessions on Mathematical Physics, Probability, Analysis, Representations and Ring Theory, and Algebra and Set Theory.

JOHN HERZOG, *Secretary-Treasurer*

APRIL MEETING OF ROCKY MOUNTAIN SECTION

The 62nd Annual Meeting of the Rocky Mountain Section was held April 27-28 at the University of Denver, Chairman John Hodges, University of Colorado presiding. There were 117 in attendance including Professor Peter Hilton, First Vice President of MAA and Professor Duane Porter, University of Wyoming, Section Governor.

Professor Hilton presented the banquet address on "Teaching Applied Mathematics." The theme of the 62nd meeting was Applied Mathematics. There was a panel discussion on this theme moderated by Professor Ottis Rechard of Denver University. Participants were: Professor Peter Hilton; Professor John Maybee, University of Colorado; and Herbert Greenberg, University of Denver.

The following papers were contributed:

Generalized Convexity and Applications, Stanley Gudder, DU
Measuring the Pointedness of a Convex Body, James Fickett, CU
How to Impress the Administration, David Ballew, SDSM
The Cubic Equation Revisited, F. Max Stein, CSU
Math Anxiety in College Students, Nancy Angle, UCD
A Mixed Integer Programming Model of Biological Systems, Wilbur Miller, USC
Problems with Potential, Roger Opp, SDSM
Computer Graphics in the Calculus Classroom, Austin R. Brown, CSM
The Mathematics Students Study and Need in Wyoming, Melfried Olson, UW
Confessions of an Applications Addict, William Ramaly, Ft. Lewis College
On Cooking a Turkey, Robert Fisk, CSM
A Stochastic Model for Resource Allocation, S. K. Sengupta, SDSM
Early History of the Mathematics Department at the University of Colorado, Burton Jones, CU
Doubly Stochastic Matrices in Accounting, Dean Lucas, Martin Marietta
Personalized Instruction Systems, Matthew Hassett, AFA & ASU

The Business Meeting was convened on Saturday, April 28 by Professor Hodges with forty-one in attendance. The officers for 1979-80 are: Chairperson, Professor Laurel Rogers, University of Colorado at Colorado Springs; Chairperson-elect, Professor William Ramaly, Ft. Lewis College; Vice Chairperson, Allan Skillman, Casper College; Program Chairperson, David Rearick, University of Colorado; Past Chairperson, Professor John Hodges, University of Colorado; Secretary-Treasurer, David Ballew, South Dakota School of Mines and Technology.

The minutes and treasurer's report were approved. The proposed changes in the Bylaws to increase registration fees and free more funds for the Annual Meeting passed. Professor Duane Porter presented the Governor's report, discussed the High School Mathematics Contest and the MAA Headquarters building. Professor Robert Vunovich presented the report on the High School Mathematics Contest. The Section voiced its appreciation to Professor Vunovich for his work on the contest. Professor Peter Hilton represented the MAA at the meeting and discussed future joint meetings of MAA and AMS. Professor Rebekka Struik of the University of Colorado presented the report on the High School Lectureship Program.

DAVID BALLEW, *Secretary-Treasurer*

SPRING MEETING OF THE NORTH CENTRAL SECTION

The spring meeting of the North Central Section was held at the College of St. Teresa in Winona, Minnesota, on April 27-28, 1979. Professor Joe Konhauser presided at the business meeting. Reports were given on the 1979 Summer Seminar on Statistics at the University of Minnesota, Duluth and on the MAA High School Contest results in Minnesota and North Dakota. Recognition was given to the high scorers on the Putnam Examination from the North Central Section.

The membership elected Steve Hilding, Gustavus Adolphus College President-Elect, Roger Avelsgaard, Bemidji State University, Member at Large on Executive Committee, and re-elected Charles Heuer, Concordia College, Moorhead as Secretary-Treasurer.

An invited address entitled *The Ideal Situation for Continuous Functions*, was given on Friday evening by Professor Don Mattson of Moorhead State University. On Saturday an invited address was given by Professor R. P. Boas of Northwestern University. His talk was entitled *The Harmonic Series*.

The following contributed papers were given:

A Minimum Competency Requirement in Mathematics for College Students, Francis Hatfield, Mankato State University
Matrix Examples for an Abstract Algebra Course, John Schue, Macalester College
A Converse Problem in Differential Equations, Warren Shreve, North Dakota State University
Elementary Integrals, Hubert Walczak, College of St. Thomas
Dissipation in Lotka-Volterra Systems, Tom Haigh, St. Johns University
Some Geometrical Aspects of the Fundamental Theorem of Calculus, Helen Skala, College of St. Teresa
The Greatest Integer Function and the Golden Ratio, Gerald Bergum (presented by Kenneth Yocom), South Dakota State University
A Generalization of Cramer's Rule, Sylvan Burgstahler, University of Minnesota, Duluth

CHARLES V. HEUER, *Secretary-Treasurer*

APRIL MEETING OF THE ALLEGHENY MOUNTAIN SECTION

The annual spring meeting of the Allegheny Mountain Section of the MAA was held at Westminster College, New Wilmington, Pennsylvania, on April 27-28, 1979. There were approximately 75 members of the Association and 35 students present from the member schools. The host was Barbara Faires, Mathematics Department of Westminster College.

Friday evening, April 27, 1979, was highlighted by two invited presentations and a contributed papers session for both students and faculty. The invited speakers were Anthony LoBello, Allegheny College, and Fred Roberts, Rutgers University. The title of Professor LoBello's presentation was *Observations on the History of Mathematics on the Occasion of the Einstein Centennial*. The invited talk by Professor Roberts was *Interval Graphs, Traffic Phasing, Ship Building, and Mobile Radio Frequency*.

There were three invited talks Saturday morning. Caulton Irwin, West Virginia University, presented a paper entitled *Mathematical Aspects of Energy Modeling*. T. R. Nealigh, Westminster College, presented a paper, designed primarily for students, entitled *The Trials and Tribulations of the*

Actuarial Exams. The last invited talk was presented by Leonard Gillman, University of Texas—Austin, and Treasurer of the MAA. The title of his talk was *The Dog with the Cocked Head.*

In addition to the invited talks, a panel discussion of the PRIME-80 conference was presented by Fred Bell, University of Pittsburgh; John Atkins, Bethany College; and Kenneth Whipkey, Westminster College. The 15-minute contributed talks Friday evening were as follows:

FACULTY:

Bayes Sequential Estimation of the Arrival Rate of a Poisson Process, Bradley Novic, Duquesne Univ.
Discrete Population Models, Richard A. Moore, Carnegie-Mellon University
Math Anxiety: Training Teachers to Recognize and Deal with It, Beverly K. Michael, Penn State-New Kensington.

STUDENT:

Fundamentals of Biological Pollutant Degrader Model, Jonathan Spencer, Allegheny College
Modeling Cost Functions, Carol Stoves, Westminster College
A Model for a Concentration Quenching of Flourescent in Chlorophyll a-Solutions, Dave Housman, Allegheny College
Haldane Model of Natural Selection, Bob Morganstern, Westminster College

The business meeting of the Allegheny Mountain Section was called to order by Chairman Richard McDermot, Allegheny College. Francis Hall, Pennsylvania State University-Fayette Campus presented the report on the high school contest in western Pennsylvania. Nicholas Ford, Pennsylvania State University-Fayette Campus replaced Professor Hall as Coordinator for western Pennsylvania. Frank Hiergeist, West Virginia University, presented the report on the high school contest in West Virginia. Melvin Woodard, Indiana University of Pennsylvania, Governor of the Section, gave a report on the status of the new MAA Building. The list of the top five students from the Allegheny Mountain Section was read: Joshua Bernoff, Pennsylvania State University, David L. Housman, Allegheny College; Randall Henry, University of Pittsburgh; Steven R. Costenable, Carnegie-Mellon University; and Margaret Lepley, Carnegie-Mellon University. The section voted to give each of these competitors a student membership in the MAA.

Professor Leonard Gillman gave a short report from the Mathematical Association of America office. Officers elected were Allan Krall, Pennsylvania State University, Chairman; Francis Hall, Penn State-Fayette Campus, Second Vice-Chairman; and Barbara Faires, Westminster College, Coordinator of Student Program. The First Vice-Chairman will be Frank Hiergeist, West Virginia University. Benjamin Haytock, Allegheny College was appointed to coordinate the high school lecturer program for western Pennsylvania. Next year's meeting will be at West Virginia Wesleyan College, Buckhannon, West Virginia.

JOHN W. MILSOM, *Secretary-Treasurer*

MAY MEETING OF THE MICHIGAN SECTION

The Michigan Section held its 1979 annual meeting at the University of Detroit on May 4 and 5. Local arrangements were handled by Mike Skaff and Bob Kane of the University of Detroit. Approximately 160 persons attended. Professors Delio Koo of Eastern Michigan University and Katherine Price of Highland Park Community College arranged the excellent program, highlighted by the following invited addresses:

How to be a Mathematician, Paul Halmos, Indiana University
Subnormal Operators: A Survey, Allen Shields, University of Michigan
Placement Tests in Practice, Phillip Miles, University of Wisconsin
Crystallography: A Case for an Undergraduate Course, Richard Phillips, Michigan State University
Algebra and Computers, Garrett Birkhoff, Harvard University
Applications in the Two Year College Curriculum, Marvin Bittinger, Indiana University-Purdue University at Indianapolis

Each day of the meeting included a panel discussion. The first, *Forming a Bridge Between Industry and the Classroom,* was moderated by George Feeman of Oakland University. Participants were Joseph R. Assenzo of Upjohn Company, Allen V. Butterworth of General Motors Research Laboratories, Lowell Hansen of Wayne State University, Edward Moylan of Ford Motor Company, and Chester Tsai of Michigan State University. The second panel of Douglas Hall, Michigan State University, and Christopher Hee of Eastern Michigan University, responded to Professor Miles' talk on placement tests. Contributed papers were:

Use of Calculators in Algebra and Trigonometry, Richard Hall, Michigan State University
Enrichment Ideas for the Freshman Mathematics Survey Course, John Ginther, Eastern Michigan University
How to Do Arithmetic, Jane Robertson
The Geometry of the Range Values of a Matrix, Mary Embry-Wardrop, Central Michigan University
Continuity for Vector-Valued Functions: A Non-Classical Approach, I. E. Schochetman, Oakland U
A Curve Fitting Problem from Chemistry, Garnet Smith Hauger, Spring Arbor College
Extensions and Applications of the Symmetric Successive Over Relaxation (SSOR), Lala B. Krishna, Oakland University
Unified Combinatorial Theory, Judith Longyear, Wayne State University
Analysis of Digital Processing Networks Based on Functional Relations, Sarma Vishnubhotia
Jesuit Mathematicians, 1550 to 1800, John O'Neill, S.J., University of Detroit

Two sessions of student papers included:

Wallpaper Design, A Study of 2-D Crystal Structure, Carl Toren, Hope College
Montague Grammar and Formal Linguistics, Neal Oliver, University of Michigan
An Artistic Approach to Infinity Using the Tektronix 4051, John Gibson, Hope College
δs, εs, and Linear Approximations, Robert Elieff and Virginia Wyman, University of Michigan-Flint

A Friday evening banquet was highlighted by a banquet address *Inventions as Elegant Solutions, Sometimes,* given by Jacob Rabinow of the National Bureau of Standards.

Officers of the Section for 1979-80 are Delia Koo, Eastern Michigan University, Chairperson; Harold Slaby, Wayne State University, Vice-Chairperson; Joel Cohen, Oakland Community College, Vice-Chairperson; Robert Chaffer, Central Michigan University, Secretary-Treasurer; and Yousef Alavi, Western Michigan University, Governor.

ROBERT CHAFFER, *Secretary-Treasurer*

MAY MEETING OF THE SEAWAY SECTION

The Seaway Section held its spring meeting at S.U.C. at Oneonta, Oneonta, New York, May 4-5, 1979. Approximately seventy people were in attendance. Section Chairman Violet Larney presided. On Friday the Section's Executive Committee and Departmental Representatives met to consider future activities of the Section. This was followed by a banquet with Ross Honsberger of the University of Waterloo as speaker. His talk was titled: *Some Mathematical Morsels.* The Saturday Sessions included two invited addresses: Professor Christopher Nevison, Colgate University spoke as a representative of U.M.A.P. on *Modules in Applied Mathematics: Some Examples and Sources for Materials.* Professors A. R. Bednarek, University of Florida was the Twelfth Annual Harry M. Gehman Lecturer and spoke on *Relational Structures and Theoretical Computer Science.* The following contributed papers were also presented:

The Optimal Assignment Problem, An Application of Elementary Mathematics to Economics, W. H. Reynolds, S.U.C. Cortland

A Model of Population Diffusion, A. C. Green, S.U.C. Buffalo

Optimal Capital Accumulation in a One Sector Economy, B. L. Warren, SUNY Binghamton

Coexistence of Species Competing for Shared Resources, C. W. Kohls, Syracuse University

A Heine-Borel Theorem is Equivalent to a Parallel Postulate, R. B. Kollgrove, S.U.C. Geneseo

New York State's New Curriculum for High School Mathematics, and *The College-Bound Student,* R. Escobales, Canisius College

A Cute Linear-Algebra Proof of the Four-Square Theorem, C. Small, Queens University

Testing: The New Frontier for Consumers, R. D. Larsson, Schenectady Count C.C.

At the business session of the Section the following officers were elected under the newly revised election scheme: Chairperson (two years); Howard Bell, Brock University. First Vice-Chairperson (one year); Kenneth Magill, SUNY Buffalo. Second Vice-Chairperson (two years); Harriette Stephens, SUNY College at Canton. Also Mabel Montgomery, S.U.C. Buffalo is the newly elected Section Governor. Continuing in service: Dennis Martin, S.U.C. Brockport, High School Contest Chairman. Donald Trasher, S.U.C. Geneseo, Secretary-Treasurer and Newsletter Editor.

The Section recognized Geoffrey Mess of the University of Waterloo as the winner of the Section's prize for highest score in the Putnam Competition.

D. W. TRASHER, *Secretary-Treasurer*

MAY MEETING OF THE METROPOLOITAN NEW YORK SECTION

The 38th Annual Meeting of the Metropolitan-New York Section of the MAA was held at Adelphi University on Saturday, May 5, 1979, with approximately 150 persons in attendance. Professor Robert J. Bumcrot of Hofstra University, Chairperson of the Section, presided at the meeting. Dean Clifford Stewart, Dean of the Faculty of Adelphi University delivered the address of welcome. The business meeting began with the presentation of reports from the following: Section Governor Harold Shapiro of New York University, Section Committee for Two-Year Colleges, Editor William Orr of the Section Newsletter, Director Howard Kleiman of the Speakers Bureau, and the Treasurer, Howard Kleiman, Chairperson, Sandra Pulver of the MAA High School Contests (by Dr. Harry Ruderman), and Chairperson Theresa Barz of the MATH Fair Committee. The following officers were elected: Chairperson of Section, Professor Godfrey L. Isaacs, Lehman College, CUNY; Vice-Chairperson for Four-Year Colleges, Professor Theresa J. Barz, St. Johns University; Vice-Chairperson for Two-Year Colleges, Professor Leo M. Levine, Queensborough Community College, CUNY; Vice-Chairperson for High Schools, Dr. Harry D. Ruderman, Hunter College High School; Secretary, Professor Lily E. Christ, John Jay College of Criminal Justice, CUNY; Treasurer, Professor Howard Kleiman, Queensborough Community College, CUNY.

The following awards consisting of the MAA membership were presented: (1) The Charles Salkind Award to the highest regional scorer in the MAA High School Mathematics Contest to Mr. Louis LoBalsamo; (2) The Section Award to the highest regional scorer in the William Lowell Putnam Mathematical Competition to Mr. Boris A. Datskovsky, Columbia University. The 39th Annual Meeting will be held on Saturday, May 4, 1980 at Mercy College.

The two invited lectures were: *Finitely Presented Groups and Recursive Functions,* by Gilbert Baumslag, City College, and *A Pedagogical Potpourri: Exercises in the Art of Survival,* by Professor Warren Page, New York City Community College. The afternoon session concluded with the following contributed papers given in four parallel sessions: *Some Results on the Informal Power Series,* by Joseph Arkin New York Academy of Sciences; *A New Proof of Uniform Continuity,* by David M. Bloom, Brooklyn College of CUNY; *The Role of Triple Algebras in the Development of Quaternions,* by Alan Chutsky, Queensborough Community College; *Choosing Polynomial Coefficients to Force Zeros in a Two-Person Game,* by Michael W. Ecker, Pennsylvania State U; *Generalization of a Counterexample in Braid Theory,* by Betty Jane Gassner, Interactive Market Systems; *An Alternative Approach to Graphing,* by Martin E. Glashman, Bard College; *The Computer in Mathematics Courses,* by Tom Hasiotis, Pete Herron, Rochelle Meyer and Art Shindhelm, Suffolk County Community College; *Discrete and Continous Models in Demography: Part 2,* by John Impagliazzo, SUNY at Farmingdale; *Orthogonality Relations for Matrices,* by Maurice Machover, St. John's University; *Discrete and Continuous Models in Demography: Part 1,* by Walter Meyer, Adelphi University; *Two Ideals on the Line—A Comparative Study,* by Jay Schiffman, St. John's University; *Individualized Instruction in Finite Linear Mathematics,* by Sylvia Svitak, Adelphi University; *How to Join a Flat Earth and an Infinitely Distant Sun,* by Aaron R. Todd, St. Johns University; *The Use of Certain Separability Measures in Normal Classification,* by Marvin Yablon, John Jay College of Criminal Justice/CUNY and John T. Chu, Polytechnic Institute of New York.

LILY E. CHRIST, *Secretary*

CALENDAR OF FUTURE MEETINGS

Sixty-third Annual Meeting, San Antonio, Texas, January 5-7, 1980.

Sixtieth Summer Meeting, University of Michigan, Ann Arbor, August 18-20, 1980.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, West Virginia Wesleyan College, Buckhannon, April 25-26, 1980.

EASTERN PENNSYLVANIA AND DELAWARE, Drexel University, Philadelphia, November 17, 1979.

FLORIDA, Jacksonville University, Jacksonville, March 7-8, 1980.

ILLINOIS, John A. Logan College, Carterville, April 25-26, 1980.

INDIANA

INTERMOUNTAIN, Utah State University, Logan, Utah, late April or early May 1980.

IOWA, Simpson College, Indianola, April 18-19, 1980.

KANSAS, March or April. Deadline for papers January 1.

KENTUCKY, Western Kentucky University, Bowling Green, April 11-12, 1980.

LOUISIANA-MISSISSIPPI, Louisiana Tech University, Ruston, February 15-16, 1980.

MARYLAND-DISTRICT OF COLUMBIA-VIRGINIA, Prince Georges Community College, Largo, Maryland, November 10, 1979.

METROPOLITAN NEW YORK, Mercy College, Dobbs Ferry, May 4, 1980.

MICHIGAN, Hope College, Holland, May 2-3, 1980.

MISSOURI, late March/early April. Deadline for papers January 31.

NEBRASKA, April.

NEW JERSEY, Essex County College, Newark, November 3, 1979.

NORTH CENTRAL, end of October and April. Deadline for papers October 1 and April 1.

NORTHEASTERN, University of Hartford, Connecticut, November 18, 1979.

NORTHERN CALIFORNIA, Naval Postgraduate School, Monterey, February 23, 1980.

OHIO

OKLAHOMA-ARKANSAS, Westark Community College, Fort Smith, Arkansas, March 28-29, 1980.

PACIFIC NORTHWEST, Central Washington University, Ellensburg, Washington, June 20-21, 1980.

ROCKY MOUNTAIN, University of Colorado, Boulder, March 28-29, 1980.

SEAWAY, first Saturday in November and Saturday in late April. Deadline for papers 6 weeks before meeting.

SOUTHEASTERN, Appalachian State University, Boone, North Carolina, April 11-12, 1980.

SOUTHERN CALIFORNIA, University of California, Riverside, November 9-10, 1979.

SOUTHWESTERN, usually in April. Deadline for papers 2 weeks before meeting.

TEXAS, East Texas State University, Commerce, April 4-5, 1980.

WISCONSIN, University of Wisconsin, Milwaukee, March 28-29, 1980.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, San Francisco, January 3-8, 1980.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES

AMERICAN MATHEMATICAL SOCIETY, San Antonio, Texas, January 3-6, 1980.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Amherst, Massachusetts, June 23-26, 1980.

ASSOCIATION FOR COMPUTING MACHINERY, Kansas City, Missouri, February 12-14, 1980.

ASSOCIATION FOR SYMBOLIC LOGIC, New York City, December 28-29, 1979.

ASSOCIATION FOR WOMEN IN MATHEMATICS

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Seattle, Washington, April 16-19, 1980.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Shoreham Hotel, Washington, D.C., May 5-7, 1980.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION, Radisson Muehlebach, Kansas City, Missouri, November 1-3, 1979.

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS, Denver Marriott, Denver, Colorado, November 12-14, 1979.

MATHEMATICALLY SPEAKING...

ARITHMETIC, Second Edition

Sandra Preis, George Cocks—
both of Northampton County Area Community College

Programmed text containing clear presentation of arithmetic operations with emphasis on skill building. Numerous exercises prepare the student to take Beginning Algebra, Technical Math and Business Math courses. This new edition incorporates relevant applications of problems, increased drills; exponents introduced early and used throughout. Students learn by writing in the text following clear explanations and carefully designed programmed format.

1980 **650 pp. (est.)** **Paper**

ALGEBRA PROGRAMMED, PART 3, Second Edition

Robert Alwin, Robert Hackworth, Joseph Howland—
all of St. Petersburg Jr. College

Freshman level, programmed text for self-instructional course in Developmental Math or Developmental Studies. This new Part 3 replaces old Parts 3 and 4 of the first edition and ALGEBRA TEXT INTERMEDIATE. Revision features more material on polynomials, simplification of radicals, equations for graphing parabolas, systems of equations, and word problems.

1980 **560 pp. (est.)** **Paper**

ALGEBRA FOR COLLEGE STUDENTS, Second Edition

Max A. Sobel, Montclair State College
Norbert Lerner, State University of N.Y.—Cortland

Basic treatment of fundamental skills and concepts of elementary and intermediate algebra for the freshman level student. This new edition features revised exercises, caution items, easier reading style, and new material on variation and third-order determinants. Provides prerequisites necessary for further courses in mathematics.

1980 **512 pp. (est.)** **Cloth**

ELEMENTARY STATISTICS

Richard I. Levin, David Rubin—
both of University of North Carolina

For the Introductory/Elementary Statistics course in mathematics, psychology, political science or engineering at the freshman/sophomore level. Teaches subject with an easy to read, easy to understand style, using non-statistical jargon, and applied current examples.

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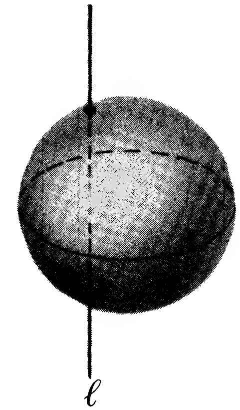
HURLEY

Intermediate Calculus: Multivariable Functions and Differential Equations with Linear Algebra

Clarifying the concepts of calculus and differential equations through the use of linear algebra, this new text is ideal for the average calculus student. It incorporates a generous amount of practical applications and fully worked examples, and covers infinite series, a topic that few texts include. This class-tested book is written in a highly readable style and is designed to follow any first year text. An **Instructor's Manual** contains essays, answers to even-numbered exercises, fully worked exams, and detailed week-by-week outlines.
By **James F. Hurley,** University of Connecticut. Ready Feb. 1980.
Order #4851-7.



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Geometry: An Exercise in Reasoning

Taking a fresh approach to geometry, Seydel emphasizes deductive reasoning and geometric relationships through applied problems and direct presentation. Students will enjoy the text's informal style, and can quickly learn from its many original exercise problems. For the highly motivated student, a large and well researched selection of outstanding problem sets is included. Discussions include transformation, historical topics, and topics of recreational value. A **Solutions Manual** is available.
By **Ken Seydel,** Skyline College, San Bruno, Calif. Ready Jan. 1980. 416 pp. About 740 ill. **Order #8070-4.**

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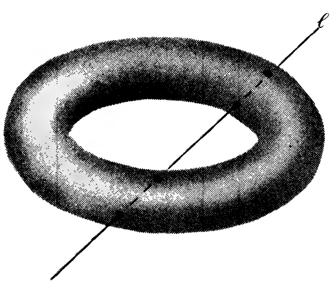

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Modern Intermediate Algebra for College Students 3rd Edition

Updated and reorganized to increase teachability, this new edition offers a clear and lucid presentation of all standard topics of intermediate algebra. Fully worked examples illustrate its concepts, graded problem sets are referenced whenever possible to illustrative examples, and "real world" applications and historical notes add interest. End of chapter review exercises and a diagnostic test with completely worked answers in the appendix make this comprehensive text almost self-teaching. An **Instructor's Manual** is available. By **Suzanne M. Shelley,** Sacramento City College, Calif. Ready Feb. 1980. About 610 pp. **Order #8217-0.**

MANOUGIAN

Elementary Algebra

Giving students the best possible preparation for subsequent courses in mathematics and related subjects, this new text introduces mathematical concepts with simple examples of real life situations. Sections begin with objectives, and include self tests, numerous examples, and graded problems, and a set of exercises. Common mistakes are highlighted, and chapters conclude with chapter reviews and programmed chapter tests. Emphasizing the use of the metric system, this stimulating text also includes interesting historical notes. An **Instructor's Manual** is available.
By **M.N. Manougian,** University of South Florida. Ready Jan. 1980. About 400 pp. **Order #6013-4.**

New for 1980

Algebra for College Students

BERNARD KOLMAN, Drexel University, and ARNOLD SHAPIRO, Pennsylvania State University

Philosophically, this new text assumes that 1) it is never possible to oversimplify an idea in elementary mathematics; and 2) an idea has not been learned until it has been used in a meaningful application. That's why the authors offer step-by-step explanations with frequent progress checks, a thorough treatment of word problems, over 280 fully-worked examples, more than 3200 exercises, and lots of interesting applications to business, economics, and other fields. (*Instructor's Manual available.*)

Available in January, 1980 400 pages

Beginning Algebra

CHARLES P. MCKEAGUE, Cuesta College

A softbound version of the popular text ELEMENTARY ALGEBRA, BEGINNING ALGEBRA is designed for use as a workbook in self-paced or P.S.I. courses. Its easy-going, informal manner eases students into the study of the traditional basics with the aid of margin exercises, 370 worked examples including intermediate steps and explanations, *Common Mistakes* sections, a strong emphasis on word problems, plus abundant hints, warnings, and reassuring remarks. (*Instructor's Resource Manual available.*)

Available in January, 1980 350 pages, paper

Linear Algebra and Its Applications
Second Edition

GILBERT STRANG, Massachusetts Institute of Technology

A revision of the text that started a revolution in the way linear algebra is taught, this new edition continues to stress applications and computational aspects of linear algebra. Substantially rewritten chapters on *Gaussian Elimination and The Theory of Simultaneous Linear Equations* provide a more gradual, fully developed introduction to basic concepts of matrix operations and vector spaces. (*Instructor's Manual available.*)

Available in January, 1980 416 pages

Elementary Linear Programming with Applications

BERNARD KOLMAN, Drexel University, and ROBERT E. BECK, Villanova University

This new textbook for students of business, mathematics, computer science, and engineering surveys the techniques of linear programming and its extensions to integer and network programming. It demonstrates how these techniques can be applied to various models, illustrating the technique of applied mathematics in solving real-world problems.

Elementary Linear Programming with Applications takes a norigorous approach. Each idea is carefully motivated and illustrated by examples. The authors provide a brief introduction to operations research as well as a quick review of linear algebra. (*Solutions Manual available.*)

Available in December, 1979 350 pages

Principles of Mathematical Modeling

CLIVE L. DYM, University of Massachusetts, and ELIZABETH S. IVEY, Smith College

This new primer on the science of mathematical modeling conveys the fundamental ideas underlying model building to the freshman or sophomore level student. It is organized into two parts: foundations and applications. The authors begin by examining dimensional analysis, scaling, and elementary concepts of approximation of functions and curves. They then develop a series of models and discuss their origin, validity, and meaning. Linear and nonlinear models of a pendulum, traffic flow models, exponential models, operations research models, and optical and acoustical waves are presented. (*Solutions Manual available.*)

Available in December, 1979 250 pages

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Washington, D.C. 20036

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THE AMERICAN MATHEMATICAL MONTHLY

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Meetings of Its Sections

Meetings of Its Sections

ERRATA

Vol. 85

p. 716, lines 6 and 7 should read

$$\int_{-\infty}^{\log c} e^x dx + c|\log c|, \text{ i.e.,}$$

$$c + c|\log c|, \text{ i.e., } c(1-\log c).$$

line 26 should read

$$\lim_{x\to 0} \frac{1+\log\dfrac{1}{x+e}}{-x}$$

p. 717, line 4 should read

$$\frac{f(e)-f(e+x)}{-x}$$

line 5. Read "$x\to e$" for "$x\to 0$."

p. 724. Delete the second paragraph (beginning "If Riesz . . .").

Vol. 86

p. 33, lines 3–4. See this volume, p. 571, for a correction.
p. 248, line 9, for "had been a lawyer" read "had studied law."
p. 249, line 16, for "Australia," read "Great Britain."
line 6 from below, for "Russia" read "Poland."
p. 390, the formula displayed on line 11 should read
p. 593, E 1243. The Comment is by Ralf Winkel ("a" not "o").

$$f'(a)\approx\frac{1}{12h}[f(a-2h)-8f(a-h)+8f(a+h)-f(a+2h)].$$

DECEMBER

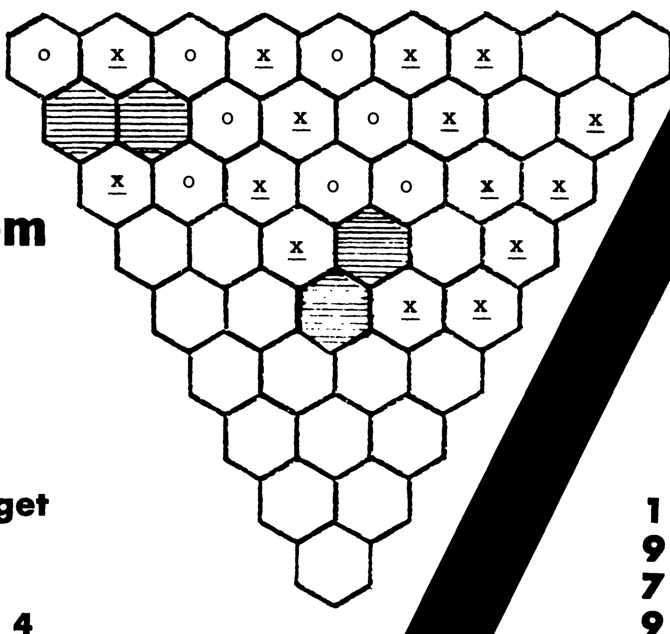
THE AMERICAN MATHEMATICAL MONTHLY

Volume 86, Number 10

How Grassmann Invented Linear Algebra

The Real Projective Plane Without Continuity

The Game of Hex and Brouwer's Fixed-Point Theorem



"Mathematics Today"— A Review from the Target

Detailed contents on cover 4

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HERMANN GRASSMANN AND THE CREATION OF LINEAR ALGEBRA

DESMOND FEARNLEY-SANDER

There is a way to advance algebra as far beyond what Vieta and Descartes have left us as Vieta and Descartes carried it beyond the ancients. ...We need an analysis which is distinctly geometrical or linear, and which will express *situation* directly as algebra expresses *magnitude* directly.

—Leibniz [**16**, p. 382]

1. Introduction. From Pythagoras to the mid-nineteenth century, the fundamental problem of geometry was to relate numbers to geometry. It played a key role in the creation of field theory (via the classic construction problems), and, quite differently, in the creation of linear algebra. To resolve the problem, it was necessary to have the modern concept of real number; this was essentially achieved by Simon Stevin, around 1600, and was thoroughly assimilated into mathematics in the following two centuries. The integration of real numbers into geometry began with Descartes and Fermat in the 1630's, and achieved an interim success at the end of the eighteenth century with the introduction into the mathematics curriculum of the traditional course in analytic geometry. From the point of view of analysis, with its focus on functions, this was entirely satisfactory; but from the point of view of geometry, it was not: the method of attaching numbers to geometric entities is too clumsy, the choice of origin and axes irrelevant and (in view of Euclid) unnecessary. Leibniz, in 1679, had mused upon the possibility of a universal algebra, an algebra with which one would deal directly and simply with geometric entities. The possibility is already suggested by a perusal of Euclid. For example, if D is a point in the side BC of a triangle ABC, then

$$\frac{BD}{BC} = \frac{ABD}{ABC};$$

this ancient theorem begs to be proved by simply multiplying numerator and denominator on the left by A. The geometric algebra of which Leibniz dreamed, and in which the concept of real number is thoroughly assimilated, was created by Hermann Grassmann in the mid-nineteenth century.

Grassmann looked on geometry as it might well be considered today but is not, as being applied mathematics: In his view, there is a part of mathematics, linear algebra, that is applicable to a part of the physical world, chalk figures on a blackboard or objects in space; and geometry, as the business of relating the two, does not belong to mathematics pure and simple. One may say without great exaggeration that Grassmann invented linear algebra and, with none at all, that he showed how properly to apply it in geometry. Linear algebra has become a part of the mainstream of mathematics, though Grassmann gets scant credit for it; but its application to geometry, affine and Euclidean, is remembered only in a half-baked version in which the notion of vector is all important and the notion of point is unnecessary. The reason is that there are other applications of linear algebra which are of greater practical importance (though they are no more interesting) than geometry—namely, within mathematics, to function spaces, and, within physics, to forces and other vectorial entities. [And yet, oddly enough, Grassmann's geometry *is* better suited to physics, since, for example, it distinguishes the notions of (polar) vector and axial vector; modern physics texts (such as Feynman [**9**]), in attempting to explain

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the physical distinction between the two types of entity, are handicapped by the fact that in the accepted model both have the same mathematical representation.]

2. Grassmann's life. A biography of Grassmann, by Friedrich Engel, may be found in the *Collected Works* [**13**, III.2] and also, more briefly, in Michael Crowe's scholarly work [**6**], the main modern source in English of which I am aware.

Hermann Gunther Grassmann was born in Stettin in 1809, lived there most of his life, and died in 1877. He was one of twelve children and, after marrying at the age of 40, fell short of his father in himself siring only eleven. He spent three years in Berlin studying theology and philology. He had no university mathematical training, nor did he ever hold a university post, though he repeatedly sought one. His life was spent as a schoolteacher.

His major mathematical works are *A Theory of Tides*, which is a kind of thesis written in 1840 in the hope of improving his status as a teacher and which was unpublished until the appearance of the *Collected Works* between 1894 and 1911; a book known briefly as the *Ausdehnungslehre* (literally "Theory of Extension"), which was published in 1844 and almost totally ignored, though it was drawn to the attention of Möbius, Gauss, Kummer, Cauchy, and others; and the *Ausdehnungslehre* of 1862, which was a new work on the same subject, rather than a new edition, and which met an equally cold reception. (Both are included in [**13**].) His many papers include important contributions to physics as well as to mathematics. He also wrote textbooks in mathematics and languages, edited a political journal for a time, and produced a translation of the *Rig Veda* and a huge commentary on it which, according to *Encyclopaedia Britannica*, is still used today. For his work in philology he received, in the last year of his life, an honorary doctorate. His achievements in mathematics were virtually unrecognized, and it has taken a century for their importance to become clearly visible.

While I intend to devote my attention to his linear algebra and geometry, there are two other contributions of Grassmann to mathematics which may be mentioned. In an arithmetic text [**12**] published in 1861, he defined the arithmetic operations for integers inductively and he proved their properties—commutativity, associativity, distributivity. He thus anticipated in its most important aspects Peano's treatment [**19**] of the natural numbers, published 28 years later. Peano generously acknowledges this, but in the naming game by which History distributes fame to the creators of mathematics Peano is a winner, Grassmann a loser. Dedekind, who published a similar development [**7**] of the natural numbers in 1888, makes no mention of Grassmann.

A feature of Grassmann's work, far in advance of the times, is the tendency toward the use of implicit definition—in which a mathematical entity is characterized by means of its formal properties rather than being obtained by an explicit construction. For example, in the *Ausdehnungslehre* of 1844 he comes very close indeed to the abstract notion of a (not necessarily associative) ring; what is lacking is the language of set theory. This is the second contribution I wanted to mention. Incidentally, the first formal definition of a ring was given by Fraenkel [**10**] in 1915.

3. The invention of linear algebra. From the beginning, Grassmann distinguished linear algebra, as a formal theory independent of any interpretation, from its application in geometry. However, in the first *Ausdehnungslehre*, the algebra is intermixed with its geometric interpretation—indicating, very interestingly, how he came upon the ideas. Those who did read his work in the late nineteenth century found it easier to follow the 1862 *Ausdehnungslehre*, in which, in modern style, the full development of the mathematical theory precedes its application, and in outlining his linear algebra I shall mainly follow the latter work; it should be borne in mind, though, that some of the ideas have their origin as much as two decades previously.

The definition of a linear space (or vector space) came into mathematics, in the sense of becoming widely known, around 1920, when Hermann Weyl [**22**] and others published formal definitions. In fact, such a definition had been given thirty years previously by Peano [**18**], who was thoroughly acquainted with Grassmann's mathematical work. Grassmann did not put down

a formal definition—again, the language was not available—but there is no doubt that he had the concept. Beginning with a collection of "units" $e_1, e_2, e_3, \ldots$, he effectively defines the free linear space which they generate; that is to say, he considers formal linear combinations $\Sigma\alpha_i e_i$, where the α_i are real numbers, defines addition and multiplication by real numbers by setting

$$\Sigma\alpha_i e_i + \Sigma\,\beta_i e_i = \Sigma(\alpha_i + \beta_i)e_i$$

and

$$\alpha(\Sigma\alpha_i e_i) = \Sigma(\alpha\alpha_i)e_i,$$

and formally proves the linear space properties for these operations. (At the outset, it is not clear whether the set of units is allowed to be infinite, but finiteness is implicitly assumed in some of his proofs.) He then develops the theory of linear independence in a way which is astonishingly similar to the presentation one finds in modern linear algebra texts.

He defines the notions of subspace, independence, span, dimension, join and meet of subspaces, and projections of elements onto subspaces. He is aware of the need to prove invariance of dimension under change of basis, and does so. He proves the Steinitz Exchange Theorem, named for the man who published it **[20]** in 1913 (and who, incidentally, defined a linear space in terms of "units" in the same way Grassmann did). Among other such results, he shows that any finite set has an independent subset with the same span and that any independent set extends to a basis, and he proves the important identity

$$\dim(U+W) = \dim U + \dim W - \dim(U\cap W).$$

He obtains the formula for change of coordinates under change of basis, defines elementary transformations of bases, and shows that every change of basis (equivalently, in modern terms, every invertible linear transformation) is a product of elementaries.

4. Products. In a paper [**11**] published in 1855, Grassmann defines a product of elements of a linear space by setting

$$(\Sigma\alpha_i e_i)(\Sigma\,\beta_j e_j) = \Sigma\alpha_i\beta_j e_i e_j,$$

and he proves distributivity. (In this paper the scalars are explicitly allowed to be complex.) If the $e_i e_j$ are themselves linear combinations of the e_i, we have here the concept of an algebra. Instead of following this path (though he did later observe that the algebra of quaternions is a special case), Grassmann singles out particular products by "equations of condition"

$$\Sigma\xi_{ij}e_i e_j = 0,$$

and, observing as a disadvantage of this notion that it lacks invariance under change of basis, he proceeds to characterize those products whose conditioning equations are invariant under various substitutions.

Grassmann's declared motive for publishing this paper was to claim priority for some results that had been published by Cauchy. The interesting story is related by Engel. In 1847 Grassmann had wanted to send a copy of the *Ausdehnungslehre* to Saint-Venant (to show that he had anticipated some of Saint-Venant's ideas on vector addition and multiplication), but, not knowing the address, Grassmann sent the book to Cauchy, with a request to forward it. Cauchy never did so. And six years later Cauchy's paper [**4**] appeared in *Comptes Rendus*. Grassmann's comment was that, on reading this, "I recalled at a glance that the principles which are there established and the results which are proved were exactly the same as those which I published in 1844, and of which I gave at the same time numerous applications to algebraic analysis, geometry, mechanics and other branches of physics." An investigating committee of three members of the French Academy, including Cauchy himself, never came to a decision on the question of priority.

In the two *Ausdehnungslehren*, Grassmann singles out for special attention those products, which he calls *linear products*, for which the conditioning equations are invariant under change

of basis. He shows that, apart from two trivial cases, there are only two possible types of linear product: it must be the case that

$$\text{either (1) } e_ie_j = e_je_i$$

$$\text{or (2) } e_ie_j = -e_je_i \text{ (and, in particular, } e_i^2 = 0)$$

for all i and j. Whereas in the 1855 paper he does not consider higher order products, in the *Ausdehnungslehren* he examines in detail products of the second type, extended, by imposing associativity, to allow multiplication of products of the original simple units (such a product being called a *compound unit*) and, by imposing distributivity, to allow multiplication of linear combinations of compound units (such a combination being called a *form*). Condition (2) entails certain relations among compound units of higher order (for example that $e_1e_3e_2 = -e_1e_2e_3$), but it is assumed that no other relations hold. Dimension plays a role here, since, for example, in the three-dimensional case we have only a single independent third-order unit, say $e_1e_2e_3$, and this forces independence of the three units e_1e_2, e_2e_3, and e_3e_1 of order 2, because

$$\xi_1e_1e_2 + \xi_2e_2e_3 + \xi_3e_3e_1 = 0$$

implies, when we multiply by e_3, that $\xi_1e_1e_2e_3 = 0$ and hence $\xi_1 = 0$ (and, similarly, $\xi_2 = 0 = \xi_3$).

This multiplication is nowadays called *exterior multiplication*. Half a century later, in treating Grassmann's ideas at length in his *Universal Algebra* **[23]**, A. N. Whitehead explicitly excludes forms of mixed degree like $e_1 + e_1e_2$ on the rather metaphysical ground that they are "meaningless"; he thus admits arbitrary products of linear combinations of simple units, but not arbitrary linear combination of products. Whitehead's objection itself would have been meaningless to Grassmann and, although he never explicitly brings in such forms, neither, so far as I can see, does he explicitly exclude them. If I am right about this, then Grassmann has the full exterior algebra, while Whitehead's presentation (like many modern treatments of exterior products) restricts consideration to a graded linear space (a far less tidy structure than an algebra).

The full development of exterior algebra as Grassmann did it (in particular the essential invariance under change of basis) is complicated and must be omitted. Perhaps the most important fact is that elements $a_1, a_2, \ldots, a_k$ of the original linear space are linearly independent if and only if $a_1a_2 \cdots a_k \neq 0$; Grassmann proves this in the modern way and gives the following application (with notation precisely as I have written it) to a system of n linear equations in n unknowns,

$$\begin{gathered}
\alpha_1^{(1)}x_1 + \alpha_2^{(1)}x_2 + \cdots + \alpha_n^{(1)}x_n = \beta^{(1)} \\
\alpha_1^{(2)}x_1 + \alpha_2^{(2)}x_2 + \cdots + \alpha_n^{(2)}x_n = \beta^{(2)} \\
\cdots \\
\alpha_1^{(n)}x_1 + \alpha_2^{(n)}x_2 + \cdots + \alpha_n^{(n)}x_n = \beta^{(n)}.
\end{gathered}$$

Introducing (independent) units $e^{(1)}, e^{(2)}, \ldots, e^{(n)}$ and quantities

$$\begin{gathered}
a_1 = \alpha_1^{(1)}e^{(1)} + \alpha_1^{(2)}e^{(2)} + \cdots + \alpha_1^{(n)}e^{(n)} \\
a_2 = \alpha_2^{(1)}e^{(1)} + \alpha_2^{(2)}e^{(2)} + \cdots + \alpha_2^{(n)}e^{(n)} \\
\cdots \\
a_n = \alpha_n^{(1)}e^{(1)} + \alpha_n^{(2)}e^{(2)} + \cdots + \alpha_n^{(n)}e^{(n)}
\end{gathered}$$

and

$$b = \beta^{(1)}e^{(1)} + \beta^{(2)}e^{(2)} + \cdots + \beta^{(n)}e^{(n)},$$

we have

$$b = x_1a_1 + x_2a_2 + \cdots + x_na_n,$$

and so $ba_2a_3 \cdots a_n = x_1a_1a_2 \cdots a_n, \ldots$ Thus if $a_1a_2 \cdots a_n \neq 0$ we have the unique solution

$$x_1 = \frac{ba_2a_3 \cdots a_n}{a_1a_2a_3 \cdots a_n}, \; x_2 = \ldots, \ldots .$$

(Equivalently, x_1 is the number obtained by dividing the (non-zero) determinant of the matrix $(\alpha_j^{(i)})$ into the determinant of the matrix obtained from $(\alpha_j^{(i)})$ by replacing each $\alpha_1^{(i)}$ by $\beta^{(i)}$.) This is Cramer's rule [5]. The same elegant derivation (but without the double subscript-superscript notation) is given in the first *Ausdehnungslehre*. It is one of the techniques that occurs in the above-mentioned paper of Cauchy.

5. Inner products. Grassmann derives the concept of inner product from that of exterior product in a very interesting way. Working in the algebra generated by the simple units $e_1, e_2, \ldots, e_n$ (subject, as always, to (2)), he defines the *supplement* $|E$ of a compound unit E as $+1$ or -1 times the product of those of the simple units that are not factors of E, the sign $+$ or $-$ being chosen in such a way that

$$E|E = +e_1e_2 \cdots e_n.$$

For example, in the three-dimensional case,

$$|e_1 = e_2e_3 \quad \text{and} \quad |e_1e_3 = -e_2.$$

The supplement is extended to linear combinations of compound units of the same order by linearity:

$$|\Sigma \alpha_j E_j = \Sigma \alpha_j |E_j.$$

(The map $A \to |A$ is nowadays called the *Hodge star-operator*.) If E and F are forms of the same order, then necessarily $E|F$ is a multiple of $e_1e_2 \cdots e_n$; indeed, since the forms of order n make up a one-dimensional linear space, one may identify them with the scalars, and this Grassmann does, setting $e_1e_2 \cdots e_n = 1$. Although $E|F$ makes sense for forms E and F of different order, it is only in the case where both orders are the same (in particular, where they are both 1) that $E|F$ is a number. Noting that $E|F$ is linear in both E and F, Grassmann calls $E|F$ the *inner product* of E and F. Its restriction to the original linear space (the space of forms of order 1) is indeed an inner product in the modern sense, since, as he shows,

$$e_i|e_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j, \end{cases}$$

and hence

$$\Sigma \alpha_i e_i | \Sigma \beta_j e_j = \Sigma \alpha_i \beta_i.$$

He calls $\sqrt{a|a}$ the *numerical value* of a. The notion of a complete orthonormal set is introduced, and it is shown that such a set must be independent and that in theorems involving the inner product the original system of units may be replaced by any such set. The notions of orthogonal complement and orthogonal projection are investigated; the *Gram-Schmidt Process* is at least implicitly involved in this.

6. Linear transformations. For the linear transformation that carries the basis elements $e_1, e_2, \ldots, e_n$ to $b_1, b_2, \ldots, b_n$, respectively, Grassmann writes

$$Q = \frac{b_1, b_2, \ldots, b_n}{e_1, e_2, \ldots, e_n},$$

and considers Q to be a generalized quotient. While there are obvious problems with this notation, it does have a certain elegance; for example, if $b_1, b_2, \ldots, b_n$ are independent then the inverse of Q is

$$\frac{e_1, e_2, \ldots, e_n}{b_1, b_2, \ldots, b_n}.$$

He effectively obtains the matrix representation of Q by showing that it may be written

$$Q=\Sigma\alpha_{r,s}E_{r,s}$$

where

$$E_{r,s}=\frac{0,\ldots,0,e_s,0,\ldots,0}{e_1,\ldots,e_r,\ldots,e_n}.$$

The *determinant* of Q is defined to be the number

$$\frac{b_1b_2\cdots b_n}{e_1e_2\cdots e_n},$$

eigenvalues and *eigenvectors* are introduced (though different terms are used), and the fact that the eigenvalues are roots of the characteristic polynomial is demonstrated as follows. Suppose that

$$Qx=\rho x$$

where $x=\Sigma\xi_i e_i\neq 0$. Writing $c_i=(\rho-Q)e_i$, we see that $\Sigma\xi_i c_i=0$; thus $c_1,c_2,\ldots,c_n$ are dependent and hence

$$[(\rho-Q)e_1][(\rho-Q)e_2]\cdots[(\rho-Q)e_n]=0.$$

Since $e_1e_2\cdots e_n\neq 0$, this is equivalent to the vanishing of the determinant of the linear map $\rho-Q$. It is shown that the eigenvectors corresponding to distinct eigenvalues are independent, and the *spectral theorem* for a symmetric Q is proved. Turning to a general linear transformation Q, Grassmann shows, in effect (by constructing an appropriate basis), that the whole space may be decomposed as the direct sum of invariant subspaces W_ρ where ρ ranges through the characteristic roots of Q, and each W_ρ is the kernel of $(Q-\rho)^k, k$ being the algebraic multiplicity of ρ. While the *spectral theorem* had been proved by Weierstrass **[21]** in 1858 (and, for the case of n distinct eigenvalues, by Cauchy **[3]** in 1829), it appears that Grassmann was the first to prove the latter result, which is sometimes called the *primary decomposition theorem*; it goes part of the way to obtaining Jordan's canonical form, published in 1870 **[15]** (the remaining step being to reduce the nilpotent maps $Q-\rho$: $W_\rho\to W_\rho$).

7. Geometry. In the *Ausdehnungslehre* of 1844, Grassmann describes the geometric considerations which led him to the theory that we now call linear algebra. Musing on the formula

$$AB+BC=AC, \tag{3}$$

which one might find in old geometry texts, to describe a relationship between lengths that holds for collinear points A, B and C, with B between A and C, he realized that the formula remains valid regardless of the order of the three collinear points, *provided that one sets*

$$BA=-AB; \tag{4}$$

for example, if C lies between A and B then (3) follows from the fact that

$$AB=AC+CB=AC-BC.$$

Over many years Grassmann carefully investigated the consequences of (4), which is the special property that defines an exterior algebra. His development of geometry is complicated, and we shall give here an oversimplified presentation that brings one rapidly to the heart of the matter; those who regard it as criminal to attempt a modern paraphrase of old mathematics should read no further. To avoid (merely notational) complications we shall, like Grassmann, restrict ourselves to three dimensions.

Beginning with the basic material of geometry, numbers and points, we permit them to be combined by formal operations of sum and product, assuming the elementary algebraic rules for such operations, but subject to the condition that (4) holds for all points A and B and also that

$$\alpha A=A\alpha \tag{5}$$

for all real numbers α and points A. From (4) we deduce that every point A has $A^2=0$; thus the square of a point is a number, and we explicitly assume that it is not a point:

$$0 \text{ is not a point.} \tag{6}$$

We need a rule by which to interpret geometrically the entities which occur in this formal algebra. For a pair of points A and B, and positive real numbers α and β with $\alpha+\beta=1$, write

$$P=\alpha A+\beta B.$$

Then immediately we have

$$AP=\beta AB, PB=\alpha AB \quad \text{and} \quad AP+PB=AB;$$

these formulas suggest that P should be interpreted as the unique point which divides the line segment from A to B in the ratio β to α. We do so interpret P; it is here, via the bijection $\alpha \to \alpha A+\beta B$ between $[0,1]$ and the line segment, that numbers enter geometry, and *all* other geometric interpretations follow from this one. To give an immediate example, the interpretation of $P=\alpha A+\beta B$ with $\alpha \leqslant 0$ and $\beta>0$ and $\alpha+\beta=1$ is forced by the fact that, equivalently

$$B=\left(-\frac{\alpha}{\beta}\right)A+\left(\frac{1}{\beta}\right)P,$$

where $-\alpha/\beta \geqslant 0, 1/\beta>0$ and $(-\alpha/\beta)+1/\beta=1$. The *line* through A and B is the set of all $\alpha A+\beta B$ with $\alpha+\beta=1$, and accordingly we assume that

$$\text{if } A \text{ and } B \text{ are points and } \alpha+\beta=1 \text{ then } \alpha A+\beta B \text{ is a point.} \tag{7}$$

To sum up, we consider a ring Ω, in which the number 1 is a unit element and which is generated by $R \cup P$, where R is the set of real numbers and P is a set whose elements are called *points*, subject to the conditions (4), (5), (6), and (7); equivalently, Ω may be regarded as an algebra generated by P. Dimensionality comes in with the assumption that

$$\Omega \text{ is generated, as an algebra, by four elements of } P \text{ but not by three,} \tag{8}$$

and, finally, ensuring non-triviality of multiplication, that

$$\text{there exist points } A, B, C, D \text{ with } ABCD \neq 0. \tag{9}$$

Existence and uniqueness of such a structure are proved in [**24**]. We then have a model for the geometry of three-dimensional space; the multiplication is an exterior product and Grassmann's abstract theory may now be brought to bear in a situation where geometric interpretation is possible.

The difference of two points is a *vector*; here the interpretation is forced by the identity

$$B-A=C-D \Leftrightarrow \tfrac{1}{2}(A+C)=\tfrac{1}{2}(B+D).$$

The sum of a vector X and a point A is a point, since

$$A+X=B \Leftrightarrow X=B-A.$$

And a product of a number and a vector is a vector, since for $X=B-A$ we have $\alpha X=P-A$ where $P=(1-\alpha)A+\alpha B$; this also entails that αX is to be interpreted as having the same direction as X and α times its length.

Here is a classic theorem which with just these ideas becomes trivial: if, in a triangle with vertices A, B, and C, the points D and E, respectively, divide the side from A to B and the side from A to C in equal ratios, then the line segment from D to E is parallel to the one from B to C and the ratio of their lengths is the appropriate number. The proof is one line:

$$D=\alpha A+\beta B,\ E=\alpha A+\beta C \Rightarrow D-E=\beta(B-C).$$

The hypotheses (8) and (9) imply that the space of all fourth order forms is one-dimensional: if A, B, C, and D are independent points then any product $A'B'C'D'$ is a multiple of $ABCD$ and one may show that the ratio must be interpreted as the ratio of the oriented volumes of the associated tetrahedra. In particular, $ABCD=A'B'C'D'$ means that the two tetrahedra have the

same orientation and volume. This in turn forces one to interpret $ABC=A'B'C'$ as meaning that the associated triangles are coplanar and have the same orientation and area; and $AB=A'B'$ as meaning that the associated line segments lie on the same line and are equal in length and direction.

We may now prove the converse of the above result about triangles in exactly the way that it was done by Euclid. If the line segments from D to E and from B to C are parallel, then

$$\frac{DB}{AB}=\frac{DBC}{ABC}=\frac{EBC}{ABC}=\frac{EC}{AC},$$

where we have used the fact that for a suitable real α

$$DBC=[E+\alpha(B-C)]BC=EBC.$$

Again, here is the proof that the diagonal of a parallelogram bisects it:

$$D=C+A-B\Rightarrow ABD=-ABC.$$

These examples give an indication, if no more, of the power of Grassmann's geometric interpretation of linear algebra. They are results of affine geometry, but by introducing an inner product one gets equally transparent algebraic proofs of the theorems of Euclidean geometry and trigonometry. (Incidentally, Grassmann, with his inclination toward abstraction and generality, displays little interest in proving known results of geometry, and these proofs are mine; many other examples are given by Forder [**8**].)

8. Contemporary and later developments. Though in 1844 he had been unaware of their work, Grassmann later acknowledged that in some respects his theory was anticipated, in particular, by the concept of vector addition of Bellavitis [**1**] and others, and by the barycentric calculus of Möbius [**17**]. But no one had approached the elegant simplicity of the formula

$$(C-B)+(B-A)=C-A$$

which, for Grassmann, forces the interpretation of the sum of two vectors; and Möbius had not exploited the full possibilities of his notation

$$P=\alpha A+\beta B+\gamma C,$$

which, even today, in his brilliant [**2**], is dismissed by Boyer as inferior to the homogeneous coordinate representation $P=(\alpha,\beta,\gamma)$. The key to the difference between Möbius and Grassmann is that, whereas for Möbius the equation $\alpha A+\beta B=\alpha' A+\beta' B$ entails merely that $\alpha:\beta=\alpha':\beta'$, for Grassmann it implies that $\alpha=\alpha'$ and $\beta=\beta'$; the one concept is appropriate to projective geometry, the other to Euclidean.

The story of Hamilton's invention of the quaternions [**14**] in 1843 and of the subsequent influence of Hamilton and of Grassmann in the emergence of vector analysis is well told by Crowe. But vector analysis in Crowe's sense of the term is a subject that has ceased to exist, or should have; it has been absorbed by linear algebra. It would not be easy to estimate the relative influences of the two men in the development of linear algebra as we know it, but there is no doubt that Grassmann in his own work came far closer to it than Hamilton or any of his contemporaries. Even in those cases where forerunners may be discerned, his results, and especially his methods, were highly original. All mathematicians stand, as Newton said he did, on the shoulders of giants, but few have come closer than Hermann Grassmann to creating, single-handedly, a new subject.

9. Conclusion. The genius of Descartes revealed itself in his decision to drop the ancient convention that a product of line segments is a rectangle. In Grassmann's geometry a product of line segments is again a higher-dimensional object. It is a return to Euclid, but to Euclid with a difference, the difference that had been dreamed of by Leibniz. But Grassmann's geometry (as distinct from his linear algebra) has been largely forgotten. Perhaps this is because, at the moment when it might have been remembered, Hilbert, with his immense prestige, closed the

book of Euclid by showing that Euclid's program, rigorously carried through, was too tedious for anyone to bother with. Grassmann's way is not tedious; properly done, it is simple, direct, and powerful, and perhaps the book should be opened again.

I conclude with a quotation (as translated in [6, p. 89]) from the preface to the 1862 *Ausdehnungslehre*:

> I remain completely confident that the labour I have expended on the science presented here and which has demanded a significant part of my life as well as the most strenuous application of my powers, will not be lost. It is true that I am aware that the form which I have given the science is imperfect and must be imperfect. But I know and feel obliged to state (though I run the risk of seeming arrogant) that even if this work should again remain unused for another seventeen years or even longer, without entering into the actual development of science, still that time will come when it will be brought forth from the dust of oblivion and when ideas now dormant will bring forth fruit. I know that if I also fail to gather around me (as I have until now desired in vain) a circle of scholars, whom I could fructify with these ideas, and whom I could stimulate to develop and enrich them further, yet there will come a time when these ideas, perhaps in a new form, will arise anew and will enter into a living communication with contemporary developments. For truth is eternal and divine.

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THE GAME OF HEX AND THE BROUWER FIXED-POINT THEOREM

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1. Introduction. The application of mathematics to games of strategy is now represented by a voluminous literature. Recently there has also been some work which goes in the other direction, using known facts about games to obtain mathematical results in other areas. The present paper is in this latter spirit. Our main purpose is to show that a classical result of topology, the celebrated Brouwer Fixed-Point Theorem, is an easy consequence of the fact that Hex, a game which is probably familiar to many mathematicians, cannot end in a draw. This fact is of some practical as well as theoretical interest, for it turns out that the two-player, two-dimensional game of Hex has a natural generalization to a game of n players and n dimensions, and the proof that this game must always have a winner leads to a simple algorithm for finding approximate fixed points of continuous mappings. This latter subject is one of considerable current interest, especially in the area of mathematical economics. This paper has therefore the dual purpose of, first, showing the equivalence of the Hex and Brouwer Theorems and, second, introducing the reader to the subject of fixed-point computations.

I should say that over the years I have heard it asserted in "cocktail conversation" that the Hex and Brouwer Theorems were equivalent, and my colleague John Stallings has shown me an argument which derives the Hex Theorem from familiar topological facts which are equivalent to the Brouwer Theorem. The proof going in the other direction only occurred to me recently, but in view of its simplicity it may well be that others have been aware of it. The generalization to n dimensions may, however, be new.

In the next section we will present the relevant facts about Hex. In Section 3 we prove the equivalence of the Hex and Brouwer Theorems. The general Hex Theorem and fixed-point algorithm are presented in the final section.

2. Hex. For a brief history of the game of Hex the reader should consult [**2**]. The game was invented by the Danish engineer and poet Piet Hein in 1942 and rediscovered at Princeton by John Nash in 1948. It was produced commercially by Parker Brothers in 1952 but has been "out of print" in this country for many years. However, the game is still very popular and much played in other countries, including France, Germany, and Israel.

Figure 1 represents an 11×11 Hex board. The rules of the game are very simple. As in tic-tac-toe, the two players move alternately, marking any previously unmarked hexagon or *tile* with an x or o respectively. The game has been won by the x- (o-) player if he/she has succeeded in marking a *connected* set of tiles which meets the boundary regions X and X' (O and O'). (A set S of tiles is connected if any two members h and h' of S can be joined by a *path* $P=(h=h^1,h^2,\ldots,h^m=h')$ where h^i and h^{i+1} are adjacent.)

In Figure 1 neither player has won and it is the o-player's move, but the x-player has a sure win in 3 moves if he plays on the shaded tiles, as the reader will easily verify. The potentially winning connected set is indicated by the tiles marked by the underlined x's.

The appealing feature of Hex is that, unlike tic-tac-toe, it can never end in a draw, because, as explained in [**2**], "one player can block the other only by completing his own chain." To be a bit more precise let us state:

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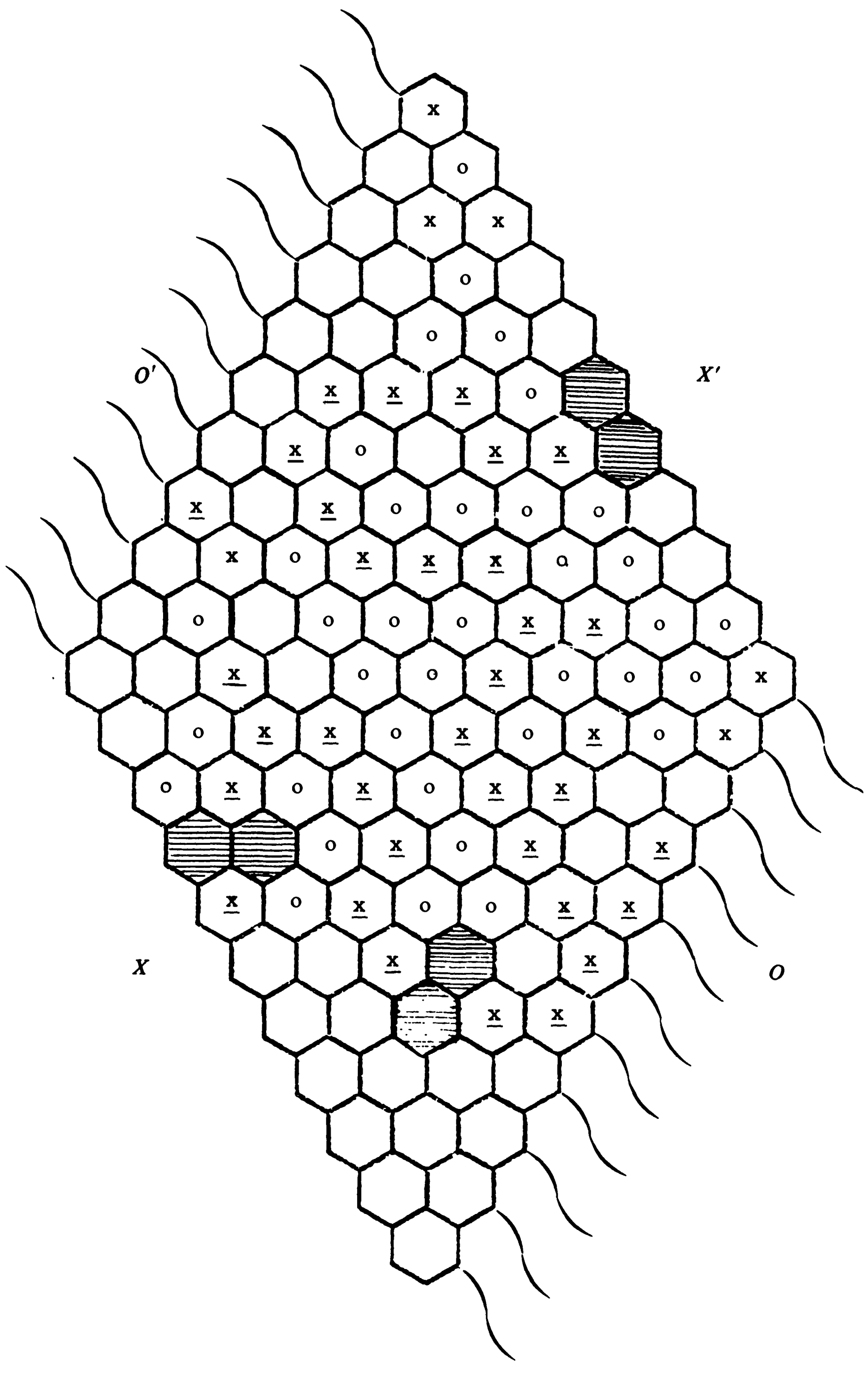

FIG. 1

HEX THEOREM. *If every tile of the Hex board is marked either* x *or* o, *then there is either an* x-*path connecting regions X and X' or an* o-*path connecting regions O and O'.*

The theorem is indeed intuitively quite obvious. Imagine, for example, that the X-regions are portions of opposite banks of the river "O" (as suggested by the ripples in the figure) and that the x-player is trying to build a dam by putting down stones. It is quite clear that he will have succeeded in damming the river only if he has placed his stones in a way which enables him to walk on them from one bank to the other. Nevertheless, this intuitively obvious fact is mathematically "deep," in that, as we shall see, it leads to a very short proof of the Brouwer Theorem.

Of course, intuitively obvious theorems which are hard to prove are nothing new in topology. The most celebrated case is the Jordan Curve Theorem, and it turns out that this theorem too is related to Hex; that is, one can strengthen the Hex Theorem by appending at the end of the statement the words "but not both." This latter property is, if possible, even more obvious than the fact that one of the players must win; for clearly—to continue the land-water analogy—if the x-player succeeds in constructing a causeway from X to X', he will in the process have dammed the river and prevented any flow from O to O'. The connection with the Jordan Theorem is evident. We wish to emphasize, however, that the analysis to follow and the relation to the Brouwer Theorem depends only on the no-draw property. The fact that it is not possible for both players to construct winning sets is special to two dimensions and the argument involved in proving it will not be given here, since for our purposes it is irrelevant. (Recently D. Lichtenstein has shown me a neat, purely combinatorial proof of "but not both," but the methods used—induction on the size of the board—are quite different from those to be used here to prove "at least one"; so once again it appears that these two results are mathematically quite distinct.)

How then does one prove the (weak) Hex Theorem? The only published proof I know of is in the elementary text **[1]**. The proof there is quite rigorous, though "informal," meaning that it uses English and pictures rather than symbols. It runs to some four pages of fine print and invokes (unnecessarily, I believe) the Jordan Curve Theorem. The proof to be presented here is closely related to that of **[1]** but is considerably shorter. We will also remain informal in this section so as to emphasize the intuitive geometrical ideas which also form the basis of the formal n-dimensional proof of Section 4.

Here then is our proof. Suppose the board has been covered by x's and o's as in Figure 2. By an X-face we will mean either a tile marked x or one of the regions X or X'. An O-face is defined analogously.

We consider the edge graph Γ of the Hex board to which additional edges ending in vertices u, u', v, v' have been added to separate the four boundary regions, as shown in the figure. We now present an algorithm for finding a winning set on the completely marked board. We shall make a tour along Γ, starting from the vertex u and following the simple rule of always proceeding along an edge which is the common boundary of an X-face and an O-face. Note that the edge from u has this property since it separates regions X and O. The key observation is that this touring rule determines a unique path; for suppose one has proceeded along some edge e and arrives at a vertex w. Two of the three faces incident to w are those of which e is the common boundary, hence one is an X-face, the other an O-face. The third face incident to w may be either an X-face or an O-face, but in either case there is exactly one edge e' which satisfies the touring rule. Figure 2 makes this clear (in Section 4 the situation will be completely "arithmetized" so that no picture will be needed).

Again one could make the seemingly stronger touring rule that the traveler always keep an X-face on his "right" and a O-face on his "left," but this would involve getting into the quite complex notion of orientation, which is not needed for our proof. The proof depends now on the fact that our touring rule guarantees that *we will never revisit any vertex*. On the other hand, since there are only a finite number of vertices of the graph, the tour must terminate; but the

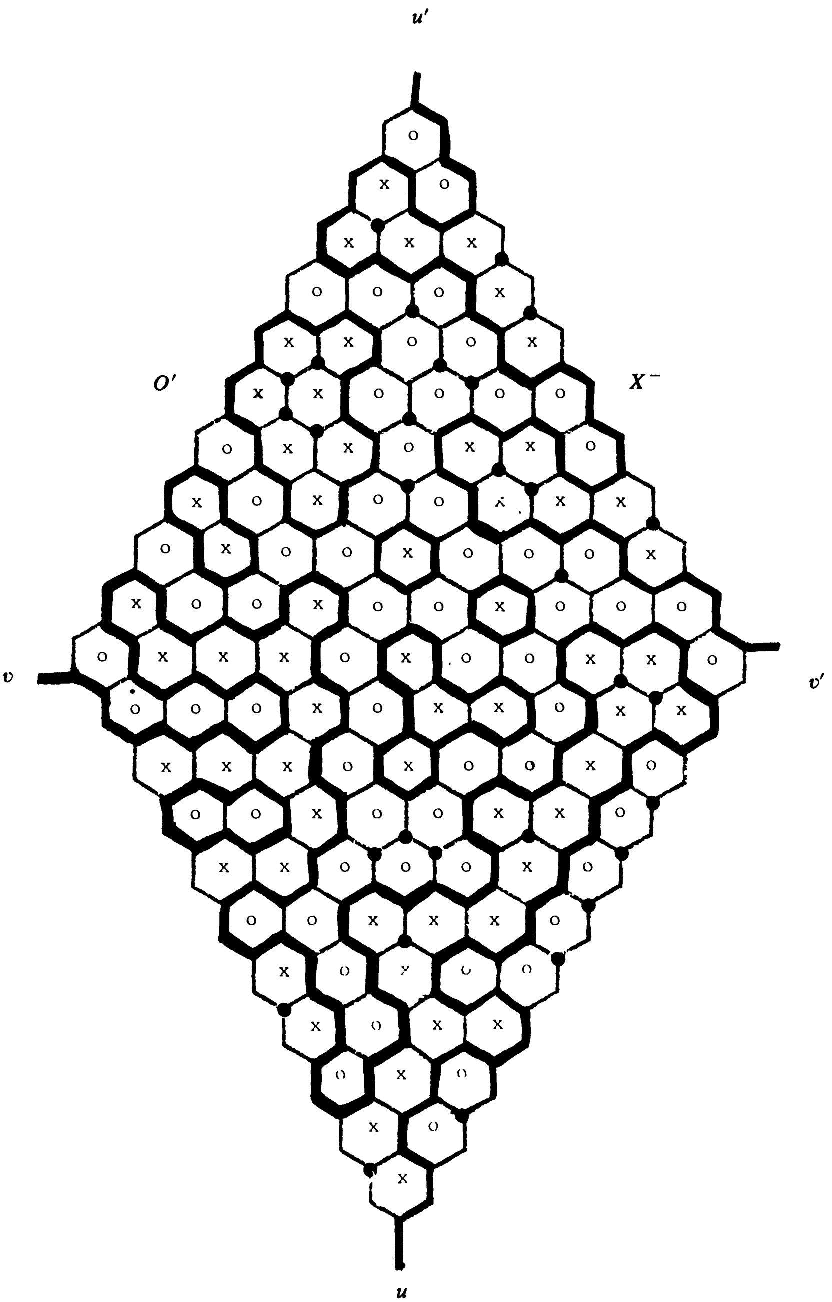

FIG. 2

only possible terminals are the vertices u', v, and v' (actually u' is not a possibility because of the "right-left" condition; but, again, we need not prove this more difficult fact). Now note that each of these three vertices is incident to one of the regions X' or O' (or both in the case of u'). If, say, the terminal vertex is incident to X', then the X-player has a winning set; for clearly the set of X-faces which are incident to our edge tour is a connected set and meets both X and X'.

Before concluding the section we wish to point out that the crucial feature of our algorithm is the italicized statement in the above paragraph which guarantees that the procedure cannot "cycle." In fact, the result which is the basis for all "fixed-point-chasing" algorithms is the following obvious fact from graph theory.

GRAPH LEMMA. *A (finite) graph whose vertices have degree at most two is the union of disjoint subgraphs, each of which is either* (i) *an isolated vertex*, (ii) *a simple cycle*, (iii) *a simple path.*

We will not insult our readers' intelligence by presenting a proof of this simple fact but suggest that they take a few minutes to see how it goes. Induction on the number of edges is recommended. The connection with the Hex algorithm should be clear. If we consider only subgraph Γ' of Γ consisting of edges which separate an X-face and an O-face, then clearly the hypothesis of the Graph Lemma is satisfied and the conclusion shows that the component of Γ starting from u must end up at one of the other degree-one vertices u', v, v'.

Figure 2 shows the graph Γ' for the completely marked board. It consists in this case of 6 simple cycles, 30 isolated vertices, and the two simple paths, in this case from u to v' and from u' to v.

3. The Equivalence of the Hex and Brouwer Theorems. For analytic purposes it is convenient to use a different but equivalent model for the Hex board. Actually, when John Nash rediscovered the game in 1948, he thought of it as being played on a checkerboard where two squares were considered to be adjacent if they were next to each other horizontally, vertically, or along a positively sloping diagonal. It is clear that this is combinatorially equivalent to the more aesthetic hexagon arrangement. This representation is easily arithmetized in a way which will generalize to n dimensions.

Notation.

Z^n denotes the lattice points of R^n. For $x \in R^n$, $|x| = \max_i x_i$. For $x \neq y \in R^n$, $x < y$ if $x_i \leqslant y_i$ for all i. The points x and y are called *comparable* if $x < y$ or $y < x$.

The (two-dimensional) *Hex board B_k of size k* is a graph whose *vertices* consist of all z in Z^2 with $(1,1) \leqslant z \leqslant (k,k)$. Two vertices z and z' are *adjacent* (span an edge) in B_k if

$$|z - z'| = 1, \tag{3.1}$$

$$z \text{ and } z' \text{ are comparable.} \tag{3.2}$$

Figure 3 gives the graphical representation of a Hex board of size 5. The boundary edges are labeled by the points of the compass N, S, E, W and consist of the vertices $z = (z_1, z_2)$ of B_k with $z_2 = k$, $z_2 = 0$, $z_1 = k$, $z_1 = 0$, respectively. We think of the *horizontal (vertical)* player as trying to make a path connecting E and W (N and S). We can now restate

HEX THEOREM. *Let B_k be covered by two sets H and V. Then either H contains a connected set meeting* E *and* W *or V contains a connected set meeting* N *and* S.

A purely combinatorial proof of this theorem generalized to n dimensions will be given in the next section. In this section we will show that it is equivalent to

BROUWER FIXED-POINT THEOREM. *Let f be a continuous mapping from the unit square I^2 into itself. Then there exists $x \in I^2$ such that $f(x) = x$.*

We first show, that "Hex" implies "Brouwer." Let $f : I^2 \to I^2$ be given by $f(x) = (f_1(x), f_2(x))$. From compactness of I^2 it suffices to show that for any $\varepsilon > 0$ there exists $x \in I^2$ such that

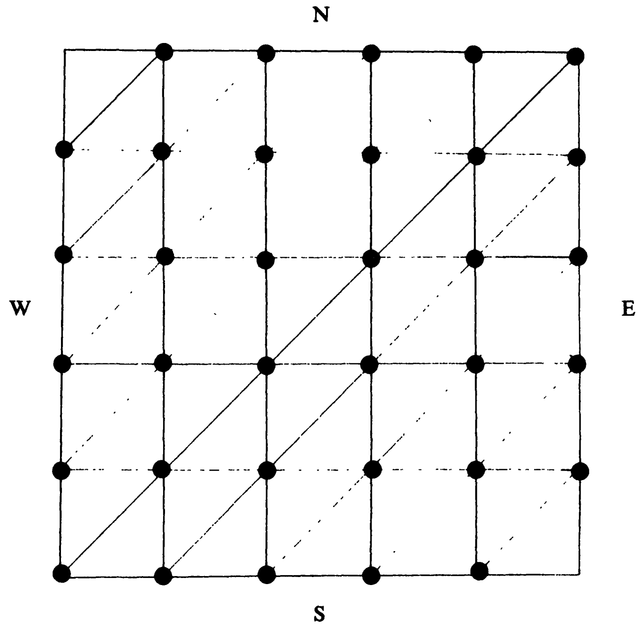

FIG. 3

$|f(x)-x|<\varepsilon$. By uniform continuity of f we know that, given $\varepsilon>0$, there is a $\delta>0$ such that $\delta<\varepsilon$ and if $|x-x'|<\delta$ then $|f(x)-f(x')|<\varepsilon$.

Now consider the Hex board B_K where $1/k<\delta$. We will define four subsets H^+, H^-, V^+, V^- of B_K as follows:

$$H^+=\{z|f_1(z/k)-z_1/k>\varepsilon\}$$
$$H^-=\{z|z_1/k-f_1(z/k)>\varepsilon\}$$
$$V^+=\{z|f_2(z/k)-z_2/k>\varepsilon\}$$
$$V^-=\{z|z_2/k-f_2(z/k)>\varepsilon\}.$$

Intuitively, a vertex z belongs to H^+, H^-, V^+, V^- according as z/k is moved by f at least ε units to the right, left, up, or down, respectively.

The theorem will be proved if we can show that these four sets do not cover B_k; for, if vertex z lies in none of them, then $|f(z/k)-z/k|<\varepsilon$. The key observation is now that the (disjoint) sets H^+ and H^- (V^+ and V^{-1}) are not *contiguous* (a pair of subsets A and B of a graph are called contiguous if there exists $a\in A$ and $b\in B$ where a and b are adjacent). That is, if $z\in H^+$ and $z'\in H^-$, then

$$f_1(z/k)-z_1/k>\varepsilon$$

and

$$z_1'/k-f_1(z'/k)>\varepsilon.$$

Adding gives

$$f_1(z/k)-f_1(z'/k)+z_1'/k-z_1/k>2\varepsilon, \tag{3.3}$$

but $z_1'/k-z_1/k<\delta<\varepsilon$ by the choice of δ and k, so

$$z_1'/k-z_1'/k>-\varepsilon. \tag{3.4}$$

Adding (3.3) and (3.4) gives

$$f_1(z/k)-f_1(z'/k)>\varepsilon, \tag{3.5}$$

which shows that z and z' are not adjacent; for if they were we would have $|z/k-z'/k|=1/k<\delta$, which would contradict the choice of δ.

Similarly V^+ and V^- are not contiguous. Now let $H=H^+\cup H^-$, $V=V^+\cup V^-$, and suppose Q is a connected set lying on H. From the previous paragraph, Q must lie entirely in H^+ or H^-. But note that H^+ does not meet E since f maps into I^2, and hence no point on the right boundary can be mapped to the right. Similarly, H^- does not meet W, so Q cannot meet both E and W. Similarly, V contains no connected set meeting both N and S. By the Hex Theorem, therefore, the sets H and V do not cover B_k, completing the proof. ■

The proof that "Brouwer" implies "Hex," based on a suggestion of John Stallings modified by one of Michael Todd, makes use of the fact that the Hex board B_k gives a *triangulation* of the $k\times k$ square I_k^2 in R^2. It is easy to prove the geometrically obvious fact that each point of I^2 is uniquely expressible as a convex combination of some set of (at most three) vertices, all of which are (pairwise) adjacent. These sets of vertices are the edges and triangles of Figure 3. The reader who is not familiar with this can see how it goes by working out an algorithm which for each point $x=(x_1,x_2)$ in I_k^2 picks out the set of mutually adjacent vertices in whose convex hull x lies.

We will make use also of the fact that any mapping f from B_k into R^2 extends to a continuous *simplicial* (or *piecewise linear*) map $\hat{f}$ on I_k^2. Namely, if $x=\lambda_1z^1+\lambda_2z^2+\lambda_3z^3$ (where the λ_i are non-negative numbers summing to 1) then by definition $\hat{f}(x)=\lambda_1f(z^1)+\lambda_2f(z^2)+\lambda_3f(z^3)$.

Assume then that B_k is partitioned by two sets H and V, and again define four sets as follows: let $\hat{W}$ be all vertices connected to W by an H-path and let $\hat{E}=H-\hat{W}$. Let $\hat{S}$ be all vertices connected to S by a V-path and let $\hat{N}=V-\hat{S}$. From the definition it is clear that $\hat{W}$ and $\hat{E}$ ($\hat{N}$ and $\hat{S}$) are not contiguous. Our proof is by contradiction assuming there is no H-path from E to W and no V-path from N to S. Let e^1 and e^2 be the unit vectors of R^2 and define the mapping f from B_k into itself by

$$\begin{aligned} f(z) &= z+e^1 \text{ for } z\in\hat{W}\\ &= z-e^1 \text{ for } z\in\hat{E}\\ &= z+e^2 \text{ for } z\in\hat{S}\\ &= z-e^2 \text{ for } z\in\hat{N}. \end{aligned}$$

For each of the four possibilities above one must verify that $f(z)$ is indeed in I_k^2. For example, $z+e^1$ is not in B_k only if $z\in$E; but by the assumption that there is no H-path from W to E, we see that $\hat{W}$ does not meet E. It is also true (but for a different reason) that $\hat{E}$ does not meet W. The reader should check the remaining two cases as well.

We now extend f simplicially to all of I_k^2 and obtain a contradiction by showing that f has no fixed point. This is a consequence of the following simple algebraic fact.

LEMMA 1. *Let z^1, z^2, z^3 be vertices of any triangle $\triangle$ in R^2 and let $\hat{\rho}$ be the simplicial extension of the mapping ρ defined by $\rho(z^i)=z^i+v^i$ where v^1, v^2, v^3 are given vectors. Then f has a fixed point if and only if* 0 *lies in the convex hull of v^1, v^2, v^3.*

■ Let $x=\lambda_1z^1+\lambda_2z^2+\lambda_3z^3$. Then $\hat{\rho}(x)=\lambda_1(z^1+v^1)+\lambda_2(z^2+v^2)+\lambda_3(z^3+v^3)$ and x is fixed if and only if $\lambda_1v^1+\lambda_2v^2+\lambda_3v^3=0$. ■

We apply this Lemma to the mapping f. Again the key fact is the non-contiguousness of $\hat{W}$ and $\hat{E}$ ($\hat{S}$ and $\hat{N}$) which implies that if one looks at three vertices of any triangle of mutually adjacent vertices it never happens that one of these vertices is translated by e^i and another by $-e^i$. This means that the three vertices are translated by vectors which lie in a single quadrant of R^2 and hence do not have 0 in their convex hull. We have thus obtained a fixed-point-free mapping, which contradicts the Brouwer Theorem. ■

4. The n-dimensional Hex Theorem. The formal definition of an n-dimensional Hex board is a direct generalization of the one used in Section 3 for the case $n=2$.

DEFINITION. The *n-dimensional Hex board of size k*, H_k^n consists of all vectors *(vertices)* $z=(z_i,\dots,z_n)\in Z^n$ such that $1\leqslant z_i\leqslant k$, $i=1,\dots,n$. A pair of vertices z and z' are *adjacent* if

$$|z-z'|=1 \tag{4.1}$$

and

$$z \text{ and } z' \text{ are } \textit{comparable} \tag{4.2}$$

(meaning again $z_i\geqslant z_i'$ for all i or $z_i'\geqslant z_i$ for all i).

To avoid typographical clutter we will henceforth omit subscripts and superscripts and denote H_k^n simply by H.

For each i we define

$$H_i^-=\{z|z\in H,z_i=1\}$$
$$H_i^+=\{z|z\in H,\,z_i=k\}.$$

A *labeling* of H is a mapping L from H to $N=\{1,2,\dots,\ n\}$. We can now state for the third time

HEX THEOREM. *For any labeling L there is at least one $i\in N$ such that $L^{-1}(i)$ contains a connected set which meets H_i^- and H_i^+.* (Such a set will be called a *winning i-set.*)

We remark that the proof that the n-dimensional Hex and Brouwer Theorems are equivalent is obtained by a purely mechanical generalization of the two-dimensional proof of the previous section. From the game-theoretic point of view n-person Hex is probably not of much interest. Aside from the mechanical problem of designing a practical board there is the usual difficulty of multi-person games that players may form coalitions, and there seems to be no very satisfactory theory as to how these should be formed. It is of some interest to note that in view of the Hex theorem various variants of ordinary Hex are possible. For example, the rule could be that the first player who connects his opposite faces is the loser rather than the winner, or that player i wins if player $i+1$ connects his opposite faces.

To prove the theorem we define the *augmented Hex board* $\hat{H}$ to be all $z\in Z^n$ with $0\leqslant z_i\leqslant k+1$. We further define

$$F_i^+=\{z|z\in\hat{H},\,z_i=k+1\}$$
$$F_i^-=\{z|z\in\hat{H},\,z_i=0\}.$$

We refer to the sets F_i^+,F_i^- as the *faces* of $\hat{H}$.

For our proof we need some further terminology.

Let e^i be ith unit vector in R^n and let e be the n-vector, all of whose coordinates are 1. A *simplex* of R^n (more usually an n-simplex) is an $(n+1)$-tuple of *vertices* $\sigma=(z^0,\dots,\ z^n)$ where $z^i\in Z^n$ and

$$z^{i+1}-z^i=e^r \quad \text{for some } r\in N \tag{4.3}$$

and

$$z^{i+1}-z^i\neq z^{j+1}-z^j \quad \text{for } i\neq j. \tag{4.4}$$

Notice that for $\sigma\subset\hat{H}$ every pair of z^i and z^j in σ are adjacent from (4.1) and (4.2). The *i-facet* of σ is the n-tuple $\tau^i=(z^1,\dots,z^{i-1},\ z^{i+1},\dots,z^n)$. An important simplex in what follows will be $\sigma^0=(0,e^1,e^1+e^2,\dots,e)$. Note that all vertices of this simplex lie in H, and its n-facet $\tau^0=(0,\dots,e^1+e^2+\cdots+e^{n-1})$ lies in F_n^-.

For $0<i<n$ the *i-neighbor* of σ is the simplex $\tilde{\sigma}$ whose vertices are the same as those of σ except that z^i is replaced by $\tilde{z}^i=z^{i-1}-z^i+z^{i+1}$. The vertex $\tilde{z}^i$ is called the *mate* of z^i (with respect to σ). The reader should verify that σ satisfies (4.3), (4.4) and also verify the *symmetry property* that $\tilde{\sigma}$ is the i-neighbor of σ if and only if σ is the i-neighbor of σ and that σ and $\tilde{\sigma}$

intersect in their common i-facet. We also define the 0-neighbor of σ to be $\tilde{\sigma}=(z^1,\ldots,z^n,\tilde{z}_0)$ where $\tilde{z}_0=z^1-z^0+z^n$ and the n-neighbor of σ to be $\tilde{\sigma}=(\tilde{z}^n,z^0,\ldots,z^{n-1})$ where $\tilde{z}^n=z^{n-1}-z^n+z^0$. Again z^0 and $\tilde{z}^0$ and also z^n and $\tilde{z}^n$ are called mates, and if $\tilde{\sigma}$ is the 0-neighbor of σ then σ is the n-neighbor of $\tilde{\sigma}$, and $\sigma\cap\tilde{\sigma}$ is the n-facet of σ and the 0-facet of $\tilde{\sigma}$.

We now extend the labeling L to $\hat{H}$ by defining it on the faces of $\hat{H}$ using the *lexicographic rule*.

$$\begin{aligned} L(z) &= \mathrm{Min}\,\{i \mid z\in F_i^-\} \quad \text{if } z\in\cup F_i^- \\ &= \mathrm{Min}\,\{i \mid z\in F_i^+\} \quad \text{otherwise.} \end{aligned} \tag{4.5}$$

We need one more definition. A simplex σ or facet τ is called *completely labeled*, abbreviated c.l., if L maps σ or τ onto all of N. Note that the simplex σ^0 and its n-facet τ^0 are c.l. since, from (4.5), vertices of τ^0 have labels $1,2,\ldots,n$. We now define a graph Γ whose nodes are all c.l. simplices in $\hat{H}$. A pair of such simplices σ and $\tilde{\sigma}$ are defined to be adjacent if they are neighbors and their intersection is a c.l. facet. The crucial observation is as in Section 2.

LEMMA. *Every node σ of Γ has degree at most* 2.

■ Let $\sigma=(z^0,\ldots,z^n)$ be c.l. Then it has exactly two nodes z^i, z^j which have the same label. Thus $\tilde{\sigma}$ is a c.l. neighbor of σ if and only if it is the i or j neighbor of σ. (For any other neighbor the facet corresponding to their intersection would not be c.l.) ■

We now observe that the simplex σ^0 has exactly one c.l. neighbor. That is, suppose $L(e)=i>1$. Now the n-neighbor of σ^0 is $\tilde{\sigma}^0=(-e^n,0,\ldots,e^1+\cdots+e^{n-1})$ which is not in $\hat{H}$ since $-e^n$ is negative. The other vertex of σ^0 with label i is $e^1+\cdots+e^{i-1}$ and its mate is $e^1+\cdots+e^{i-2}+e^i$ which is in $\hat{H}$ so σ^0 has degree 1. (Exercise: What if $L(e)=1$?)

Applying the Fundamental Graph Lemma of Section 2 we see that σ^0 is the initial node of a simple path $P=(\sigma^0,\sigma^1,\ldots,\sigma^m)$. We claim

LEMMA. *The* c.l.*-facet of σ^m lies on some face F_i^+ of $\hat{H}$.*

■ Let $\sigma^m=(z^0,\ldots,z^n)$. Since σ^m has only one neighbor it must be that for some i the mate $\tilde{z}^i$ of z^i is not in $\hat{H}$. Now for $0<i<n$ this cannot happen, for $z^{i-1}<\tilde{z}^i<z^{i+1}$ and since z^{i-1} and z^{i+1} are in $\hat{H}$ so is $\tilde{z}^i$. Suppose that σ^m has no 0-neighbor; this means that $\tilde{z}^0=z^1-z^0+z^n$ is not in $\hat{H}$. Let $z^1-z^0=e_r$. Then $\tilde{z}^0$ is not in $\hat{H}$ only if $z_r^n=k+1$; but then from (4.4) we see that $z_r^i=k+1$ for all $i>0$, so the 0-facet of σ^m lies on F_r^+ as claimed.

It remains to consider the possibility that σ^m has no n-neighbor which would imply $\tilde{z}^n=z^0-z^n+z^{n-1}$ is not in $\hat{H}$. If $z^n-z^{n-1}=e_r$ then this means $z_r^0=0$; but again by (4.4) this means that $z_r^i=0$ for $i<n$, so the n-facet τ of σ^m lies in F_r^-. But from (4.5) τ can only have labels $i\leqslant r$, so since τ is c.l. it follows that $r=n$ and $\tau\in F_n^-$. To show that this is impossible we show that if $\tau=(z^0,\ldots,z^{n-1})$ is c.l. and lies in F_n^- then $\tau=\tau^0$. We claim first that $z^0=0$; for if, say, $z^0>0$, then the $z_r^i>0$ for all i, and r could not be a label of τ. Likewise $z^1=e^1$, for if z^1 were any other unit vector then no vertex of τ would get label 2. The same argument shows that $z^2-z^1=e^2$ and so on. On the other hand σ^m cannot have τ^0 as a facet because then P would be a cycle, whereas we know it is a path. ■

The Hex Theorem is now proved, because if σ^m has a c.l. facet on F_i^+ then there is a winning i-set. One merely chooses the vertices labeled i in each simplex of the sequence P, and note that they form a connected set. Further the vertex $e^1+\cdots+e^{i-1}$ of σ^0 lies on F_i^- and has label i so this set meets F_1^- and F_1^+. ■

The above proof may seem a bit involved, but its important feature is that it is entirely constructive and leads to a very simple algorithm for finding a winning i-path. More important, as we shall see in a moment, it gives an effective way of finding *"almost fixed points"* of mappings.

Here is the algorithm. Given the labeling L, examine the label of the vertex e. If $L(e)=i$, bring in the mate $\tilde{z}^i$ of the vertex z^i (with respect to σ^0). Call the new simplex σ^1. Now examine the label of $\tilde{z}^i$. There is exactly one other vertex of σ^1 having this label. Replace it by its mate, etc. Each time a new vertex is brought into a simplex σ^k the other vertex of σ^k with the same label is dropped and replaced by its mate. As our proof shows, one eventually constructs in this way a winning i-path.

By combining the ideas of this section with those of Section 3 one is able to locate for any continuous function f of I^n into itself points which are moved by arbitrarily small amounts. Let us be somewhat more precise and say the point x in the n-cube I^n is *moved in the direction i* if $|f(x)-x|=|f_i(x)-x_i|$ (in case of ties choose the smallest subscript). To find a nearly fixed point choose a Hex board H_k^n. The larger the value of k, the better the approximation will be. The label $L(z)$ of the vertex z is then defined to be the direction in which z/k is moved under f. Of course it is not necessary to calculate the labels of all the $n!k^n$ points of H_k^n. One simply follows the Hex algorithm, calculating the label of a vertex only if it is brought into some simplex of the path P. The Hex theorem guarantees that at some point one must encounter two vertices, z and z', which are adjacent and such that both are moved in the direction i but the first is moved *forward*, that is, $f_i(z)-z_i \geqslant 0$; and the second is moved *backward*, $f_i(z)-z_i \geqslant 0$. This follows from the fact that points on the back (front) i-face of the n-cube cannot move backward (forward). But, clearly, if two nearby points are moved in opposite directions then neither can be moved very far if k is large. (The statement may be made precise by the usual ε-δ uniform continuity argument.)

For more information on the subject of fixed-point chasing the text [3] is recommended.

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THE "REAL" PROJECTIVE PLANE WITHOUT CONTINUITY

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The title of this article refers to the elegant treatise *The Real Projective Plane* by H. S. M. Coxeter [1], in which the main theorems of classical plane projective geometry are proved synthetically from axioms of incidence and order. The only non-elementary axiom is the axiom

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of continuity 10.11. Analytically, this axiom states that the plane is coordinatized by the field **R** of real numbers. The purpose of this note is to replace this axiom by more elementary axioms which suffice for all the classical theory. Analytically, what we show is that **R** can be replaced by an arbitrary Euclidean field K. Synthetically, we introduce *the conic axiom* stating that a line through an interior point of a conic is a secant.

1. Classical incidence theory. Most of the theorems of classical projective geometry follow from incidence axioms alone, without use of order. Coxeter wrote a second textbook [**2**] in which he dropped the order and continuity axioms, expanding the incidence axioms to include the fundamental theorem ("a projectivity is uniquely determined by its effect on three points") and the theorem that the diagonal points of a quadrangle form a triangle. He dropped Desargues' theorem from his list of axioms; but instead of giving Hessenberg's involved planar proof of it, he deduced it from axioms of three-dimensionality. It is well known that a model of these axioms can be coordinatized by a commutative field K of characteristic$\neq 2$ [**6**].

Abandoning the real numbers necessitates a re-definition of certain concepts so that the theory still holds. Thus a projectivity must be defined as a composite of perspectivities; with this definition, it is easy to prove that a projectivity preserves the harmonic relation among four collinear points, but the converse does not hold if K admits non-trivial automorphisms. Collineations, correlations, and polarities must be assumed by definition to induce projectivities on the one-dimensional forms. It follows that all these transformations are represented in coordinates by non-singular matrices over K.

A conic is defined as the locus of self-conjugate points of a hyperbolic polarity; analytically, it is the locus of zeros of a non-singular indefinite quadratic form. If point P is on conic Γ, intersecting each line through P a second time with Γ gives a one-to-one correspondence between the pencil of lines through P and the points on Γ. This correspondence can be used to transfer to Γ the structure of one-dimensional projective space, and Steiner's theorem guarantees that the notion of projectivity defined on Γ by this transfer is independent of the choice of P. This structure will play a vital role in what follows.

A conic Γ is uniquely determined by three points, A, B, C, on it and the tangents at two of them (say, at B and C) [**1**, 6.53]. If D is the intersection of those tangents, and D' is similarly defined for points A', B', C' on conic Γ', the unique collineation $ABCD \to A'B'C'D'$ maps Γ onto Γ'. This projective equivalence allows us to choose the equation of Γ and the three points on it conveniently for calculations. It also shows that a projectivity on a conic extends uniquely to a collineation of the whole plane. When $\Gamma = \Gamma'$, that collineation has an invariant line—its *axis*—with the property that the cross-joins AB' and $A'B$ of any two pairs of corresponding points on Γ meet on the axis; the pole of the axis is called the *center*, and the pole has the dual property with respect to pairs of corresponding tangents to Γ. The fixed points (if any) of the projectivity on the conic are the common points of the axis and the conic [**1**, 7.33]. The projectivity is an involution exactly when the collineation is the harmonic homology whose center-axes are pole-polar with respect to the conic; then a line joining any pair of corresponding points on Γ passes through the center.

Define a point P not on Γ to be *exterior* to Γ if P lies on a tangent to Γ (it then lies on two tangents); otherwise P is called *interior*. An exterior point is the pole of a secant; the polar of an interior point does not meet Γ and is called an *exterior line*. From the relation mentioned above between fixed points and points of intersection of the axis with the conic, we see that a projectivity on Γ has zero, one, or two fixed points (i.e., is elliptic, parabolic, or hyperbolic, respectively) according as its center is interior to, on, or exterior to Γ. An involution cannot be parabolic [**1**, 4.63].

The incidence axioms used do not suffice to develop the full classical theory of fixed points of projectivities and interior-exterior of a conic. Without further assumptions, it is possible for an exterior line to pass through an interior point (e.g., in the projective plane coordinatized by the field **Q**, of rationals, if Γ is the conic $2x^2+y^2=z^2$, the line at infinity is exterior but the point at

infinity on the y-axis is interior). To develop the Euclidean theory of conics, e.g., to prove that a central conic has axes, we need to know that the elliptic involution of perpendicular lines and the involution of conjugate lines through the center of the conic have a common pair; over $\mathbf{Q}$, the hyperbola $x(x-2y)=1$ has no axes. Thus the next theorem is of interest.

THEOREM 1. *The following statements are equivalent:*
(i) *Any line through an interior point of a conic is a secant.*
(ii) *An elliptic involution and another involution have a common pair.*

Proof. Assume (i). Transfer the two involutions to a conic and join their centers by a line l. One center is interior since one involution is elliptic; so, by (i), it meets the conic in points which are the common pair [**1**, 7.56]. Conversely, if we assume (ii) and line l goes through a point O_1, interior to a conic Γ, choose another point O_2 on l not on Γ, and consider the elliptic involution with center O_1 and the involution with center O_2. Their common pair of points on Γ lie on l.

It seems quite natural to take statement (i) as an additional axiom (which we might call *the conic axiom*) in order to complete the classical theory without recourse to the full continuity axiom. Not only does it imply the existence of axes of central conics, but it also allows the construction of the foci by intersecting the major (respectively, transverse) axis with a suitable circle [**1**, 9.74 and 9.75].

The next result shows what restriction is placed on the field K by this new axiom.

THEOREM 2. *Given a field K of characteristic $\neq 2$, the conic axiom holds in the projective plane coordinatized by K if and only if*
(i) *x or $-x$ is a square for all $x\in K$, and*
(ii) *x^2+y^2 is a square in K for all $x,y\in K$.*

Proof. Suppose K satisfies these conditions. We may assume the conic is the unit circle $x^2+y^2=z^2$, whose envelope of tangents has equation $u^2+v^2=w^2$. The point at infinity on line $[u,v,w]$ is $(v,-u,0)$, and condition (ii) ensures that it is exterior. A finite point $(a,b,1)$ lies on a tangent exactly when the equation

$$u^2+v^2=(au+bv)^2$$

has a solution $(u,v)\neq(0,0)$, which it has exactly when $a^2+b^2-1\in K^2$. Given (a,b) interior, condition (i) tells us $1-a^2-b^2\in K^2$; a point on a line through (a,b) can be represented parametrically as $(a+tc,b+td)$, and it lies on the unit circle for some value of t exactly when

$$4\left[(ac+bd)^2+(c^2+d^2)(1-a^2-b^2)\right]\in K^2$$

which is the case by condition (ii).

Conversely, suppose the conic axiom holds. Then the line at infinity has no points interior to the unit circle, so condition (ii) holds in K. To verify condition (i), we use part (ii) of Theorem 1. Suppose $-x\notin K^2$, $x\neq 1$. Then the matrix

$$\begin{bmatrix} 0 & x \\ -1 & 0 \end{bmatrix}$$

represents an elliptic involution, and if we multiply it by the involutory matrix

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

we get the matrix

$$\begin{bmatrix} 0 & x \\ 1 & 0 \end{bmatrix}$$

which according to Theorem 1 (ii) has an eigenvector; as its characteristic polynomial is t^2-x, we see that $x\in K^2$.

Note: Neither of the conditions in Theorem 2 implies the other. A field satisfying (ii) is called *Pythagorean*. The power series field $\mathbf{R}\{t\}$ is Pythagorean but contains neither $\sqrt{t}$ nor $\sqrt{-t}$. A finite field of odd characteristic satisfies (i) exactly when -1 is not a square, but then it does not satisfy condition (ii) because -1 is a sum of squares and not a square. *Thus the conic axiom fails for finite planes.*

The main example we have in mind for Theorem 2 is a *Euclidean* field: an ordered field in which every positive is a square. Examples other than $\mathbf{R}$ are the field of numbers constructible from rationals by straightedge and compass and the non-Archimedean field

$$\bigcup_{n=0}^{\infty} \mathbf{R}\{t^{1/2^n}\}.$$

2. **Ordered Planes.** Let us now assume the order axioms [**1**, 3.1], which is equivalent to assuming the field K ordered. The conic axiom is then equivalent to the assumption that K is Euclidean, but let us not make that assumption before examining what has been done without it (and without continuity).

The order axioms refer to the separation relation $//$ between pairs of collinear points. It is connected to the sense $S(ABC)$ on the line determined by three collinear points by the equivalence [**1**, 3.32]

$$AB//CD \Leftrightarrow S(ABC) \neq S(ABD).$$

A projectivity is called *direct* or *opposite* according as it preserves or reverses sense; analytically, it is direct if and only if its matrix has a positive determinant.

A hyperbolic involution, being the correspondence of harmonic conjugates with respect to its fixed points, is opposite [**1**, 3.41 and 4.63]. Two pairs in an opposite involution cannot separate each other [**1**, proof of 3.42]. Hence if $AA'//BB'$, the involution $(AA')(BB')$ is direct elliptic. If, on the other hand, AA' does not separate BB', an easy argument shows that the involution $(AA')(BB')$ is opposite, but we cannot prove it hyperbolic without a further axiom: consider the field $\mathbf{Q}$ and the opposite elliptic involution

$$\begin{bmatrix} 0 & 2 \\ 1 & 0 \end{bmatrix}.$$

(The proof in [**1**, 3.62] uses continuity; we will prove it from the conic axiom.)

Transferring the involution to a conic Γ, where it has a center, we have shown that, if four points on Γ satisfy $AA'//BB'$, then the center $AA' \cdot BB'$ is interior to Γ. (It is not possible to prove the converse for general ordered planes, and Veblen and Young re-defined the notion of "interior" to get the converse *by definition* and thereby develop a decent theory [**7**, Chapter 5].)

Let Q, R be two points on conic Γ and A an exterior point on secant QR. Point A determines two open segments on line QR: the segment QR/A of points separated from A by Q and R, and the complementary open segment containing A. The proof in [**1**, 6.32] shows that every point on segment QR/A is interior to Γ, but Coxeter's claim that the segment containing A consists entirely of exterior points is not valid without a further axiom (cf. the example preceding Theorem 1).

THEOREM 3. *In an ordered projective plane coordinatized by an ordered field K, the following statements are equivalent*:

1. *K is Euclidean.*
2. *A line through an interior point of a conic is a secant* (*the conic axiom*).
3. *An elliptic involution has a common pair with any other involution.*
4. *Every opposite projectivity is hyperbolic.*
5. *If Q, R lie on a conic Γ and A on QR is exterior to Γ, then every point on the open segment containing A ending at Q and R is exterior to Γ.*

6. *If A,A',B,B' are four points on a conic Γ such that $AA'\cdot BB'$ is interior to Γ, then $AA'//BB'$.*

7. *Pairs of an elliptic involution always separate each other.*

Proof. Theorems 1 and 2 show that the first three statements are equivalent. $6\Leftrightarrow 7$ follows by transferring the elliptic involution $(AA')(BB')$ to a conic, using our result that ellipticity is equivalent to $AA'\cdot BB'$ being interior. To prove $2\Rightarrow 5$: consider any point X on the open segment complementary to QR/A; its harmonic conjugate X' with respect to QR lies on QR/A and we have seen that X' is interior, so the polar x' of X' is an exterior line through X. By the conic axiom, X must be an exterior point.

To prove $5\Rightarrow 6$: given $O=AA'\cdot BB'$ interior to Γ, let $P=AA\cdot BB'$, where AA is the tangent at A. Then P is exterior, so by 5, $PO//BB'$, whence $(AP)(AO)//(AB)(AB')$, i.e., $(AA)(AA')//(AB)(AB')$, so by definition $AA'//BB'$.

$4\Rightarrow 7$ follows from our observation that an involution $(AA')(BB')$ for which AA' does not separate BB' is opposite.

To prove $2\Rightarrow 4$: Suppose the opposite projectivity relates A to A' and A' to A''. Since the projectivity is opposite, $S(AA'B)\neq S(A'A''B')$ for any B. If $A''=A$, the projectivity is an involution and we get $AB'//BA'$ (possibly after re-labeling B and B' as B' and B). If $A''\neq A$, choose C such that $AA'//A''C$ and B in AA''/C. Then $S(AA'B)\neq S(A'A''B')$ forces B' into $A'A''/C$ so that again $AB'//BA'$. Now transfer the projectivity to a conic. The point $AB'\cdot BA'$ is interior and lies on the axis of the projectivity. By the conic axiom, the axis meets the conic in two points which are fixed under the projectivity.

Finally, $7\Rightarrow 1$: Given $x>0$, the involution of matrix

$$\begin{bmatrix} 0 & x \\ 1 & 0 \end{bmatrix}$$

is opposite; hence as we have seen, two pairs in this involution could not separate each other. By 7, the involution must be hyperbolic, which means that the eigenvalues $\pm\sqrt{x}$ of the matrix belong to K.

Note 1. In proving this theorem, we proved all the geometric implications synthetically. This shows how, by assuming the conic axiom instead of the continuity axiom, Coxeter could re-write Chapters 1–9 of **[1]** without introducing any obnoxious analytic geometry!

Note 2. The circle axiom (which makes sense only for planes with a metric or other additional structure) is the special case of the conic axiom for conics that are circles. It plays an analogous role in developing elementary Euclidean or hyperbolic geometries without recourse to continuity: see **[3]**, **[4]**.

Note 3 (inspired by a suggestion of Coxeter). Assume only the incidence axioms, as in section 1. Given a polarity π, points P and Q are said to be *accessible* to each other if either $P=Q$ or if P and Q are harmonic conjugates with respect to a pair of points R, S conjugate under π. In other words, the harmonic homology with center R and axis $\pi(R)$ (through S) interchanges P and Q. Let G be the group generated by all harmonic homologies whose center and axis are pole and polar, respectively, under π (the group of π-motions). Accessibility is an equivalence relation, the equivalence classes being the orbits of G. If π is hyperbolic, the points on the conic Γ defined by π form one orbit, and the points exterior to Γ form another. Call *the accessibility axiom* the statement that *the interior of* Γ *is an orbit*; since all conics are projectively equivalent and the conic determines the polarity, the validity of this axiom for one Γ implies its validity for all Γ.

Let K be the coordinate field of the plane and (xx) a quadratic form determining π **[1**, p. 201**]**. The coset of (xx) modulo K^2 is called the *form-value* of the point (x), and two points are accessible if and only if they have the same form-value. Points on Γ have the form value $\{0\}$, points exterior to Γ have the same form-value, and all cosets are form values (consider $x_1x_2-x_3^2$). *The accessibility axiom is therefore equivalent to the requirement that the group of*

squares have index $\leqslant 2$ *in the multiplicative group of* K. In particular, the accessibility axiom holds in all finite projective planes.

If we now assume the order axioms, we see that the accessibility axiom is equivalent to the field K being Euclidean, hence, by Theorem 3, equivalent to the conic axiom. However, without the order axioms, these axioms are not equivalent, since we have seen that the conic axiom fails in all finite planes.

The details of these assertions are easily worked out using the material on accessibility in [2].

The interior of a conic over a Euclidean field is the model for plane hyperbolic (Bolyai-Lobachevsky) geometry. In the language of that model, the accessibility axiom states that there exists a reflection interchanging two given points or, equivalently, that every segment has a midpoint [3].

Note 4. S. Schuster has informed me of reference [5], where Ostrom assumes only that the group of squares has index 2 in the multiplicative group of K and defines a separation relation $AB//CD$ for collinear points by the condition that, for some (hence every) conic Γ through A and B, one of C, D is interior to Γ and the other exterior. He shows that $AB//CD$ if and only if the cross ratio (AB, CD) is not a square, and that some of the order axioms hold for this relation.

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A MINIMAL-PATH ALGORITHM FOR THE "MONEY CHANGING PROBLEM"

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1. Introduction. Let $a_1, \ldots, a_k$ be fixed positive integers; the set of linear combinations of these, with non-negative integer coefficients:

$$n = \sum_{i=1}^{k} x_i a_i \qquad (x_i \in \mathbb{Z}^+) \tag{1}$$

is a semi-group $S = S(a_1, \ldots, a_k)$; it represents the values of "change" that can be made with "coins" in denominations $a_1, \ldots, a_k$. Assuming the $\mathrm{GCD}(a_1, \ldots, a_k) = 1$, all but finitely many

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positive integers belong to S. The number of these *omitted* values is denoted by $\omega=\omega(S)$, while $\kappa=\kappa(S)$ is the *conductor*: all $n\geqslant\kappa$ belong to S, but $\kappa-1$ does not.

This note shows that a determination of κ and ω can be made by constructing, in a graph with weighted edges, a path of minimal weight from one vertex to all the others. Consequently, algorithms for such paths apply, e.g., one by Dijkstra [2], as improved on by Johnson [4].

Our method runs in a time of order $a_1(k+\log_2 a_1)$, which is usually much better than Wilf's "circle of lights" [6] of order $k\kappa$, which in turn improved on Heap and Lynn's [3] of order $a_k^3\log_2\kappa$. For the test case $S(271,277,281,283)$ [3] with $\kappa=13023$ Wilf's method requires the better part of an hour on a personal computer, while ours takes only about 10 percent of that: a few minutes. On the other side, Wilf requires a_k *bits* of storage while we need $2a_1$ *words*, and our program is longer.

Our weighted graph (see section 2 for details) is constructed on vertices $p=0,\ldots,a_1-1$, and the minimum weight n_p of paths from 0 to p is exactly the smallest element in S congruent to p modulo a_1. According to Brauer [1] (see also section 3), then

$$\kappa=\max\{n_p\}-a_1+1, \qquad \omega=\sum_p\lfloor n_p/a_1\rfloor. \tag{2}$$

2. Minimal paths. Denote by $\operatorname{mod}(x)$ the unique number such that $\operatorname{mod}(x)\equiv x \pmod{a_1}$ and $0\leqslant\operatorname{mod}(x)<a_1$.

Consider the directed graph G (with multiple edges and loops) as follows. Its vertices are $0,\ldots,a_1-1$, and for every vertex p and every i $(1\leqslant i\leqslant k)$ there is a directed edge $p\to\operatorname{mod}(p+a_i)$. The *weight* of this edge is a_i. A *path to* p is a directed path from 0 to p in G; its weight is the sum of the weights of the edges.

The following two lemmas form the basis for our algorithm. They are evident upon inspection.

LEMMA 1. *Let $n\in\mathbb{Z}^+$ and $p=\operatorname{mod}(n)$. Then $n\in S$ iff there is a path to p of weight n. More precisely, the paths to p of weight n are in one-to-one correspondence with the number of ways to write n as a sum of (repeated) terms $a_1,\ldots,a_k$ as follows: for every representation of n in the form* (1), *and every permutation of x_1 objects "a_1", x_2 objects "a_2",$\ldots$,x_k objects "a_k" there is one such path.*

LEMMA 2. *For each p $(0\leqslant p<a_1)$ n_p is the minimal element in S with $\operatorname{mod}(n_p)=p$. For every representation of n_p as a sum* (1), *the coefficient x_1 is zero.*

The weight n_p of a minimal path to p $(p\neq 0)$ is obtained as the minimum of n_r+a_i for vertices r with a path $r\to p$ of weight a_i:

$$n_p=\min\{n_r+a_i \mid r=\operatorname{mod}(p-a_i),\ 2\leqslant i\leqslant k\}.$$

In this minimum, vertices r with $n_r>n_p$ make no contribution. It follows that, if N_p is a set of weights of paths to p, which satisfies

$$N_p=\{n_r+a_i \mid r=\operatorname{mod}(p-a_i),\ n_r<\min N_p,\ 2\leqslant i\leqslant k\} \tag{3}$$

then $n_p=\min N_p$. This is the basis for the following algorithm. It uses a process of scanning and visiting. Vertices satisfying (3) are scanned, in increasing order of n_p; during visits to a vertex q, new weights are inserted into N_q. The set of visited, but not yet scanned, vertices is denoted by V. Initially, $V=\{0\}$, and $N_p=\varnothing$ except $N_0=\{0\}$.

Algorithm scan-and-visit

(A) [Initial states] $V\leftarrow\{0\}$; $N_0\leftarrow\{0\}$; $N_p\leftarrow\varnothing$ for $p\neq 0$.

(B) [Find next vertex to be scanned] If V is empty (no more vertices are to be scanned), then Exit. Else, let $p\in V$ be such that $\min N_p$ is minimal. Then n_p is this value. Remove p from V.

(C) [Scan p] For each i $(2\leqslant i\leqslant k)$ let $q=\operatorname{mod}(p+a_i)$; then insert $N_q\Leftarrow n_p+a_i$ (there is a path to q of weight n_p+a_i), and if $q\notin V$ then insert $V\Leftarrow q$. (Now q has been visited.) To (B)

The algorithm shows that if p has been visited at least once, then N_p is non-empty, and the members of N_p are weights of paths to p. Furthermore, when p is about to be scanned ($\min N_p$ is the smallest $\min N_q$ for $q \leqslant V$), then (3) holds.

To make the algorithms *simpler*, we do not carry the whole sets N_p, but for each p we retain only the currently smallest element of N_p.

To make the algorithm *more precise*, we have to outline how to find the smallest element of the set $\{\min N_p | p \in V\}$. Note that the $\min N_p$ are subject to updating.

The data structure we use here is a *heap*, see [5] for a detailed discussion. Briefly, a heap of size l is a binary tree with vertices $1,\ldots,l$, vertex 1 being the root. The "children" of vertex i are the vertices $2i$ and $2i+1$ (if $\leqslant l$), and the value stored at any "parent" does not exceed that of its "children." Through appropriate sub-algorithms one can *insert* an element into a heap, *update* an element anywhere in the heap, and *remove* the smallest element. Our heap will hold no more than a_1 elements, and the total labor associated with each element is of order $\log_2 a_1$.

The working storage consists of an array h (heap) of size a_1 which holds the current values of $\min(N_p)$ in an order compatible with the heap structure, and an array e of size a_1; e_p points to the current location in h where $\min N_p$ is stored.

A program to calculate κ and ω, in (IBM 5100) BASIC, is contained in the Appendix. Instructions $150-330$ constitute the main loop.

To identify integers as belonging to S, it suffices to save the n_p; then $m \in S$ iff $m \geqslant n_{\mathrm{mod}(m)}$. To obtain a decomposition of such m in a_1 steps, also save for each p an a_i such that a decomposition (1) of n_p has $x_i > 0$. To obtain the decomposition in k steps, also save the corresponding x_i.

3. Closing remarks.

1. We prove the first formula in (2). By the maximality of, say, n_p in $\{n_q | 0 \leqslant q < a_1\}$, the smallest element n_q of S in the residue class of q $(\bmod a_1)$ for $q \neq p$ is less than n_p, hence the number in $[n_p - a_1 + 1, n_p - 1]$ congruent to q belongs to S. Thus, $n_p - a_1 \notin S$ and the set $[n_p - a_1 + 1, n_p]$ of a_1 consecutive numbers is in S.

2. We derive a bound for κ. Let (1) be a representation of n_p (the maximal one), then a path to p of weight n_p is obtained (walking backward) by reducing the coefficients x_i one by one, one unit at a time, until all are zero. Each reduced sum is an n_q, and since these n_q decrease, the q are distinct. It follows that $\Sigma x_i \leqslant a_1 - 1$. Since $a_i \leqslant a_k$ we have

$$\kappa = n_p - a_1 + 1 \leqslant (\Sigma x_i) a_k - a_1 + 1 \leqslant (a_1 - 1) a_k - a_1 + 1 = (a_1 - 1)(a_k - 1).$$

It follows easily that the bound is sharp for $k = 2$, since $x_1 = 0$.

I am pleased to thank Professor Wilf for our discussions on this subject.

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Appendix

```
0010 REM CONDUCTOR / OMITTED VALUES
0020 DIM A(20),E(250),H(250)
0030 PRINT ' NUMBER OF GENERATORS, PLEASE'
0040 INPUT K
0050 PRINT ' GENERATORS, IN INCREASING ORDER'
0060 MAT INPUT A(K)
0070 A1=A(1)
0080 DEF FNM(X)=X-A1*INT(X/A1)
0090 REM L =  SIZE OF HEAP  H , O = # OMITTED
0100 REM N0 = WT.MIN.PATH TO VX SCANNED NEXT
0110 L,O,N0=0
0120 REM F(P) POINTS TO  H  UNLESS P SCANNED
0130 REM ' (E(P)=-1) OR NOT VISITED (E(P)=0)
0140 MAT E(A1)=(0)
0150 FOR I=2 TO K
0160 REM N = WT. NEW PATH TO  Q = MOD(N)
0170 N=N0+A(I)
0180 Q=FNM(N)
0190 REM IF Q=0 OR E(Q)<0 Q HAS BEEN SCANNED
0200 IF Q=0 GOTO 0280
0210 IF E(Q)<0 GOTO 0280
0220 REM IF E(Q)=0 THIS IS FIRST VISIT
0230 IF E(Q)>0 GOTO 0260
0240 GOSUB 0380
0250 GOTO 0280
0260 IF N≥H(E(Q)) GOTO 0280
0270 GOSUB 0520
0280 NEXT I
0290 O=O+INT(N0/A1)
0300 IF L=0 GOTO 0340
0310 REM GET NEXT VX. TO BE SCANNED
0320 GOSUB 0550
0330 GOTO 0150
0340 PRINT ' COND =';N0-A1+1;'  OMIT =';O
0350 STOP
0360 REM SUBR:INSERTS NEW ENRTY INTO HEAP
0370 REM      ADJUSTS POINTERS IN  E
0380 L=L+1
0390 U=L
0400 T=INT(U/2)
0410 IF T=0 GOTO 0480
0420 H1=H(T)
0430 IF N≥H1 GOTO 0480
0440 H(U)=H1
0450 E(FNM(H1))=U
0460 U=T
0470 GOTO 0400
0480 H(U)=N
0490 E(Q)=U
0500 RETURN
0510 REM SUBR: UPDATES ENTRY  N  IN A HEAP
0520 U=E(Q)
0530 GOTO 0400
0540 REM SUBR: REMOVES SMALLEST  N0  FROM HEAP
0550 N0=H(1)
0560 H0=H(L)
0570 L=L-1
0580 IF L=0 GOTO 0730
0590 T=1
0600 U=2*T
0610 IF L<U GOTO 0710
0620 IF L=U GOTO 0650
0630 IF H(U+1)≥H(U) GOTO 0650
0640 U=U+1
0650 H1=H(U)
0660 IF H0≤H1 GOTO 0710
0670 H(T)=H1
0680 E(FNM(H1))=T
0690 T=U
0700 GOTO 0600
0710 H(T)=H0
0720 E(FNM(H0))=T
0730 RETURN
0740 END
```

MATHEMATICAL NOTES

EDITED BY DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

Material for this department should be sent to Professor Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis, MO 63121.

Advice to prospective authors: The editors have recently been receiving about ten times as many Mathematical Notes as can be used. It will simplify our work if authors will submit only especially interesting manuscripts. Mathematical Notes should be short papers of one to four printed pages which give new insights, new and improved proofs of old theorems, brief bits of mathematical folklore that have not found a home in the literature, or (occasionally!) new results that are not too technical. The topics should be of wide current interest. Papers that have already been rejected by a research journal are only very rarely suitable as Mathematical Notes.

R.P.B.

COMMENTS AND COMPLEMENTS

DEBORAH TEPPER HAIMO AND FRANKLIN TEPPER HAIMO

The editors periodically receive comments on various papers that appear in the Notes sections of the MONTHLY. We take this opportunity to share some of our correspondence and to point out errors, priorities, duplications, variations of proofs, additional references, and other items of interest brought to our attention.

Set theory. H. H. Schneider's note *Some remarks concerning a definition of ordered pairs* (this MONTHLY, 84 (1977) 636–638) prompted Jürgen Schmidt, of the University of Houston, to refer us to his book on set theory published in Mannheim in 1966 and to inform us of his own related work.

Paul S. Schnare, in his note *The maximal T_0 (respectively, T_1) subspace lemma is equivalent to the axiom of choice* (this MONTHLY, 75 (1968) 761), proves that MT_0 (the property that every topological space has a maximal T_0-subspace) is equivalent to AC (the axiom of choice) as is MT_1. S. D. McCartan and D. M. G. McSherry of Belfast add DT_0 (the property of having a T_0-subspace dense in the sense of intersecting nontrivially all open subsets), DT_1, CT_0 (the property of having a T_0-subspace codense, in the sense of intersecting nontrivially all closed subsets), and CT_1 to the list of equivalents of AC.

Combinatorics. David Singmaster's note *An elementary evaluation of the Catalan numbers* (this MONTHLY, 85 (1978) 366–368) generated a number of comments. He himself adds the observation that M. Percsy, *Sur le nombre de parenthésages d'un produit de n facteurs* (Bull. Soc. Math. Belg., 27 (1975) 167–170), gives a proof similar to that of his own, but using ordinary Polish strings, in which n adjacent pairs OX are successively deleted from a sequence of $n+1$ O's and n X's. The remaining O then corresponds to the unique initial point of a well-formed shift of the original sequence.

Dana Richards of the University of Illinois notes that the results given in Singmaster's note have appeared before in greater generality and with equally simple proofs. Aside from the Sands paper to which Singmaster himself refers, Richards cites G. M. Bergman's *Terms and cyclic permutations* (Algebra Universalis, 8 (1978) 129–130), where no fewer than three proofs are given, the first, a simple induction, ostensibly the same as Singmaster's if the OX deletion is chosen from an OOX substring. Bergman's is a more general but analogous result that allows the operators to have arbitrary arity not restricted to 2 or k. It is a corollary of a result in G. N.

Raney's *Functional composition patterns and power series reversion* (Trans. Amer. Math. Soc., 94 (1960) 441–451) but the proof is not difficult. On the other hand, a generalization of Raney's result given by L. Takács in *Combinatorial Methods in the Theory of Stochastic Processes* (Wiley, New York, 1967, p.1) has a more obscure derivation.

A variation of Singmaster's development using a probability argument from the classic "ballot problem" is offered by Richard A. Gibbs, of Fort Lewis College, Durango, CO. He considers the set of all $\binom{2n+1}{n}$ strings of $n+1$ O's and n X's, and partitions this set into the three subsets:

S_1: Those strings where every initial string has more O's than X's, namely, Singmaster's well-formed strings;

S_2: Those strings beginning with an X;

S_3: The rest, that is, those strings beginning with an O where some initial string has an equal number of O's and X's.

Since there are $n+1$ O's and n X's $n/(2n+1)$ of all of the strings begin with an X and hence

$$|S_2| = \frac{n}{2n+1}\binom{2n+1}{n}.$$

Each element of S_2 must have an initial string with an equal number of X's and O's. If, for each string, the shortest such initial string is selected and the X's and O's in it are interchanged, a unique element of S_3 is obtained, so that $|S_3| = |S_2|$. Hence

$$C_n = |S_1| = \binom{2n+1}{n} - \frac{2n}{2n+1}\binom{2n+1}{n}$$
$$= \binom{2n+1}{n} \Big/ (2n+1).$$

Number theory. T. Daignières, of Paris, contributes a postscript to Robert Miller's note *A game with n numbers* (this MONTHLY, 85 (1978) 183–185) by observing that, if n is odd, the non-constant sequence $a_1, a_2, \ldots, a_n$ is never mapped on $(0, 0, \ldots, 0)$ by S^q, q any integer $\geqslant 1$, a fact which he establishes by contradiction.

Ray Davis, of Lexington, KY, notes that if $X = 235$ and $Y = 268$ then $X^4 + Y^4 = (300.999508)^4$, so that when $n = 4$ Fermat's Last Theorem just misses being false to within 0.0002% of $Z = 301$. Davis develops a method for generating other near misses.

As an addendum to Robert E. Dressler's *A lower bound for $\pi(n)$* (this MONTHLY, 82 (1975) 151–152), Barry W. Hill-Tout, of Brown University, shows that the constant C in Dressler's inequality $\pi(n) \geqslant \log_2(n) + C$ is > -1. His calculations, following the pattern of Dressler, are

$$\Sigma \frac{1}{p^2} < \Sigma_{n \geqslant 2} \frac{1}{n^2} - \frac{1}{4^2} < \int_1^\infty \frac{dx}{x^2} - \frac{1}{16} = \frac{15}{16}$$

so that

$$C > \log_2\left(1 - \frac{15}{16}\right) = \log_2\left(\frac{1}{16}\right) = -4,$$

or, even better,

$$\Sigma \frac{1}{p^2} < \Sigma_{n \geqslant 1} \frac{1}{n^2} - \frac{1}{1^2} - \frac{1}{4^2} - \frac{1}{6^2} - \cdots - \frac{1}{18^2}$$
$$= \frac{\pi^2}{6} - 1 - \frac{1}{4^2} - \frac{1}{6^2} - \cdots - \frac{1}{18^2},$$

so that

$$C > \log_2\left(2 - \frac{\pi^2}{6} + \frac{1}{4^2} + \cdots + \frac{1}{18^2}\right) \simeq \log_2(0.50235) > -1.$$

In his note *Galileo sequences, a good dangling problem* (this MONTHLY, 79 (1972) 67–69), K. O. May defines a Galileo sequence GS to be a sequence $\{a_n\}$ of positive integers such that (1) $a_{2n-1}+a_{2n}=qa_n$ for some positive integer q. He notes that a strictly increasing GS for $q\geqslant 5$ is given by $a_1=1$ and (2) $a_{2n}=[qa_n/2]+1, a_{2n-1}=[(qa_n-1)/2]$. The inaccuracy of this chacterization of a GS for $q=5$ when according to (1) $a_2=4$ whereas (2) gives $a_2=3$ was pointed out by D. Zeitlin in *A family of Galileo sequences* (this MONTHLY, 82 (1975) 819–822). Richard Johnsonbaugh, of Chicago State University, writes that with $a_1=1$, $a_2=q-1$ and (2) defining a_n for $n\geqslant 2$ an increasing GS is obtained and May's results hold.

Both Kenneth S. Williams, of Carleton University, and William C. Waterhouse, of the Pennsylvania State University, cite F. Tano's *Sur quelques théorèmes de Dirichlet* (J. Reine Angew. Math., 105 (1889) 160–169) as a paper containing a slightly more general version, but with roughly the same proof, of the result in Morris Newman's *A note on an equation related to the Pell equation* (this MONTHLY, 84 (1977) 365–366). Williams gives the additional reference of L. E. Dickson's *History of the Theory of Numbers*, vol. 2, p. 386 (Washington, D.C., 1920), whereas Waterhouse goes on to say that the result is also a corollary of theorems on the 2-component of the strict ideal class group. It is included, for example, in the theorem at the end of L. Rédei and H. Reichardt, *Die Anzahl der durch 4 teilbaren Invarianten der Klassengruppe eines beliebigen quadratischen Zahlkörpers* (J. Reine Angew. Math., 170 (1933) 69–74).

Algebra. Joel Brenner, of Palo Alto, CA, notes that his paper *Zolotarev's theorem on the Legendre symbol* (Pacific J. Math., 45 (1973) 413–414) is pertinent to recent work of Patrick Morton, *A generalization of Zolotarev's theorem* (this MONTHLY, 86 (1979) 374–375). Brenner appeals to properties of circulant matrices, while Morton uses group-theoretic methods to relate the parity of the permutation of $\mathbb{Z}_p$ obtained by multiplying by a to the residue symbol (a/p).

A. H. M. Hoare, of Birmingham, England, writes that Bruce C. McQuarrie pointed out an error in the proof of Theorem 2 of his note *On finite abelian groups* (this MONTHLY, 75 (1968) 40–41). Apart from a t for k in the second line, the statement $(d,m_k)=1$ in the third and fourth lines is false in general. It can, however be corrected by applying the given argument to the images H_{d^1} and G_{d^1} of H and G, respectively, under the endomorphisms $x\rightarrow d^1x$ where $d^1=(m_k,n_k)$. The original monomorphism of H into G gives the monomorphism of H_{d^1} into G_{d^1}. These groups have expressions as direct sums of cyclic groups of order $n_i^1=n_i/(d^1,n_i)$ and $m_j^1=m_j/(d^1,m_j)$, respectively, satisfying $(d,n_i^1)=d$ for $i=1,\dots,k$, and $(d,m_j^1)\ =\ 1$ for $j=k,\dots,r$ where $d=n_k^1=n_k/(m_k,n_k)$.

Analysis. In their note *On Cauchy's inequality for Hermitian matrices* (this MONTHLY, 85 (1978) 486–487), E. Deutsch and H. Hochstadt give a simple, elementary proof of Cauchy's inequalities for the eigenvalues of a Hermitian matrix. Leon Gerber, of St. John's University, Jamaica, NY, writes to point out that their method of proof is similar to that used by H. Wilf on pages 26–27 of his book *Mathematics for the Physical Sciences* (Wiley, New York, 1962). Harold Falk, of City College, NY, notes that their treatment appears on pages 68–71 of *Matrices and Tensors* (Pergamon, New York, 1963) by G. G. Hall, a part of the *International Encyclopedia of Physical Chemistry and Chemical Physics*. Commenting in greater detail, William Stenger, of Ambassador College, Pasadena, CA, states that this particular proof was originally contained in results given by A. Weinstein in 1962–63. He notes that, actually, Weinstein's proof was more general in that he considered a compact self-adjoint operator on a Hilbert space (not just a Hermitian matrix), and more extensive, in that he gave not only the inequalities but also the necessary and sufficient conditions for equality to hold. He remarks as an aside that the special case of finite-rank operators has been thoroughly discussed in several books and expository papers and recalls that Weinstein's results were subsequently extended to the case of unbounded self-adjoint operators of Schrödinger-type.

Reference is also made in the Deutsch-Hochstadt note to the Courant-Fischer theorem.

Stenger deplores the considerable confusion that exists in the literature on this matter since, strictly speaking, there really is no such theorem. He points out that there are, in fact, two complementary principles, one based on inequalities of Poincaré and another based on inequalities of Weyl. While the two principles are in a sense equivalent on finite-dimensional spaces, the equivalence is more subtle than a simple reindexing of the eigenvalues in opposite order. Surprisingly, the two principles are not equivalent on infinite-dimensional spaces, even for operators of finite rank. A comprehensive discussion of these inequalities, principles, and related results is contained in the book by A. Weinstein and W. Stenger, *Methods of Intermediate Problems for Eigenvalues* (Academic Press, New York, 1972).

Paul Putter, of Pennsylvania State University, refers to the argument of H. Anton included in the *Comments and complements* article (this MONTHLY, 84 (1977) 804) for establishing that $\lim_{x\to\infty}(\ln x)/x=0$. He offers a version that is reasonable to introduce in a calculus class as an application of the fact that a function with a positive derivative must be monotonic increasing. The proof consists of the observation that since e^x has a positive derivative, it is monotonic increasing. Now $(d/dx)e^x=e^x$ and $e^x>e^0=1$ for $x>0$. Hence $f(x)=e^x-x$ is also strictly monotonic increasing on $(0,\infty)$, since $f'(x)=e^x-1>0$ for $x>0$. It follows that, for $x>0, f(x)>f(0)=1$ or $e^x>x+1>x$. Similarly, $g(x)=e^x-x^2/2$ is strictly monotonic increasing for $x>0$ since $g'(x)=e^x-x=f(x)>1>0$. Hence, for $x>0$, $g(x)>g(0)=1$ or $e^x>x^2/2+1>x^2/2$. It follows that $e^x/x\to\infty$ as $x\to\infty$.

C. E. Edwards, Jr., of the University of Georgia, notes that in A. P. French's paper *The integral definition of the logarithm and the logarithmic series* (this MONTHLY, 85 (1978) 580–582) his derivation of the logarithmic series $\log(1+x)=x-x^2/2+x^3/3-\cdots$ using the binomial expansion is that given by Euler in Chapter 7 of his text *Introductio in Analysin Infinitorum* (1748). Euler introduces an "infinitely small" number ε and an "infinitely large" number N such that the product $N\varepsilon$ is the (finite) number $y=\log(1+x)=N\varepsilon$. Then

$$1+x=e^y=e^{N\varepsilon}=\left[(1+\varepsilon)^{1/\varepsilon}\right]^{N\varepsilon}=(1+\varepsilon)^N.$$

Hence

$$1+\varepsilon=(1+x)^{1/N} \quad \text{or} \quad \varepsilon=(1+x)^{1/N}-1$$

so that

$$\log(1+x)=N\varepsilon=N\left[(1+x)^{1/N}-1\right],$$

which is Euler's version of $\log x=\lim_{n\to\infty}n(x^{1/n}-1)$. Edwards comments that French's derivation of the latter limit by a direct appeal to the integral defining $\log x$ is different not only from Euler's but also from standard derivations of that formula. Richard Johnsonbaugh, of Chicago State University, offers the additional information that French's principal result was contained in M. J. Pascual's *Note on* $\int_a^x t^y$ (Math. Mag., 35 (1962) 175) and reprinted in T. M. Apostol et al., eds., *Selected Papers on Calculus* (MAA, 1969, p. 309).

H. Kestelman of University College, London, observes that the lemma given by P. L. Walker in his note *On Lebesgue integrable derivatives* (this MONTHLY, 84 (1977) 287–288) can be slightly improved as follows.

LEMMA. *Let g be a real and continuous function on $[a,b]$ and let $E\subset[a,b]$. If* (i) *$g(E)$ has no interior points,* (ii) *for every $x\in[a,b)\setminus E$ there are arbitrarily small positive δ such that $g(x+\delta)>g(x)$, and* (iii) *g has no intervals of constancy, then g increases strictly in $[a,b]$.*

For the proof, choose any p, q, with $a\leqslant p<q\leqslant b$. Suppose the conclusion is false and $g(p)>g(q)$. By (i), η can be so chosen that $g(p)>\eta>g(q)$ and $\eta\notin g(E)$. If m is the greatest x in $[p,q]$ with $g(x)=\eta$, then $m\notin E$ and so $g(x)>\eta$ for some $x\in(m,q)$; but this implies $g(x)=\eta$ for some $x>m$. It follows that $g(p)\leqslant g(q)$. Since $g(v)\neq g(p)$ for some $v\in(p,q)$, by (iii), the possibility $g(p)=g(q)$ would imply that $g(v)\leqslant g(q)$ so that $g(v)<g(p)$, a contradiction. Hence $g(p)>g(q)$.

Lajos Takács proposes that on page 35, line 5 from the bottom, of his note *An increasing continuous singular function* (this MONTHLY, 85 (1978) 35–37) the equation $\lim_{n\to\infty}F(y_n)-F(x_n)=0$ be replaced by $\lim_{n\to\infty}\max_{0\leqslant k_n\leqslant n+1}\rho^{k_n}(1+\rho)^{-n}=0$. In connection with Takács's note, attention should be directed as well to a related paper by James G. Mauldon, *Continuous functions with zero derivatives almost everywhere* (Oxford Quarterly J. Math., (2) 17 (1966) 257–262). One of the functions therein defined found an application in J. H. Conway's *On Numbers and Games* (Academic Press, London, 1976, p. 84).

Bruce Berndt, of the University of Illinois, asserts that the method used by E. L. Stark in his note *Application of a mean value theorem for integrals to series summation* (this MONTHLY, 85 (1978) 481–483) to evaluate $\zeta(2)$ is very well known and cites, as an example, a problem on page 196 of A. Ostrowski's *Vorlesungen über Differential-und-Integralrechnung*, Drittes Band (Verlag Birkhäuser, Basel, 1954). In that proof, the integral is shown to tend to zero by a simple integration by parts rather than by the second mean value theorem. Berndt's note *Elementary evaluations of* $\zeta(2n)$ (Math. Mag., 48 (1975) 148–154) is relevant in this connection.

Peter D. Johnson, of Beirut, has obtained results addressing the same questions (raised by A. Ostrowski) as those in Wolfgang Walter's note *A counterexample in connection with Egorov's theorem* (this MONTHLY, 84 (1977) 118–119). He provides, however, two references not included in the Walter paper, namely, H. L. Royden's *Real Analysis*, 2nd edition (Macmillan, New York, 1968), and Walter Rudin's *Real and Complex Analysis*, 2nd edition (Tata McGraw-Hill, New Delhi, 1974).

Peter K. F. Kuhfittig, of the Milwaukee School of Engineering, provides the following commentary on the note *The Schauder fixed-point theorem for nonexpansive mappings*, by W. G. Dotson and W. R. Mann (this MONTHLY, 84 (1977) 363–364). He states that they have shown that if C is a compact convex subset of a Banach space, then a nonexpansive self-mapping T of C has a fixed point. Their use of the Banach contraction principle suggests the following short argument: Assume without loss of generality that $0\in C$ and define $T_r=(1-r)T\colon C\to C, 0<r<1$. Since T_r is contractive, there exists for every r a unique point $x_r=T_rx_r$. Now

$$\|x_r-Tx_r\|=\|(1-r)Tx_r-Tx_r\|=r\|Tx_r\|\leqslant rd(C)$$

where $d(C)$ is the diameter of C. Hence $\mathrm{Inf}_{x\in C}\|x-Tx\|=0$. Since $\|x-Tx\|$ is continuous and C compact, there exists a $z\in C$ such that

$$\|z-Tz\|=\operatorname*{Inf}_{x\in C}\|x-Tx\|$$

whence $Tz=z$.

Klaus Floret, of SUNY at Buffalo, responds to the problem posed at the end of the note *Whitley's technique and* K_δ*-subspaces on Banach spaces*, by G. J. O. Jameson (this MONTHLY, 84 (1977) 459–461), that asks for a description of the infinite-dimensional K_δ-spaces of m. He establishes that (i) the $\sigma(m,l^1)$-closed subspaces of m are all K_δ-subspaces and (ii) the $\sigma(m,l^1)$-closed subspaces of m are, in their norm-topology, exactly the duals of separable Banach spaces. Reference is made to G. Köthe, *Topological Vector Spaces* I (Springer-Verlag, New York, 1969), sections 20.8 and 22.4.

The note *Relation of the conjugate harmonic functions to f(z)*, by E. V. Laitone (this MONTHLY, 84 (1977) 281–283), generated from Ralph Boas, of Northwestern University, the observation that the result is treated by a simpler but more sophisticated technique on page 126 of H. Cartan's *Elementary Theory of Analytic Functions of One or Several Complex Variables* (Addison-Wesley, Reading, Mass., 1963).

Statistics. The paragraph on Analysis in the *Comments and complements* article (this MONTHLY, 84 (1977) 804) with reference to a Tchebyshev inequality prompted William Kruskal, of the University of Chicago, to share a March 20, 1952, letter he wrote to A. W. Kimball about

the latter's paper *On dependent tests of significance in the analysis of variance* (Ann. Math. Statist., 22 (1951) 600–602). He notes that the following more general theorem than that of Kimball occurred to him; namely, that if X is a non-constant univariate random variable with any distribution whatever, and if f and g are two strictly increasing functions of the real variable x (except possibly over intervals in which X falls with zero probability), then assuming that $E[f(x)]$ and $E[g(x)]$ exist, it follows that

$$E[f(X)g(X)] > E[f(X)]\cdot E[g(X)].$$

He provided a very simple proof at the time and states that, as he examines the result again, it seems transparent when viewed in the form $\operatorname{Cov}(X), f(X) \geqslant 0$ for f weakly monotone increasing. For if X_1, X_2 are independent copies of X and if $(X_1 - X_2)(f(X_1) - f(X_2)) \geqslant 0$, then the expectation is non-negative, and the result follows on expanding and simplifying. He remarks that the result surely must precede Tchebyshev and asserts that, if Gauss and Laplace didn't know it, he's prepared "to eat [his] history books."

James C. Frauenthal's and Noreen Goldman's note, *Demographic dating of the Nukuoro society* (this MONTHLY, 84 (1977) 613–618), placed the Nukuoro Atoll in the South Pacific. Graham Lord, of Université Laval, advises that its geographical location is 155° E 4° N, which is in the North Pacific Ocean.

Computation. Lawrence J. Cohen, of Port Jefferson Station, NY, points out that the integer 140,541, which occurs in numbered item (10) of N. J. Fine's article *Infinite products for kth roots* (this MONTHLY, 84 (1977) 629–630), should read 140,451.

A. Nijenhuis, of Dartmouth College, writes that the value judgments on computing time, as stated in George H. Brown's note *On Halley's variation of Newton's method* (this MONTHLY, 84 (1977) 726–728), are subject to question. Brown states that "a pair of interactions of Newton's method is, in general, more efficient than a single iteration of Halley's variation." Nijenhuis gives examples where Halley's method plus modern computing equipment could be preferable.

LOCAL CONDITIONING OF PARAMETRIC FORMS USED TO APPROXIMATE CONTINUOUS FUNCTIONS

DAVID W. KAMMLER AND ROBERT J. MCGLINN

1. Introduction. Suppose that we wish to numerically find parameters $\mathbf{p} \in \mathbb{R}^n$ which make the approximating function $Y(\mathbf{p}, t)$ a best uniform approximation to a given continuous function $F(t)$ on $[0,1]$, e.g., by using a suitable variant of the Remez algorithm. The accuracy with which we can determine the optimal parameters depends on the local conditioning of $Y(\mathbf{p}, t)$ and on the size of the unit roundoff u of the computer we are using (e.g., $u \approx 2 \cdot 10^{-16}$ when we work in double precision on the IBM 370-158). In this note we present an elementary perturbation analysis of this conditioning problem (suitable for classroom presentation in a beginning course in numerical analysis) and illustrate the results with two commonly used polynomial models. A similar but less elementary analysis is given in [6]. A very nice alternative analysis for polynomial models is presented in [3].

2. Local Conditioning. Assuming that Y is sufficiently smooth we can write

$$Y(\mathbf{p} + \alpha \mathbf{p}^*, t) = Y(\mathbf{p}, t) + \alpha H(\mathbf{p}, \mathbf{p}^*, t) + o(\alpha)$$

as $\alpha \to 0^+$ where

$$H(\mathbf{p}, \mathbf{p}^*, t) = \sum_{i=1}^{n} p_i^* \phi_i(\mathbf{p}, t)$$

and

$$\phi_i(\mathbf{p}, t) = \partial Y(\mathbf{p}, t)/\partial p_i, \qquad i = 1, \ldots, n.$$

Thus the linear mapping $S(\mathbf{p}):\mathbf{p}^*\to H(\mathbf{p},\mathbf{p}^*,-)$ from $\mathbb{R}^n$ to the tangent space of $Y(\mathbf{p},-)$ governs the way in which a small perturbation in $\mathbf{p}$ alters Y, and (assuming the ϕ_i's are independent) the inverse map $S^{-1}(\mathbf{p}):H(\mathbf{p},\mathbf{p}^*,-)\to\mathbf{p}^*$ governs the way in which the parameters $\mathbf{p}$ must be altered in order to bring about a desired perturbation in $Y(\mathbf{p},-)$.

Suppose now that $\mathbf{p}$ is optimal. We can expect our numerical process to terminate at some nearby point $\mathbf{p}+\mathbf{p}^*$ with $H(\mathbf{p},\mathbf{p}^*,-)$ being so small that

$$\|H(\mathbf{p},\mathbf{p}^*,-)\|\leqslant\|Y(\mathbf{p},-)\|\cdot u.$$

For simplicity we use the uniform norm on $C[0,1]$. (In some cases one might wish to use an alternate norm, e.g., the sup norm taken over the set of extremal points of $F-Y$.) This being the case we obtain the first order bound

$$\begin{aligned}\|\mathbf{p}^*\|&=\|S^{-1}(\mathbf{p})(H(\mathbf{p},\mathbf{p}^*,-)\|\\&\leqslant\|S^{-1}(\mathbf{p})\|\,\|Y(\mathbf{p},-)\|\cdot u\end{aligned}$$

for the error in our computed approximation to $\mathbf{p}$. If $\mathbf{p}\neq\mathbf{0}$ we obtain the relative error bound

$$\|\mathbf{p}^*\|/\|\mathbf{p}\|\leqslant C\cdot u \tag{1}$$

where

$$C=\|S^{-1}(\mathbf{p})\|\,\|Y(\mathbf{p},-)\|/\|\mathbf{p}\|$$

provides a relative measure of the local conditioning of $Y(\mathbf{p},-)$. If Y is linear we can further majorize (1) by taking

$$C=\|S^{-1}\|\cdot\|S\|,$$

thereby obtaining a local measure of the conditioning which is independent of the parameters.

In cases of practical importance we use the max norm on $\mathbb{R}^n$ and compute

$$\begin{aligned}\|S(\mathbf{p})\|&=\max\{\|H(\mathbf{p},\mathbf{p}^*,-)\|/\|\mathbf{p}^*\|:\mathbf{p}^*\neq\mathbf{0}\}\\&=\max\{|\phi_1(\mathbf{p},t)|+\cdots+|\phi_n(\mathbf{p},t)|:0\leqslant t\leqslant 1\}.\end{aligned}$$

We also have

$$\begin{aligned}\|S^{-1}(\mathbf{p})\|&=\max\{\|\mathbf{p}^*\|/\|H(\mathbf{p},\mathbf{p}^*,-)\|:\mathbf{p}^*\neq\mathbf{0}\}\\&=\max\{\|\mathbf{q}_i\|/\|H(\mathbf{p},\mathbf{q}_i,-)\|:i=1,\dots,n\}\end{aligned} \tag{2}$$

where $\mathbf{q}_i=(q_{i1},\dots,q_{in})$ is chosen so that $q_{ii}=1$ and

$$\begin{aligned}\|H(\mathbf{p},\mathbf{q}_i,-)\|&=\|\phi_i(\mathbf{p},-)+\sum_{k\neq i}q_{ik}\phi_k(\mathbf{p},-)\|\\&=\min\left\{\|\phi_i(\mathbf{p},-)+\sum_{k\neq i}p_k{}^*\phi_k(\mathbf{p},-)\|:p_k{}^*\in\mathbb{R},\,k\neq i\right\}.\end{aligned}$$

We can thus obtain $\|S^{-1}(\mathbf{p})\|$ by finding a best uniform approximation to ϕ_i by $\phi_1,\dots,\phi_{i-1},\phi_{i+1},\dots,\phi_n,i=1,\dots,n$ and then using (2). In cases where these approximation problems must be solved numerically we have used the subroutine DAPMM from IBM's Scientific Subroutine Package [**8**, pp. 283–288] based on Barrodale and Young's L_∞ linear approximation algorithm [**2**, pp. 300–304] (which does not require the basis functions to satisfy the Haar condition).

3. Examples. Students in a beginning numerical analysis course will usually attempt to build polynomial models using the canonical parametrization

$$Y(\mathbf{p},t)=p_1+p_2t+\cdots+p_nt^{n-1}.$$

In this case $\phi_i(\mathbf{p},t)=t^{i-1}$ so that

$$\|S\|=\max\{1+t+\cdots+t^{n-1}:0\leqslant t\leqslant 1\}=n.$$

Suppose now that

$$T_k(2t-1)=a_{k0}+a_{k1}t+\cdots+a_{kk}t^k$$

where $T_k(t)=\cos[k\cos^{-1}(t)]$ is the usual Chebyshev polynomial. Since $T_{n-1}(2t-1)$ alternates $n-1$ times on $[0,1]$ we may use Descartes' Rule of Signs [5, p. 43, # 36] to infer that each a_{ki} is nonzero and that $t^{k-1}-T_{n-1}(2t-1)/a_{n-1,k-1}$ is the unique best approximation to t^{k-1} by a linear combination of $t^{i-1}, i\neq k, i=1,\ldots,n$. Thus,

$$\begin{aligned}\|S^{-1}\|&=\max\{\|\mathbf{q}_k\|/\|H(\mathbf{q}_k,-)\|:k=1,\ldots,n\}\\&=\max\{|a_{n-1,k}|:k=0,1,\ldots,n-1\}.\end{aligned}$$

Using a well-known expression [7, p. 32] for T_n together with the identity

$$T_n(2t-1)=T_{2n}(\sqrt{t}\,),\qquad 0\leqslant t\leqslant 1$$

it follows that

$$|a_{n,n-k}|=\frac{2n}{2n-k}\binom{2n-k}{k}2^{2n-2k-1}, n=1,2,\ldots, k=0,1,\ldots,n. \tag{3}$$

For a given $n\geqslant 1$ the moduli given in (3) increase monotonically from 2^{2n-1} when $k=0$ up to the desired maximum and then decrease monotonically to 1 when $k=n$, and so the largest modulus occurs for the smallest nonnegative integer k for which

$$|a_{n-1,n-k-1}/a_{n-1,n-k}|=\frac{(n-k-1)(2n-2k-3)}{2(k+1)(2n-k-3)}\leqslant 1$$

or equivalently for which

$$n-\frac{13}{8}-\frac{1}{8}\sqrt{32(n-1)^2-7}\leqslant k. \tag{4}$$

The rapid growth of $\|S^{-1}\|$ and hence the condition number C is seen in Table 1 where $\|S\|$, $\|S^{-1}\|$, and C are listed for $n=1,2,\ldots,10$. In particular, we can expect a relative error of order $10^7\cdot u$ when approximating by a ninth degree polynomial. More generally, Stirling's approximation for the factorial (cf. [1, p. 257]) can be used in conjunction with (4) to show that

$$C\sim\sqrt{n-1}\cdot(1+\sqrt{2}\,)^{2n-2}/2^{1/4}\cdot\sqrt{\pi}\qquad\text{as}\quad n\to+\infty$$

with this approximation having a relative error of less than 5% for $n\geqslant 10$.

TABLE 1.
Local Conditioning for the Canonical Polynomial Parametrization

n	$\|S\|$	$\|S^{-1}\|$	C
1	1	1	1
2	2	2	4
3	3	8	24
4	4	48	192
5	5	256	1280
6	6	1280	7680
7	7	6912	48384
8	8	39424	315392
9	9	212992	1916928
10	10	1118208	11182080

In contrast, we consider the parametrization

$$Y(\mathbf{p},t)=p_1T_0(2t-1)+\cdots+p_nT_{n-1}(2t-1)$$

(which can be evaluated almost as easily as the canonical parametrization, cf. [7, pp. 125–126]).

We find $\phi_i(\mathbf{p},t)=T_{i-1}(2t-1)$ so that

$$\|S\|=\max\left\{\sum_{i=0}^{n-1}|T_i(2t-1)|:0\leqslant t\leqslant 1\right\}=n.$$

In this case it follows from the orthogonality of the Chebyshev polynomials that

$$|q_{in}|=\left|\frac{2-\delta_{i1}}{\pi}\cdot\int_0^1\frac{H(\mathbf{q},t)\cdot T_{i-1}(2t-1)}{\sqrt{t(1-t)}}dt\right|$$

so that $\|S^{-1}\|\leqslant 2$. We have actually computed $\|S^{-1}\|$ for $n=1,2,\ldots,10$, and these values along with the corresponding values of $\|S\|$ and C are listed in Table 2.

TABLE 2.
Local Conditioning for the Chebyshev Polynomial Parametrization.

n	$\|S\|$	$\|S^{-1}\|$	C
1	1	1.00	1.00
2	2	1.00	2.00
3	3	1.00	3.00
4	4	1.15	4.60
5	5	1.15	5.75
6	6	1.21	7.26
7	7	1.21	8.47
8	8	1.23	9.84
9	9	1.23	11.07
10	10	1.24	12.40

For even moderately large values of n the relative error bound $C\cdot u$ is much larger for the canonical polynomial parametrization than it is for the Chebyshev parametrization. For this reason the Chebyshev parametrization is a much better parametrization for the space of polynomials of degree $n-1$ or less.

The analysis outlined earlier is of course applicable to nonlinear parametrizations as well as to linear parametrizations. Indeed, in [4] we have investigated the local conditioning for the parametrizations of the space of real valued exponential sums of order n or less given by

$$Y(\mathbf{a},\boldsymbol{\lambda},t)=a_1e^{-\lambda_1 t}+\cdots+a_ne^{-\lambda_n t}$$

and by

$$Y(\mathbf{b},\mathbf{c},t)=\mathbf{v}^T\cdot\exp(Bt)\cdot\mathbf{c}$$

where $\mathbf{b}$, $\mathbf{c}$ have components $b_i, c_i, i=1,\ldots,n$, where

$$v_i=\delta_{i1},\qquad i=1,\ldots,n,$$

and where B is the $n\times n$ companion matrix with elements

$$B_{ij}=\delta_{i+1,j}-b_j\delta_{in},\qquad i,j=1,\ldots,n.$$

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PERFECT QUADRILATERALS

R. D. Nelson

M. V. Subbarao [1] (see also [2]) showed that the number of triangles whose integer-valued sides add up to λ times their area is finite for all positive λ. (In fact the number of such triangles is zero if $\lambda > 4\sqrt{3}$.) He suggested it would be interesting to examine a similar problem for the quadrilateral.

It is easy to show that the number of quadrilaterals whose integer-valued sides add up to λ times their area is infinite for all positive values of λ. M. J. Marsden [3] remarked that the number becomes finite for each λ if the problem is restricted to *cyclic* quadrilaterals, whose vertices lie on a circle.

In this article we examine the number $N(\lambda)$ of *cyclic* quadrilaterals whose integer-valued sides a,b,c,d add up to λ times their area. If λ is integral such quadrilaterals will be called "perfect." We show there are 11 perfect cyclic quadrilaterals.

Theorem. (a) *$N(\lambda)$ is finite for all positive λ*; (b) *$N(\lambda)=0$ for $\lambda>4$.*

Proof (a). For each λ we seek positive integers a,b,c,d such that

$$a+b+c+d=\lambda\sqrt{(s-a)(s-b)(s-c)(s-d)}\,, \tag{1}$$

where $2s=a+b+c+d$, subject to the restrictions

$$b+c+d>a,\quad c+d+a>b,\quad d+a+b>c,\quad a+b+c>d. \tag{2}$$

Now let

$$A=2s-2a,\quad B=2s-2b,\quad C=2s-2c,\quad D=2s-2d \tag{3}$$

so that $A+B+C+D=4s$; then equation (1) becomes

$$4(A+B+C+D)^2=\lambda^2ABCD, \tag{4}$$

and the restrictions (2) become $A,B,C,D>0$.

If (1) has a solution in integers then (4) will have a corresponding solution in which A,B,C,D and $2s$ will be either all odd or all even because of relations (3).

Thus to solve (1) we need only consider those solutions of (4) in which $A,B,C,D,2s$ are all odd or all even. Also, since $B+C+D-A=4a$, we also require that $B+C+D>A$, etc. If, under these restrictions, (4) has only a finite number of solutions (A,B,C,D) for each λ, then $N(\lambda)$ is also finite for each λ.

Assume $A\geqslant B\geqslant C\geqslant D$, and suppose A,B,C,D satisfy (4); then

$$D^2\leqslant CD=\frac{4(A+B+C+D)^2}{\lambda^2AB}.$$

Now $A(2B+C+D)^2-B(A+B+C+D)^2=(A-B)((B+C+D)^2-AB)\geqslant 0$, since $B+C+D>A$ and $A\geqslant B$. So we have

$$\frac{4(A+B+C+D)^2}{\lambda^2 AB}\leqslant\frac{4(2B+C+D)^2}{\lambda^2B^2}\leqslant\frac{64}{\lambda^2}.$$

Thus, $CD\leqslant 64/\lambda^2$ and $D\leqslant 8/\lambda$.

These inequalities imply that for a given λ, C and D are restricted to a finite number of pairs and values. Since $A\geqslant B$ and $B+C+D>A$ give $A-B<C+D$, then for each possible pair (C,D) it follows that $A-B$ can only take the values $0,2,4,\dots,C+D-2$. Now write $A=B+k$ and substitute in (4) to obtain

$$4(2B+k+C+D)^2=\lambda^2BCD(B+k).$$

For any pair (C,D) we have, on letting $k=0,2,\dots,C+D-2$, a finite number of quadratics in B to examine for integral solutions. Thus B, and A, can take only a finite number of values and so for each λ equation (4) has only a finite number of solutions under the stated restrictions.

Thus $N(\lambda)$ is always finite.

Proof (b). If $\lambda\geqslant 4$ then either $D=1$, $C=1$ or 3 or $D=2$, $C=2$.

In each case we seek a solution, remembering that $A,B,C,D,2s$ are to be all odd or all even, that $2s=\frac{1}{2}(A+B+C+D)$, and that $A-B<C+D$.

(i) $D=1$, $C=1$: Here A,B are odd and $A-B<2$ so $A=B$. But now $2s$ will be even so no solution is possible.

(ii) $D=1$, $C=3$: Here A,B are odd and $A-B<4$. $2s$ is only odd when $A=B$. Then we have

$$\lambda^2=\frac{4(2B+4)^2}{3B^2}\leqslant\frac{400}{27}<16$$

since $B\geqslant 3$. So no solutions exist.

(iii) $D=2$, $C=2$: Here A,B are even and $A-B<4$. $2s$ is only even when $A=B$ and then

$$\lambda^2=\frac{(2B+4)^2}{B^2}\leqslant 16$$

since $B\geqslant 2$. So we have the solution $A=B=C=D=2$ when $\lambda=4$ and no others. Thus, $N(4)=1$ and $N(\lambda)=0$ for $\lambda>4$.

COROLLARY. *There are* 11 *perfect cyclic quadrilaterals.*

Proof. When $\lambda=1,2,3$ there are no solutions in which A,B,C,D are all odd, for such a solution would imply λ^2 was a multiple of 16 (from (4)) which is absurd. Accordingly, we replace A,B,C,D by $2\alpha,2\beta,2\gamma,2\delta$ and seek solutions in positive integers of

$$(\alpha+\beta+\gamma+\delta)^2=\lambda^2\alpha\beta\gamma\delta$$

subject to the restrictions that $\alpha+\beta+\gamma+\delta$ is even and $\beta+\gamma+\delta>\alpha$, etc. Assuming $\alpha\geqslant\beta\geqslant\gamma\geqslant\delta$, the analysis of the theorem gives $\delta\leqslant 4/\lambda$, $\gamma\delta\leqslant 16/\lambda^2$ and $\alpha-\beta<\gamma+\delta$. The method is then to let (γ,δ) be a possible pair, write $\alpha=\beta+k$ where $k=0,1,2,\dots,\gamma+\delta-1$ and inspect the quadratic $(2\beta+k+\gamma+\delta)^2=\lambda^2\beta\gamma\delta(\beta+k)$ for integral solutions. The results are as follows:

λ	$(\alpha,\ \beta,\ \gamma,\ \delta)$	$(a,\ b,\ c,\ d)$
4	(1, 1, 1, 1)	(1, 1, 1, 1)
3	(2, 2, 1, 1)	(1, 1, 2, 2)
2	(2, 2, 2, 2)	(2, 2, 2, 2)
1	(10, 10, 9, 1)	(5, 5, 6, 14)
1	(8, 5, 5, 2)	(2, 5, 5, 8)
1	(6, 6, 3, 3)	(3, 3, 6, 6)
1	(4, 4, 4, 4)	(4, 4, 4, 4)

These solutions correspond to 11 non-congruent cyclic quadrilaterals.

REMARK. If the restrictions such as $\beta+\gamma+\delta>\alpha$ are replaced by $\beta+\gamma+\delta\geqslant\alpha$ the method gives additional solutions corresponding to the 6 "perfect" triangles of [1]:

$\lambda=2$, $(3,4,5)$; $\lambda=1$, $(6,8,10)$, $(5,12,13)$, $(9,10,17)$, $(7,15,20)$ and $(6,25,29)$.

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RESEARCH PROBLEMS

EDITED BY RICHARD GUY

In this Department the Monthly presents easily stated research problems dealing with notions ordinarily encountered in undergraduate mathematics. Each problem should be accompanied by relevant references (if any are known to the author) and by a brief description of known partial results. Manuscripts should be sent to Richard Guy, Department of Mathematics and Statistics, The University of Calgary, Calgary, Alberta, Canada, T2N 1N4.

MONTHLY RESEARCH PROBLEMS, 1969–79

RICHARD K. GUY

This section has survived a decade; though occasional absences over the past year may suggest we've decayed. Those who carefully scan the December issues of this MONTHLY in odd-numbered years should note that we also did some updating in April 1978. As already mentioned [1971, 1114, and 1973, 1120] we don't publish *solutions* (occasionally one may appear in the Notes section) but these two-yearly articles give information on progress with published Research Problems. Correspondence is welcome.

References in brackets are to years and pages of this MONTHLY. References in parentheses are to articles cited at the end: years, tbp (to be published) or wrc (written communication).

Klee [1969, 180] asked if every polygonal region is illuminable from some point. Rauch (1978) gives bounded, but *not* polygonal, regions which require an infinity of sources for their illumination.

In further answer to Kronk's question [1969, 809], Simone Gutt (1977) gives infinite families of hypohamiltonian graphs, as do Collier and Schmeichel (1977, 1978), who also give an exponential lower bound for the number of such graphs of order p.

The tree-packing problem of Ringel, mentioned by Duke [1969, 1128], and the closely related conjecture of Kotzig and Ringel, that all trees are graceful (in the language of Golomb) or have a β-valuation (Rosa's terminology), and other problems in this area, e.g., Golomb [1974, 499], Cahit [1976, 35] have prompted a great rash of papers, but the original problems remain open. Bermond (1978a) gives a survey with a good bibliography; with Kotzig and Turgeon (1976) he applies the problem to radioastronomy; and with Brouwer and Germa (1976) relates it to the

Langford-Skolem problem. Koh et al. (1977, 1978a, 1978b, 1978c, 1979, tbpa, tbpb) give methods of obtaining graceful trees from one or two smaller ones. Pastel and Raynaud (1978) show that "olive trees" are graceful. Rao Hebbare (1976) showed that cycles C_n are graceful just if $n=0$ or 3, mod 4. Hoede and Kuiper (1978) showed that wheels are graceful, and Bodendiek et al. (1976, 1977a, 1977b) obtained other infinite families of graceful graphs containing circuits. Kotzig and Turgeon (1976) showed that the only graceful regular graphs of valence v consisting of c complete components, have $c=1$ and $v<4$. See also Bermond et al. (1975, 1978b), Bloom and Golomb (1978), Cahit (1977), Chen et al. (1977a, 1977b, 1978), Chung and Huang (tbp), Cornejo (tbpa, tbpb), Gyárfás and Lehel (1978), Hoede (1976), Köhler (1978), Kundu (1979), Rogers (1978), Rosa (1977), Shiloach (tbp) and Smyth (tbp).

Dumont and Foata (1976) extend the results of Carlitz and of Riordan and Stein [1971, 1117; 1975, 996; 1977, 808] which settled Gandhi's conjecture [1970, 505].

Fischer (1975) has a further paper on Fejes Tóth's illumination problem [1970, 869; 1975, 996].

In [1977, 808] it was reported that Lawrence (1977) had settled a conjecture of Grünbaum [1970, 1088]. Coincidentally this was independently achieved by Borodin and Kostochka (1977) and by Catlin (1978).

Dekking (tbp) has another paper on Brown's problem [1971, 886] on non-repetitive sequences. He says that the (4,2) problem "remains an interesting open problem."

In connection with Herda's problem [1971, 888], Falconer (1977) proves a conjecture of Alexander (1975) by showing that if $\mathcal{C}$ is a plane rectifiable curve such that for every point x on $\mathcal{C}$ there is a point of $\mathcal{C}$ at distance at least 1 from x, then $\mathcal{C}$ has length at least π, the value being attained just if $\mathcal{C}$ bounds a convex set of constant width 1. This generalizes theorems of Herda and Chakerian [1974, 146, 153].

For progress on Murty's problem [1971, 1000] on magic graphs, see the papers of Doob (1978) and Sedláček (1976a, 1976b).

Goodey and Woodcock (1978) give an answer to Peterson's problem [1972, 505] concerning self-intersections and curves of constant width.

Schmidt and Wang (1977) quote Brenner and Cummings [1972, 626] on Hadamard's maximum determinant problem.

Katchalski (1977) gives a characterization theorem for three-dimensional convex polytopes Q having the property that there is a polytope P, isomorphic to Q, all edges of which can be cut by a pair of planes that miss all vertices. This gives an affirmative solution to a conjecture of Grünbaum [1972, 890; 1975, 998].

Anshel (1978, and see references there) has written further papers on the equality problem for vector addition systems which are relevant to Nash's reachability problem [1973, 292]. Neil Immerman and Jan van Leeuwen write that the announcement [1977, 808] of the existence of a complete decision procedure was premature, so that the general problem is still open.

Gyires (1976, 1977, 1978) has further papers on the van der Waerden conjecture, discussed by Merris [1973, 791].

Neil Sloane (wrc) corrects my remarks [1975, 1000] concerning Golomb's problem [1974, 499] by observing that Wichman's formula for the maximum number $E(n)$ of edges in a graceful graph on n nodes, $E(n)\geqslant(n^2+n-2-k^2)/3$, where k is the difference between $n+2$ and the nearest multiple of 6, is best known, except in the following cases: $E(6)\geqslant 13, E(7)\geqslant 17; E(8)\geqslant 23, E(13)\geqslant 58$.

Readers may wish to compare the papers of Györy et al. (tbp) and Schäffer (1956) with the problem of Edgar [1974, 758].

Scott and Singmaster point out our error [1977, 810] in stating that Tung-Po Lin's paper settles Scott's area-perimeter problem [1974, 884]. This has now been settled by Arkinstall and Scott himself (1979) and Croft (1979) gives a more general answer by maximizing A/P^k

$(0<k<2)$. The answers are found among an entertaining collection of cushions, cigars, cigarillos, diamonds, hexagons, and lozenges. A related paper is by Singmaster and Souppouris (1978).

A paper which should have been referred to earlier in connection with Alter's problem [1975, 632] is that of Jucys (1976).

Markowsky (1979) shows that the semilattice extension problem of Arbib and Manes [1975, 744] is NP-hard, so that no simple solution is likely to be forthcoming.

The problem of R. H. Fox, repeated by Ringel and by Alpert and Gross [1975, 835] continues to generate papers, although it was solved much earlier [see 1977, 810]: Pisanski (1977), Ruscitti (1976, 1977).

The complete reference to Taylor (1977), who answered a question of Leech [1975, 923] is now given. In connection with the problem of Leech and Haselgrove (1977, 198), see the paper of Pulleyblank (1976).

The complete reference to Fredman and Weide (1978), who, among others, settled Klee's question [1977, 284] on the computation of the measure of the union of intervals, is now given.

In the notation of Simmons and Abbott [1977, 633], Usiskin (tbp) proves that $f_4(6)=4$ and obtained the better bounds $f_4(7)\leqslant 8, f_4(8)\leqslant 11, f_4(9)\leqslant 15$.

In relation to the problem of Grünbaum and Shephard [1978, 37], see their paper (1977) and that of Harborth (1977).

Peter Montgomery (wrc) gives the matrix

$$\frac{1}{12}\begin{bmatrix} 8 & 2+2x & 2-2x \\ 2+2y & 5-x-y & 5+x-y \\ 2-2y & 5-x+y & 5+x+y \end{bmatrix}$$

which has permanent $5/18$ for all real x and y. So any convex linear combination of the three doubly stochastic matrices

$$\frac{1}{12}\begin{pmatrix} 8 & 4 & 0 \\ 4 & 3 & 5 \\ 0 & 5 & 7 \end{pmatrix} \qquad \frac{1}{12}\begin{pmatrix} 8 & 0 & 4 \\ 4 & 5 & 3 \\ 0 & 7 & 5 \end{pmatrix} \qquad \frac{1}{12}\begin{pmatrix} 8 & 4 & 0 \\ 0 & 5 & 7 \\ 4 & 3 & 5 \end{pmatrix}$$

also has $5/18$. This gives a permanental triple in answer to Wang's question [1978, 188]. Brenner and Wang (tbp) have a more recent paper on this problem.

David Larue (wrc) proves that $|S_n|$, in the notation of Kotzig and Laufer [1978, 364], is even for $n>1$, by making the following pairing between sigma permutations which are invariant under negative reversals: if u is one such, then $-f-u$ is another and distinct one, with $-f-(-f-u\ )$ being u again. Put $-f-u=v$. Then $f+v=-u$, a permutation. Since $v=-(f+u)$, v itself is a permutation, so v is a sigma permutation. Since $u+v=f$, when $n>1$, f contains an odd component, so that $u\neq v$. Finally, negative reverse $v=$ reverse negative $-f-u$ $=$ reverse $f+u=$ reverse $f+$ reverse $u=-f-u=v$. Colin Mallows and Neil Sloane observe that $|S_n|$ is sequence 666 in Sloane's Handbook (1973) and refer to Bennett and Potts (1967), who give

$$|S_{15}|=659632,$$

but who assert that the problem is exceedingly difficult.

Buell and Williams [1978, 483] asked for an octic reciprocity law of Scholz type, and themselves (tbp) found one. See also Leonard and Williams (1977).

Additional references relevant to McCarty's problem [1978, 578] on queen squares are Bruen and Dixon (1975), Fine (1965), Goldstein (1979), Hansche and Vucenic (1973), Klarner (1978) and Kløve (1977).

Thank you to the many correspondents who have sent comments, references, offprints, and preprints, and to Richard Nowakowski for valuable help in preparing this article.

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CLASSROOM NOTES

Edited by Deborah Tepper Haimo and Franklin Tepper Haimo

Material for this department should be sent to Professor Deborah Tepper Haimo, Department of Mathematical Sciences, University of Missouri, St. Louis, MO 63121.

Advice to prospective authors: The editors have recently been receiving about **ten times** as many Classroom Notes as can be used. It will simplify our work if authors will submit only especially interesting manuscripts.

R.P.B.

MEASURE AND CARDINALITY

James M. Briggs and Thomas Schaffter

The purpose of this paper is to describe how the set theory axioms we accept affect Lebesgue measure on the real line. The results are generally known among set theorists and easy to state but rather hard to get at for the non-specialist. A course in real analysis would seem to be an appropriate place to introduce students to these results.

One of the first properties learned about Lebesgue measure on the real line is that countable sets of real numbers are measurable and they have measure zero. This is, of course, an immediate consequence of the facts that singleton sets have measure zero and that Lebesgue measure is countably additive. To the student who knows there may be sets of real numbers of cardinality strictly between $\aleph_0$ (the cardinality of the natural numbers) and c (the cardinality of the reals), the question naturally arises as to what can be said about their measurability. This question can be answered from two perspectives: (1) what *must* hold in the reals, and (2) what *might* hold in the reals. More precisely, the distinction is between (1) what is implied by our basic axioms, i.e., what can be proved from them, and (2) what is consistent with and what is independent of the axioms. Of course, it is not sufficient to consider just the usual axioms for the reals (i.e., the axioms for a complete, ordered field); we must also consider axioms of set theory. Clearly, the answers given to the above questions will depend on what set theory axioms we use. There is consensus among mathematicians that the axioms of Zermelo-Fraenkel set theory (ZF) are acceptable but there are controversies concerning the Axiom of Choice (AC) and the Continuum Hypothesis (CH).

From the work of Gödel [3] and Cohen [2] we know that both of the statements, AC and CH, are independent of ZF, i.e., neither can be proved or refuted in ZF. Another way to put it is that there are models of ZF in which AC (respectively, CH) holds and models in which AC (CH) fails. For those who believe in an objective reality of the objects of mathematics there is strong intuitive evidence for AC. (Given a set of nonempty sets, "surely" there is a set containing one element from each. Why wouldn't there be? For a discussion of this and contrary views of mathematics see Bernays [1]. For a similar discussion related to CH see Gödel [4]. There is no such evidence for CH (why should c be the first uncountable cardinal?). Accordingly, most mathematicians unhesitatingly use AC but do not take a stand on CH. Thus, our original query (what can be said in ZF+AC (ZFC) about the measurability of sets of reals with cardinality between $\aleph_0$ and c) should be meaningful to most mathematicians.

It is well known among persons specializing in the foundations of mathematics that if there are any sets of real numbers of cardinality between $\aleph_0$ and c, all such sets which are measurable have measure zero. We present an elementary proof of this fact. Next we discuss what is known about the possibility of non-measurable sets in this "in between" land. Finally, for those who consider AC up for debate, we comment on what can be said in the absence of this axiom.

THEOREM. *If E is a measurable set of reals of cardinality less than c, then E has measure* 0.

Proof. We show that if E has positive measure then the cardinality of E is c. Accordingly, suppose $mE>0$ and let F be a closed subset of E of positive measure. Then there are two disjoint, closed intervals I_0, I_1 of length less than mF which intersect F in sets of positive measure. Now within I_0 there exist two disjoint, closed intervals I_{00}, I_{01} of length less than $m(F \cap I_0)$ which also intersect F in sets of positive measure. Similarly, we can find intervals I_{10} and I_{11} within I_1 with the corresponding properties.

Continuing in this manner, we produce at the nth step 2^n disjoint, closed intervals $I_{j_1,\ldots,j_n}$, $j_i \in \{0,1\}$, each of which intersects F in a set of positive measure. In the limit each infinite sequence $j_1, j_2, \ldots$ from $\{0,1\}$ corresponds to an infinite sequence $I_{j_1}, I_{j_1 j_2}, \ldots$ of nested, closed intervals, all of which intersect F. By the completeness of the reals the intersection of such a sequence is non-empty, and hence determines at least one point of F. The points of F produced by these intersections are clearly distinct, and therefore this process identifies $2^{\aleph_0}=c$ elements of F, giving us the desired result.

At this point we are assuming AC. However, we note that it is not needed for the constructions in this proof: the interval selection clearly could be made without its use; its use can also be avoided when choosing a point from each intersection of nested intervals by requiring the lengths of the intervals to go to zero, thereby producing singletons.

Thus, in all models of the reals (within ZFC) where CH fails (i.e., where sets of reals of "in between" cardinality exist), all *measurable* sets of "in between" cardinality have measure zero. But can there be *non-measurable* sets of "in between" cardinality? The classical proof of the existence of a non-measurable set, due to Vitali (1905), is of no help since it produces a set of cardinality c (see [5, pp. 69–70]). However, Robert Solovay has provided the answer. On the one side he has shown that there are models of ZFC in which CH fails where every set of cardinality less than c is measurable. Hence, assuming that sets of "in between" cardinality exist, we cannot prove, using ZFC, that any such sets are non-measurable. In the terminology of Foundations, ZFC+ $\neg$CH cannot prove there are non-measurable sets of "in between" cardinality.

On the other side, Solovay has shown that there are models of ZFC in which CH fails where there are sets of cardinality $\aleph_1$ (the first uncountable cardinal) which are not measurable. Hence, ZFC cannot prove there are no non-measurable sets of the kind in question (see [9]).

Thus, just as we cannot prove or refute (in ZFC) that there are sets of reals of "in between" cardinality, we cannot prove or refute that among such sets (if there are any) some are non-measurable. Stated differently, just as ZFC does not decide the Continuum Hypothesis, ZFC does not decide whether all sets of reals of cardinality less than c are measurable.

Finally, we comment on what can happen in the absence of AC. Some form of the axiom must be retained in order to preserve a reasonable measure. In the absence of AC it can happen that the set of reals is a countable union of countable sets (Lévy [7]), forcing us into a measure which is either trivial or not countably additive. A weak form of AC, which is sufficient to preserve the main properties of Lebesgue measure is the principle of Dependent Choices (DC). DC states that, given a binary relation R on a set S such that for all $x \in S$ there is a $y \in S$ for which xRy, then starting with a particular $x_0 \in S$, a countable number of choices can be made, relative to R, each one depending on the previous; more exactly, given $x_0 \in S$ there exists a countable sequence $x_1, x_2, \ldots$ such that $x_0 R x_1 R x_2 \cdots$. This statement implies the Countable Axiom of Choice (every *countable* collection of nonempty sets has a choice function) and is clearly implied by AC. (See Jech [**6**, p. 79] for a proof of the first implication.)

Solovay has shown in [**8**] that if there is a model of ZF which contains an inaccessible cardinal (i.e., if the existence of an inaccessible cardinal is consistent with ZF), then there is a model of ZF+DC in which every set of reals is measurable. (Roughly, an inaccessible cardinal is an uncountable cardinal which cannot be reached from below by taking unions or power sets; i.e., if $\aleph$ is inaccessible, then the union of fewer than $\aleph$ sets, each of cardinality less than $\aleph$, has cardinality less than $\aleph$, and the power set of a set of cardinality less than $\aleph$ has cardinality less than $\aleph$. Many set theorists believe it is quite unlikely that the existence of such a cardinal is inconsistent with ZF; however, the question is open at this time.) Thus, if one is willing to give up AC, which most mathematicians are not, it is consistent to assume that there are no non-measurable sets of reals at all (assuming the consistency of the existence of an inaccessible cardinal). Solovay's motivation here was not to present the possibility of considering a theory of reals in which all sets were measurable, but rather to show the necessity of AC in proving the existence of a non-measurable set: "Of course, the axiom of choice is true, and so there are non-measurable sets [**8**, p. 3]." However, if one is not quite so certain about the truth of AC, the above interpretation is possible.

In the same paper Solovay shows, under the inaccessible cardinal assumption, that there are models of ZFC, some in which CH holds and some in which it fails, where all *definable* sets of reals are measurable. The adequacy of an interpretation of a concept such as *definability* within the formalism of set theory is always open to question; however, Solovay's translation as "definable from a sequence of ordinals" seems to come very close to capturing the notion. This is very strong evidence that we will never be able to exhibit explicitly a non-measurable set.

In summary, we have seen that if the Axiom of Choice is accepted it can be proved that there are non-measurable sets of reals of cardinality c; and if there are uncountable sets of reals of cardinality less than c which are measurable, they must have measure zero. Further, just as it cannot be proved or refuted (from the ordinary axioms of set theory, including the Axiom of Choice) that such uncountable sets of reals exist, it also cannot be proved or refuted that among these sets (assuming that there are some) there are non-measurable ones.

We have also seen that if one is willing to give up the axiom of choice, it cannot be proved that there are any non-measurable sets of reals at all (assuming the consistency of the existence of an inaccessible cardinal). Furthermore, there are models of ZFC, some in which CH holds and some in which it fails, in which all *definable* sets of reals are measurable. Thus it is very unlikely we will ever be able to produce explicitly a non-measurable set.

Acknowledgment: We thank Lòóy Simonoff for his contributions.

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THE MATRIC EQUATION $X^2 = A$

W. R. UTZ

The purpose of this note is to give a useful procedure for solving the matric equation $X^2 = A$.

In response to the curiosity of sophomores in linear algebra courses I have accumulated a number of exercises and theorems involving polynomial equations in matrices suitable for an elementary course. Chapter 7 of a book [5] of C. C. MacDuffee provides a summary of basic results and is a good source of references for matric equations up to 1933. It is adequate for enriching an elementary course. Matric equations is still an active research field (cf. [2], [4], for example) but I am unaware of a recent survey. According to [6, Chapter 2], the solution of the matric equation $\sum_{i=0}^{n} A_i X^i = 0$ is equivalent to the search for right divisors $I\lambda - X$ of $\sum_{i=0}^{n} A_i \lambda^i$. In this connection the paper [3] is of interest.

Returning to the simple quadratic $X^2 = A$, a classical theorem [1, p. 299] ensures a (nonsingular) solution of the equation if A is nonsingular. I now describe a method, once suggested to me by W. E. Roth, that reduces the problem of solving $X^2 = A$ to linear equations regardless of the rank of A.

If X is a solution of the equation (examples for which there is no solution are easily devised), then

$$X^2 - \lambda^2 I = A - \lambda^2 I;$$

hence

$$(X - \lambda I)(X + \lambda I) = A - \lambda^2 I.$$

If $\phi(\lambda)$ denotes the characteristic function of a solution, then

$$\phi(\lambda)\phi(-\lambda) = \det(A - \lambda^2 I) \tag{*}$$

and so the characteristic function of a solution, if one exists, must be a divisor of $\det(A - \lambda^2 I)$. For each of the possible solutions $\phi(\lambda)$ of (*) one secures the equation $\phi(X) = 0$ in which all even powers of X may be replaced by powers of the known matrix A to reduce the equation to a linear matrix equation.

As an example, consider the equation

$$X^2 = \begin{bmatrix} 9 & 0 & -8 \\ 5 & 4 & -5 \\ 0 & 0 & 0 \end{bmatrix}$$

For this A,

$$\phi(\lambda)\phi(-\lambda) = \det(A - \lambda^2 I) = -\lambda^2(\lambda^2 - 9)(\lambda^2 - 4)$$

and the possible characteristic functions of solutions are $\phi_1(\lambda)=\lambda(\lambda-3)(\lambda-2)\phi_1(-\lambda), \phi_2(\lambda)=\lambda(\lambda+3)(\lambda-2), \phi_2(-\lambda), \phi_3(\lambda)=\lambda(\lambda+3)(\lambda+2), \phi_3(-\lambda), \phi_4(\lambda)=\lambda(\lambda-3)(\lambda+2), \phi_4(-\lambda)$. In this problem the solutions given by the last four functions are the solutions given by the first four functions.

Consider

$$\phi_1(X_1)=X_1^3-5X_1+6X_1=AX_1-5A+6X_1=(A+6I)X_1-5A=0.$$

This linear equation has the unique solution

$$X_1=\begin{bmatrix} 3 & 0 & -8/3 \\ 1 & 2 & -7/6 \\ 0 & 0 & 0 \end{bmatrix},$$

which satisfies the given equation. Similarly, $\phi_2(X_2)=0$ gives

$$X_2=\begin{bmatrix} -3 & 0 & 8/3 \\ -5 & 2 & 25/3 \\ 0 & 0 & 0 \end{bmatrix}.$$

These two matrices and their negatives are the four solutions of the given quadratic equation.

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MATHEMATICAL EDUCATION

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A SMALL GROUP STRATEGY FOR ENHANCED LEARNING

WARREN PAGE

Classroom learning need not be dull or routine. In his article [2], Julian Weinglass describes teaching mathematics to small groups. Gersting and Kuczkowski, who experimented with this approach in a variety of classes over a five-year period, elaborate further in [1]. Both articles give compelling arguments for adopting this approach.

This article offers an especially interesting and highly successful classroom-tested variation on the same theme and specifically demonstrates how to realize some of its objectives and advantages. Surprisingly, it turns out that this approach can succeed only if the students' competitiveness is controlled once they become enthusiastic and involved.

Timeliness and Overall Intent. The strategy described here is an effective way to summarize topics and review for an examination. Students are prepared to cope with the material, and their time is utilized efficiently. The overall intent is a cooperative review, in the sense that many questions are raised and answered concurrently. This is far more efficient than the instructor's responding to one question at a time—particularly since not all students feel free to ask questions and, in most instances, a large percentage of the class already knows the answer.

Specific Objectives. The following objectives are attainable through the strategy outlined in the next section.

1. Maximize learning through student questions and answers. Work in small group settings is more relaxed; the authoritarian aspect is removed, and students are encouraged to raise issues since they no longer feel that they are holding the class back.

2. Ensure active participation of all students. In particular, motivate the weaker students to work beyond their normal level, encourage the better students to serve as tutors or peer guides, and make it more attractive for all students to probe further into their understanding of the topics.

3. Transcend the simplistic mechanics of "how things are done" by having students discover "why such procedures work." In the process, provide students with additional insight on how they can learn and teach themselves through properly posed questions.

4. Highlight student interaction and feelings of academic fellowship through a positive learning process. Thus demonstrate that mathematics is indeed a lively, creative, and meaningful means of communication.

The Strategy. Divide the class into groups of equal (or almost equal) size—4, 5, or 6, each with roughly the same proportion of stronger, weaker, and average students. Select four or five problems, each involving a few operations, and a fixed time during which the groups will work on these problems. A group is considered to have a problem "worked out" when each member of the group understands the problem and is prepared to explain its solution to the class at the blackboard. At the end of the specified time interval, all work stops and the groups are ranked according to the number of worked-out problems. (A tie is easily broken by giving one more problem to the tied groups.) Group I (the highest-ranked group) is first given the opportunity to become the winner, subject to the procedures outlined below. The winning group is rewarded as agreed upon beforehand (for example, each group member receives the equivalent of "100" on a quiz, each group member earns 5 points on the forthcoming test for which the class is reviewing, or each group member accumulates 2 points toward his final grade). There are no losers and no penalties for anyone.

It will quickly become clear that our definition of "worked out" increases a group's interest in seeing that each of its members thoroughly understands the problems. The depth of understanding is revealed by

Procedure 1. The instructor chooses a particular problem worked out by group I and one individual from group I to work with that problem at the blackboard. Once at the blackboard, the student is permitted no assistance from his group. He first explains his group's solution and then attempts to "defend" the problem by answering all questions from students outside group I. Permissible questions, refereed by the instructor for appropriateness, include: clarification of all aspects of the student's solution, questions concerning alternate approaches or seemingly unrelated solutions (as when different procedures yield distinct expressions for the same solution), and the nature of the solution when certain aspects of the problem are changed. Another important rule is that members of other groups who see a mistake in the blackboard solution cannot point it out immediately; they must give the student the opportunity to discover and correct the error by first asking, "Are you sure that your solution, or part of your solution, is correct?" This question is also used effectively, by those who know the correctness of the blackboard solution, to test the student's conviction in his answers.

Variation: Require that the student explain his group's solution, and then allow the whole group to defend the problem.

The instructor, as referee, does not become involved in the group problem-solving or defense. Occasionally, however, a student at the blackboard responds to an unusual question with an equally unusual, plausible, and incorrect answer. If no one in the class takes up this response or corrects it, the instructor should do so—in which case the student at the blackboard is not disqualified.

The defense frequently raises interesting questions for the instructor as well as for the class. Exploring the problem's overall structure increases understanding and enhances learning at many levels. Above all, the defense involves every student in the class because of

Procedure 2. The student at the blackboard successfully defends his problem if he correctly answers all questions (and corrects all errors) raised by members of other groups. The instructor then proceeds to select another problem worked out by group I and a different individual from group I to explain and defend this problem. Group I is declared the winner and rewarded accordingly if each of its worked-out problems is successfully defended by the chosen members of its group. Group I is totally disqualified as soon as one of its worked-out problems is not successfully defended—as, for example, when any nonmember of group I raises and provides the correct answer to a permissible question which is neither answerable nor correctly answered by the group I student at the blackboard. With group I disqualified, the opportunity to become the winner passes on to group II (their worked-out problems, which have already been considered as part of group I's worked-out problems, are now easily defended and dispatched). If group II fails to defend its worked-out problems successfully, the chance to win goes to group III, and so on down the ranking.

This procedure ensures that everyone stays tuned in the hope of eventually advancing his group to the winner's position. As the opportunity to try to win moves down the ranking, there are fewer problems to defend and a greater likelihood of winning. Accordingly, attentive students who persevere are rewarded commensurately.

Concluding Remarks. A few sessions with this strategy will loosen up the class, increase its familiarity with the procedures, and ensure that all the objectives (1)–(4) are realized. Group composition should be changed for each session so that students get to know and work with others in the class. On rare occasions there will be no group winner. This essentially means that there was a good amount of individual learning. From the instructor's perspective, this class was a winner.

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AN EXPANDED DOMAIN FOR OBJECTIVES IN MATHEMATICS EDUCATION

NEIL A. DAVIDSON AND RONALD L. MCKEEN

Specificity has been hailed by some as the savior in education. The specificity resulting from precise statements of desired educational outcomes has been praised as one of the single most important advances in instruction in several decades. There has been a quest for clarity and precision in stating objectives; it has been embodied in the movement toward basing curriculum

PROBLEMS AND SOLUTIONS

The editors receive more problems than they can use, but they are always on the lookout for interesting and novel problems. The task of selection will be much easier if proposers will keep the following guidelines in mind:

Send all proposed problems, in duplicate if possible, to Prof. Vladimir Drobot, Department of Mathematics, University of Santa Clara, Santa Clara, CA 95051. Please include solutions and any information that will help the editors, including reasons why the problem is interesting. Problems in well-known textbooks and results in generally accessible sources are not acceptable.

Solutions should be sent to the addresses given at the head of each problem set.

An asterisk (*) indicates that the proposer did not supply a solution. If you submit a problem without a solution, you should tell the editors whether you know (or somebody else knows) how to solve the problem. If you are uncertain, the problem might be more suitable for the Research Problems section; see the instructions given there.

Proposers are asked to aim for the same audience as for the rest of the MONTHLY; *a rule of thumb is to think of people who have had at least a year of graduate work in Mathematics, very likely some years ago. Proposals that can be both understood and solved by high school students, or by those in the first two years of college, are usually more appropriate for the Mathematics Magazine or the Two-Year College Mathematics Journal. No precise line can be drawn between elementary and advanced problems, but in general an elementary problem will involve notions no more difficult than those encountered in the first year of graduate work and should be solvable by methods common at the same level. An advanced problem may deal with more advanced concepts or require more sophisticated techniques for its solution.*

A problem that takes half a page or more to state, or that involves several ad hoc definitions, is usually unsuitable. The statement of a problem should avoid specialized jargon and symbols whenever the English language is more readily understood: for example, "f is a continuous function" is preferable to "$f \in C$."

Before submitting a problem, be sure that it says exactly what you intend it to say. Solvers are not telepathic and will usually solve what was proposed, not what you had in mind.

PROBLEMS DEDICATED TO EMORY P. STARKE

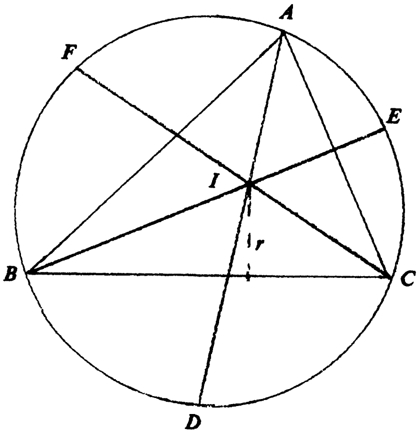
Solutions of these problems dedicated to E. P. Starke should be mailed to Prof. A. P. Hillman, Department of Mathematics and Statistics, University of New Mexico, Albuquerque, NM 87131, by April 30, 1980. To facilitate consideration, type with double spacing in the format used below. If acknowledgment is desired, include a self-addressed card, label, or envelope.

S 22. *Proposed by Edward T. H. Wang, Wilfrid Laurier University, Waterloo, Ontario, Canada, and Roy Westwick, University of British Columbia.*

Let V and W be two vector spaces over the same field. Suppose f and g are two linear transformations $V \to W$ such that for every $x \in V$, $g(x)$ is a scalar multiple (depending on x) of $f(x)$. Prove that g is a scalar multiple of f.

S 23. *Proposed by Jack Garfunkel, Flushing, N.Y., and Leon Bankoff, Los Angeles, Calif.*

Prove that the sum of the distances from the incenter of a triangle ABC to the vertices does not exceed half of the sum of the internal angle bisectors, each extended to its intersection with the circumcircle of triangle ABC. (See the figure.)



ELEMENTARY PROBLEMS

Solutions of these Elementary Problems should be mailed in duplicate to Dr. J. L. Brenner, 10 Phillips Road, Palo Alto, CA 94303 (U.S.A.), by April 30, 1980. Please type with double spacing and place the solver's name and mailing address on each sheet. If acknowledgment is desired, include a self-addressed card or label.

E 2803. *Proposed by L. R. Shenton, Frank Bowman, and H. K. Lam, University of Georgia.*

Prove

(a) $\int_0^{\pi/4} g(\theta)d\theta = \pi^2/24$,

(b) $\int_0^{\pi/6} g(\theta)d\theta = \pi^2/32$,

where $g(\theta) = \arctan[(\cos 2\theta)/(2\cos^2\theta)]^{\frac{1}{2}}$.

E 2804*. *Proposed by Harry D. Ruderman, Hunter College Campus School.*

Let k be a positive integer and S_k be the set of integers j expressible in the form

$$j = k|ab| + a + b,$$

where a, b, run through the nonzero integers. Find the cardinality of the set of positive integers not in S_k.

(The case $k=6$ was problem E 969 [1951, 338] proposed by Solomon W. Golomb.)

E 2805. *Proposed by Wells Johnson, Bowdoin College, Maine.*

Let the integer $r \geqslant 0$ be given. Show that each of the numbers $(2^{2^r})^n - 1$ has at least $2r+1$ distinct prime factors if $n > 2^r$, with the lone exception $r=1, n=3$, when $4^3 - 1 = 3^2 \cdot 7$.

E 2806. *Proposed by F. S. Cater, Portland State University.*

Let S denote a topological space in which every compact set is closed, and let x and y be distinct points of S.

(1) Prove that x and y have disjoint neighborhoods if each of x and y has a countable local base.

(2) Show by example that x and y need not have disjoint neighborhoods if each element of S, other than x, has a countable local base.

E 2807. *Proposed by Solomon W. Golomb, University of Southern California.*

Let a and r be fixed positive constants with $r>1$. For each positive integer k there is a smallest positive integer $n=n(k)$ which satisfies $(n+a)^k \leqslant rn^k$. Show that $\lim n(k)/k$ as $k\to\infty$ exists and evaluate this limit.

E 2808. *Proposed by P. Henrici, Zentrum, ETH, Zürich, Switzerland.*

Let $p(z)=a_0+a_1z+\cdots+a_kz^k$, where the a_i are complex numbers and $a_0\neq 0$. Ordinary iteration applied to p in the form

$$q_{n+1}=\frac{-a_0}{a_1+q_n(a_2+q_n(a_3+\cdots+q_na_k)\cdots)} \tag{1}$$

may or may not produce a sequence $\{q_n\}$ that converges to a zero of p. Show, however, that if (1) is replaced by

$$q_{n+1}=\frac{-a_0}{a_1+q_n(a_2+q_{n-1}(a_3+\cdots+q_{n-k+2}a_k)\cdots)} \tag{2}$$

then for almost all choices of starting values $(q_1,q_2,\ldots,q_{k-1})$ the sequence $\{q_n\}$ converges to the zero of smallest modulus of p, if p has a single such zero.

SOLUTIONS OF ELEMENTARY PROBLEMS

A Recursive Real Sequence

E 2721 [1978, 496]. *Proposed by Allen Emerson, Austin, Texas.*

Let a_0, $a_1>0$ and define a_n $(n\geqslant 2)$ recursively by

$$a_n=\sqrt{a_{n-1}}+\sqrt{a_{n-2}}\,.$$

Show that (a_n) is convergent and compute its limit.

Solution by M. L. J. Hautus, Eindhoven University of Technology, the Netherlands, and T. S. Bolis, State University College at Oneonta, New York (independently).

THEOREM. *Let $g:R_+^k\to R_+$ be increasing with respect to each of its arguments. Let the function $f:R_+\to R_+$ be defined by $f(x)=g(x,\ldots,x)$. Suppose that for some $\alpha>0$,*

$$f(x)>x \quad \text{if } 0<x<\alpha \quad \text{and} \quad f(x)<x \quad \text{if } x>\alpha. \tag{*}$$

Let $a_0>0,\ldots,a_{k-1}>0$, and define

$$a_n=g(a_{n-1},\ldots,a_{n-k}), \qquad n=k,\,k+1,\ldots\,.$$

Then $\lim a_n=\alpha$ $(n\to\infty)$.

Proof. Let $m=\min\{a_0,\ldots,a_{k-1},\alpha\}$ and let $M=\max\{a_0,\ldots,a_{k-1},\alpha\}$. We show that $m\leqslant a_n\leqslant M$ for all n by induction. The inequalities hold for $n=0,1,\ldots,k-1$. Suppose they hold for $n<N$. Then

$$a_N=g(a_{N-1},\ldots,a_{N-k})\geqslant g(m,\ldots,m)=f(m)\geqslant m.$$

The last inequality holds since $m\leqslant\alpha$. In the same way we show $a_N\leqslant f(M)\leqslant M$. Thus the sequence (a_n) is bounded. Let $a=\liminf a_n$, and let $A=\limsup a_n$. Then

$$a=\liminf g(a_{n-1},\ldots,a_{n-k})\geqslant g(\liminf a_{n-1},\ldots,\liminf a_{n-k})=f(a).$$

Since $a\geqslant f(a)$, it follows from (*) that $a\geqslant\alpha$. Similarly $A\leqslant f(A)$, and so $A\leqslant\alpha$. Since $a\leqslant A$, $a=A=\alpha=\lim a_n$. □

Now take $k=2$, $g(x_1,x_2)=x_1^{\frac{1}{2}}+x_2^{\frac{1}{2}}$; we obtain $\lim a_n=4$.

REMARK 1. Let $g(x_1,\dots,x_k)=c_1x_1^s+\cdots+c_kx_k^s$, where $c_i\geqslant 0$, and $0<s<1$. For arbitrary positive numbers $a_0,a_1,\dots,a_{k-1}$, the recursively defined sequence $a_n=c_1a_{n-1}^s+c_2a_{n-2}^s+\cdots+c_ka_{n-k}^s$ has $\lim a_n=\alpha$, where α is given by $\alpha^{1-s}=\Sigma c_i$.

REMARK 2. More generally, take a finite set S of k-vectors $t=(t(1),\dots,t(k))$ with positive (nonnegative) real components, and with $0<\Sigma t(i)<1$ in the case of each k-vector t of S. Define $g(x_1,\dots,x_k)=\Sigma_{t\in S}c_tx_1^{t(1)}x_2^{t(2)}\cdots x_k^{t(k)}$, all coefficients c_t being positive ($c_t>0$). The existence of the number α in (*) follows from the fact that the graph of $f(x)-x$ is concave for $x>0$. Thus this graph crosses the x-axis exactly once for $x>0$, namely, at $x=\alpha$. Hence the correspondingly defined sequence has limit α.

REMARK 3. Several solvers, including J. L. Baker, of Vancouver, B.C., Canada, and R. Bojanic, of Ohio State University, Columbus, showed in the course of their proofs that the rate of convergence of a_n is geometric. Specifically, they showed that $|a_n-4|\leqslant Mq^n$, where $M>0$ and $0<q<1$.

There were 74 other solvers, including the proposer.

Finite Sets and Arithmetic Progressions

E 2730 [1978, 594]. *Proposed by R. L. Graham, Bell Laboratories, Murray Hill, N.J.*

Describe all finite sets A of real numbers with the property that any two elements of A belong to some 3-term arithmetic progression in A.

Solution by C. R. Hampton, College of Wooster. Since the property is invariant under linear transformations, we may assume that 0 is the least element and 1 is the greatest element of A. Clearly $1/2$ must be an element of A. Note that $A_0=\{0,1/2,1\}$ and $A_1=\{0,1/3,1/2,2/3,1\}$ are two examples of sets having the desired property. No others exist (as is shown below), except for the empty set and the one-element sets, which satisfy the condition vacuously.

Suppose there exists another A and that there is a c in $A-A_1$. We may as well assume that c is in the interval $(0,1/2)$ and that c is the element nearest to but not equal to $1/3$. It is then clear that $\frac{1}{2}(1+c)$ and $\frac{1}{4}(1+c)$ are elements of A. If $c>1/3$ then $c>\frac{1}{4}(1+c)>\frac{1}{3}$ and if $c<1/3$ then $c<\frac{1}{4}(1+c)<1/3$. In each case we have a contradiction of the choice of the element c as nearest to $1/3$.

Also solved by M. Albertson & S. Wagon, A. Blum, W. Boucher, R. Breusch, K. Brown, J. Bryant & R. Gilmer, F. S. Cater, E. T. Dixon, M. Eisner, G. Ehrlich, L. I. Foster, M. Goldberg, G. Gripenberg (Finland), J. W. Grossman, C. Hurd, E. L. Isaacson, T. Jager, E. Johnston, C. A. Jones, M. Josephy (Costa Rica), L. Keener, L. Kuipers (Switzerland), V. Laszlo (Czechoslovakia), P. W. Lindstrom, O. P. Lossers (Netherlands), J. G. Mauldon, N. Miku (Netherlands), A. Nijenhuis, R. Patenaude, R. F. Poubel (Brazil), C. C. Rousseau, M. Ruxton, Santa Clara Problem Solving Group, A. J. Schwenk, T. Sckiguchi, K. Singh (India), W. Staton, F. B. Strauss, University of South Alabama Problem Group, R. Whittekin, and the proposer.

Characterization of a Polynomial

E 2731 [1978,681]. *Proposed by Bruce Reznick, Duke University.*

Characterize all polynomials which satisfy $P(x,y)=P(y,x)=P(x,x-y)$.

Solution by Duane M. Broline, Auburn University. First, suppose that the coefficients are in a field K of characteristic other than 2. Let ψ_1 and ψ_2 be the maps defined by $\psi_1(x)=y$, $\psi_1(y)=x$, $\psi_2(x)=x$, and $\psi_2(y)=x-y$. Upon composition, these maps form a group G. Since $\psi_1^2=\psi_2^2=(\psi_1\psi_2)^6=1$, the group G has order 12. Let $K(x,y)$ be the field of rational functions in x and y, considered as commuting variables. Since G acts as a group of field automorphisms on $K(x,y)$, the problem is equivalent to determining the subfield S of $K(x,y)$ which is fixed by G. We claim that $S=K(t_1,t_2)$ where $t_1=\frac{1}{2}(x^2+y^2+(x-y)^2)=x^2+y^2-xy$ and $t_2=x^2y^2(x-y)^2$.

A straightforward computation shows that the set of images of x (or y) under G is $\{x,y,x-y,y-x,-x,-y\}$. Hence, $K(t_1,t_2)\subseteq S$. Let $\mathrm{Gal}(K(x,y),S)$ be the Galois group of $K(x,y)$ with respect to S. A standard result of Galois Theory [see, e.g., Herstein, *Topics in Algebra*, 1st ed., p. 199] shows that $12\leqslant|\mathrm{Gal}(K(x,y),S)|\leqslant|K(x,y)\colon S|$. Hence $12\leqslant|K(x,y)\colon K(t_1,t_2)|$ with equality only if $K(t_1,t_2)=S$. Let $F=K(t_1,t_2)$.

The polynomial $q(T)=(T-x)(T+x)(T-y)(T+y)(T-x+y)(T+x-y)=T^6-t_1T^4+t_1^2T^2-t_2$ is in $F[T]$ and has x for a root. Thus $|F(x)\colon F|\leqslant 6$. The polynomial $r(T)=(T-y)(T-x+y)=T^2-xT+x^2-t_1$ is in $F(x)[T]$ and has y for a root. Thus $|F(x,y)\colon F(x)|\leqslant 2$. Since $F(x,y)=K(x,y)$ we have $12\leqslant|K(x,y):F|\leqslant|F(x,y):F(x)|,|F(x):F|\leqslant 12$ and $|K(x,y):F|=12$ as desired.

The proof above works when characteristic of K is 2, with the following modifications: (i) $|G|=6$, since $(\psi_1\psi_2)^3=1$; (ii) $t_1=x^2+y^2+xy$; (iii) $t_2=xy(x+y)$; (iv) $q(T)=(T+x)(T+y)(T+x+y)=T^3+t_1+T+t_2$; (v) $r(T)=(T+y)(T+x+y)=T^2+xT+t_1+x_2$.

Also solved by Irl Bivens, Robert Breusch, Stephen Bronn & Gilbert Orr, Grupo Cam (Venezuela), Matthew Chen, Jeffrey Cohen, Paul Cull, J. Gillis (Israel), Robert Gilmer, Graceland College Mathematics Research Group, Eli Isaacson, L. Kuipers (Switzerland), J. H. van Lint & J. W. Nienhuys (Netherlands), O. P. Lossers (Netherlands), Andy Magid, J. G. Mauldon, Richard Mueller, David Robbins, Nan-Shan Shou (Canada), Joseph Silverman, Wolfe Snow, and the proposer.

Editorial note. Robert Gilmer points out that the problem statement is ambiguous: the coefficient domain is not specified. He shows that if the coefficients come from a commutative ring R with identity, then the polynomials with the described properties form a subring S of $R[x,y]$, where $S=R[x^2-xy+y^2,x^2y^2(x-y)^2,\{axy(x-y)|a\in I\}]$, where I is the ideal of R consisting of all elements $r\in R$ such that $r=-r$. J. H. van Lint and J. W. Nienhuys point out that the method for solving this problem is treated in detail in N. J. A. Sloane's paper "Error-Correcting Codes and Invariant Theory: New Applications of a Nineteenth Century Technique," this MONTHLY, 84 (1977), 82–107. D. Ž. Djoković notes that this problem is a special case of a well-known theorem of Chevalley on invariants of a finite group generated by reflections (in the present case, the dihedral group of order 12). See Bourbaki, Groupes et Algèbres de Lie, Chapter 5, §5, Ex. 1, p. 135, for a more general exercise.

Labeling Chessboard Squares

E 2732 [1978, 681]. *Proposed by Peter Sjögren, University of Uppsala, Sweden.*

It is easy to see that one can label the squares of an $n\times n$ chessboard by integers from 1 to n^2 so that the difference between labels of neighboring squares does not exceed n. Is this best possible? (Two squares are neighbors if they share a common side.)

Solution by D. M. Broline, Auburn University. Yes. Let $R_k=\{i|$ the ith row contains a square whose label is $\leqslant k\}$ and $C_k=\{i|$ the ith column contains a square whose label is $\leqslant k\}$. Since $R_{n^2}=C_{n^2}=\{1,\dots,n\}$, there exists a k such that either R_k or C_k is equal to $\{1,\dots,n\}$ and neither R_{k-1} nor C_{k-1} is equal to $\{1,\dots,n\}$.

Suppose $R_k=\{1,\dots,n\}$. Since $R_{k-1}\neq\{1,\dots,n\}$, the square labeled k is the only one in its row labeled with a number in $\{1,\dots,k\}$. If there is a row, all of whose labels are in $\{1,\dots,k-1\}$, then $C_{k-1}=\{1,\dots,n\}$, contrary to our choice of k.

Let $B=\{$squares whose label is $\leqslant k\}$. The above paragraph shows that each row of the checkerboard contains a square which is not in B. In particular, there are at least n squares not in B which are neighbors of one or more squares of B.

If the difference of neighboring squares is to be $\leqslant d$, each of these neighbors must be labeled by a number in $\{k+1,k+2,\dots,k+d\}$. Therefore, $n\leqslant d$, as required.

Also solved by M. Chen, Robert T. Cunningham, P. Eitner, D. Hensley, C. Hurd, O. P. Lossers (Netherlands), M. J. Rentmeesters, Steve Ricci, M. Skalsky, A. Stenger, P. Vojta, and the proposer.

Editor's comments. P. Eitner points out that this problem is equivalent to a special case of a result of J. Chvátlova in "Optimal Labelling of a Product of Two Paths," Discrete Math., 11 (1975) 249–253.

C. Hurd lists the first mn integers and inserts two partitioning division marks so that S_1, the first part, includes the smallest integers, $|S_2|=m-1$, and $|S_3|=|S_1|$ or $|S_3|=|S_1|-1$. Then he observes that, on an $m\times n$ grid, at least one member of S_1 must be adjacent to at least one member of S_3. (Example: $m=4$, $n=6$, $S_1=\{1,\ldots,11\}$, $S_2=\{12,13,14\}$, $S_3=\{15,\ldots,24\}$. The members of S_2 cannot form an effective barrier.) This argument generalizes to k-dimensional rectangular grids, and even to certain irregularly shaped grids.

Infinitely Many Subsets of [0, 1] With the Same Non-zero Length and Small Pairwise Intersections

E 2733 [1978, 682]. *Proposed by Jim Fickett, University of Colorado.*

Let S_i, $i=1,2,\ldots,m$ be subsets of $[0,1]$; each S_i is a finite union of disjoint intervals and let $l(S_i)$ be the sum of the lengths of these intervals. Assume that $l(S_i)=\varepsilon$, $l(S_i\cap S_j)\leqslant\varepsilon^2$ $(i\neq j)$, where $\varepsilon>0$ is fixed. How large can m be?

Solution by Eli L. Isaacson, New York University. For $0<\varepsilon\leqslant 1$ we will show that m can be infinite by constructing an infinite sequence of sets S_i which satisfy $l(S_i)=\varepsilon$ and $l(S_i\cap S_j)=\varepsilon^2$. (For $\varepsilon>1$ there are clearly no sets $S_i\subset[0,1]$ which satisfy $l(S_i)=\varepsilon$.)

First we describe an infinite sequence of partitions of $[0,1)$. Let

$$I_0^0=[0,1)$$
$$I_0^1=[0,\varepsilon), I_1^1=[\varepsilon,1).$$

Note that $[0,1)=I_0^0=I_0^1\cup I_1^1$. Now assume that the nth partition

$$I_0^0=I_0^n\cup I_1^n\cup\cdots \text{ into } 2^n \text{ intervals } I_j^n$$

has been defined and define the next partition as follows: Divide each of the intervals I_j^n into two subintervals whose lengths are in the ratio $\varepsilon:1-\varepsilon$. That is, for $0\leqslant j\leqslant 2^n-1$,

$$\begin{gathered} I_j^n=I_{2j}^{n+1}\cup I_{2j+1}^{n+1} \\ l\left(I_{2j}^{n+1}\right)=\varepsilon l(I_j^n), l\left(I_{2j+1}^{n+1}\right)=(1-\varepsilon)l(I_j^n). \end{gathered} \tag{*}$$

In the continuation, the following abbreviations are used: $s(n)=2^n-1$; $r(i,j,k)=2^kj+i$; $\bigcup_i'$ means union over even values of i. Now for $n\geqslant 0$ define $S_{n+1}=\bigcup_{j=0}^{s(n)}I_{2j}^{n+1}$. To simplify the notation, set $l_j^n=l(I_j^n)$. Then, by (*)

$$l(S_{n+1})=\sum_{j=0}^{s(n)}l_{2j}^{n+1}=\varepsilon\sum l_j^n=\varepsilon l([0,1))=\varepsilon.$$

Also, from the first equation of (*), we obtain, by induction, the relations (**) for all $k\geqslant 0$:

$$\begin{gathered} I_j^n=\bigcup_{i=0}^{s(k)} I_{r(i,j,k)}^{n+k}, \\ l_j^n=\sum_{i=0}^{s(k)} l_{r(i,j,k)}^{n+k} \end{gathered} \tag{**}$$

Hence $I_{2j}^n \cap S_{n+k} \bigcup{}'^{s(k)}_{i=0} I^{n+k}_{r(i,j,k+1)} = \bigcup^{s(k-1)}_{i=0} I^{n+k}_{r(2i,j,k+1)}$. Thus, by (*), $l(I_{2j}^n\cap S_{n+k})=\sum_{i=0}^{s(k-1)}l_{r(2i,j,k+1)}^{n+k}=\varepsilon\sum_{i=0}^{s(k-1)}l_{r(i,j,k)}^{n+k-1}$. But by (**), this last sum is εl_{2j}^n. Thus finally, $l(S_n\cap S_{n+k})=\sum_{j=0}^{s(n-1)}l(I_{2j}^n\cap S_{n+k})=\sum_{j=0}^{s(n-1)}\varepsilon l_{2j}^n=\varepsilon\sum_{j=0}^{s(n-1)}l_{2j}^n=\varepsilon l(S_n)=\varepsilon^2$.

Also solved by the University of South Alabama Problem Group, and by the proposer.

The USA Problem Group noted that the statement can be generalized to require $l(S_{n(1)}\cap S_{n(2)}\cap\cdots\cap S_{n(m)})=\varepsilon^m$ for $n(1)<n(2)<\cdots<n(m)$, and that this is the best possible, according to problem E 2362 [1973; 810–811].

Exponential of a Matrix

E 2734 [1978, 682]. *Proposed by Melvin Hausner, Courant Institute, New York University.*

Let $A=(a_{ij})$ be a real square matrix such that $a_{ij}>0$ for $i\neq j$. Show that all entries of e^A are positive.

Solution by O. P. Lossers, Eindhoven University, Netherlands. Let σ be such that $A+\sigma I$ is positive. Now obviously $e^{A+\sigma I}$ is positive, so then e^A must be positive. For $-\sigma I$ commutes with $A+\sigma I$ and

$$e^A=(e^{-\sigma I})(e^{A+\sigma I})=(e^{-\sigma}I)(e^{A+\sigma I})=e^{-\sigma}e^{A+\sigma I}.$$

Also solved by Alan Berger, Richard Bittman, Wayne Boucher, J. M. Cohen, C. G. Cullen, Emeric Deutsch, Thomas Foregger, E. J. Gallopoulous, Clark Givens, Gustav Gripenberg (Finland), Grupo Cam (Venezuela), Doug Hensley, A. A. Jagers (Netherlands), H. Kestelman (England), J. R. Kuttler, Detlef Laugwitz, Joel Levy, Peter Lindstrom, Allan Macdonald & John Remmers, M. Marcus, J. G. Mauldon, Jose Luis de Miguel (Spain), Gary O'Brien, Wolfe Snow, Gerson Sparer, Allen Stenger, L. Van Hamme (Belgium), and the proposer. K. W. Lau (Hong Kong) found the result on p. 176 of "Introduction to Matrix Analysis," 2nd ed., by Richard Bellman.

ADVANCED PROBLEMS

Solutions of these Advanced Problems should be sent to Professor Roger C. Lyndon, Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, by April 30, 1980. To facilitate their consideration, please type (in duplicate, with double spacing) and place the solver's mailing address on the solution sheets. If acknowledgment is desired, include a self-addressed card.

6282*. *Proposed by David P. Robbins, Hamilton College.*

It is known (Wilansky, Topology for Analysis, p. 113) that any two countable metric spaces with no isolated points are homeomorphic. Give an explicit homeomorphism between the space of rational numbers r with $0<r<1$ and that of rationals t with $0\leqslant t\leqslant 1$.

6283. *Proposed by Gordon R. Feathers, North Carolina State University at Raleigh.*

It is well known that a strongly closed convex subset of a Banach space is weakly closed. Is the same true of a strongly closed star-shaped subset?

6284. *Proposed by William P. Wardlaw, U. S. Naval Academy.*

Let R be a finite ring with more than one element and with no nonzero nilpotent element. Show that R is a direct sum of fields.

(This generalizes Wedderburn's theorem that a finite division ring is a field.)

SOLUTIONS OF ADVANCED PROBLEMS

Power Sums in Finite Fields

6201 [1978, 203]. *Proposed by Daniel D. Anderson, University of Missouri, Columbia.*

Let $GF(p^n)$ be the finite field of order p^n. For which positive integers k is every element of $GF(p^n)$ a sum of kth powers?

Composite solution. Most solvers gave a version of the following solution.

PROPOSITION. *For k a positive integer, every element of the finite field $F=GF(p^n)$ is a sum of kth powers if and only if k is not divisible by $(p^n-1)/(p^m-1)$ for m less than n and dividing n.*

Proof. Let x be a generator of the multiplicative group of F. Then k has the required property if and only if x^k generates F; that is, if x^k does not generate any proper subfield of F. This holds

if and only if, for each m less than n and dividing n, $x^{k(p^m-1)} \neq 1$, that is, p^n-1 does not divide $k(p^m-1)$.

Solutions by J. Alonso; L. Carlitz; Jeffrey Mitchell Cohen; Columbia University Problem Group; Lenny Jones & Blair Spearman; Ernst Kani (West Germany); O. P. Lossers (Netherlands); jointly by the following ten people: M. Bailey, M. Boyle, M. Czaja, C. Gerson, W. Gremmel, S. Keeler, V. Ndakbo, G. Sigurdsson, C. Thompson & J. Treiman; Kenneth W. Spackman; Edward T. Wong; and the proposer.

Charles Small notes that the proposition shows that, for given k, every element of all sufficiently large finite fields is a sum of kth powers, and that, in fact, he has shown (Proc. Amer. Math. Soc., 65 (1977), 35–36) that in every sufficiently large finite field each element is a sum of two kth powers.

Inequality for Finite Groups

6202 [1978, 203]. *Proposed by A. A. Jagers, Twente University of Technology, Netherlands.*

Let S be a set of generators of a finite group G. For $g \in G$, let $m(g)$ be the least number of terms in a representation of g as a product of elements of S. Let $n_1, n_2, \ldots, n_k$ be the degrees of the irreducible characters of G. Prove that $m(g) \leqslant n_1 + n_2 + \cdots + n_k - 1$.

Solution by I. M. Isaacs, University of Wisconsin. In the group algebra $\mathbb{C}[G]$ let α be the sum of the elements of S. Then $g \in G$ can be represented as a product of m elements of S iff g lies in the support of α^m.

Now $\mathbb{C}[G]$ is the direct sum of algebras of $n_i \times n_i$ matrices for $1 \leqslant i \leqslant k$ and hence is isomorphic to a subalgebra of the full ring of $N \times N$ matrices where $N = \Sigma n_i$. Identify $\mathbb{C}[G]$ with this subalgebra. Since the minimal polynomial of the "matrix" α has degree at most N, it follows that every power α^m is a $\mathbb{C}$-linear combination of $1, \alpha, \alpha^2, \ldots, \alpha^{N-1}$. Since S generates G, each $g \in G$ lies in the support of some $\alpha^m = \Sigma_{i=1}^{i=N-1} c_i \alpha^i$. Thus g lies in the support of some α^i with $i \leqslant N-1$. The result follows.

Also solved by the proposer, who remarks that this is really a result on Cayley (di)-graphs, and refers to N. Biggs, *Algebraic Graph Theory*, Cambridge Univ. Press, 1973, pp. 12, 13, 106, 146.

Supremum of Projections

6228. [1978, 686]. *Proposed by Ivan Vidav, University of Ljubljana, Yugoslavia.*

Let A be a C^*-algebra with unit 1, and let e and f be two projections of A such that $e+f$ is invertible in A. Show that $e \cap f = 2e(e+f)^{-1}f$. ($e \cap f$ is the supremum of the set of all projections $h \in A$ such that $h \leqslant e$ and $h \leqslant f$.)

Solution by O. P. Lossers, Eindhoven University of Technology, the Netherlands. Let $p = 2e(e+f)^{-1}f$, then $p \in A$. We have to prove:

(i) p is a projection, i.e., p is self-adjoint and $p^2 = p$.

(ii) $p \leqslant e$ and $p \leqslant f$, i.e., $ep = pe = p$ and $fp = pf = p$.

(iii) $p = e \cap f$.

Proof. (i) and (ii) Since

$$2e(e+f)^{-1}f = 2e(e+f)^{-1}((e+f)-e) = 2e(1-(e+f)^{-1})e,$$

p is self-adjoint, and $pe = ep = p$. Since

$$2e(e+f)^{-1}f = 2((e+f)-f)(e+f)^{-1}f = 2f(1-(e+f)^{-1})f,$$

$pf = fp = p$. Therefore we can write $2p = (e+f)p = p(e+f)$. So

$$p^2=2pe(e+f)^{-1}f=ep(e+f)(e+f)^{-1}f=epf=p.$$

Conclusion: p is a projection and $p\leqslant e$, $p\leqslant f$.

(iii) Let h be a projection, $h\leqslant e$ and $h\leqslant f$; we shall prove $h\leqslant p$, i.e., $hp=ph=h$. We have $2h=(e+f)h=h(e+f)$, so

$$ph=e(e+f)^{-1}f2h=e(e+f)^{-1}(e+f)h=eh=h;$$

and

$$hp=2he(e+f)^{-1}f=h(e+f)(e+f)^{-1}f=hf=h.$$

Conclusion: $p=e\cap f$.

Also solved by Jun Ichi Fujii (Japan), T. D. Morley, and the proposer. Fujii states that a solution is given by the proposer (I. Vidav, Proc. Amer. Math. Soc., 65 (1977) 297–298). Morley remarks: "Via representability theorems, this problem follows immediately from the *parallel sum* construction of Anderson and Duffin [**1**, Theorem 8] (finite dimensions), Fillmore and Williams [**3**, Theorem 4.3], or Anderson and Trapp [**2**, Theorem 11(c)]."

1. W. N. Anderson, Jr., and R. J. Duffin, Series and parallel addition of matrices, J. Math. Anal. Appl., 26 (1969) 576–594.

2. W. N. Anderson, Jr., and G. E. Trapp, Shorted operators II, SIAM J. Appl. Math., 28 (1975) 60–71.

3. P. A. Fillmore and J. P. Williams, On operator ranges, Advances in Math, 7 (1971) 254–281.

REVIEWS

EDITED BY J. ARTHUR SEEBACH, JR., AND LYNN A. STEEN
with the assistance of the mathematics departments of St. Olaf, Carleton, and Macalester Colleges
COLLABORATING EDITOR FOR FILMS: SEYMOUR SCHUSTER, CARLETON COLLEGE

We invite readers to submit reviews of significant recent college-level mathematics books. We especially encourage reviews based on classroom use, or comparative reviews of several related books. Reviews should ordinarily not exceed two pages (per book) typed double spaced. Manuscripts of reviews as well as books submitted for review should be sent to: Book Review Editor, American Mathematical Monthly, St. Olaf College, Northfield, MN 55057.

Mathematics Today: *Twelve Informal Essays*. Edited by Lynn Arthur Steen. Springer-Verlag, New York, 1978. viii+367 pp. $12. (Telegraphic Review, March 1979.)

Double helix in the sky tonight,
Throw out the hardware, let's do it right.
—Steely Dan, *Aja*.

Mathematics Today is a collection of twelve essays on theoretical and applied mathematics by distinguished practitioners of their respective fields. The essays are designed, writes Jacob T. Schwartz, chairman of the committee that produced the book, "to convey to the intelligent nonmathematician something of the nature, development, and use of mathematical concepts, particularly those that have found application in current scientific research" (p. v).

Now this is a noble aim, especially in an age like ours when rock bands sing of double helices, literary critics invoke analogies between modern fiction and Lie algebras, and nearly everyone, it seems, who wants to appear *au courant* has ready-made opinions about the metaphysical implications of the unified field theory or of black hole singularities. Mathematics, in other words—*real* mathematics, and not just the half-remembered indignities of sophomore trigonom-

etry—is being used and abused on a nearly unprecedented scale by humanists, philosophers, poets, and novelists. *Mathematics Today* is a much-needed attempt to maximize the usefulness and minimize the abuse, simply by having some very good mathematicians explain, as clearly as they can, what they are really about (I remember the enthusiasm of a literary colleague of mine for the concept of the "Lie algebra," since he thought it referred to a calculus of universal untruth rather than to the family name of a nineteenth-century theorist: at the very least the book makes that kind of silliness less likely).

But though the aim may be noble, the question still has to be asked, Is it fulfilled? *Does* the book manage to explain to the "intelligent nonmathematician" the range, complexity, and spirit of current mathematics? I am, rather definitively, a nonmathematician: and this MONTHLY has kindly elected me intelligent enough to respond, on behalf of my fellow (unnumbered?) nonmathematicians, to the collection's success or failure.

If cogency of presentation and the pure excitement of abstract ideas in action are reasonable standards of success for a book like this one, then *Mathematics Today* is a success of a very high and happy order. Almost every essay in the series has its share of fire. And it can even be said that the relative proportions of heat and light in the essays vary directly with the courage of the convictions of the respective writers: that is, the more unself-consciously a given writer in this volume is willing to abandon himself to his enthusiasm for his chosen area of research—without apologies, without oversimplifying condescension—the more readable the essay is. Paradoxically, then, some of the initially most unapproachable-seeming of the essays—Jonathan Alperin's "Groups and Symmetry," Kenneth Appel and Wolfgang Haken's "The Four Color Problem," or Martin Davis's "What Is a Computation?"—are also among the richest and most elegant (in both the mathematical and the stylistic sense) in the volume.

To be sure, this paradox is helped along by the "informal" quality of the book as a whole, and by the courtesy of its typography. Nothing (and here I speak with authority) frightens a nonmathematician more or more quickly than the sight of equations arrayed in the iron and invincible phalanx of proof. And the editor has carefully set apart the most daunting of those temptations to despair, in boxes edged by orange lines and with suitably cautionary titles like "Tijdeman's Near Solution of Catalan's Problem" (p. 59), "Bjerknes's Meteorological Manifesto" (p. 137), and "An Unsolvable Word Problem" (pp. 258–259). Not that the proofs, as presented, are finally all that difficult to follow. But it is, well, *considerate* of the editor to put them where they are. The effect is more or less like that of horror films of thirty years ago: when you heard organ music in a minor key, you knew you could either look or not look at what came next, but you knew you had a choice.

To this extent, anyhow, the contributors to *Mathematics Today* have followed the advice of Steely Dan: they have thrown out (or thrown apart) the hardware, and have done it right. The amazing and delightful thing is how much remains *after* the hardware is thrown out, how resonant and memorable are most of the discussions even after they have been stepped down to the presumed level of the reasonably prudent layman.

The book is divided into four major parts. After a fine and succinct essay by Lynn Arthur Steen on the intention and genesis of the volume as a whole, Part One consists of a piece called "Mathematics—Our Invisible Culture," by Allen L. Hammond. Unfortunately, this is the weakest essay in the book, being mainly a transcript of Mr. Hammond's interview with three practicing mathematicians of different specialities, different ages, and different attitudes toward the nature of the art. His interviewees say some interesting things about the business—or the life—of doing mathematics, but Hammond's main purpose in the essay seems to be simply to demonstrate that mathematicians are people, too, and that they think of their work as passionately as do poets or the better sort of dentists. The problem is that we already *knew* that or would not be reading *Mathematics Today* in the first place. Well done as it may be as a Sunday-supplement piece of human-interest writing, Hammond's essay is just the wrong one with which to begin this volume—not only because it is beside the main point of the volume

itself, but because it establishes an initial tone of cheap-shot, *People*-magazine journalism that the rest of the essays far surpass.

After the unfortunate Part One, Part Two of the book consists of five essays on the more (or more or less) theoretical aspects of contemporary mathematical thought: Ian Richards's "Number Theory," Jonathan Alperin's "Groups and Symmetry," Roger Penrose's "The Geometry of the Universe," Philip Thompson's "The Mathematics of Meteorology," and Appel and Haken's "The Four Color Problem." This is by far the most intensely exciting part of the book, and the most likely to worm its way into the hearts and minds of humanists seeking to understand the conceptual underpinnings of the n-dimensional universe we increasingly realize we inhabit. I have already cited Alperin's and Appel/Haken's pieces, but it is impossible not to remark that Richards's essay is also a gem, combining rigor and charm in a way that makes most so-called "literary" writers of my acquaintance look like amateurs of the written word.

Part Three is a collection of five essays on "applied" mathematics: Ronald Graham's "Combinatorial Scheduling Theory," David S. Moore's "Statistical Analysis of Experimental Data," Martin Davis's "What Is a Computation?" Jacob Schwartz's "Mathematics as a Tool for Economic Understanding," and Frank C. Hoppensteadt's "Mathematical Aspects of Population Biology." Interesting as the essays are in this section, they lack something of the power of Part Two—with the exception of the Davis essay—and I think the reason for that has something rather fundamental to do with the aura of the whole book. We—those of us who do not really understand mathematics but want desperately to—are not really as interested or as impressed with the numerous demonstrations of mathematics' triumphant relevance to the "real" world. That, after all, is a story we have all heard for a long time now. What rivets our attention is rather the unfolding, on its own terms and perhaps even in its own strange heterocosm, of this most sublime and most exigent accomplishment of human thought. Numerous essays in *Mathematics Today* refer to the famous, perhaps epochal, debate between Hilbert and Gödel: Hilbert with his ferocious faith in the fitting of the products of mind to the structure of external reality, and Gödel with his radical insistence on the unprovability of certain intuitively true propositions. It is an insoluble problem of epistemology which of these two thinkers is the more formalistic, the more resigned to mathematics' inevitable closure from some Platonic idea of the "really real." But it is undeniable that across this central axis the debate, not only of contemporary mathematical thought but of much contemporary fiction and philosophy, takes place. And it is also true that contemporary thought forces most of us to be fundamentally sympathetic to Gödel, to a kind of "uncertainty principle" within the palace of reason itself, and to the glories of a mathematics that, whatever its applications to post-Einsteinian physics or the universes of quanta, is nonetheless at heart an autonomous human creation whose most expansive wonders are those of its own self-consciousness.

Part Four of *Mathematics Today* is called "The Relevance of Mathematics," by Felix E. Browder and Saunders Mac Lane. But despite its title, this essay should probably have been the introductory one to the volume. For in attempting (successfully and perspicuously) to demonstrate the "relevance" of mathematical understanding to the history and development of human science, Browder and Mac Lane also indicate how much our concept of the physically "real" is, in fact, a projection and reformulation of the shape of our most private and most distinctively subjective perceptions of the real as *inward*. And that is simply to say that the great and dazzlingly articulated point of *Mathematics Today* is, in the most serious sense, another affirmation of the poetic quality of reason itself: a better statement than most of our poets have given us of the concept that reason, when truest to its own complexity, is inevitably and eternally creative and world-making.

FRANK McCONNELL, Northwestern University Department of English

This review was solicited and edited by Ralph Boas.

TELEGRAPHIC REVIEWS

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T = textbook P = professional reading
S = supplementary reading L = undergraduate library purchase
13 to 18 = freshman to second year graduate level usage
1 to 4 = appropriate time in semesters to cover text

Asterisks (*) or question marks (?) denote special positive or negative emphasis, respectively. Publishers are denoted by standard abbreviations; complete addresses may be found in *Books in Print*.

GENERAL, S**, P, L**, *Mathematical Plums*. Ed: Ross Honsberger. Dolciani Math. Expos., No. 4. MAA, 1979, ix + 182 pp, $14. [ISBN: 0-88385-304-3] Ten elementary but vigorous essays that will tantalize the curious and astonish the indifferent: dividing cake, Ramsey numbers, remarkable sequences, Kepler's conics, Skewes number, and more. Excellent source material for mathematics clubs, and a motivation for text-jaded students. LAS

GENERAL, T(13), S, *Logic and Boolean Algebra*. Kathleen Levitz, Hilbert Levitz. Barron's Educ Ser, 1979, viii + 132 pp, $3.50 (P). [ISBN: 0-8120-0537-6] An attractive, informal introduction to the propositional calculus, using truth tables, switching circuits and elementary set theory; leads to the more abstract generalization provided by Boolean algebra. Not enough for a full course, but useful as supplementary reading even by high school students. LCL

GENERAL, S*, L, *Puzzles, Games, and Paradoxes*. R.J. Duffin. Carnegie-Mellon U. (Dept. of Math., Pittsburgh, PA), 1979, 41 pp, (P). 269 questions--some old, some new--for a problems seminar, most (but not all) from elementary mathematics. No answers or hints. LAS

GENERAL, L***, *Mathematics Magazine: 50 Year Index*. Ed: J. Arthur Seebach, Lynn Arthur Steen. MAA, 1978, xxvi + 165 pp, $8 (P). [ISBN: 0-88385-423-5] A brief history followed by title and author indexes for the period 1926-1977. In a warm and thoughtful account, Edwin F. Beckenbach traces the journal from its first year of existence as a membership campaign folder in the Deep South through two name changes, several financial crises and seven editors to its present polished format. Titles, sometimes augmented, are folded at key words to provide access by subject. For those titles which appear more than once, complete bibliographic information is given in each appearance. In the author index each article appears with complete information under the first named author. Co-authors are listed with "see also" references to the first named author. This painstakingly prepared and immaculately typed index will send those readers fortunate enough to have access to earlier volumes scurrying to read the contributions of prominent and influential mathematicians of the 1930's and 1940's. A must purchase for every reader of the journal. JK

GENERAL, P, *Proceedings Bicentennial Congress Wiskundig Genootschap*. Ed: P.C. Baayen, D. van Dulst, J. Oosterhoff. Math Centrum, 1979. *Part I*, Math. Centre Tracts, No. 100, xiv + 212 pp, Dfl. 26 (P). [ISBN: 90-6196-168-8]; *Part II*, Math. Centre Tracts, No. 101, 215 pp, Dfl. 26 (P). [ISBN: 90-6196-169-6]. These bicentennial proceedings include the Brouwer Memorial Lecture (1975) by Armand Borel "On the development of Lie group theory," as well as a large number of papers from the Congress on a very wide range of topics. JAS

GENERAL, P, *A Guide to Undergraduate Science Course and Laboratory Improvements*. NSF, xii + 150 pp, (P). A comprehensive subject index keyed to a project roster of LOCI (Local Course Improvement) and ISEP (Instructional Scientific Equipment Program) grants awarded during 1976-1978. LAS

GENERAL, P, *Transactions of the Moscow Mathematical Society, 1979, Issue 1*. AMS, 1979, iii + 267 pp, $48 (P). Translation of Volume 35, 1976. LAS

GENERAL, P, L*, *CRC Handbook of Mathematical Sciences, Fifth Edition*. Ed: William H. Beyer. CRC Pr, 1978, xii + 982 pp, $49.95. [ISBN: 0-8493-0655-8] A revision of the 1975 *Fourth Edition* with new material in number theory, analysis, numerical methods and astrodynamics. No longer a compact "hand" book, it now resembles an unabridged dictionary. LAS

GENERAL, L, *Mathematical Brain-Teasers*. J.A.H. Hunter. Dover, 1976, ix + 111 pp, $2 (P). [ISBN: 0-486-23347-2] One hundred ten problems in story form and forty alphametics. Except for correction of misprints and a new brief appendix on the Boolean approach to "inferential" problems, this Dover edition, first published in 1976, is Hunter's *Math Brain Teasers* originally published by Bantam Books in 1965. JK

GENERAL, S, *Against Infinity: An Anthology of Contemporary Mathematical Poetry*. Ed: Ernest Robson, Jet Wimp. Primary Pr, 1979, 90 pp, $8.95 (P); $17. [ISBN: 0-934982-00-7; 0-934982-01-5] Poetry, experimental writing, and computer compositions on various mathematical themes, by 51 writers, scientists, poets and artists. LAS

GENERAL, P, L, *List of 16mm Films on Mathematical Subjects*. David Singmaster. Open U, 23 pp, (P). 570 films, coded by subject, level and distributor, prepared from a continually updated computer file. Emphasizes films available in the United Kingdom, but omits most Open University films (because they are closely tied to specific courses). LAS

GENERAL, P**, L*, *T_EX: Tau Epsilon Chi, A System for Technical Text*. Donald E. Knuth. AMS, 1979, 200 pp, $8.80 (P). [ISBN: 0-8218-0209-7] A literate, well organized, and occasionally witty user's guide to Knuth's new system of linearizing mathematical text for computer composition. AMS is investigating the development of a T_EX-based system whereby authors "typeset" their own papers on their own institutions' computers, and invites users of this volume to join an AMS-sponsored user's group. LAS

GENERAL, L*. *Mathematical Diversions.* J.A.H. Hunter, Joseph S. Madachy. Dover, 1975, vii + 178 pp, $2.25 (P). [ISBN: 0-486-23110-0] Unabridged, virtually unchanged from D. Van Nostrand's 1963 edition. A cut above many of its kind. Recommended for every school library. JK

BASIC, T(13). *Algebra for College Students.* Nancy Myers. D. Van Nostrand, 1979, x + 446 pp, $14.95 (P). [ISBN: 0-442-25625-6] "Cookbook" treatment of intermediate algebra where examples and rules provide primary explanation of most concepts. Review of background concepts inserted where needed. All exercises and self-tests have answers. A few omissions and inaccuracies (e.g., if (x-2)(x-1) = 0 then x-2 = 0 and x-1 = 0). MW

PRECALCULUS, T(13: 1). *College Algebra with Trigonometry, Second Edition.* Raymond A. Barnett. McGraw, 1979, xvi + 523 pp, $15.95. [ISBN: 0-07-003809-0]; *College Algebra, Second Edition,* 1979, xvi + 403 pp, $14.95. [ISBN: 0-07-003778-7] Informal yet sound expository style and open format ensure student comprehension. Many examples, each followed by related problems with answers to check understanding. Clumsy trigonometry introduction due to ambivalence between angle and unit circle development. Numerous exercises, graded according to difficulty. More application problems in this edition. (*First Edition,* TR, December 1974.) MW

PRECALCULUS, T(13: 1). *Understanding College Algebra.* Gene R. Sellers. Merrill, 1979, ix + 338 pp, $12.95 [ISBN: 0-675-08306-0]; *Understanding Algebra and Trigonometry,* xi + 516 pp, $13.95. [ISBN: 0-675-08294-3] Attractive format, wide margins containing topic headings and questions to students. Chapter summaries and self-test exercises keyed to relevant sections. "More practical than theoretical" approach means theorems often justified on basis of not-very-clear examples. Study guide and test book available. MW

EDUCATION. *Guidelines for the Preparation of Teachers of Mathematics, Second Edition.* NCTM, 1979, 31 pp, $2.15 (P). Revision of the 1973 edition (TR, April 1974); includes extensive lists of specific competencies arranged in three sections: Academic and Professional Knowledge, Professional Competencies and Attitudes, and Institutional Responsibilities. LAS

EDUCATION. *Ideas from the Arithmetic Teacher.* George Immerzeel, Bob Wells. NCTM, 1979, ii + 142 pp, $5.40 (P). [ISBN: 0-87353-143-4] A collection of reprints from the Ideas section of the *Arithmetic Teacher.* The activities, appropriate for students in grades four through eight, involve computation, fractions, problem solving, geometry and measurement. JNC

HISTORY, P, L*. *Elie Cartan-Albert Einstein: Letters on Absolute Parallelism 1929-1932.* Ed: Robert Debever. Trans: Jules Leroy, Jim Ritter. Princeton U Pr, 1979, xvii + 233 pp, $20. [ISBN: 0-691-08229-4] Original (German and French) correspondence with facing English translations, supplemented by the editor's historical and mathematical preface (in French) and various explanatory footnotes (also in French). A fascinating glimpse into the minds of two of this century's premier scientists as they struggled to express and compromise their frequently clashing physical and mathematical perspectives. LAS

HISTORY, P, L. *A History of the Mathematics Departments of the University of Colorado.* Burton W. Jones, Wolfgang Thron. (Math. Dept., U. of Colorado, Boulder, CO 80309), 1979, viii + 102 pp, (P) A brief, undocumented account of the first century (1876-1976) of mathematics at Boulder, with extensive photographs, lists of faculty, and graduate degrees. A useful document that records not just Colorado details, but also the extensive intercourse between Colorado and other departments. LAS

HISTORY, P, L. *Stefan Banach.Oeuvres, Volume II: Travaux sur L'Analyse Fonctionnelle.* Stefan Banach. PWN, 1979, 470 pp. [ISBN: 83-01-00393-6] Concluding volume of Banach's works (*Volume I* was published in 1967; TR, May 1968): reprint of *Théorie des Opérations Linéaires;* an extensive survey (in English) by A. Pelczyński of aspects of present Banach space theory; and 17 of Banach's papers on functional analysis. LAS

HISTORY, P. *Gustav Herglotz, Gesammelte Schriften.* Hans Schwerdtfeger. Vandenhoeck & Ruprecht, 1979, xl + 652 pp, DM 128. [ISBN: 3-525-40720-3] This volume includes, in addition to Herglotz's mathematical papers, several eulogies by Herglotz and six essays by active mathematicians about the influence of Herglotz's work, which spans the period 1902-1953. JAS

HISTORY, P, L. *Henri Cartan: Oeuvres (Collected Works).* Ed: R. Remmert, J.-P. Serre. Springer-Verlag, 1979 [ISBN: 0-387-09189-0; 3-540-09189-0]. *V. I,* xxix + 538 pp; *V. II,* vi + 381 pp; *V. III,* vi + 548 pp, $125 set. The first two volumes contain all Cartan's papers on analytic functions, arranged chronologically; the third volume contains all further papers. The *Oeuvres* do not include the 1948-64 *Séminaires de l'Ecole Normal Supérieure.* Volume I includes a one page *curriculum vitae,* and a brief introduction to the papers written by Cartan himself (in 1973). Each volume begins with a frontispiece photograph of Cartan--in 1945, 1960, and 1975. LAS

HISTORY, P, L. *Selected Papers of Abraham Robinson.* Ed: H.J. Keisler, *et al.* Yale U Pr, 1979. *V. 1: Model Theory and Algebra,* xxxvii + 694 pp, $45 [ISBN: 0-300-02071-6]; *V. 2: Nonstandard Analysis and Philosophy,* xlv + 582 pp, $40 [ISBN: 0-300-02072-4]; *V. 3: Aeronautics,* xxxii + 270 pp, $30 [ISBN: 0-300-02073-2] This selection, covering Robinson's most important papers, documents a career of extraordinary breadth. Each volume begins with a superbly written biography by George Seligman that outlines Robinson's remarkable life and career; with a topical introduction by one of the editors; and with frontispiece photographs from 1941, 1970, and 1972. Each volume concludes with a complete bibliography, and a list of Robinson's doctoral students. LAS

HISTORY, P, L. *André Weil: Oeuvres Scientifiques (Collected Papers).* Springer-Verlag, 1979 [ISBN: 0-387-09330-5; 3-540-09330-5]. *V. I (1926-1951),* xviii + 574 pp; *V. II (1951-1964),* xii + 561 pp; *V. III (1964-1978),* xii + 465 pp, $85 set. Papers are arranged chronologically, each volume concluding with commentary by Weil on some of its contents. Frontispieces portray Weil in 1929, 1953, and 1977. LAS

COMBINATORICS, T(14-15: 1), S, L. *An Introduction to Computational Combinatorics.* E.S. Page, L.B. Wilson. Comp. Sci. Texts, No. 9. Cambridge U Pr, 1979, vi + 218 pp, $24.50; $8.50 (P). [ISBN: 0-521-22427-6; 0-521-29492-4] An introduction to those areas of combinatorics that are especially important for advanced computation. The techniques described in detail include formation and solution of difference equations, ordering and generation of elementary configurations, search procedures and selection algorithms. Includes many interesting and challenging exercises. CEC

NUMBER THEORY, T*(17: 2), S, P, L. *Rational Quadratic Forms.* J.W.S. Cassels. London Math. Soc. Mono., No. 13. Acad Pr, 1978, xvi + 413 pp, $36.25. [ISBN: 0-12-163260-1] A self-contained text which deals with quadratic forms over the rationals. The first half develops the theory of rational or integral coefficients. The second half treats spinor genera, reduction theory and automorphs, and the composition of binary forms. A solid introduction to the theory. Includes exercises and a substantial list of references. CEC

LINEAR ALGEBRA, T(16-17: 1, 2), S, L. *Compléments d'Algèbre Linéaire.* Léonce Lesieur, Roger Témam, Jean Lefebvre. Librarie Armand Colin, 1978, 379 pp, (P). Advanced linear algebra intended to follow the two books *Algèbre Générale* and *Algèbre linéaire et Géométrie* previously published in this series (Librarie Armand Colin--collection U). Good indexes and a significant collection of exercises (with solutions) as well as extensive problems provide a very substantial introduction to the usual material on canonical forms, duality, and quadratic forms. Also includes material on classical groups (for geometry) and some numerical methods. JAS

LINEAR ALGEBRA, T*(16-17: 1, 2), S, P, L*. *Linear Algebra and Geometry.* David M. Bloom. Cambridge U Pr, 1979, xiii + 625 pp, $34.50; $18.95 (P). [ISBN: 0-521-21959-0; 0-521-29324-3] An ambitious and innovative work which integrates linear algebra and higher analytic geometry. The author assumes that the student has already studied abstract algebra. A solid advanced course in linear algebra along with connections to affine and Euclidean geometry, isometries, quadric surfaces and other topics. Lots of good drawings and exercises. CEC

LINEAR ALGEBRA, T(16-18: 1), S, P, L. *Generalized Inverses of Linear Transformations.* S.L. Campbell, C.D. Meyer, Jr. Fearon-Pitman, 1979, xi + 272 pp, $42. [ISBN: 0-273-08422-4] Three kinds of generalized inverse (Moore-Penrose, (i,j,k), Drazin) for a matrix are developed, then each is used to solve several linear algebra problems for which it is particularly suitable. Applications are emphasized, and include problems in such areas as electrical engineering, statistics, Markov chains, and differential equations. Index, exercises, and extensive bibliography. JS

ALGEBRA, P. *Localisations and Grothendieck Categories.* L. Budach, R.-P. Holzapfel. VEB, 1975, 217 pp. A general treatment of those methods of commutative algebra and algebraic geometry which can be extended to Grothendieck categories. Focuses on geometric categories and extension and localization. JAS

ALGEBRA, T(15-16: 1), L. *Einführung in die Algebra.* Erich Lamprecht. Birkhäuser, 1978, ix + 270 pp, sFr. 16.80 (P). [ISBN: 3-7643-0943-1] Groups, rings, and fields together with computational linear algebra to provide an applications-oriented abstract algebra course. Lots of exercises, good subject and notation indexes make this an interesting candidate for a specialized course. It's brief but sophisticated. JAS

ALGEBRA, P. *Lecture Notes in Mathematics-697: Topics in Algebra.* Ed: M.F. Newman. Springer-Verlag, 1978, xi + 229 pp, $12.50 (P). [ISBN: 0-387-09103-3; 3-540-09103-3] Invited lectures and "splinter group talks" from the Eighteenth Summer Research Institute of the Australian Mathematical Society held in Canberra at the Australian National University on January 9-February 17, 1978. JAS

ALGEBRA, T(15-16: 1, 2), S, L. *Classical Galois Theory with Examples, Third Edition.* Lisl Gaal. Chelsea Pub, 1979, viii + 248 pp, $9.95. [ISBN: 0-8284-1268-5] A reissue "with various small improvements" of this classic text printed on "long-life" paper. (*First Edition,* TR, August/September 1971; *Second Edition,* TR, February 1974.) JAS

ALGEBRA, T(17: 1, 2), S, P, L. *Semigroups and Combinatorial Applications.* Gerard Lallement. Wiley, 1979, xi + 376 pp, $27.50. [ISBN: 0-471-04379-6] This advanced text presents those parts of the theory of semigroups that are directly related to automata theory, algebraic linguistics and combinatorics. Along with an introduction to the theory of algebraic semigroups, topics include languages, codes, decidability and algorithms, the star operation on languages, rational prefix codes and combinatorial questions related to the Burnside problem and MacMahon's master theorem. Many challenging exercises and an excellent list of references. CEC

ALGEBRA, P, L. *Von Neumann Regular Rings.* K.R. Goodearl. Fearon-Pitman, 1979, xvii + 369 pp, $46. [ISBN: 0-273-08400-3] A comprehensive development of the major ring-theoretic aspects of this popular class of rings. Introductory chapters on the fundamentals are followed by detailed studies of several classes of regular rings, e.g., abelian, unit-regular, directly finite, self-injective, continuous. Open problems. TRS

ALGEBRA, S(18), P. *Lecture Notes in Mathematics-716: The Theory of Lie Superalgebras, An Introduction.* M. Scheunert. Springer-Verlag, 1979, x + 271 pp, $14.30 (P). [ISBN: 0-387-09256-0; 3-540-09256-0] Lie superalgebras are not Lie algebras but are closely related in essence, and their present theory and development rests heavily on the classical Lie theory. The central theme here is the classification (due largely to V. Kac) of the simple Lie superalgebras (classical and non-classical), finite-dimensional over an algebraically closed field of characteristic zero. Some further discussion of related ideas; super-derivations, nilpotent, solvable, etc. Index, extensive bibliography. JS

ALGEBRA, T(17-18: 2), S, P, L. *Universal Algebra, Second Edition.* George Grätzer. Springer-Verlag, 1979, xviii + 581 pp, $29.80. [ISBN: 0-387-90355-0] The text proper is essentially that of the 1968 *First Edition* (TR, October 1969; ER, October 1970) but the book has been significantly updated and amplified by addition of seven appendices (143 pages) and a supplemental bibliography (58 pages). Appendices include discussion of problems posed in the *First Edition* as well as contributed articles on congruence varieties, equational logic, primality, and equational compactness. JS

ALGEBRA, T(18: 1), S, P, L. *Positively Ordered Semigroups.* M. Satyanarayana. Lect. Notes in Pure and Appl. Math., V. 42. Dekker, 1979, v + 102 pp, $12.50 (P). [ISBN: 0-8247-6810-8] A summary of some of the recent results in the theory of positively ordered semigroups, including some new results by the author. Chapters on elementary theory, totally ordered semigroups, naturally ordered semigroups and anomalous pairs. List of unsolved problems, bibliography, index. JS

FINITE MATHEMATICS, T(13: 1), S. *Mathematics for Business and Economics.* Rochid J. Elias. D. Van Nostrand, 1979, ix + 342 pp, $16.95. [ISBN: 0-442-21757-9] Reasonable coverage of mathematical techniques used in business and economics. Emphasis on techniques with minimal theory. Suitable for business programs that assume minimum mathematical background. Good economics examples, clear sketches. Suitable for its intended audience. WC

CALCULUS, T(13), L. *Basic Calculus with Applications.* Donald R. Williams, Thomas J. Woods. Wadsworth, 1979, xi + 548 pp, $16.95. [ISBN: 0-534-00685-X] Written for students of management, biology, behavioral and social sciences, the approach is intuitive. Omits trigonometric functions, but includes exponential functions. The authors use an extraordinary number of "real world" examples and exercises. The approach is very applications-oriented and appropriate for the intended audience. TAV

COMPLEX ANALYSIS, P**. *Quasiconformal Mappings and Riemann Surfaces.* Samuil L. Krushkal. V.H. Winston, 1979, xii + 319 pp, $24.95. [ISBN: 0-470-26695-3] Draws together recent work by the author and other Russian mathematicians, exploiting the connection between quasiconformal mappings and the theory of Riemann surfaces. The claim that the book begins at the beginning assumes a running start followed by quick acceleration. Extensive bibliography, no index. AWR

DIFFERENTIAL EQUATIONS, T(15-16), S. *Introduction to Dynamic Systems: Theory, Models, and Applications.* David G. Luenberger. Wiley, 1979, xiv + 446 pp, $19.95. [ISBN: 0-471-02594-1] The theory is tersely written, the models are all well known, most of the "classics" are here, and the applications are varied enough to stimulate any student's interest. This text blends the classical differential equation approach with the state space approach. Much of the last half is devoted to control theory. Problem sets fit in well. TLS

DIFFERENTIAL EQUATIONS, P. *Partial Differential Equations and Geometry.* Ed: Christopher I. Byrnes. Lect. Notes in Pure and Appl. Math., V. 48. Dekker, 1979, x + 323 pp, $35 (P). [ISBN: 0-8247-6775-6] Proceedings of a 1977 Park City, Utah, conference on partial differential equations and geometry--13 abstracts, 15 papers. For researchers in partial differential operators over real and complex manifolds and for those interested in the interconnections among potential theory, real and complex geometry, partial differential equations and stochastic processes. No. 48 in a distinguished series of lecture notes. Reproduced by direct photocopy of author's typewritten manuscript. JK

NUMERICAL ANALYSIS, T(15: 1), L. *Elementary Numerical Analysis.* W. Allen Smith. Har-Row, 1979, x + 470 pp, $16.95. [ISBN: 0-06-046312-0] A very nice presentation of the standard topics of an undergraduate course in numerical analysis. Presumes only elementary calculus and offers flexible use of computers and scientific calculators. Many worked examples and over 700 problems. Some Basic and Fortran programs appear in the appendix. TRS

FUNCTIONAL ANALYSIS, T(17-18: 2), S, L. *A Course of Applied Functional Analysis.* Arthur Wouk. Wiley, 1979, xvii + 443 pp, $24.95. [ISBN; 0-471-96238-4] An introductory text, distinctive for its diverse applications to numerical analysis, optimal control theory, and constrained optimization. Functional problems are attacked using approximate and interative methods such as: contractions, projections, steepest descent, Newton's methods, convexity, separation theorems, convex duality, and Kuhn-Tucker theory. Exercises. TRS

FUNCTIONAL ANALYSIS, T(17-18: 2), S. *Applied Functional Analysis: A First Course for Students of Mechanics and Engineering Science.* J. Tinsley Oden. P-H, 1979, xvi + 426 pp, $21.95. [ISBN: 0-13-040162-5] Developed from the author's courses for beginning graduate students, this text introduces abstract and linear algebra, real analysis, metric spaces, Banach and Hilbert spaces, and linear operator theory. Emphasizes mathematical concepts (not "tools") and provides abundant examples from mechanics. Exercises. TRS

FUNCTIONAL ANALYSIS, P. *Lecture Notes in Mathematics-696: Special Functions, Probability Semigroups, and Hamiltonian Flows.* Philip J. Feinsilver. Springer-Verlag, 1978, vi + 112 pp, $9 (P). [ISBN: 0-387-09100-9; 3-540-09100-9] The author introduces a generator corresponding to a process with "stationary independent increments." This construct arises in probability and also in many of the special functions from physics. A working paper, with many lines for further development. LCL

FUNCTIONAL ANALYSIS, T(17-18: 2). *Modern Methods in Topological Vector Spaces.* Albert Wilansky. McGraw, 1978, xiii + 298 pp, $34.50. [ISBN: 0-07-070180-6] An exceptionally well-written text, combining historical comments with current research questions. Stresses the role of duality. One nice feature is a constant reappraisal of major theorems, with an eye toward strengthening them as the material merits. Tables are provided outlining roles of basic properties. Problem sets are long and varied. TLS

FUNCTIONAL ANALYSIS, P**, *Equations of Evolution.* Hiroki Tanabe. Trans: N. Mugibayashi, H. Haneda. Fearon-Pitman, 1979, xii + 260 pp, $42. [ISBN: 0-273-01137-5] Prerequisites for this book in the area of differential equations in Banach spaces are said to be elementary linear operators in Banach spaces and distributions. A grasp of semigroups of operators and familiarity with elliptic boundary value problems would also help. Translated from Japanese. AWR

OPTIMIZATION, T(17-18: 2), P*, L*, *Theory of Extremal Problems.* A.D. Ioffe, V.M. Tihomirov. Stud. in Math. and Its Appl., V. 6. North-Holland, 1979, xii + 460 pp, $78. [ISBN: 0-444-85167-4] Translated from the 1974 Russian edition. For practitioners of the theory of extremal problems. Concise, but clearly written, survey and unified exposition of fundamentals, including differential calculus in Banach spaces, convex analysis, necessary conditions, existence of solutions and sufficient conditions, with treatment of extremal problems in calculus of variations, optimal control, linear and nonlinear programming. Numerical methods are omitted. Some applications to geometrical optics, geometry and control theory. Over 100 problems--several solved in text--in calculus of variations including ancient ones in geometry and mechanics. Well over 300 items in extensive bibliography. Good index. High priced. JK

ANALYSIS, P, *Lecture Notes in Mathematics-695: Measure Theory Applications to Stochastic Analysis.* Ed: G. Kallianpur, D. Kölzow. Springer-Verlag, 1978, xii + 261 pp, $14.30 (P). [ISBN: 0-387-09098-3; 3-540-09098-3] Proceedings of the Oberwolfach Conference, July 3-9, 1977. JAS

ANALYSIS, S(16-18), P, L, *Problems and Propositions in Analysis.* Gabriel Klambauer. Lect. Notes in Pure and Appl. Math., V. 49. Dekker, 1979, vii + 456 pp, $24.50 (P). [ISBN: 0-8247-6887-6] An impressive, diverse collection of problems "of an unusual and beautiful nature, designed to stimulate interest" taken from four areas: arithmetic and combinatorics (117), inequalities (115), sequences and series (152), real functions (115). Problems tend to be nontrivial, and each is followed immediately by a complete succinct solution, occasionally with remarks by the author. No index or bibliography.JS

ANALYSIS, P, *Harmonic Analysis in Euclidean Spaces.* Ed: Guido Weiss, Stephen Wainger. AMS, 1979. Proc. of Symp. in Pure Math., V. XXXV. *Part 1*, xxvi + 460 pp [ISBN: 0-8218-1436-2]; *Part 2*, vii + 438 pp, $44.40 set. [ISBN: 0-8218-1438-9] Papers from the 1978 AMS summer symposium held at Williams College. *Part 1* contains papers dealing with analysis in R^n (real, complex, and Hardy spaces); *Part 2* contains papers on analysis in other settings (several complex variables, differential operators, martingales, Lie groups, functional analysis). Begins with a dedication (with photograph and professional biography) to Nestor Rivière. LAS

ANALYSIS, P, *Lecture Notes in Mathematics-728: Non-Commutative Harmonic Analysis.* Ed: Jacques Carmona, Michèle Vergne. Springer-Verlag, 1979, 244 pp, $12.50 (P). [ISBN: 0-387-09516-0; 3-540-09516-0] Proceedings, with one exception, of a colloquium held at Marseille-Luminy in June 1978. LAS

ANALYSIS, P, *Applied Nonlinear Analysis.* Ed: V. Lakshmikantham. Acad Pr, 1979, xx + 726 pp, $38.50. [ISBN: 012-434180-2] Papers from an April 1978 conference at the University of Texas at Arlington, emphasizing reaction-diffusion equations, optimization theory, constructive techniques in numerical analysis, and applications to physical and life sciences. LAS

ANALYSIS, P, *Unitary Group Representations in Physics, Probability, and Number Theory.* George W. Mackey. Benjamin/Cummings, 1978, xiv + 402 pp, $19.50 (P); $31.50. [ISBN: 0-8053-6703-9; 0-8053-6702-0] A published version of the "Oxford Notes," lectures given at Oxford University in 1966-1967, plus several paragraphs of "Notes and References." These notes tend to supplement the Chicago notes (TR, May 1977); applications are heavily emphasized (necessary background material in quantum physics and advanced number theory is included). Mackey's Berkeley notes (summer 1965) are best seen as a condensed first draft. Detailed proofs are not given, except in the very beginning; however, much effort has been made to motivate theorems and explain their significance. LCL

DIFFERENTIAL GEOMETRY, S(18), P, *Non-Spherical Principal Series Representations of a Semisimple Lie Group.* Alfred Magnus. Memoirs No. 216. AMS, 1979, vi + 52 pp, $6.40 (P). [ISBN: 0-8218-2216-0] Necessary and sufficient conditions for a nonspherical principal series representation of a semisimple Lie group to be irreducible or cyclic are obtained and described "in terms of parameters determined by certain rank one subgroups of G. A sufficient condition for such a representation to be unitary is found, and the condition is shown to be necessary in the rank one case." Bibliography, no index. JS

GEOMETRY, L, *Geometry Problems My Students Have Written.* Ruth Carwell Kespohl. NCTM, 1979, viii + 87 pp, $5.80 (P). [ISBN: 0-87353-142-6] Imaginative and amusing problems with much more appeal than the usual textbook variety. Solutions included. JNC

TOPOLOGY, P, *Studies in Algebraic Topology.* Ed: Gian-Carlo Rota. Acad Pr, 1979, xi + 263 pp, $28. [ISBN: 0-12-599152-5] Nine papers with a mostly geometric orientation published as *Advances in Mathematics, Supplementary Studies,* Volume 5. JAS

TOPOLOGY, P, *Lehrbuch der Topologie.* W. Rinow. VEB, 1975, 724 pp. A no-nonsense abridged encyclopaedia of point set and algebraic topology (the latter mostly homology and cohomology theory). The content and style are classical; a sixteen page index and a nine page bibliography make this a potentially useful reference work for the essentials of topology. JAS

PROBABILITY, P, *Probability Measures on Locally Compact Groups.* Herbert Heyer. Ergebnisse der Math., B. 94. Springer-Verlag, 1977, x + 531 pp, $58. [ISBN: 0-387-08332-4; 3-540-08332-4] A presentation of a well-developed core of recent work growing out of a synthesis of probability and harmonic analysis. JAS

STATISTICS, P, *Dynamic Programming and Its Applications.* Ed: Martin L. Puterman. Acad Pr, 1978, xv + 410 pp, $19.50. [ISBN: 0-12-568150-X] Proceedings of the international conference held at the University of British Columbia on April 14-16, 1977. JAS

APPLICATIONS (ENGINEERING), P. *Integral Equation Methods in Potential Theory and Elastostatics.* M.A. Jaswon, G.T. Symm. Acad Pr, 1977, xiv + 287 pp, $25.50. [ISBN: 0-12-381050-7] The first part describes the theory of integral equations in potential theory and elastostatics, providing a unifying account of scalar and vector potential theory, while the second part presents numerical methods for solving the integral equation formulations. Examples drawn from electrostatics, potential fluid flow, heat conduction, and the stretching and bending of thin plates. TRS

APPLICATIONS (MODELLING), P. *Numere Aleatoare.* Ion Săcuiu, Dan Zorilescu. Editura Academiei (Romania), 1978, 251 pp, (P). Numerical generation of random numbers having a given distribution, with applications to numerical problems and modelling in economics, industry, and natural phenomena. JAS

APPLICATIONS (PHYSICS), S(17-18), L. *Gravitational Curvature: An Introduction to Einstein's Theory.* Theodore Frankel. Freeman, 1979, xviii + 172 pp, $8.95 (P); $18.50. [I SBN: 0-7167-1062-5] A high-level introduction for someone who has had a course or so in differential geometry: the machinery of forms and tensors is used without hesitation or (pictorial) exegesis. The result is a mathematically clean presentation of the central core of this classical application of differential geometry. It does not overwhelm the beginner with Penrose-Hawking singularities, connections, or other current research directions. JAS

APPLICATIONS (PHYSICS), T(13: 1), S*, L*. *General Relativity from A to B.* Robert Geroch. U of Chicago Pr, 1978, xi + 225 pp, $11.95. [ISBN: 0-226-28863-3] A beautiful novel whose hero is the universe, heroine, mathematical physics (disguised as Einstein), and the villain, Aristotle. A book to be read, enjoyed, and thought about like a novel. The mathematical technique is at the level of college algebra, but there is sufficient substance to make the book suitable for courses such as the liberal arts course at the University of Chicago out of which it grew. JAS

APPLICATIONS (PHYSICS), P. *Field Theory, the Renormalization Group, and Critical Phenomena.* Daniel J. Amit. McGraw, 1978, xiv + 336 pp, $23. [ISBN: 0-07-001575-9] Written by a statistical physicist to describe his "conversion to the very elegant and efficient new faith, with the intention of demystifying the formalism...", this book presents (without proofs) a survey of the syntheses currently being achieved between statistical physics and particle physics. JAS

APPLICATIONS (PHYSICS), T(18: 2), P, L. *Waves in Fluids.* James Lighthill. Cambridge U Pr, 1978, xv + 504 pp, $37.50. [ISBN: 0-521-21689-3] A comprehensive introduction to the science of waves in liquids and gases. Four substantial chapters on sound waves, one-dimensional waves in fluids, water waves, and internal waves. Includes applications to problems in noise abatement, circulatory physiology, hydraulics, oceanography and meteorology. Exercises. TRS

APPLICATIONS (PHYSICS), P. *A Computational Method in Plasma Physics.* Frances Bauer, Octavio Betancourt, Paul Garabedian. Springer-Verlag, 1978, viii + 144 pp, $14.80. [ISBN: 0-387-08833-4; 3-540-08833-4] Authors present a numerical method for computation and analysis of the equilibrium and stability of plasma in toroidal geometry. Includes a careful discussion of continuous and then discrete variational models and a complete listing of the Fortran program which implements the numerical method. TRS

APPLICATIONS (PHYSICS), T(17-18). *A Course in Mathematical Physics-1: Classical Dynamical Systems.* Walter Thirring. Trans: Evans M. Harrell. Springer-Verlag, 1978, xii + 258 pp, $19.80. [ISBN: 0-387-81496-5; 3-540-81496-5] First volume of a proposed four volume set. After a fast review of the necessary tools from manifold theory, the author concentrates on Hamiltonian systems, non-relativistic and relativistic motion and the structure of space and time. Many exercises, including solutions, as well as some unsolved problems. TLS

APPLICATIONS (PHYSICS), P. *Computational Fluid Dynamics.* Herbert B. Keller. SIAM-AMS Proc., V. XI. AMS, 1978, v + 177 pp, $18. [ISBN: 0-8218-1331-5] The proceedings (seven out of nine presentations) of the Symposium on Computational Fluid Dynamics held April 14-15, 1977 in New York City. JAS

APPLICATIONS (PHYSICS), P. *Many Degrees of Freedom in Field Theory.* Ed: L. Streit. Plenum Pr, 1978, vii + 248 pp, $27.50. [ISBN: 0-306-35730-5] This book constitutes Volume 30, Series B of the NATO Advanced Study Institutes Series and presents the proceedings of the part of the 1976 International Summer Institute of Theoretical Physics (University of Bielefeld, August 23 to September 4, 1976) which dealt with the topic of the title. The rest of the proceedings occur in Volume 31, *Many Degrees of Freedom in Particle Theory.* JAS

APPLICATIONS (PHYSICS), P. *New Frontiers in High-Energy Physics.* Ed: Behram Kursunoglu, *et al.* Stud. in Nat. Sci., V. 14. Plenum Pr, 1978, ix + 670 pp, $59.50. [ISBN: 0-306-40037-5] Proceedings of the 1978 *Orbis Scientiae* at the University of Miami in which "the editors are pleased to submit to the readers the state of the art in high energy physics as it appears at the beginning of 1978." JAS

APPLICATIONS (PHYSICS), P. *Lecture Notes in Mathematics-676: Differential Geometrical Methods in Mathematical Physics II.* Ed: K. Bleuler, H.R. Petry, A. Reetz. Springer-Verlag, 1978, vi + 626 pp, $27.50 (P). [ISBN: 0-387-08935-7; 3-540-08935-7] The proceedings of the 1977 conference in Bonn. Part I is *Lecture Notes in Mathematics-570* (TR, August/September 1978). JAS

Reviewers Whose Initials Appear Above

Richard J. Allen, St. Olaf; William Carlson, St. Olaf; Judith N. Cederberg, St. Olaf; Clifton E. Corzatt, St. Olaf; Joseph Konhauser, Macalester; Loren C. Larson, St. Olaf; A. Wayne Roberts, Macalester; Thomas R. Savage, St. Olaf; John Schue, Macalester; J. Arthur Seebach, Jr., St. Olaf; Lynn Arthur Steen, St. Olaf; Timothy L. Strotman, St. Olaf; T.A. Vessey, St. Olaf; Martha Wallace, St. Olaf.

NEWS AND NOTICES

EDITED BY PAUL A. HAEDER, University of Nebraska at Omaha

Readers are invited to contribute to the general interest of this department by sending news items to Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036

SUGGESTION BOX

Members of the MAA are encouraged to send in suggestions, questions, etc., about the operation of the Association. Communications will be referred to the appropriate officer of the Association for answering; from time to time, those of general interest may also be answered in one or both of the official journals. Communications should be addressed to: Suggestion Box, Mathematical Association of America, 1529 Eighteenth Street, N.W., Washington, D.C. 20036

PERSONAL ITEMS

Dr. Leila Bram, 52, died on September 7, 1979. She was a member of the Association for fourteen years. She was, in turn, Mathematician, Head of the Mathematics Bureau, and Head of the Mathematics Program of the Office of Naval Research for almost 25 years during the period 1951-79.

FACULTY EXCHANGE CENTER

The Faculty Exchange Center, a non-profit, faculty-administered program, helps to arrange college and university faculty exchanges on this continent and overseas where the language of instruction is English. The roster for the academic year 1979-80, released this fall, is made up of over 500 professors from all parts of the United States and from 23 foreign countries. More than 155 American and 60 foreign institutions of higher learning are now represented in the membership of the Center.

Upon registration, a faculty member will receive gratis a list of the members in his discipline as it appears in the fall 1979 directory. For more information send a stamped, self-addressed envelope to: Faculty Exchange Center, 952 Virginia Avenue, Lancaster, PA 17605.

COMBINATORICS CONFERENCE

The Eleventh Southeastern Conference on Combinatorics, Graph Theory and Computing will be held at Florida Atlantic University in Boca Raton, March 3-7, 1980. The Conference aims to promote better understanding of modern applied mathematics, combinatorics and computer science. This year, the instructional lecture series will be given by Professor Paul Erdos of the Hungarian Academy of Sciences, Dr. Lester R. Ford, Jr. of the General Research Corporation, Professor Charles C. Lindner of Auburn University and Professor Carsten Thomassen of Aarhus University.

This year's Conference will also feature special sessions, on the afternoons of March 5 and 6 and the morning of March 7, devoted to applications of combinatorics, graph theory and computing to transportation. These sessions will include survey talks by speakers from government and industry on problems dealing with network analysis, routing and design. Contributed fifteen-minute talks of either survey or original research nature are solicited for these sessions. Selected papers will be included in a special supplement to the Proceedings. It is anticipated that these sessions will be made possible by partial support for the Conference by the United States Department of Transportation through the Research and Special Projects Administration, Transportation Systems Center in Cambridge, Massachusetts. Further details on the Conference may be had by writing to Professor Frederick Hoffman, Department of Mathematics, Florida Atlantic University, Boca Raton, Florida 33431. The telephone is (305) 395-5100, Ext. 2746 and 2758.

SYMPOSIUM ON MULTIPLE-VALUED LOGIC

The International Symposium on Multiple-Valued Logic will hold its 10th annual conference on the scenic campus of Northwestern University. Meetings will be held at the Norris University Center overlooking Lake Michigan. The Symposium is co-sponsored by the IEEE Computer Society and Northwestern, and will be held June 3-5, 1980. You are invited to submit original survey, tutorial, or research papers on any subject in the area of multiple-valued logic including, but not exclusively limited to: *Algebraic and Formal Aspects, Circuit/Device Implementation, Fault Detection and Diagnosis, Fuzzy Logic, Logic Design and Switching Theory, Philosophic Aspects,* and *Probabilistic and Variable-Valued Systems.* For further information, contact: Dr. Jon T. Butler, Symposium Chairman, ISMVL-80, Department of Electrical Engineering and Computer Science, Northwestern University, Evanston, IL 60201. Telephone: (312) 492-5628.

[*CORRECTION*: An item printed on page 613, August-September, 1979, should have read, "Associate Professor and Chairman Edward W. Brande, S.J., Fordham University, has been appointed Professor of Mathematics and Academic Vice President, Saint Peter's College, Jersey City, New Jersey."]

EDUCATIONAL COMPUTING CONFERENCE

The Second National Educational Computing Conference (NECC/2) will be held June 23-25, 1980, at the Holiday Inn, Downtown at Scope, Norfolk, Virginia. It is aimed at providing a broad and rich forum for discussion between individuals, at all levels, and from all institutions with interests in educational computing. Building on the success of the first NECC at the University of Iowa in 1979, the planned sessions and related activities will be of benefit to both experienced computer users and new users and will stress the practical nature of such computer use.

As a cooperative venture undertaken by 15 professional organizations interested in educational computing, NECC/2 has at least the following four major objectives: (1) presentation in one forum of all major work regarding computers in education in the United States; (2) interaction between individuals involved at all levels in various aspects of computer uses in education; (3) development of better liaison and cooperation between the various professional groups involved with computer uses in education; and (4) production of a proceedings documenting the status of computers in education in the United States.

Authors are invited to submit papers describing actual experiences with computer use in the classroom or the consequences of such use upon the educational process in general. Papers submitted should report concrete results or be survey or tutorial papers which include a synthesis and thorough evaluation. Generally, papers that describe projects presented at previous conferences are not considered unless substantial new information can be reported, and in this case a brief synopsis of the earlier paper with clear indication of the new information should be provided.

In addition to the type of sessions that will normally come from contributed papers, there will be a limited number of special sessions involving invited speakers who will participate in several survey and tutorial sessions, as well as special panels. These sessions will focus on the current status of educational use of computers at the national level and projections for the future. Requests for additional information, or questions regarding the NECC/2 should be addressed to the General Chairman: Professor Gerald L. Engel, Computer Science Department, Christopher Newport College, Newport News, Virginia 23606. Phone: (804) 599-7065.

SYMPOSIUM ON TRANSITION AND TURBULENCE

The Mathematics Research Center at the University of Wisconsin-Madison will hold a conference on recent developments and trends in the study of transition and turbulence in fluids and their applications on October 13-15, 1980. The main program will consist of about 12 lectures, and the invited speakers include P. Bradshaw, G. L. Brown, F. Busse, R. C. DiPrima, M. Gaster, J. Laufer, H. W. Liepmann, J. T. C. Liu, J. C. Lumley, H. Maguib, S. Orszag, P. Saffman, J. T. Stuart, A. M. Yaglom. The conference will emphasize late transition phenomena and new directions in turbulence research and the relation between them. A detailed program will be available in May and further information may be obtained from Mrs. Gladys Moran, Mathematics Research Center, University of Wisconsin, 610 Walnut Street, Madison, Wisconsin 53706.

NRC RESEARCH PROGRAMS

The National Research Council is pleased to announce the NRC Research Associateship Programs for 1980. These programs provide opportunities for research in many fields of *Atmospheric and Earth Sciences, Chemistry, Engineering, Environmental Sciences, Life Sciences, Mathematics, Physics,* and *Space Sciences.*

Approximately 250 new awards will be made on a competitive basis to recent recipients of the doctorate and, in some programs, to senior investigators as well. Certain programs are open to non-U.S. nationals as well as to U.S. citizens. Stipends (subject to income tax) will be individually determined and will begin at $18,000 a year. Grants will be provided for family relocation and for limited professional travel during tenure. Postmark deadline for applications is *January 15, 1980.* Awards will be announced in April.

The NRC administers the Research Associateship Programs in cooperation with selected federal research organizations which have laboratories at about 65 geographic locations in the United States. Prospective applicants may request information and application forms from the *Associateship Office, JH 608-P, National Research Council, 2101 Constitution Avenue, NW, Washington, DC 20418. Telephone (202) 389-6554.*

VIDEOTAPED INTRODUCTORY FORTRAN COURSE

Color videotaped course comprised of ten 25-minute lectures designed to provide a working knowledge of FORTRAN and an introduction to the power of computer operating systems is now available from Colorado State University. Material covered is equivalent to a university course in programming, and encompasses flow charting, programming techniques and a repertoire of FORTRAN statements. Topics covered include: Arithmetic and logical operations; Various formats for reading and writing information; Loops; Predefined and user-defined subprograms; 1- and 2-dimensional arrays; and Attacking and debugging programs. A 107-page Study Guide supplements the course. The Lecturer is Director of Computer Utilization in the CSU College of Agricultural Sciences. For more information contact: W. L. Somervell, ERG Director, Christman Field, Building 1000, Colorado State University, Fort Collins Colorado 80523.

OHIO STATE ACTUARIAL SCIENCE PROGRAM

The Ohio State University, with financial support from the Griffith Foundation for Insurance Education, has established a number of courses on mathematical topics which support the first four actuarial examinations. These can be taken by graduate students while they are working on the M.S. or Ph.D. degree in mathematics, or can be taken in partial fulfillment of the requirements for an undergraduate major in actuarial science. The program is administered by the Department of Mathematics. For detailed information and application forms write to Robert B. Brown, Actuarial Science Program, The Ohio State University, Department of Mathematics, 231 West 18th Avenue, Columbus, Ohio 43210.

SIMULATION OF LARGE SYSTEMS

The Interdisciplinary Centre for Mathematization at the University of Bielefeld is arranging a survey conference on the 'simulation of large systems' in the summer of 1980. The main interest lies in the description of the macro-behaviour of large systems based on the micro-behaviour of their elements.

The conference is to be seen within the framework of a continued mathematical education for users of mathematics and also as a part of an organized exchange of experiences. Interested parties are invited to send us contributions for (A) non-technical case studies (1000 words) and/or (B) encyclopedic/survey articles (3000 words) which will be used in the preparation and the accomplishment of the conference, as well as in the edition of a newsletter on the same theme.

Languages accepted for the contributions are English, French and German; the deadline is 31 January, 1980. The best contributions will be selected by a jury and will be given awards as follows: (A) 500, 300 and 200 DM, (B) 1000, 600 and 400 DM. Please write for detailed information to: USP Mathematisierung, Universität Bielefeld, Postfach 8640, 4800 Bielefeld 1, West Germany.

BAIL I CONFERENCE

An international conference on *Boundary and Interior Layers—Computational and Asymptotic Methods* will be held in Dublin, Ireland from 3rd to 6th June, 1980 under the auspices of the Numerical Analysis Group. Invited speakers include A. Acrivos (Stanford), K. Aziz (Calgary), K. Coates (Houston), N. S. Bakhvalov (Moscow), A. E. Berger (Silver Spring), A. Brandt (Rehovot), T. Cebeci (Long Beach), P.Concus (Berkeley), J. E. Flaherty (Troy), D. K. Garling (Albuquerque), P. M. Gresho (Livermore), T.J.R. Hughes (Pasadena), A. Jameson (New York), R.B. Kellogg (College Park), J. L. Lions (Le Chesnay), M. Mimura (Kobe), K. W. Morton (Reading), J. D. Murray (Oxford), R. E. O'Malley (Tucson), S. A. Orszag (Cambridge), D. W. Peaceman (Houston), G.I. Shishkin (Sverdlovsk), R. L. Taylor (Berkeley), T. Todd (Houston), A. Veltman (Amsterdam).

Relevant topics include artificial viscosity, asymptotic expansions, boundary layers, continuation methods, convection-dominated flows, degeneration of the differential equation type, flows in porous media with small capillary pressure, fluid flows with large Reynolds number, free boundary techniques, internal layers, large and small parameters, plasma flow with large Hartman number, multigrid methods, shocks, singular perturbations, transition layers, upwinding.

Contributed papers are solicited from biologists, chemists, engineers, mathematicians, physicists and other scientists on computational or asymptotic methods for problems involving any of the above, or related, topics. The preliminary version of such a paper should be submitted as soon as possible, and in any case not later than 14th April, 1980. It *must* be accompanied by a separate one-page abstract. The final version of an accepted contribution should be delivered to the registration desk on the first day of the conference. It must be typed according to our instructions on special paper supplied by us and should be at most five pages long.

Communications should be addressed to the *Bail I Conference* at 39 Trinity College, Dublin 2, Ireland, telephone (01) 772941 ext. 189 or 1949, telex no. 5442 or 31166, telegraphic address TRINITY DUBLIN.

WASTE MANAGEMENT '80

The University of Arizona College of Engineering will conduct a symposium on March 10-14, 1980, at the Tucson Community Center, Tucson, Arizona.

The theme is embodied in the topics:

The State of Waste Disposal Technology
Mill Tailings
Risk Analysis Models

Details on costs, hotel reservations, etc., may be obtained from Dr. Roy G. Post, Nuclear Engineering Department, University of Arizona, Old Engineering Building, Tucson, Arizona 85721.

POSTDOCTORAL FELLOWSHIP

The University of Windsor, Department of Mathematics, Windsor, Ontario, Canada, announces a postdoctoral fellowship in the area of Linear Partial Differential Equations. The fellowship will extend from September 1, 1980, to August 31, 1981. The closing date for applications is January 31, 1980. Applicants should send a curriculum vitae, a list of publications and reprints (or photostatic copies), and arrange for letters of recommendation from two or three referees.

Applications should be sent to: Professor E. Kreyszig
Department of Mathematics,
University of Windsor,
Windsor, Ontario N9B 3P4,
CANADA

BERLIN CONFERENCE 1981

The University of Berlin is planning a conference of the Association for Mathematical Physics in (West) Berlin from August 26 to September 4, 1981. For further information please write to R. Schrader and R. Seiler, Institut für Theoretische Physik, Freie Universität Berlin, Arnimallee 3, 1000 Berlin 33.

MATHEMATICAL ASSOCIATION OF AMERICA

Official Reports and Communications

BENEFIT INCREASE ON MAA GROUP LIFE INSURANCE PLAN

Based on favorable experience, a 20% benefit increase was given at no additional charge to all members under age 60 and insured under our Group Life Insurance Plan as of October 1, 1979. This will increase the member coverage options available to a minimum of \$12,000 and a maximum of \$120,000, in multiples of \$12,000.

When an insured member attains age 60, each \$12,000 benefit unit reverts to a \$10,000 unit and further reduces to a \$5,000 unit on attainment of age 65 (coverage terminates at age 70).

Eligible members are able to choose for their spouses benefit options from \$5,000 to \$35,000, in multiples of \$5,000 (not to exceed 50% of member coverage) and each unmarried dependent child is able to be insured for up to \$2,500.

The Association does not incur any expense, nor does it derive any financial benefit in connection with the Group Life Insurance Plan. It has been made available solely as a service and membership benefit to our members. Other plans of insurance available as part of the MAA Group Insurance Program are Accidental Death and Dismemberment, Excess Major Medical, Disability Income, and In-Hospital.

For more information regarding the MAA Life Insurance Plan, as well as the other coverages in the MAA Group Insurance Program, contact the Administrator, MAA Group Insurance Program, 1707 L Street N.W., Washington, D.C. 20036 (Telephone (202) 296-8030).

INSTITUTIONAL REPRESENTATIVES

The MAA Institutional Representative at each college and university is the contact person between the Association and the faculty and students at the institution. The representatives encourage membership and active participation in section activity, publicize section meetings, and provide feedback to the Executive Board about the mathematical interest and concerns of their colleagues on campus.

The term of office for an institutional representative is three years, beginning the year after the election of the Section Governor. Representatives are nominated by the Chairman of their department and officially appointed by the Section Governor. Traditionally the incumbent is reappointed unless the Chairman nominates someone else. If the present representative is no longer at the institution, or is unable to continue in office, the Department Chairman should send the name of a new nominee to the Section Governor by March 1, 1980.

U.S. MATH OLYMPIAD TEAM

Eight of the nation's most brilliant high school mathematics students won positions on the U.S. team for the 21st International Mathematical Olympiad. Selection of the U.S. team began in March with the Annual High School Mathematics Contest. Over 360,000 students in the U.S. and Canada participated.

About 100 students with top contest scores were invited to compete in the Eighth U.S.A. Mathematical Olympiad, the final round of team selection. The eight winners went on to represent the U.S. at the International Olympiad on July 2 and 3 in London, England.

The members of the team were: Michael Finn, Annandale, Virginia; Bruce K. Smith, San Rafael, California; Michael Larsen, Lexington, Massachusetts; Lawrence E. Penn, Great Neck, New York; Ronald

F. Kaminsky, Albany, New York; Mark G. Pleszkoch, Manassas, Virginia; Randy L. Elk, North Huntington, Pennsylvania; Richard G. Agin, Chicago, Illinois.

This is the sixth U.S. team to enter the IMO. Since joining in 1974, the U.S. teams have consistently scored amongst the top three. In 1978 they placed second, and in 1977 they placed first in the competition.

Sponsors of the USAMO and the Annual High School Mathematics Contest are:

Mathematical Association of America (MAA)
Society of Actuaries (SA)
The National Council of Teachers of Mathematics (NCTM)
Mu Alpha Theta (MATH)
The Casualty Acturial Society (CAS)

SUMMER MEETING OF THE NORTHEASTERN SECTION

The summer meeting of the Northeastern Section was held at the University of Maine in Orono on June 22-23 at the conclusion of the Short Course on mathematical problems in biology. Chairman Donald B. Small presided. On Friday evening, June 22, Dr. Stanley J. Bezuszka, S.J., presented a talk entitled *Number Theory for Non-Mathematics Majors*. A pizza party followed hosted by the University of Maine Department of Mathematics. On Saturday morning the following talks were presented:

Math Anxiety, Jean Smith, Middlesex Community College

A Recursive Algebraic Model of a Generalized Information System, William G. Madison, Universal Systems, Inc., Arlington, Virginia

Self-Paced Precalculus, Nancy Myers, Bunker Hill Community College

Simple Tools and Subtle Results: Statistical Sleuthing in the Shaping of Public Policy, Arlene S. Ash, Boston University.

At the business meeting that followed, the regional chairpersons for the Annual High School Contest were recognized for their contribution to this activity. They are: Donald A. Gibbs (Connecticut), Alfred B. Harper (Maine), Owen B. Koppang (New Hampshire), J.E.H. Johansson (Vermont), Murray Abramson (Massachusetts/Rhode Island), and W.H.S. Crawford (Atlantic Provinces). Mr. Gibbs will be succeeded as chairperson for Connecticut by Mr. C. Douglas Henck. The other chairpersons will continue to serve for 1980.

A proposed change in the by-laws was presented and discussed. Under the proposal a representative from the two-year colleges, elected to a two year term at the November meeting in even-numbered years, will be added to the Executive Committee. Also, the term of the vice-chairman will be shortened to one year with election to this office occurring in even-numbered years. The duties of a vice-chairman will be performed in odd-numbered years by the immediate past chairman. Chairmen will be elected in odd-numbered years to serve a two year term of office.

ROGER L. COOKE, *Vice-Chairman*

MATHEMATICAL ASSOCIATION OF AMERICA

FIFTY-NINTH SUMMER MEETING OF THE ASSOCIATION

The Fifty-Ninth Summer Meeting of the Mathematical Association of America was held at the University of Minnesota, Duluth, from Tuesday, August 21, to Thursday, August 25, 1979, in conjunction with meetings of the American Mathematical Society, the Association for Women in Mathematics, and Pi Mu Epsilon. There were registered 833 persons, including 581 members of the MAA.

Sessions of the Association were held on Tuesday morning and afternoon, Wednesday morning and Thursday afternoon. These sessions were held in the Marshall Performing Arts Center and in Bohannon Hall, Room 90.

Presiding at the three Earle Raymond Hedrick Lectures by Professor Mary Ellen Rudin were President Dorothy L. Bernstein and First Vice-President Peter J. Hilton; at the lecture by Professor Thomas L. Saaty, Professor Duane Anderson; at the lecture by Professor Doris Schattschneider, Professor J. Arthur Seebach; at the lecture by Ms. Constance Reid, Professor Warren Loud; at the lecture by Professor Donald G. Saari, Professor Sabra Anderson; at the lecture by Professor Jay R. Goldman, Professor Sylvan Burgstahler; at the lecture by Professor John A. Wheeler, Professor Thomas A. Carnevale; at the lecture by Professor Edith H. Luchins, Professor James L. Nelson; at the lecture by Professor S. M. Ulam, Professor Harlan A. Hewitt; at the lecture by Professor D. H. Lehmer, Professor Seymour Schuster.

FIRST SESSION OF THE ASSOCIATION

The Earle Raymond Hedrick Lectures: "The Intersection of Set Theory and Topology", Lecture I, by Professor Mary Ellen Rudin, University of Wisconsin.

The first lecture was concerned with a general discussion of the kinds of mathematical problems which may be set theoretic in nature and thus may have one answer in one model of Zermelo-Frankel set theory and a different answer in another model.

"Priorities, Hierarchies, and Behavioral Systems", by Professor Thomas L. Saaty, University of Pennsylvania.

Hierarchical structures play a fundamental role in representing complex social, economic, political and other kinds of problems. To solve such problems with limited resources and with proper emphasis, we need to determine their priorities as a measure of their influence on the system in which they arise. The Analytic Hierarchy Process (AHP) determines the impact of the elements in a level on each element in the immediately adjacent upper level. Hierarchical impact is derived by a composition principle using the principal eigenvectors of pairwise comparison matrices which are positive and reciprocal. Inconsistency in the results is provided. Many applications of the theory have been made in physics and economics (for validation purposes), and in operations research. Examples of the latter applications have been to develop a transport energy to industries and money to projects according to priority.

"Tiling the Plane with Congruent Convex Pentagons", by Professor Doris Schattschneider, Moravian College.

A funny thing happened...Martin Gardner described to his readers a mathematician's "complete" list of types of convex pentagons which can tile the plane...and two amateurs proved that the list was incomplete. Accompanied by a slide presentation, the speaker gave the history of the problem of tiling the plane with congruent convex pentagons, the story of the newest discoveries, and the questions that remain unanswered.

SECOND SESSION OF THE ASSOCIATION

Hedrick Lecture II, by Professor Mary Ellen Rudin.

The second lecture presented a number of specific examples of abstract topological problems which have been solved recently through the recognition of their set theoretic nature and the development of interesting set theoretic tools. In the final lecture such a theorem was proved.

"The Answer To The Question I Am Always Asked", by Ms. Constance Reid.

Constance Reid, the author of Hilbert (Springer 1970) and Courant in Goettingen and New York (Springer 1976), told how she, a non-mathematician, came to write the life of the great mathematician David Hilbert.

"Geometry and Apportionment Problems", by Professor Donald G. Saari, Northwestern University.

Apportionment problems concern the division of goods among several parties. Examples include the division of a firm's profits among partners where the percentages of profits are determined by prior agreement, and the apportionment of Congressional representatives to states according to population figures. Complications arise when the distribution is to be made in discrete units, e.g. profits in units of $10,000 and states allotted integer number of representatives. For example, it may be to the financial advantage of two partners to add their own money to the firm before profit division. Also, by increasing house size, some state may lose a representative (Alabama paradox). The geometric reason for these paradoxes, which is in terms of a flow in a torus, was discussed. A division method avoiding these complications was described.

Panel Discussion: "Continuing Education of Two-Year College Mathematics Teachers: What Should the Graduate Schools Be Doing?"

A panel discussion with presentations by Professor Donald J. Albers, Menlo College, Professor John W. Jewett, Oklahoma State University and Professor James Werntz, University of Minnesota. Moderated by Professor Robert McKelvey, University of Montana.

Two-year and community colleges have become a major component of American post-secondary education, accounting for over 40% of all undergraduate enrollment. The full-time math faculty numbers 6000 and is young. If, as anticipated, enrollments do not grow, this group will constitute a stable dominant influence on mathematics in the two-year colleges for the next quarter century. Thus, continuing education for this group of teachers is of great significance. Recently the Rocky Mountain Mathematics Consortium conducted a study of the graduate training needs of TYC college mathematics teachers, based on an examination of their own views and perceptions. (The report is available from RMMC, c/o The University of Montana.) The study turned up interesting and sometimes surprising insights into the rapid changes that have occurred in the college environment, the teachers' professional lives, and their views on graduate education. The panel examined these turbulent changes, and their implications for graduate and continuing education.

THIRD SESSION OF THE ASSOCIATION

Hedrick Lecture III, by Professor Mary Ellen Rudin.

Business Meeting of the Association: Presentations of the Carl B. Allendoerfer, Lester R. Ford, and George Polya Awards.

"The Combinatorial Way of Thinking", by Professor Jay R. Goldman, University of Minnesota.

"Women and Mathematics: Fact and Fiction", by Professor Edith H. Luchins, Rensselaer Polytechnic Institute.

In attempts to separate fact from fiction, this talk surveyed data about women mathematicians, obtained mainly through questionnaires and interviews. It focused on these questions: What are the characteristics and interests of contemporary women mathematicians? Who are considered the leading women mathematicians? Is mathematics a young woman's game? What people and factors have encouraged or discouraged women, and have they been treated differently than men, as mathematics students and professionals? What role is played by math anxiety? The nature and implications of sex differences in mathematics were discussed and related to theories about difference in brain laterality and functioning.

FOURTH SESSION OF THE ASSOCIATION

"Some Mathematical Challenges of Einstein's Dynamic Geometry", by Professor John A. Wheeler, Univerity of Texas at Austin.

One area of challenge includes: (A-1) foliation by spacelike hypersurfaces of constant mean curvature, (A-2) stability with respect to choice of initial conditions, (A-3) singularities, (A-4) the general solution? (A-5) initial value data versus end point data for "steering spacetime". Still larger challenges are: (B-1) "the boundary of the boundary" in general relativity, (B-2) from harmonic map to general relativity, (B-3) the dynamics of geometry as propagation in superspace. Even deeper issues concern: (C-1) general relativity as "gauge" or "phase" theory, (C-2) supergravity as the square root of general relativity, (C-3) pre-geometry as the substratum of geometry.

"The Role of Mathematical Abstraction in the Physical Sciences", by Professor S. M. Ulam, University of Florida.

A number of recent examples of work in novel mathematical schemata suggested by problems in physics were discussed. A general formulation of branching processes involving generation of pairs (or triplets) of elements produced by pairs was sketched. Problems involving the morphology and asymptotic behavior of such systems were outlined.

"Some Recursive and Non-Recursive Algorithms and Their Applications to Combinatorics", by Professor Derrick H. Lehmer, University of California, Berkeley.

A set of combinatorial objects, for example the partitions of an integer subject to certain specifications, can be produced by the application of an algorithm. A recursive algorithm is one that, starting with an initial partition, produces the partitions one at a time each from one or more of its predecessors. The non-recursive algorithm is one that operates independently of any predecessor partition and produces the k-th partition simply as a function of k. For parallel machines the recursive or serial feature is a decided drawback. This is also true when one is making a random census of a combinatorial population too large to process in its entirety. Applications to the study of the symmetric group were given.

SPECIAL SESSIONS OF THE ASSOCIATION

Film showings were held in Marshall Performing Arts Center on Tuesday and Wednesday evenings. The following films were shown:

FIRST FILM PROGRAM: Tuesday

7:00-7:10 P.M.	Caroms-a film of the College Geometry Project
7:14-7:26 P.M.	Probability
7:30-7:55 P.M.	Modelling Pollution-a B.B.C. broadcast as part of the Open University's Foundation Course in Mathematics
8:00-8:09 P.M.	I Maximize-a film of the MAA Calculus Film Project
8:12-8:20 P.M.	Accidental Nuclear War
8:23-9:23 P.M.	Professor George Polya and Students

SECOND FILM PROGRAM: Wednesday

7:00-7:20 P.M.	Turning a Sphere Inside Out-a film of the Topology Film Project
7:23-7:35 P.M.	Flatland
7:38-7:57 P.M.	Sampling and Estimation, Inferential Statistics Part I
8:00-8:13 P.M.	Mathematics of the Honeycomb
8:15-8:40 P.M.	Iteration and Convergence-a B.B.C. broadcast as part of the Open University's Foundation Course in Mathematics
8:43-8:50 P.M.	Sierpinski's Curve Fills Space
8:54-9:20 P.M.	Donald in Mathmagic Land

MINI-COURSES

In order to help mathematicians meet the on-going need to stay creative while teaching undergraduate mathematics, the Association planned a series of mini-courses for the meeting in Duluth. The conviction that all undergraduate teachers of mathematics need to be exposed to new problems and to new teaching ideas led to this sequence of courses.

The sequence of mini-courses was open only to persons who had registered for the Joint Mathematics Meetings and paid the registration fee. The mini-courses on hand calculators had a separate registration fee of $15 and was limited to 30 participants. Calculators were provided to all participants for use during this mini-course.

The mini-courses offered in Duluth follow:

MOTIVATION AND ENRICHMENT
Tuesday, August 21

The mini-course featured talks by Professors Warren Page of New York Community College and John Niman of Hunter College. Professor Niman's presentations were entitled "Mathematics and Art" and "Mathematics and the Geoboard". Professor Page made presentations entitled "Some Alternate Modes of Instruction" and "Teaching Techniques that Enhance Creativity and Build Mathematical Muscle."

MATHEMATICAL MODELING
Wednesday, August 22

This mini-course featured talks by Professors William F. Lucas of Cornell University and Alan C. Tucker of SUNY at Stony Brook. There were also presentations on mathematical models from the collections of materials of UMAP and CUPM. Professor Lucas gave a talk entitled "Mathematical Modeling" and Professor Tucker gave a talk entitled "Problem Solving".

PRECALCULUS AND CALCULUS WITH HAND-HELD CALCULATORS
Thursday, August 23

Professor Harry P. Allen of Ohio State University presented the course. That institution has concluded the second year of a three-year NSF grant to develop a numerically-oriented calculus curriculum. The materials produced thus far provide a geometric approach which strongly supports the building and maintaining of intuition, while establishing standard calculus skills. Much of this is suitable for inclusion in precalculus courses.

The workshop covered an introduction to programming the hand-held calculators (no previous experience required). Participants were provided with TI-58 calculators for use during the workshop. The work-discussion topics included graphing, error estimates, and the definite integral.

MEETING OF THE BOARD OF GOVERNORS

The Board of Governors met at 9:00 A.M. on Monday, August 20, in the Ballroom of the Kirby Student Center. Among the items of business transacted were the following:

The Board of Governors elected Professor Paul R. Halmos of Indiana University as Editor of the MONTHLY for the term 1982-86.

The Board elected Professor Larry S. Haw of Nicholls State University as Governor of the Louisiana-Mississippi Section. Professor Haw will fill the unexpired portion of the term of Professor Thomas A. Atchison as Section Governor.

It was voted to establish the Committee on Teaching Undergraduate Mathematics as a standing committee of the Association. This committee is responsible for having produced "College Mathematics: Suggestions on How To Teach It" and will next undertake several additional projects directed toward improvements in teaching.

Tentative approval was given to publication of an Association newsletter. It is planned to initiate this newsletter in January, 1981.

The Board approved the revised By-Laws of the Intermountain and Texas Sections.

The Committee on Visiting Lecturers and Consultants presented its annual report and listed 87 visits by lecturers and 11 visits by consultants.

The Board approved cooperation with the Undergraduate Mathematics and Its Applications Project (UMAP) in publication of a journal of modules.

The revised Guidelines for Evaluation of Collegiate Mathematics Programs were approved. These Guidelines are to be distributed to college presidents, mathematics department chairmen, and accrediting agencies.

The Board of Governors approved the position of Associate Executive Director. This person will be employed full-time in the headquarters of the Association and will share responsibilities for office management, member services, computerization, fund-raising, and public relations. A committee will be appointed to write a position description and conduct a nationwide search.

The Board voted to accept the following grants:

1. Grants of $8 750 and $10 860 from the Exxon Foundation for support of a junior high school lectureship program to encourage Black students to complete the high school mathematics curriculum.

2. A grant of $9 500 from IBM for support of the 1979 Olympiad Awards Ceremony.

3. A grant of $37 500 from the Sloan Foundation for support of CUPM's Panel on a General Mathematical Sciences Program. This grant is for support of the CUPM Panel chaired by Professor Alan Tucker and Professor Tucker is due thanks for having arranged this grant.

4. A grant of $133 186 from NSF for support of the International Mathematical Olympiad in the United States in 1981.

BUSINESS MEETING OF THE ASSOCIATION

President Dorothy L. Bernstein presented the third annual set of Carl B. Allendoerfer Awards, the fourteenth set of Lester R. Ford Awards, and the third set of George Polya Awards. Present to accept these awards were Bruce C. Berndt, Richard L. Francis, Ned Glick, Kenneth I. Gross, Richard H. Plagge, and Doris Schattschneider.

The Secretary next presented his report and detailed some of the actions taken by the Board of Governors.

The Secretary thanked the Program Committee for having arranged such an excellent series of presentations. He also called attention to the series of mini-courses and invited comments by the participants. He said that he hoped the mini-courses would be well received and could be made part of all national meetings.

The Secretary also thanked Professor Mary Ellen Rudin for her fine series of lectures. He noted that these lectures had been very well attended and a highlight of the entire series of Hedrick Lectures.

Finally, the Secretary called attention to the excellent arrangements for the Duluth meeting. He said that the committee charged with these arrangements --especially the Co-Chairmen, Sylvan Burgstahler and James Nelson--had done a model job.

MEETING OF SECTION OFFICERS

The meeting of Section Officers was held on Tuesday, August 21, at 7:30 P.M. in the Ballroom of the Kirby Student Center. Dean Lester H. Lange presided. Fifty-four persons were present representing twenty-eight of the twenty-nine Sections. Members of the Committtee on Sections present were Professors David Ballew, James C. Bradford, Louis A. Guillou, John D. Neff, Alfred B. Willcox, and Dean Lange.

Dean Lange first introduced MAA President Bernstein who welcomed the Section Officers. President Bernstein said that she and all of the other officers of the Association are grateful to the Section officers for their good work on behalf of the Association.

Dean Lange said that the theme of the meeting was "what constitutes a good Section." He said that he had arranged for Past-Presidents Henry L. Alder and Henry O. Pollak and MAA Treasurer Leonard Gillman to give talks on this theme. He first called on Professor Alder who addressed the meeting as follows:

"To talk about a good Section is very similar to talking about a good teacher. It is very easy to tell when one sees one, but it is much harder to describe one. After having had the privilege of visiting in the last few years 17 of the 29 Sections of the Association, I would find it very easy to assign to every one of the Sections a grade ranging from A to F, but to describe what a good Section is, is somewhat more difficult, especially in five minutes.

Clearly there are many features which make a Section a good Section, and it is obviously not possible to list them all in a few minutes. Let me, therefore, confine my remarks to some observations resulting from my visits to Sections and list only four features which I believe are the prime ingredients guaranteeing that a Section is a good Section.

1. In a good Section, there is always strong leadership.

2. In a good Section, there is equal participation in all its activities by all constituents of the Section, mathematicians from Ph.D.-granting institutions, four-year colleges, two-year colleges, and industry and government.

3. A good Section has good meetings.

4. A good Section engages in a number of activities to advance the interests of mathematics within its region.

This, then, is one person's perspective as to what a good Section is. I realize, of course, that there is much more to it than that, and I hope that this will be brought out by the other speakers."

Leonard Gillman said that he wanted most to emphasize the need to fully involve persons from Ph.D.-granting institutions in the Section meetings. He pointed out that the University of Texas, Texas A & M, and Texas Tech are represented at meetings of the Texas Section. Professor Gillman said that, while he was not advocating that the programs of Section meetings should be research-oriented, the programs might be aided if persons from the larger institutions were involved in the planning.

As an example of university--college cooperation, Professor Gillman cited the program of the University of Texas whereby faculty positions are set aside for visiting faculty members from Texas. Persons who fill these posts need not have sabbatical leaves from their home institutions. Rather, they teach and are paid by the University of Texas.

Dr. Henry O. Pollak said that the diversity of the Sections was very enjoyable. He noted that one such diversity is the amount of other mathematical activity in the area. For example, some Sections have good and lively sessions of research papers while such an activity might not make sense for other Sections.

Dr. Pollak encouraged the Sections to discuss curriculum innovations and innovations in pedagogy, exposition, employment, and applications. He said that persons who are industrially employed are resources for applications and should not be overlooked by the Sections. Dr. Pollak also said that, in his opinion, far too few universities take seriously the need to train their graduate students to be good teachers of mathematics. He urged the Sections to take the lead in encouraging such training by having part of their programs devoted to this topic.

Dr. Pollak said that cooperative meetings with groups affiliated with NCTM is a good idea. This leads to discussions of topics in the high school--college interface. He also urged the Sections to include presentations by students in their programs and praised recent efforts of the Sections in initiating workshops and summer institutes.

Professor Samuel W. Hahn of the Committee on Sections reported on the summary of the annual report of MAA Sections. The summary lists each Section's size, number of meetings, attendance, special activities, certain financial data, and perceived strengths and weaknesses. The information for the summary was gathered by polling the Section Secretaries and the summary is available upon request from Dr. Alfred B. Willcox, 1529 Eighteenth St., N.W., Washington, D.C. 20036.

Professor Hahn noted that the number of Section Newsletters has grown from two in 1975 to twenty-two in 1980. Eight Sections mail no other materials to the members of the Section.

Approximately 4000 persons attended Section meetings during the past year. Eleven Sections reported having had book exhibits at their meetings and several charge the exhibitors and/or permit the exhibitors to sponsor social gatherings.

President Bernstein announced that MAA texts may now be sold at a 20% discount at Section meetings. She noted that Professor Kenneth R. Rebman of the Northern California Section organized a very successful booksale by announcing this policy and simultaneously distributing photo-reduced copies of the table of contents of several volumes published by the Association. Alfred B. Willcox, MAA Executive Director, said that his office would prepare brochures and would make order forms available for distribution at Section meetings. Professor Donald O. Koehler, Ohio Section, asked if there might be instead a 15% discount and a 5% rebate to Sections.

Dean Lange reported on the binder, "Handbook for Section Officers." Copies of this Handbook are available in all Sections, he said. Dean Lange also encouraged the Section Officers and any members of the Association to write to him or to Dr. Willcox concerning possible ways the Association might aid the Sections.

Professor Frederick Hoffman, Florida Section, asked for a report on workshop activities of the Sections. This led to a suggestion by Professor Donald O. Koehler that there be a swap session at the San Antonio meeting in January, 1980, on short courses and student participation. It was later suggested that there be a similar session on Section newsletters.

Elaine L. Tatham, Director of Institutional Research at Johnson County Community Junior College and Chairman-Elect of the Kansas Section, urged the Section Officers to help get students more involved in science fairs and Section meetings.

Professor Gillman encouraged the Section Officers to apply to the Fund for Aid to Sections for grants to the Sections. Typical grants are $300 and the money is available, he said.

Professor Jon Laible, Illinois Section, inquired about national collection of Section dues. There followed a discussion at which time the reasons for the Committee on Sections having voted against this proposal in 1978 were reviewed.

Professor Don H. Tucker of the Intermountain Section suggested a reduced dues schedule for secondary school teachers. He pointed out that, for most high school teachers, the Association is not a professional organization of primary importance.

The Kansas Section is beginning a project to write its history. A brief history will be distributed at the next meeting and members will be asked to make additions.

The quality of appearance of Section newsletters and programs was discussed. The importance of an attractive program was emphasized, but it was also noted that professional layout and printing can run into considerable expense.

Ways of identifying suitable speakers for the Section meetings were discussed. Professor Howard E. Bell, Seaway Section, said that the program is a topic of discussion at the annual meeting of MAA Departmental Representatives.

Professor Donald B. Small, Northeastern Section, reviewed his Section's efforts to more fully involve two-year college faculty. In particular, he said that these faculty had been invited to be guests of the Section at one of its meetings.

Professor J. Myron Hood, Southern California Section, reported on his Section's successful series of luncheon talks. Professor Koehler said that the Ohio Section had sponsored a special session of fifteen minute talks on geometry and is planning a similar session on operator theory.

DINNER IN HONOR OF TWENTY-FIVE YEAR MEMBERS

On Wednesday, August 22, there was a buffet dinner in the Moorish Room of the Hotel Duluth for those who have been members of the Association for twenty-five years or more. The dinner was preceded by a social hour and was followed by a program emceed by G. Baley Price. Greetings were brought to the group by President Dorothy L. Bernstein and Professors Murray S. Klamkin and Andrew M. Gleason were speakers. The titles of these presentations were: "From the USA Mathematical Olympiad to the William Lowell Putnam Competition" and "The Putnam Exam - Writing a Solutions Book."

ARRANGEMENTS, ENTERTAINMENT AND RECREATION

The Committee on Arrangements consisted of: Duane E. Anderson, Sabra S. Anderson, Paul T. Bateman (ex-officio), Sylvan Burgstahler (Co-Chairman), Francis G. Florey, Louis M. Friedler, Joseph A. Gallian, William J. LeVeque (ex-officio), William R. McEwen, James L. Nelson (Co-Chairman), and David P. Roselle, (ex-officio).

Registration headquarters and exhibits were in the Tweed Museum of Art. The Mathematical Sciences Employment Register operated on an informal basis throughout the meeting.

On Thursday evening, there was a walleye pike dinner at Spirit Mountain.

David P. Roselle, Secretary

ACADEMIC AND CORPORATE MEMBERS ELECTED BY THE ASSOCIATION

At its meeting in Duluth on August 20, 1979, the Board of Governors elected as institutional members the following:

Addison-Wesley Publishing Company
Arthur D. Little, Inc.
Colby College
Loyola University of Chicago
Missouri Southern State College
Northwest Community College
Pergamon Press, Inc.
Quinnipiac College
Seminole Community College
St. Mary's College
Texas Christian University
U. S. Coast Guard Academy
W. B. Saunders Company

David P. Roselle, Secretary

ACKNOWLEDGMENT

The editors wish to thank the following individuals who have refereed manuscripts between mid-1978 and mid-1979: Alexander Abian, J. D. Aczél, Henry Africk, Calvin Ahlbrandt, H. L. Alder, E. Z. Andalafte, G. E. Andrews, T. M. Apostol, J. M. Ash, R. A. Askey, Bernice Auslander, R. G. Ayoub, Bernhard Banaschewski, G. P. Barker, D. R. Barr, K. J. Barwise, P. T. Bateman, J. V. Baxley, Edward Bender, Adi Ben-Israel, D. C. Benson, Sterling Berberian, Bruce Berndt, Dorothy L. Bernstein, Garrett Birkhoff, David Blackwell, Mary L. Boas, W. M. Boothby, W. E. Boyce, Fred Brauer, P. J. Browne, A. M. Bruckner, Ellen F. Buck, R. C. Buck, Peter Bullen, R. T. Bumby, Robert Bumcrot, H. S. Butts, F. P. Callahan, G. T. Cargo, Leonard Carlitz, B. C. Carlson, David Carlson, Gulbank Chakerian, A. H. Clifford, J. E. Cochran, Earl Coddington, Robert Connelly, Jr., Howard Conner, William Connett, P. C. Consul, Mary Jane Cowles, H. S. M. Coxeter, Michael Crandall, H. T. Croft, D. W. Crowe, Frederic Cunningham, Jr., Arlene Kaplan Daniels, R. A. Dean, C. W. R. deBoor, Allen Devinatz, Michael Doob, David Drasin, Vladimir Drobot, Lester Dubins, Underwood Dudley, R. J. Duffin, R. A. Duke, B. C. Eaves, W. F. Eberlein, Albert Edrei, P. D. T. A. Elliott, M. J. Evans, R. J. Evans, Vance Faber, E. R. Fadell, Werner Fenchel, Daniel Finkbeiner, C. N. Fischer, Stephen Fisher, Harley Flanders, W. H. Fleming, Frank Forelli, R. B. Frary, Shmuel Friedland, Eric Friedlander, Laszlo Fuchs, P. R. Garabedian, A. O. Garder, Jr., Leon Gerber, P. M. Gibson, Robert Gilmer, Herman Gluck, H. J. Godwin, Casper Goffman, Michael Goldberg, Colin Graham, Ronald Graham, Branko Grünbaum, H. W. Guggenheimer, Hiroshi Gunji, J. K. Hale, K. B. Hannsgen, Louise Hay, H. T. Hayslett, Jr., Leon Henkin, Peter Henrici, Melvin Henriksen, I. N. Herstein, Edwin Hewitt, Denis Higgs, F. B. Hildebrand, B. P. Hillam, Charles Hilland, P. J. Hilton, Neil Hindman, J.-B. Hiriart-Urruty, R. E. Hodel, J. M. Horvath, Karel Hrbacek, P. D. Humke, R. A. Hunt, Stan Hurst, I. M. Isaacs, J. R. Isbell, B. Jacobson, E. C. Johnsen, Arnold Johnson, E. W. Johnson, Daniel Kahn, Ehud Kalai, Wilfred Kaplan, J. L. Kelley, Paul Kelly, Murray Klamkin, V. L. Klee, D. J. Kleitman, Morris Kline, Stuart Klugman, F. T. Kocher, Anton Kotzig, H. F. Kreimer, Kenneth Kunen, Ray Kunze, T. G. Kurtz, J. F. Ladik, Peter Lancaster, Lawrence Landweber, D. C. Lantz, Katherine P. Layton, D. H. Lehmer, Emma Lehmer, Walter Leighton, W. J. LeVeque, Lawrence Levy, C. T. Long, O. P. Lossers, Jr., Edith Luchins, Yudell Luke, S. S. Magill, M. B. Marcus, Morris Marden, J. M. Masley, J. G. Mauldon, J. E. Mazo, R. J. McEliece, Gary Meisters, N. C. Metropolis, R. S. Millman, E. C. Milner, Mabel D. Montgomery, K. R. Mount, R. C. Mullin, James Munkres, J. A. Murthy, Alex Nagel, C. H. Nevison, Morris Newman, Peter Ney, A. Nijenhuis, Ivan Niven, E. C. Nummela, C. O. Oakley, Stanley Ogilvy, J. M. H. Olmsted, Robert Osserman, J. C. Oxtoby, R. S. Palais, S. E. Payne, Jean J. Pedersen, F. R. Peterson, R. R. Phelps, Mark Pinsky, Eileen L. Poiani, M. W. Pownall, J. J. Price, M. H. Protter, J. S. Provan, L. B. Rall, R. A. Rankin, Ray Redheffer, A. C. Reynolds, Jr., John Riordan, Joel Robbin, D. P. Roselle, Alex Rosenberg, Ira Rosenholtz, Joseph Rosenstein, Joseph Rotman, H. L. Royden, Walter Rudin, D. G. Saari, Judith D. Sally, D. H. Saracino, Doris Schattschneider, I. J. Schoenberg, A. H. Schoenfeld, D. R. Scholz, Abraham Schwartz, W. R. Scott, S. M. Shah, V. L. Shapiro, M. C. Sholander, T. S. Shores, E. E. Shult, Richard Sinkhorn, N. J. A. Sloane, Malcolm Smiley, D. A. Smith, Ernst Snapper, L. J. Snell, J. G. Stampfli, R. P. Stanley, Andrew Sterrett, Gilbert Strang, C. J. Stuth, Ronald Sverdlove, M. H. Taibleson, L. F. Takacs, P. D. Taylor, Selden Trimble, T. W. Tucker, R. E. Turner, Dietrich Uhlenbrock, Peter Ungar, W. R. Utz, F. A. Valentine, H. Wegner, C. E. Weil, H. F. Weinberger, Allen Weitsman, L. R. Welch, R. R. Welland, J. G. Wendel, John Wermer, Dick Wheeden, R. J. Whitley, D. V. Widder, Albert Wilansky, R. L. Wilder, H. S. Wilf, D. J. Winter, Christoph Witzgall, Clarence Wylie, G. S. Young, Wo-sang Young, Sandy Zabell, Lawrence Zalcman, Hans Zassenhaus, Martha Zelinka, D. Zissos.

CALENDAR OF FUTURE MEETINGS

Sixty-third Annual Meeting, San Antonio, Texas, January 5–7, 1980.

Sixtieth Summer Meeting, University of Michigan, Ann Arbor, August 18–20, 1980.

The following is a list of the Sections of the Association with dates of future meetings so far as they have been reported to the Editorial Director.

ALLEGHENY MOUNTAIN, West Virginia Wesleyan College, Buckhannon, April 25–26, 1980.

EASTERN PENNSYLVANIA AND DELAWARE, Saturday before Thanksgiving.

FLORIDA, Jacksonville University, Jacksonville, March 7–8, 1980.

ILLINOIS, John A. Logan College, Carterville, April 25–26, 1980.

INDIANA

INTERMOUNTAIN, Utah State University, Logan, late April or early May 1980.

IOWA, Simpson College, Indianola, April 18–19, 1980.

KANSAS, March or April. Deadline for papers January 1.

KENTUCKY, Western Kentucky University, Bowling Green, April 11–12, 1980.

LOUISIANA–MISSISSIPPI, Louisiana Tech University, Ruston, February 15–16, 1980.

MARYLAND–DISTRICT OF COLUMBIA–VIRGINIA, Saturday before Thanksgiving and last Saturday in April.

METROPOLITAN NEW YORK, Mercy College, Dobbs Ferry, May 4, 1980.

MICHIGAN, Hope College, Holland, May 2–3, 1980.

MISSOURI, late March/early April. Deadline for papers January 31.

NEBRASKA, April.

NEW JERSEY, early November and early May.

NORTH CENTRAL, end of October and April. Deadline for papers October 1 and April 1.

NORTHEASTERN, Saturday after Thanksgiving, and third week in June in odd-numbered years.

NORTHERN CALIFORNIA, Naval Postgraduate School, Monterey, February 23, 1980.

OHIO

OKLAHOMA–ARKANSAS, Westark Community College, Fort Smith, Arkansas, March 28–29, 1980.

PACIFIC NORTHWEST, Central Washington University, Ellensburg, Washington, June 20–21, 1980.

ROCKY MOUNTAIN, University of Colorado, Boulder, March 28–29, 1980.

SEAWAY, first Saturday in November and Saturday in late April. Deadline for papers 6 weeks before meeting.

SOUTHEASTERN, Appalachian State University, Boone, North Carolina, April 11–12, 1980.

SOUTHERN CALIFORNIA, first or second Saturday in March.

SOUTHWESTERN, usually in April. Deadline for papers 2 weeks before meeting.

TEXAS, East Texas State University, Commerce, April 4–5, 1980.

WISCONSIN, University of Wisconsin, Milwaukee, March 28–29, 1980.

FUTURE MEETINGS OF OTHER ORGANIZATIONS

AMERICAN ASSOCIATION FOR THE ADVANCEMENT OF SCIENCE, San Francisco, January 3–8, 1980.

AMERICAN MATHEMATICAL ASSOCIATION OF TWO YEAR COLLEGES

AMERICAN MATHEMATICAL SOCIETY, San Antonio, Texas, January 3–6, 1980.

AMERICAN SOCIETY FOR ENGINEERING EDUCATION, Amherst, Massachusetts, June 23–26, 1980.

ASSOCIATION FOR COMPUTING MACHINERY, Kansas City, Missouri, February 12–14, 1980.

ASSOCIATION FOR SYMBOLIC LOGIC, New York City, December 28–29, 1980.

ASSOCIATION FOR WOMEN IN MATHEMATICS, San Antonio, Texas, January 3–7, 1980.

CANADIAN SOCIETY FOR HISTORY AND PHILOSOPHY OF MATHEMATICS/SOCIÉTÉ CANADIENNE D'HISTOIRE ET DE PHILOSOPHIE DES MATHÉMATIQUES

FIBONACCI ASSOCIATION

INSTITUTE OF MATHEMATICAL STATISTICS

MU ALPHA THETA

NATIONAL COUNCIL OF TEACHERS OF MATHEMATICS, Seattle, Washington, April 16–19, 1980.

OPERATIONS RESEARCH SOCIETY OF AMERICA, Shoreham Hotel, Washington, D.C., May 5–7, 1980.

PI MU EPSILON

SCHOOL SCIENCE AND MATHEMATICS ASSOCIATION

SOCIETY FOR INDUSTRIAL AND APPLIED MATHEMATICS

Meetings of Its Sections

ERRATA

Vol. 85

p. 716, lines 6 and 7 should read

$$\int_{-\infty}^{\log c} e^x dx + c|\log c|, \text{ i.e.,}$$

$$c + c|\log c|, \text{ i.e., } c(1-\log c).$$

line 26 should read

$$\lim_{x\to 0} \frac{1+\log\dfrac{1}{x+e}}{-x}$$

p. 717, line 4 should read

$$\frac{f(e)-f(e+x)}{-x}$$

line 5. Read "$x \to e$" for "$x \to 0$."

p. 724. Delete the second paragraph (beginning "If Riesz . . .").

Vol. 86

p. 33, lines 3–4. See this volume, p. 571, for a correction.
p. 248, line 9, for "had been a lawyer" read "had studied law."
p. 249, line 16, for "Australia," read "Great Britain."
line 6 from below, for "Russia" read "Poland."
p. 390, the formula displayed on line 11 should read
p. 593, E 1243. The Comment is by Ralf Winkel ("a" not "o").

$$f'(a) \approx \frac{1}{12h}[f(a-2h)-8f(a-h)+8f(a+h)-f(a+2h)].$$

THE GEHMAN LECTURES

Request for Contributions

In 1968, the Seaway Section of the MAA initiated the "Gehman Lecture Series" in honor of Harry M. Gehman who faithfully served the Association for many years as Secretary, Treasurer, and Executive Director. The Gehman Lecture is given annually by invitation at the spring meeting of the Seaway Section. Periodic publication of these lectures is planned.

The Seaway Section needs help in defraying the expenses of producing the first volume of Gehman Lectures. Friends and admirers of Professor Gehman are urged to send donations for this purpose, with checks payable to the "Gehman Lecture Fund," to

K. D. Magill
Chairman, Gehman Lecture Fund
Mathematics Department
SUNY at Buffalo
Buffalo, NY 14214

THE CARUS MATHEMATICAL MONOGRAPHS

The Monographs are a series of expository books intended to make topics in pure and applied mathematics accessible to teachers and students of mathematics and also to non-specialists and scientific workers in other fields.

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2. *Analytic Functions of a Complex Variable,* by D. R. Curtiss.
3. *Mathematical Statistics,* by H. L. Rietz.
4. *Projective Geometry,* by J. W. Young.
6. *Fourier Series and Orthogonal Polynomials,* by Dunham Jackson.
8. *Rings and Ideals,* by N. H. McCoy.
9. *The Theory of Algebraic Numbers* (Second edition), by Harry Pollard and Harold G. Diamond.
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